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Reidemeister's theorem for 3-manifolds

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The Reidemeister theorem describes the equivalence of links in terms of diagrams (this theorem was proven in detail by Alexander [1]). A diagram for a link in S^3 can be found by projecting the link to any disc recording the over or under crossings. It was shown that two links are equivalent if and only if their diagrams are related by a sequence of so called Reidemeister moves and isotopy in the disc.

This arbitrary projection is not possible in more general 3-manifolds. However, a theorem proven by Alexander [2] states that any closed orientable 3-manifold contains a link whose complement fibres over S^1 with the fibre being an orientable surface. Thus we may project a link in the manifold (after an adjustment) to one of the fibres to obtain a diagram. This process is described for a more general class of 3-manifolds, Reidemeister moves are defined for these manifolds and then the following theorem is proven.

Theorem 1. *Any two diagrams for a link in a* 3*-manifold with an open book decomposition are related by a sequence of Reidemeister moves for that manifold.*

This result was also obtained previously and independently by Richard Skora: see [3].

Invariants of linking using these moves are then discussed.

Notation. M will be allowed to be any manifold of the form $M^3 = F \times I/\sim$ where

1. $I = [-1, 1]$;
2. F is a surface with boundary;
3. $\sim$ is the equivalence relation such that (*a*) $(x,t) \sim (x,s)$ if $x \in C$ where C is a union of boundary components of F and (*b*) $(h(x), 1) \sim (x, -1)$ where $h\colon F \to F$ is a homeomorphism which is the identity on a regular neighbourhood U of the boundary.

Let $q\colon F \times I \to M$ be the quotient map and $\pi\colon F \times I \to F$ be projection $(x,t) \mapsto x$. Let

$$F_t = q(F \times \{t\}) \quad \text{for} \quad -1 < t < 1,$$
$$\bar{F} = q(F \times \{1\} \cup F \times \{-1\}),$$
$$\overline{\partial F} = q(\partial F \times I),$$
$$\bar{C} = q(C \times I),$$
$$N = q(U \times I).$$

L will denote a link in M, i.e. a PL embedding of a finite number of copies of S^1. Two links in M will be called equivalent if there is an isotopy of M taking one link to the other.

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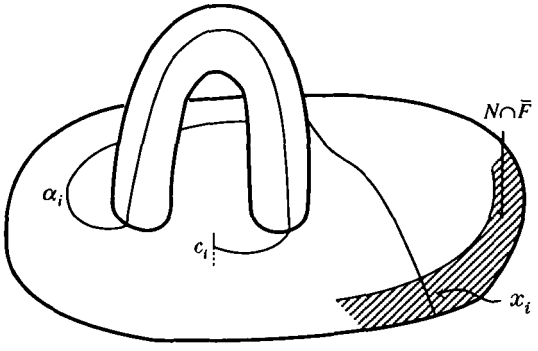


Fig. 1. An example of an α_i for a c_i where F is a punctured torus.

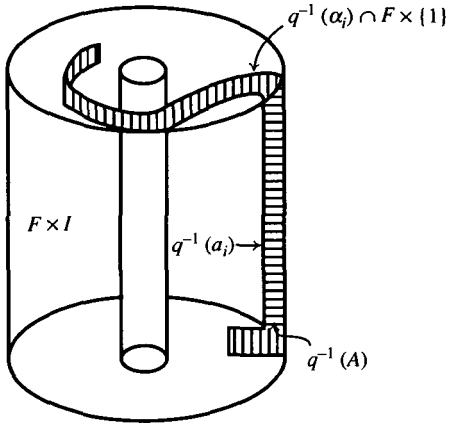


Fig. 2. An example of an a_i in $F \times I$.

Definition. Let $N(\bar{F})$ be a regular neighbourhood of $\bar{F} \backslash \bar{C}$ in $M \backslash \bar{C}$. So $N(\bar{F})$ is a D^1 bundle over $\bar{F} \backslash \bar{C}$. Let $f(p)$ be the fibre over $p \in \bar{F}$. Define a diagram for L as follows.

1. Let $\{p_1, \ldots, p_n\} = L \cap \bar{F}$. By a slight isotopy of L near $\bar{F}$, $L \cap N(\bar{F}) = \bigcup_{i=1}^{n} f(p_i)$.

2. Choose disjoint embedded arcs $\alpha_1, \ldots, \alpha_n$ in $\bar{F}$, for the $f(p_i)$ so that $\alpha_i(0) = p_i$ and $\alpha(1) \in \bar{C}$. Let $x_i \in \alpha_i \cap N$ near $\alpha_i(1)$. Let $\hat{\alpha}_i$ be α_i from $\alpha_i(0)$ to x_i. See Figure 1.

3. For $i = 1, \ldots, n$ let

$$a_i = \bigcup_{p \in \hat{\alpha}_i} \partial f(p) \cup q(\pi(q^{-1}(x_i))) \times [-1+\epsilon, 1-\epsilon]).$$

See Figure 2.

4. Notice that since $\alpha_i \backslash \hat{\alpha}_i \subseteq N$, $q(\pi(q^{-1}(\alpha_i \backslash \hat{\alpha}_i)) \times I)$ is a disc $\hat{A}_i$ in M. Let A_i be the disc

$$A_i = \bigcup_{p \in \hat{\alpha}_i} f(p) \cup q(\pi(q^{-1}(\alpha_i \backslash \hat{\alpha}_i)) \times I)$$

Then $\partial A_i = f(p_i) \cup a_i$, A_i transversely intersects $\bar{C}$ and $A_i \cap \bar{F} = \alpha_i$. Replace $f(p_i)$ with a_i by isotopy across A_i. Now $L \cap \bar{F} = \varnothing$.

5. Isotope L so that the projection $\pi' : M \backslash \bar{F} \to F_0$ given by $\pi'(x,t) \mapsto (x,0)$ is transverse on L and has at most double points on L.

This projection with crossing information will be called a diagram for L and will be denoted $D_{\alpha_i}(L)$. See Figure 3.

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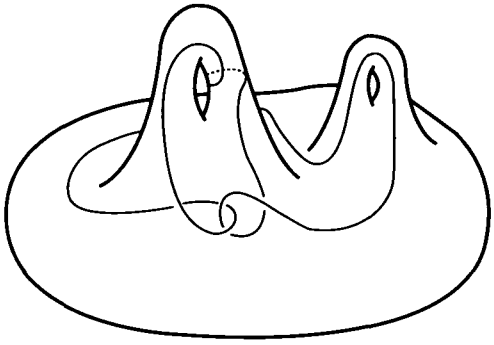
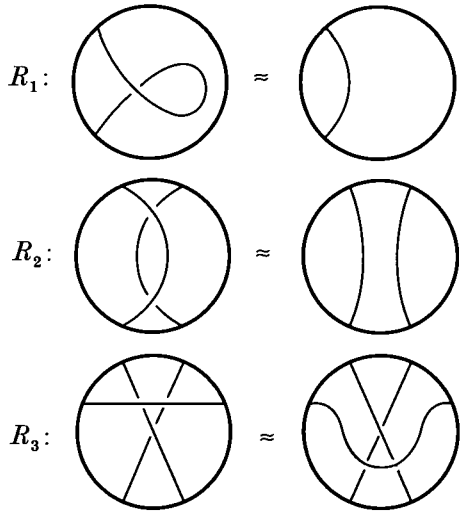


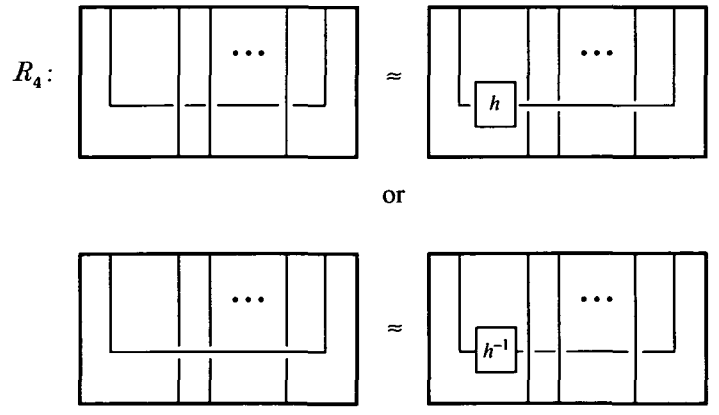
Fig. 3. An example of a diagram of a knot where F is a punctured surface of genus 2.

Definition of Reidemeister equivalence

A pair of diagrams for L are related by the R-move R_i if they are the same except for a disc where



or a portion of the diagram where

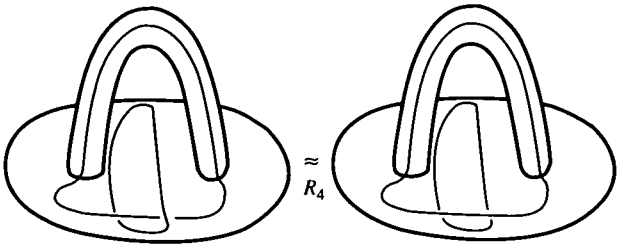


The $\boxed{h}$ indicates the image of the horizontal arc under h and the vertical sides of each rectangle are pieces of C.

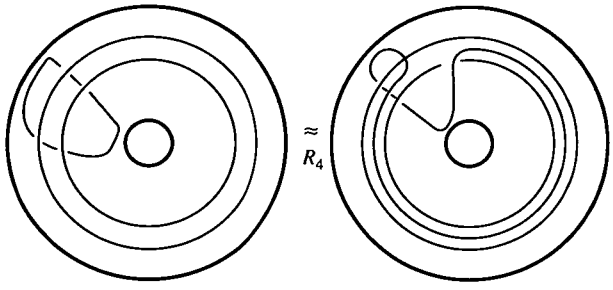
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Examples

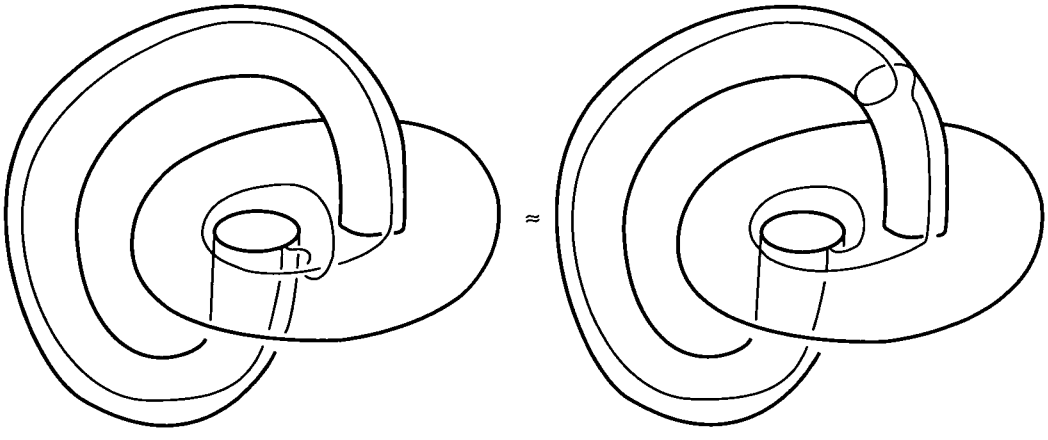
1. Let F be a punctured torus and h the identity map $\mathrm{id}: F \to F$. Then



2. Let F be an annulus and h a Dehn twist along its core. Then



3. Let F be a punctured Klein bottle and h a Dehn twist around the meridian of F.



THEOREM 2. *Any two diagrams for a link are related by a sequence of R-moves. (They are R-equivalent.)*

Note. The proof will be accomplished in two steps.

1. To show independence of the definition for a particular embedding.
2. Given two isotopic embeddings of L, to show that there are diagrams of each that are R-equivalent.

LEMMA 2·1. *Given an embedding of a link L which is transverse to $\overline{F}$ in M, any two choices of arcs α_i give R-equivalent diagrams.*

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Proof. If L does not intersect $\bar{F}$ there is nothing to prove. Suppose that L intersects $\bar{F}$ in n intersection points $\{p_i\}$. Let $f(p_1), \ldots, f(p_n)$ and $\{\alpha_i\}$ be as in the definition of a diagram. Let $c_i = f(p_i)$ for $i = 1, \ldots, n$. Suppose $\{\beta_i\}$ is some other system of arcs for the $\{c_i\}$ which we may assume to be transverse to the $\{\alpha_i\}$. Suppose $\{\alpha_i\} \cap \{\beta_i\}$ has m points. The case $m = 0$ is dealt with below. Suppose the lemma is true for all systems of arcs with $m-1$ points of intersection.

Pick an arc α_j such that for some β_k, $\alpha_j \cap \beta_k \neq \varnothing$. Let $t_1 = \max\{t | \exists s,\ \alpha_j(t) = \beta_k(s)\}$. Let s_1 be the point at which $\beta(s) = \alpha(t_1)$. Let γ be the path from $\beta_k(0)$ to $\beta_k(s_1)$ along β_k and then to $\alpha_j(1)$ along α_j, i.e. define

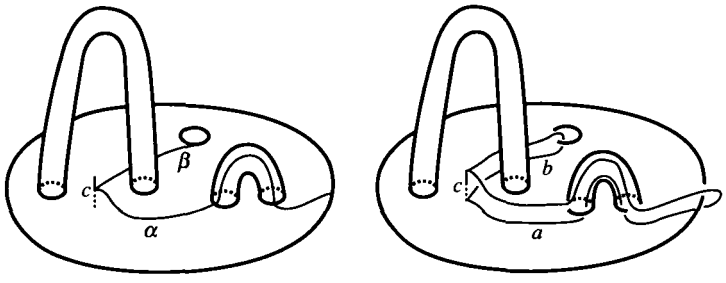
$$\gamma(t) = \begin{cases} \beta_k(t) & (0 \leqslant t \leqslant s_1), \\ \alpha_j([(1-t_1)t + (t_1 - s_1)]/(1-s_1)) & (s_1 \leqslant t \leqslant 1). \end{cases}$$

Let $\{\beta_i'\}$ be $\{\beta_i\}$ with β_k replaced with γ. The systems $\{\beta_i\}$ and $\{\beta_i'\}$ have no intersections, i.e. $m = 0$. $\{\alpha_i\}$ and $\{\beta_i'\}$ have less intersections than $\{\alpha_i\}$ and $\{\beta_i\}$. So by induction $D_{\alpha_i}(L) \approx D_{\beta_i'}(L) \approx D_{\beta_i}$.

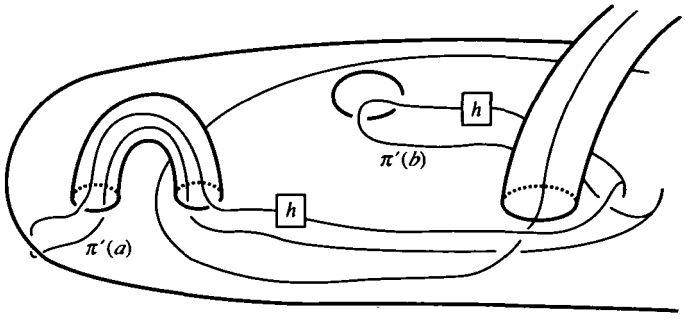
Now we take care of the case $m = 0$. Clearly it is enough to show the equivalence of two diagrams which are different only by one change of path.

Let $c = f(p)$ be the transverse arc, α and β the two paths and a and b the arcs for c using α and β respectively. Suppose $\alpha \cap \beta = p$ $(= \alpha(0) = \beta(0))$.

The following shows a particular situation where F is a twice punctured surface of genus 2.



Now replace c with a and b. After applying π' the situation becomes as follows.

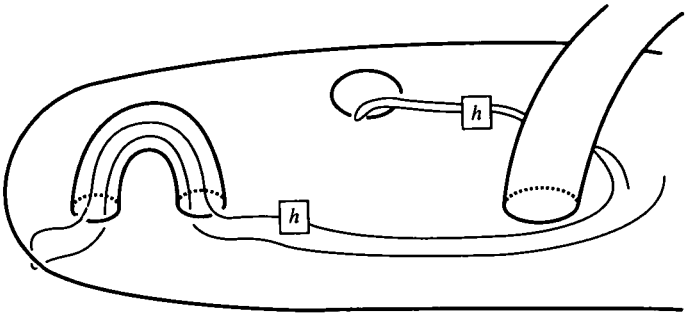


Let α be the union of two arcs $a^+ = a \cap \{F_t | t \geqslant 0\}$ and $a^- = a \cap \{F_t | t \leqslant 0\}$. Notice that $a^+ \cap a^- = a \cap F_0$. Similarly $b = b^+ \cup b^-$. Since $a \cup c$ bounds a disc then $h(\pi(q^{-1}(a^+))) = \pi(q^{-1}(a^-))$. The $\boxed{h}$ in $\pi'(a)$ and $\pi'(b)$ indicates this relation. Also the fact that $c \cup a$ and

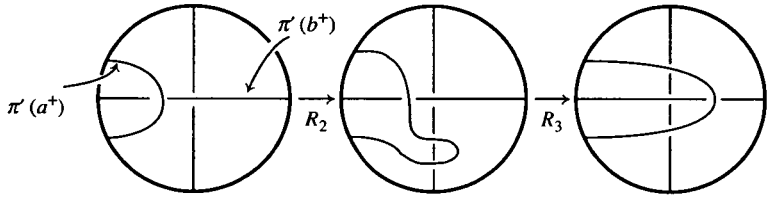
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$c \cup b$ bound discs in $M \backslash L$ implies that in $D(L)$, $\pi'(a^+)$ will consist entirely of overcrossings and $\pi'(a^-)$ will consist entirely of undercrossings.

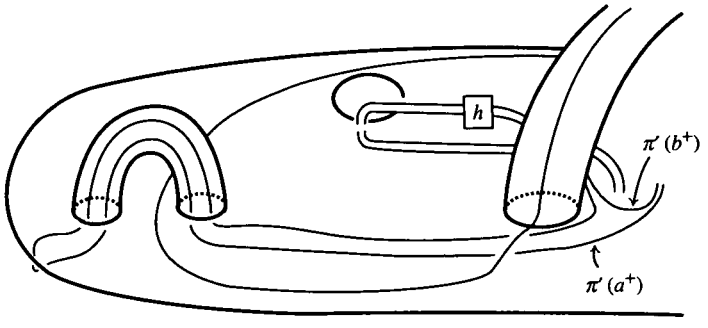
Move $\pi'(a^+)$ to the following position.



That is, $\pi'(a^+)$ now follows along $\pi'(b^+)$ to N and back along $\pi'(b^+)$. This can be accomplished using R_2 and R_3:



Now there is an arc of $\pi'(a^+)$ which is in an R_4-move position. (In the picture it is the upper part with the two $\boxed{h}$'s on it.) After the R_4-move this arc consists entirely of undercrossings. The situation is now as follows.



The part of $\pi'(a)$ which does not follow along $\pi'(b)$ can be retracted to $\pi'(b)$ using the moves R_2 and R_3. Thus $D_\alpha(L) \approx D_\beta(L)$. ▌

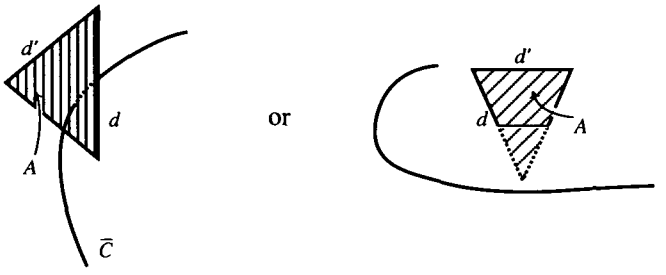
Note. Since a diagram for an embedding which is transverse to $\bar{F}$ depends only on the choice of arcs α_i this shows that any two diagrams for a particular embedding which is transverse to $\bar{F}$ are R-equivalent.

Proof of theorem. Since a diagram for a link is obtained from a link which is adjusted to be transverse to $\bar{F}$, equivalent links in M may be assumed to be transverse to $\bar{F}$. Furthermore any two isotopic links in M are isotopic through a sequence of links $L_1, \ldots, L_l$ which are identical except for arcs d and d' which bound a disc A in M. Thus we may assume that L and L' are transverse to $\bar{F}$, and that they differ by arcs d and d' which bound a disc A.

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Case 1. Suppose $A \cap \bar{F} = \varnothing$. The isotopy across A can be subdivided into isotopies across discs so that the image of π' on one of these discs is contained in a disc in F_0. Suppose A is such a disc. There are diagrams for L and L' which are identical except for arcs $\pi'(d)$ and $\pi'(d')$ which are contained in a disc in F_0. Thus $\pi'(d)$ can be moved to $\pi'(d')$ using only R_1, R_2 and R_3.

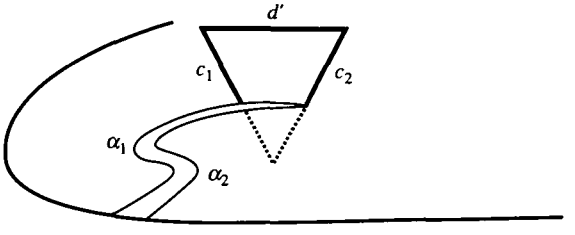
Case 2. Suppose $A \cap \bar{F} \neq \varnothing$. Then the isotopy of d to d' across A can be subdivided so that each smaller move corresponds to a move as in Case 1 or one of



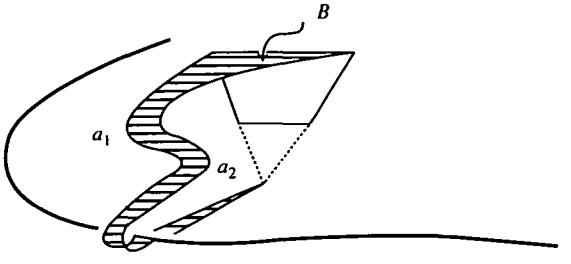
Type *a* Type *b*

Type a. By the lemma the path α for d may be chosen arbitrarily. Let $\alpha = A \cap \bar{F}$. Then a is (essentially) d' so that the two diagrams are identical.

Type b. d is made up of two smaller arcs c_1 and c_2 which are transverse to $\bar{F}$. Again we may pick arcs α_1 and α_2 for c_1 and c_2 arbitrarily. Let α_1 be an arc for c_1 and let α_2 lie in a regular neighbourhood of $\alpha_1 \cup (A \cap \bar{F})$ so that $\alpha_1 \cap \alpha_2 = \varnothing$.



Then the replacement arcs a_1 and a_2 for α_1 and α_2 together with d' lie in a 3-ball in $M \backslash \bar{F}$ and so bound a disc B in $M \backslash \bar{F}$.



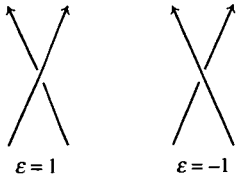
This is the situation of Case 1. ∎

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Linking in $S^2 \times S^1$

This section discusses a generalization of the linking number in S^3 to M where h is the identity and F is orientable. These manifolds include connected sums of $S^2 \times S^1$ as a special case when $C = \partial F$. Let $J \cup K$ be an oriented two component link in M. Define the linking number, $\Sigma(J,K)$, of J and K in M as follows.

1. At each crossing x, let $\epsilon(x)$ be



2. Let

$$s = \sum_{x \in J \cap K} \epsilon(x)$$

3. Let (c) be the greatest common divisor of the coefficients of $[c] \in H_1(M, \mathbb{Z})$ with respect to some basis of $H_1(M, \mathbb{Z})$.

4. Let $l = 2\gcd((J),(K))$

5. Let $\Sigma(J,K) = s \bmod (l)$.

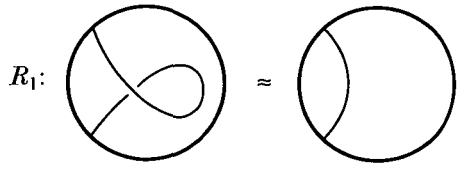
Note. For two knots J and K in S^3 the linking number $\mathrm{lk}(J,K)$ is just $\frac{1}{2}\Sigma(J,K)$.

PROPOSITION 1. *Let J and K be knots in M. Given a particular fibration of $M, \Sigma(J,K)$ is an invariant of the link $J \cup K$.*

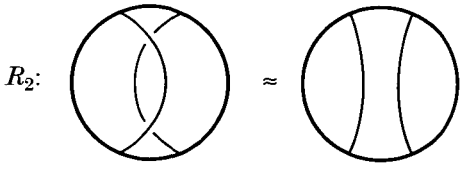
Proof. Since $h = \mathrm{id}$, $H_1(M, \mathbb{Z}) = H_1(F, \mathbb{Z})$. For $[c] \in H_1(F)$, (c) is independent of the choice of basis so (c) is an invariant of the knot c and thus $l = 2\gcd((J),(K))$ is an invariant of $J \cup K$. By Theorem 1·1 it remains to show that

$$s = \sum_{x \in J \cap K} \epsilon(x)$$

is changed only by a multiple of l under the R-moves.

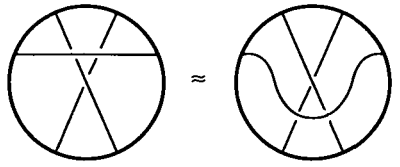


Since this does not change any x in $J \cap K$, s remains the same.

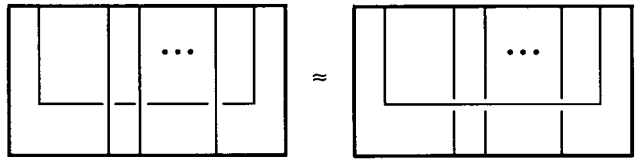


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If the arcs are from the same component s does not change and if they are from different components one crossing produces $a\ +1$ and the other $a\ -1$.

R_3: 

Again the sum of the ϵ's for both of these figures remains the same. Finally,

R_4: 

s is not effected by crossings of J or K with themselves so by the symmetry of the definition we may assume that the horizontal arc is J and the vertical arcs are K. Call the crossings $x_1, \ldots, x_m$. By mapping the disc with handles F to a wedge of circles it is evident that $\sum_{i=1}^{m} \epsilon(x_i)$ is the sum of some collection of coefficients of $[K]$. Clearly (K) divides each of these numbers in the sum and so divides $\sum_{i=1}^{m} \epsilon(x_i)$. Thus $2\gcd((J),(K))$ divides $2\sum_{i=1}^{m} \epsilon(x_i)$ which is the amount which s is changed, as was to be shown. ∎

PROPOSITION 2. *There is a one to one correspondence between the representatives of* $\Sigma(J,K)$ *in* $\mathbb{Z}_l$ *and the numbers that can be achieved as crossing numbers* s *in the diagrams for* $J \cup K$.

Proof. We need to show that s can be changed by exactly $2\gcd((J),(K))$. Let F be a surface expressed as a disc with m handles. Label the handles $H_1, \ldots, H_m$. Give $H_1(F)$ a basis so that (by an abuse of notation), in $H_1(F)$,

$$J = \eta_1 H_1 + \ldots + \eta_m H_m, \quad K = \mu_1 H_1 + \ldots + \mu_m H_m.$$

Using the algebraic properties of the greatest common divisor,

$$\begin{aligned} 2\gcd((J),(K)) &= 2\gcd(\gcd(\eta_1, \ldots, \eta_m), \gcd(\mu_1, \ldots, \eta_m)) \\ &= 2\gcd(\eta_1, \ldots, \eta_m, \mu_1, \ldots, \mu_m) \\ &= 2(u_1\eta_1 + \ldots + u_m\eta_m + v_1\mu_1 + \ldots + v_m\mu_m) \end{aligned}$$

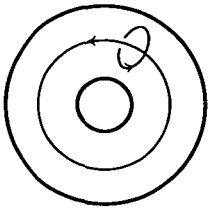
for some $\{u_1, \ldots, u_m, v_1, \ldots, v_m\} \subseteq \mathbb{Z}$. By moving any arc of J completely over (or completely under) K along the cocore of H_1 and performing an R_4-move on J, s will be changed by $2\mu_1$. Repeat this v_1 times for J. This changes s by $2v_1\mu_1$. Continuing in this way for each η_i and μ_i will change s by $2\gcd((J),(K))$. ∎

Notes. This means that in some sense Σ is a best possible generalization of the linking number to M.

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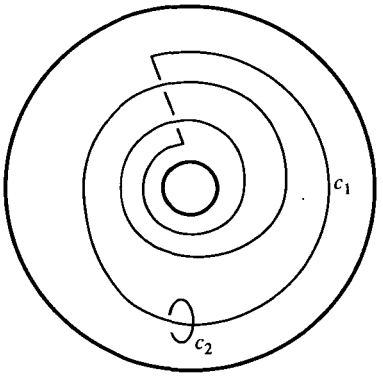
Examples

1. Let F be an annulus and L the following link.



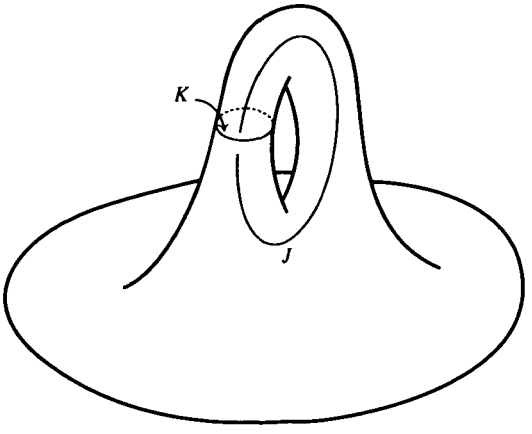
$\epsilon(x) = 1$ for both crossings. $\Sigma(J,K) = \overline{0} \in \mathbb{Z}_2$.

2. Let F be an annulus and L the link:



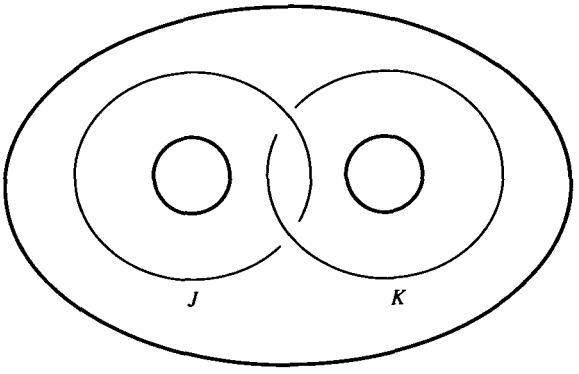
For this link $(c_1) = 3$, $(c_2) = 0$ and so $\gcd((c_1),(c_2)) = 3$. $\Sigma(c_1, c_2) = \overline{2} \in \mathbb{Z}_6$. If the component c_1 were to wind around k times and link as above then $\Sigma(c_1, c_2) = \overline{2} \in \mathbb{Z}_{2k}$.

3. This example shows that $\Sigma(J,K)$ depends on the fibration. Let F be a punctured torus and let L be the link



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$\Sigma(J,K) = \bar{1} \in \mathbb{Z}_2$ and $(J) = 1 = (K)$. The same manifold can be constructed using a twice punctured disc. The above link in M corresponds to the following link:



$\Sigma(J,K) = \bar{0} \in \mathbb{Z}_2$.

PROPOSITION 3. *If $K \cup L$ is a link in M then $\Sigma(L,K)$ is an invariant of the class $[L] \in H_1(M \backslash K)$.*

Proof. Suppose $[L] = 0$ in $H_1(M \backslash K)$. Then L bounds a non-singular 2-chain E in $M^3 \backslash K$. The triangulation of E corresponds to homeomorphisms $\sigma : \Delta \to E$ where $\Delta = \{(x,y) \in \mathbb{R}^2 | x, y \geqslant 0,\ x+y \leqslant 1\}$ with the usual restrictions on the images.

Suppose $L \cap \sigma(\Delta) \neq \varnothing$. Eliminate $\sigma(\Delta)$ as follows.

Case 1. Suppose L intersects $\sigma(\Delta)$ in exactly one face f of $\sigma(\Delta)$. f may be moved across $\sigma(\Delta)$ to the opposite two faces of $\sigma(\Delta)$ eliminating $\sigma(\Delta)$. This movement of L corresponds to doing R-moves of L across K and so leaves $\Sigma(L,K)$ fixed.

Case 2. Suppose that L intersects $\sigma(\Delta)$ in exactly two faces f_1 and f_2 of $\sigma(\Delta)$. As above move $f_1 \cup f_2$ across $\sigma(\Delta)$ to the opposite side eliminating $\sigma(\Delta)$. Again this does not change $\Sigma(L,K)$.

Case 3. Suppose $L \cap \sigma(\Delta) = f \cup \{p\}$ where p is the vertex opposite f. As in Case 1 push f across $\sigma(\Delta)$. This new L is no longer a link, however by splitting L at p in the orientation-preserving way will produce a new link L with the same linking number as the old L. Define $\Sigma(L,K)$ to be this common value.

Case 4. Suppose $L \cap \sigma(\Delta) = \partial\sigma(\Delta)$. Then $L \cap \sigma(\Delta)$ is a simple closed curve bounding a disc in $M \backslash K$ and so has linking number zero with K. Remove $\sigma(\Delta)$ from E. Notice that the result is still a link which still has the same linking number with K.

Any case of $L \cap \sigma(\Delta)$ falls into one of these four cases. We may remove all of the $\sigma(\Delta)$ from E producing $\Sigma(L,K) = 0$.

Suppose $[L_1] = [L_2]$ in $M \backslash K$. Then

$$\gcd((L_1),(K)) = \gcd((L_2),(K)) = \gcd((L_1),(L_2),(K))$$

and so

$$\begin{aligned} 0 = \Sigma(L_1 - L_2, K) &= \sum_{c \in (L_1 - L_2) \cap K} \epsilon(c) \bmod 2 \gcd((L_1),(L_2),(K)) \\ &= \sum_{c \in L_1 \cap K} \epsilon(c) \bmod 2 \gcd((L_1),(K)) + \sum_{c \in -L_2 \cap K} \epsilon(c) \bmod 2 \gcd((L_2),(K)) \\ &= \Sigma(L,K) + \sum_{c \in L_2 \cap K} -\epsilon(c) \bmod 2 \gcd((L_2),(K)) = \Sigma(L_1,K) - \Sigma(L_2,K). \end{aligned}$$

∎

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Remark. The last proposition allows one to define a Seifert form for a knot K such that $[K] = 0 \in H_1(M)$ in the following manner: K bounds an orientable surface S in M,

$$\Theta : H_1(S) \times H_1(S) \to Q/\mathbb{Z} \cup \mathbb{Z}$$

can be defined by

$$\Theta(x, y) = \begin{cases} s/l \in Q/\mathbb{Z} & \text{if } l \neq 0 \\ s/2 \in \mathbb{Z} & \text{if } l = 0, \end{cases}$$

where $\Sigma(x^+, y) = s \bmod l$. It is hopeful therefore to construct an Alexander polynomial using this linking number invariant.

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