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Some formulae for Jacobi polynomials

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1. *Introduction.* The object of this paper is to prove some formulae for Jacobi polynomials. The formulae (6), (7), (9), (10) and (13) are believed to be new.

We require the following formulae ((3), pp. 254, 257, 263) and ((1), p. 2) in our investigation:

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n(1-x)^{-\alpha}(1+x)^{-\beta}}{2^n n!} D^n[(1-x)^{n+\alpha}(1+x)^{n+\beta}] \tag{1}$$

$$= \frac{(-1)^n(1+\beta)_n}{n!}\, {}_2F_1\left[\begin{matrix}-n, 1+\alpha+\beta+n; \\ 1+\beta\end{matrix}\ \tfrac{1}{2}(1+x)\right] \tag{2}$$

$$= \frac{(1+\beta)_n}{n!}[\tfrac{1}{2}(x-1)]^n\, {}_2F_1\left[\begin{matrix}-n, -\alpha-n; \\ 1+\beta\end{matrix}\ (x+1)/(x-1)\right], \tag{3}$$

$$D^m P_n^{(\alpha,\beta)}(x) = 2^{-m}(1+\alpha+\beta+n)_m P_{n-m}^{(\alpha+m,\beta+m)}(x) \tag{4}$$

and

$$\sum_{n=0}^{\infty} P_n^{(\alpha-n,\beta-n)}(x)\, t^n = [1+\tfrac{1}{2}t(x+1)]^\alpha [1+\tfrac{1}{2}t(x-1)]^\beta. \tag{5}$$

2. The first formula to be proved is

$$\sum_{r=0}^{\min(m,n)} \frac{(-4)^{-r}}{r!}(1-x^2)^r (1+\alpha+\beta+m+2n)_r P_{n-r}^{(\alpha+r,\beta+r)}(x)\, P_{m-r}^{(\alpha+n+r,\beta+n+r)}(x)$$

$$= \binom{m+n}{m} P_{m+n}^{(\alpha,\beta)}(x). \tag{6}$$

Proof. By (1), we have

$$P_{m+n}^{(\alpha,\beta)}(x) = \frac{(-1)^{m+n}(1-x)^{-\alpha}(1+x)^{-\beta}}{2^{m+n}(m+n)!} D^{m+n}[(1-x)^{m+n+\alpha}(1+x)^{m+n+\beta}]$$

$$= \frac{(-1)^n(1-x)^{-\alpha}(1+x)^{-\beta} m!}{2^n(m+n)!} D^n[(1-x)^{n+\alpha}(1+x)^{n+\beta} P_m^{(\alpha+n,\beta+n)}(x)] \quad \text{by (1)}$$

$$= \frac{m!\,n!}{(m+n)!} \sum_{r=0}^{n} \frac{(-2)^{-r}}{r!}(1-x^2)^r P_{n-r}^{(\alpha+r,\beta+r)}(x)\, D^r[P_m^{(\alpha+n,\beta+n)}(x)]$$

$$= \frac{m!\,n!}{(m+n)!} \sum_{r=0}^{\min(m,n)} \frac{(-4)^{-r}}{r!}(1-x^2)^r (1+\alpha+\beta+m+2n)_r$$

$$\times P_{n-r}^{(\alpha+r,\beta+r)}(x)\, P_{m-r}^{(\alpha+n+r,\beta+n+r)}(x)$$

by (4). Hence the result.

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3. The second formula to be proved is

$$\sum_{n=0}^{\infty} \binom{m+n}{m} P_{m+n}^{(\alpha-n,\,\beta-n)}(x)\,t^n = [1+\tfrac{1}{2}(x+1)\,t]^{\alpha}\,[1+\tfrac{1}{2}(x-1)\,t]^{\beta}\,P_m^{(\alpha,\,\beta)}[x+\tfrac{1}{2}(x^2-1)\,t]. \quad (7)$$

Proof. Now from (6), we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} \binom{m+n}{m} P_{m+n}^{(\alpha-n,\,\beta-n)}(x)\,t^n \\
&\quad = \sum_{r=0}^{m} \frac{(1+\alpha+\beta+m)_r}{r!}\,[\tfrac{1}{4}(x^2-1)\,t]^r\,P_{m-r}^{(\alpha+r,\,\beta+r)}(x) \sum_{n=0}^{\infty} P_n^{(\alpha-n,\,\beta-n)}(x)\,t^n \\
&\quad = [1+\tfrac{1}{2}(x+1)\,t]^{\alpha}\,[1+\tfrac{1}{2}(x-1)\,t]^{\beta} \sum_{r=0}^{m} \frac{(1+\alpha+\beta+m)_r}{r!}\,[\tfrac{1}{4}(x^2-1)\,t)]^r\,P_{m-r}^{(\alpha+r,\,\beta+r)}(x) \\
&\quad = [1+\tfrac{1}{2}(x+1)\,t]^{\alpha}\,[1+\tfrac{1}{2}(x-1)\,t]^{\beta}\,P_m^{(\alpha,\,\beta)}[x+\tfrac{1}{2}(x^2-1)\,t],
\end{aligned}$$

by virtue of (5), since

$$P_n^{(\alpha,\,\beta)}(x+y) = \sum_{r=0}^{n} \frac{(1+\alpha+\beta+n)_r}{r!}\,(\tfrac{1}{2}y)^r\,P_{n-r}^{(\alpha+r,\,\beta+r)}(x). \quad (8)$$

This completes the proof of (7).

4. The third formula to be proved is

$$P_n^{(\alpha+\gamma,\,\beta+\delta)}(x) = \sum_{r=0}^{n} P_{n-r}^{(\alpha+r,\,\beta+r)}(x)\,P_r^{(\gamma-r,\,\delta-r)}(x). \quad (9)$$

Proof. By (1), we have

$$\begin{aligned}
P_n^{(\alpha+\gamma,\,\beta+\delta)}(x) &= \frac{(-1)^n\,(1-x)^{-\alpha-\gamma}\,(1+x)^{-\beta-\delta}}{2^n\,n!}\,D^n[(1-x)^{n+\alpha+\gamma}\,(1+x)^{n+\beta+\delta}] \\
&= \frac{(-1)^n\,(1-x)^{-\alpha-\gamma}\,(1+x)^{-\beta-\delta}}{2^n\,n!} \sum_{r=0}^{n} n_{c_r}\,D^{n-r}[(1-x)^{n+\alpha}\,(1+x)^{n+\beta}] \\
&\qquad \times D^r[(1-x)^{\gamma}\,(1+x)^{\delta}] \\
&= \sum_{r=0}^{n} P_{n-r}^{(\alpha+r,\,\beta+r)}(x)\,P_r^{(\gamma-r,\,\delta-r)}(x),
\end{aligned}$$

by virtue of (1).

5. The fourth formula to be proved is

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{n!}{(1+\delta)_n}\,P_n^{(\alpha-n,\,\beta-n)}(x)\,P_n^{(\gamma-n,\,\delta)}(y)\,t^n &= (1-\tfrac{1}{2}(x+1)\,t)^{\alpha}\,(1-\tfrac{1}{2}(x-1)\,t)^{\beta} \\
&\times F_1\left[1+\gamma+\delta;\ -\alpha,\ -\beta;\ 1+\delta;\ -\frac{(x+1)\,(y+1)\,t}{4-2(x+1)\,t},\ -\frac{(x-1)\,(y+1)\,t}{4-2(x-1)\,t}\right]. \quad (10)
\end{aligned}$$

For Appell's function F_1, see (2), p. 224.

Proof. Before proving (10), we give the proof of the following formula

$$F_1(\gamma;\ -\alpha,\ -\beta;\ \delta;\ x,\ y) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{(\delta)_n}\,(y-x)^n\,P_n^{(\alpha-n,\,\beta-n)}\left(\frac{x+y}{x-y}\right). \quad (11)$$

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By the definition of F_1, we have

$$\begin{aligned} &F_1(\gamma;\ -\alpha,\ -\beta;\ \delta;\ x, y) \\ &\quad = \sum_{m,n=0}^{\infty} \frac{(\gamma)_{m+n}\,(-\alpha)_m\,(-\beta)_n}{(\delta)_{m+n}\,m!\,n!}\,x^m y^n \\ &\quad = \sum_{n=0}^{\infty} \frac{(\gamma)_n\,(-\beta)_n}{n!\,(\delta)_n}\,y^n \sum_{m=0}^{n} \frac{(-n)_m\,(-\alpha)_m}{m!\,(1+\beta-n)_m}\left(\frac{x}{y}\right)^m \\ &\quad = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{(\delta)_n}\,(y-x)^n\,P_n^{(\alpha-n,\,\beta-n)}\left(\frac{x+y}{x-y}\right) \quad \text{by (3).} \end{aligned}$$

To prove (10), we start with the left-hand side of (10)

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{n!}{(1+\delta)_n}\,P_n^{(\alpha-n,\,\beta-n)}(x)\,P_n^{(\gamma-n,\,\delta)}(y)\,t^n \\ &\quad = \sum_{n=0}^{\infty}\sum_{k=0}^{n} (-1)^n \binom{n}{k} \frac{(1+\gamma+\delta)_k}{(1+\delta)_k}\,P_n^{(\alpha-n,\,\beta-n)}(x)\,[-\tfrac{1}{2}(1+y)]^k\,t^n \quad \text{by (2)} \\ &\quad = \sum_{k=0}^{\infty} \frac{(1+\gamma+\delta)_k}{(1+\delta)_k}\left(\frac{1+y}{2}\,t\right)^k \sum_{n=0}^{n} \binom{n+k}{k} P_{n+k}^{(\alpha-n-k,\,\beta-n-k)}(x)\,(-t)^n \\ &\quad = \sum_{k=0}^{\infty} \frac{(1+\gamma+\delta)_k}{(1+\delta)_k}\left(\frac{1+y}{2}\,t\right)^k [1-\tfrac{1}{2}(x+1)\,t]^{\alpha-k}\,(1-\tfrac{1}{2}(x-1)\,t)^{\beta-k} \\ &\qquad\qquad \times P_k^{(\alpha-k,\,\beta-k)}[x-\tfrac{1}{2}(x^2-1)\,t] \quad \text{by (7)} \\ &\quad = [1-\tfrac{1}{2}(x+1)\,t]^{\alpha}\,[1-\tfrac{1}{2}(x-1)\,t]^{\beta} \\ &\qquad \times F_1\left[1+\gamma+\delta;\ -\alpha,\ -\beta;\ 1+\delta;\ -\frac{(x+1)\,(y+1)\,t}{4-2(x+1)\,t},\ -\frac{(x-1)\,(y+1)\,t}{4-2(x-1)\,t}\right], \end{aligned}$$

by virtue of the formula (11). Further, if we make use of the relation ((4), p. 63)

$$P_n^{(\alpha-n,\,-\alpha-\gamma-n)}(x) = [\tfrac{1}{2}(1+x)]^n\,P_n^{(\alpha-n,\,\gamma-1)}\left(\frac{3-x}{1+x}\right), \tag{12}$$

and replace δ and t by $-\delta-\gamma-1$ and $\frac{1}{2}(1+y)\,t$ respectively, (10) is reduced to the following interesting form

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{n!}{(-\gamma-\delta)_n}\,P_n^{(\alpha-n,\,\beta-n)}(x)\,P_n^{(\gamma-n,\,\delta-n)}(y)\,t^n \\ &\quad = [1-\tfrac{1}{4}\{(x+1)\,(y+1)\}\,t]^{\alpha}\,[1-\tfrac{1}{4}\{(x-1)\,(y+1)\}\,t]^{\beta} \\ &\qquad \times F_1\left[-\delta;\ -\alpha,\ -\beta;\ -\gamma-\delta;\ \frac{2(x+1)\,t}{(x+1)\,(y+1)\,t-4},\ \frac{2(x-1)\,t}{(x-1)\,(y+1)\,t-4}\right]. \end{aligned} \tag{13}$$

If we replace x, y, t, γ, δ by $(x+1)/(x-1)$, $(y+1)/(y-1)$, $(x-1)\,(y-1)\,t$, β and α respectively and using the relation ((2), p. 238)

$$F_1[\alpha;\ \beta, \beta';\ \beta+\beta';\ x, y] = (1-y)^{-\alpha}\,2F_1\left[\alpha, \beta;\ \beta+\beta';\ \frac{x-y}{1-y}\right] \tag{14}$$

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in (13), it yields the formula

$$\sum_{n=0}^{\infty} \frac{n!}{(-\alpha-\beta)_n} (x-1)^n (y-1)^n t^n P_n^{(\alpha-n,\beta-n)}\left(\frac{x+1}{x-1}\right) P_n^{(\beta-n,\alpha-n)}\left(\frac{y+1}{y-1}\right)$$

$$= (1-xt)^{\alpha}(1-yt)^{\beta} {}_2F_1\left[-\alpha, -\beta; -\alpha-\beta; \frac{(x-1)(y-1)t}{(1-xt)(1-yt)}\right]. \tag{15}$$

(15) is due to Carlitz ((1), p. 3).

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