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A Numerical Comparison of Finite Difference and Finite Element Methods for a Stochastic Differential Equation with Polynomial Chaos

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Abstract. A numerical comparison of finite difference (FD) and finite element (FE) methods for a stochastic ordinary differential equation is made. The stochastic ordinary differential equation is turned into a set of ordinary differential equations by applying polynomial chaos, and the FD and FE methods are then implemented. The resulting numerical solutions are all non-negative. When orthogonal polynomials are used for either continuous or discrete processes, numerical experiments also show that the FE method is more accurate and efficient than the FD method.

AMS subject classifications: 60H10, 60H35, 65L12, 74S05

Key words: Stochastic differential equation, polynomial chaos, finite difference method, finite element method, non-negative solution.

1. Introduction

We consider numerical solutions of the stochastic ordinary differential equation

$$\begin{aligned} &\frac{\mathrm{d}y(t;\omega)}{\mathrm{d}t} = -k(\omega)y(t;\omega) + g(t), \quad t \in (0,T], \\ &y(0;\omega) = \hat{y}, \end{aligned} \tag{1.1}$$

where the decay rate coefficient $k(\omega)$ is a random variable, $\omega \in \Omega$ where Ω is all possible event space, and $\hat{y}$ is the initial value. For given values of k we have the exact solution $y(t) = c_0 e^{-kt} + \int_0^t e^{k(s-t)} g(s)\,\mathrm{d}s$, where the constant c_0 is determined from the initial condition. On the other hand, for a random input $k(\omega)$ the method of polynomial chaos can be invoked, where both $k(\omega)$ and $y(t;\omega)$ may be expanded in series to get the solution.

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Wiener [11] introduced the concept of chaos, when representing a Gaussian process as an expansion in Hermite polynomials. Ghanem & Spanos [3–5] combined the Hermite expansion with the finite element (FE) method to model the uncertainty encountered in various problems of solid mechanics. According to the Cameron-Martin theorem [1], a series in Hermite polynomials for any stochastic process with finite second-order moment converges. However, while the convergence is fast (exponential) for Gaussian random processes, for other stochastic processes the convergence speed of Hermite polynomial chaos it can be very slow indeed. To overcome this drawback, Tang [16] introduced an Hermite spectral method for Gaussian-type functions, and Tang & Zhou [17] considered discrete least square projections in unbounded domains with random evaluations and the application of their approach to quantification of parameter uncertainty. Further, the convergence speed is improved by using a scaling factor or polynomials other than Hermite.

Later, Ogura [8] used Charlier polynomial chaos in simulating the Poisson process, and Xiu & Karniadakis presented a generalised polynomial chaos method based on Askey polynomials [12–15]. In particular, stochastic processes with optimum trial bases from the Askey family of orthogonal polynomials reduced the dimensionality of the system, leading to exponential convergence for the error [12]. These methods usually require finite truncation and Galerkin projection. Furthermore, each member in the family of Askey polynomial chaos has distinctive mutually orthogonal weight functions that are typically consistent with the probability density function for the stochastic process, whether continuous or discrete. Thus Hermite polynomials correspond to the Gaussian distribution, Laguerre polynomials to the Gamma distribution, Charlier polynomials to the Poisson distribution and Krawtchouk polynomials correspond to the Binomial distribution, when representing different stochastic processes with different polynomial chaos in order to achieve the optimal approximation rate.

Following Xiu & Karniadakis [12], in this article we use FD and FE methods to solve the resulting ordinary differential equations numerically, and consider the effectiveness of certain time-discretization schemes. Moreover, through special handling of the orthogonal polynomials we can guarantee that all of the numerical solutions of the resulting ordinary differential equations are non-negative, and find that the FE method proves better results than either the FD method or the first-order and second-order Runge-Kutta schemes envisaged in Ref. [12]. The application of polynomial chaos is introduced in Section 2. In Section 3, we present our FD and FE methods for solving the random input inhomogeneous ordinary differential equations, and find non-negative solutions together with some error estimates. We present our numerical results for various distributions in Section 4, and our brief concluding remarks are in Section 5.

2. Application of Wiener-Askey Polynomial Chaos

We first briefly introduce some properties of the orthogonal polynomials. Since the random variable obeys different distributions associated with different orthogonal polynomials chaos, the orthogonal polynomials $\{\Phi_n,\ n = 0, 1, \cdots\}$ satisfy the following relationships.

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In the continuous case

$$\langle \Phi_i(\xi)\Phi_j(\xi)\rangle = \int \Phi_i(\xi)\Phi_j(\xi)w(\xi)\mathrm{d}\xi = h_{ij}\delta_{ij}\,, \quad i,j=0,1,\cdots\,, \tag{2.1}$$

and in the discrete case

$$\langle \Phi_i(\xi)\Phi_j(\xi)\rangle = \sum_{\xi} \Phi_i(\xi)\Phi_j(\xi)w(\xi) = h_{ij}\delta_{ij}\,, \quad i,j=0,1,\cdots\,, \tag{2.2}$$

where $\langle\cdot,\cdot\rangle$ denotes the ensemble average via the inner product in the Hilbert space of the variable ξ, $w(\xi)$ is the weighting function, and the h_{ij} are nonzero constants. In particular, if $h_{ij}=1$ then $\{\Phi_n\,, n=0,1,\cdots\}$ is a set of standard orthogonal polynomials [2, 9, 10], which can be generated via a three-term recurrence relation or by repeatedly applying the relevant differential operator [12].

Next, we apply the Wiener-Askey polynomial chaos to the numerical solution of the stochastic differential equation (1.1). The random input $k(\omega)$ and the solution $y(t;\omega)$ of Eq. (1.1) can be expanded into truncated series — i.e.

$$k(\omega) = \sum_{i=0}^{P} \alpha_i \Phi_i(\xi)\,, \qquad y(t,\omega) = \sum_{i=0}^{P} y_i(t)\Phi_i(\xi)\,, \tag{2.4}$$

which can be considered to be spectral expansions in the random dimension ξ. We can take the dimensionality of ξ to be one, and absorb $k(\omega)=\xi$ into the polynomial basis $\Phi(\xi)$. Then substituting Eqs. (2.4) into Eq. (1.1) yields

$$\sum_{i=0}^{P} \frac{\mathrm{d}y_i(t)}{\mathrm{d}t}\Phi_i = -\sum_{i=0}^{P}\sum_{j=0}^{P} \Phi_i\Phi_j\alpha_i y_j(t) + g(t)\,, \tag{2.5}$$

and a Galerkin projection of this equation onto each polynomial basis $\{\Phi_i\}$ ensures that the error is orthogonal to the functional space spanned by the finite-dimensional basis $\{\Phi_i\}$. Thus projecting with Φ_m for each $m\in\{0,\cdots,P\}$ and employing the orthogonality relation, we obtain

$$\frac{\mathrm{d}y_m(t)}{\mathrm{d}t} = -\frac{1}{e_{mm}}\left(\sum_{i=0}^{P}\sum_{j=0}^{P} e_{ijm}\alpha_i y_j(t) - e_{0m}g(t)\right), \quad m=0,1,\cdots,P\,, \tag{2.6}$$

where $e_{ijm}=\langle\Phi_i\Phi_j\Phi_m\rangle$, $e_{mm}=\langle\Phi_m\Phi_m\rangle$, $i,j,m=0,1,\cdots,P$. The values of e_{ijm} and e_{mm} can be evaluated analytically from the definition of Φ_m. Eqs. (2.6) are a set of $P+1$ coupled ordinary differential equations, and the total number $(P+1)$ of expansion terms is determined by the dimensionality of the chaos expansion n and the highest order p of the polynomial $\{\Phi\}$, where

$$(P+1) = (n+p)!/(n!\,p!)\,. \tag{2.7}$$

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When represented in a truncated series of the same form as Eqs. (2.4), the initial condition is

$$y(0,\omega)=\sum_{i=0}^{P} y_i(0)\Phi_i(\xi)=\hat{y}\,,\quad y_i(0)=\frac{\langle \hat{y}\Phi_i\rangle}{e_{ii}}\,, \tag{2.8}$$

and the coefficient of α_i is

$$k(\omega)=\sum_{i=0}^{P}\alpha_i\Phi_i(\xi)=\xi\,,\quad \alpha_i=\frac{\langle \xi\Phi_i\rangle}{e_{ii}}\,, \tag{2.9}$$

where $i=0,1,\cdots,P$. By matching coefficients in the expansions, we obtain the initial condition for each expanded equation in (2.8) to complete the system. We may now re-express the above equations (2.6) and (2.8) in the matrix form

$$\frac{\mathrm{d}Y(t)}{\mathrm{d}t}=AY(t)+Bg(t)\,, \tag{2.10}$$

where

$$Y(t)=[y_0(t),y_1(t),\cdots,y_P(t)]^T\,,\qquad B=[1,0,\cdots,0]^T$$

and

$$A=\sum_{i=0}^{P}\alpha_i A_i\,, \tag{2.11}$$

with

$$A_i=-\begin{pmatrix} \frac{e_{i00}}{e_{00}} & \frac{e_{i10}}{e_{00}} & \cdots & \frac{e_{iP0}}{e_{00}} \\ \frac{e_{i01}}{e_{11}} & \frac{e_{i11}}{e_{11}} & \cdots & \frac{e_{iP1}}{e_{11}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{e_{i0P}}{e_{PP}} & \frac{e_{i1P}}{e_{PP}} & \cdots & \frac{e_{iPP}}{e_{PP}} \end{pmatrix}_{(P+1)\times(P+1)}$$

or

$$A_i(m+1,j+1)=-\frac{e_{ijm}}{e_{mm}}\,,\qquad i,j,m=0,1,\cdots,P\,.$$

3. Numerical Methods and Quantitative Analysis

Using polynomial chaos, the stochastic differential equation produces a set of ordinary differential equations. Although various numerical methods can be used to solve the equations (2.10), including the Euler and Runge-Kutta methods, as indicated we will proceed here to implement and compare the effectiveness of FD and FE methods.

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3.1. Numerical methods

(i) Finite difference method.

With $0 = t^0 < t^1 < \cdots < t^{n+1} = T$ and $\tau = t^{i+1} - t^i$, $i = 0, 1, \cdots, n$, we construct the first-order implicit difference scheme

$$\frac{y_i^{n+1} - y_i^n}{\tau} = f_i(t^{n+1}, y_0^{n+1}, y_1^{n+1}, \cdots, y_P^{n+1}), \quad i = 0, 1, \cdots, P\,, \tag{3.1}$$

or

$$\frac{Y^{n+1} - Y^n}{\tau} = AY^{n+1} + Bg(t^{n+1})\,. \tag{3.2}$$

(ii) Runge-Kutta method (R-K).

With the same mesh points as above, the second-order Runge-Kutta scheme that can be used is

$$\begin{cases} y_i^{n+1} = y_i^n + (K_i^1 + K_i^2)/2\,, \\ K_i^1 = \tau f_i(t^n, y_0^n, y_1^n, \cdots, y_P^n)\,, \\ K_i^2 = \tau f_i(t^{n+1}, y_0^n + K_i^1, y_1^n + K_i^1, \cdots, y_P^n + K_i^1)\,, \end{cases} \tag{3.3}$$

where $i = 0, 1, \cdots, P$, or

$$\begin{cases} Y^{n+1} = Y^n + (K^1 + K^2)/2\,, \\ K^1 = \tau\,(AY^n + Bg(t^n))\,, \\ K^2 = \tau\left(A(Y^n + K^1) + Bg(t^{n+1})\right)\,. \end{cases} \tag{3.4}$$

(iii) Finite element method.

Here we choose the classical p_1-conforming FE method [6, 7]. Standard definitions are used for the Sobolev spaces $H^1([0,T])$ with the usual scalar product, hence for any $v \in H^1([0,T])$ the variational form of Eqs. (3.1) is

$$\left(\frac{\mathrm{d}y_i}{\mathrm{d}t}, v\right) = \left(f_i(t^{n+1}, y_0^{n+1}, y_1^{n+1}, \cdots, y_P^{n+1}), v\right), \quad i = 0, 1, \cdots, P\,, \tag{3.5}$$

or

$$\left(\frac{\mathrm{d}Y}{\mathrm{d}t}, v\right) = A(Y, v) + B(g, v)\,, \tag{3.6}$$

where $(y, v) = \int_a^b yv\mathrm{d}t$.

3.2. Quantitative analysis

Clearly, the properties of coefficient matrix A have a great influence on the long-time stability and convergence of numerical solutions of Eqs. (3.2) or (3.6). According to Eqs. (2.9), in addition to α_0 and α_1 other coefficients α_i are zero ($2 \le i \le P$), so $A = \alpha_0 A_0 + \alpha_1 A_1$ and

$$A_0(m+1, j+1) = -\frac{e_{0jm}}{e_{mm}}, \quad A_1(m+1, j+1) = -\frac{e_{1jm}}{e_{mm}}, \quad j, m = 0, 1, \cdots, P\,. \tag{3.7}$$

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We consider the Gaussian distribution and Poisson distribution in the continuous and discrete case respectively, to analyse the properties of matrix A. Whichever distribution k obeys, A_0 is always the negative unit matrix — i.e.

$$A_0 = \begin{pmatrix} -1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 \end{pmatrix}_{(P+1)\times(P+1)} \tag{3.8}$$

but A_1 is not a constant matrix. When k obeys the Gaussian distribution associated with Hermite polynomial chaos, the weighting function is $w(\xi) = e^{-\xi^2/2}/\sqrt{2\pi}, -\infty < \xi < \infty$, and the orthogonal polynomials satisfy the three-term recurrence relation

$$H_{n+1}(\xi) = \xi H_n(\xi) - nH_{n-1}(\xi), \quad n = 0, 1, \cdots \tag{3.9}$$

together with $H_{-1}(\xi) = 0$ and $H_0(\xi) = 1$. This generates the usual forms

$$H_0(\xi) = 1, \quad H_1(\xi) = \xi, \quad H_2(\xi) = \xi^2 - 1, \quad \cdots \tag{3.10}$$

and determines the value of $\alpha_0 = 0$ and $\alpha_1 = 1$. If $P = 4$, we use symbolic computation to obtain

$$A_1 = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & -2 & 0 & 0 \\ 0 & -1 & 0 & -3 & 0 \\ 0 & 0 & -1 & 0 & -4 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & -2 & 0 & 0 \\ 0 & -1 & 0 & -3 & 0 \\ 0 & 0 & -1 & 0 & -4 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}. \tag{3.11}$$

If we dispose of the obtained orthogonal polynomials as

$$\tilde{H}_n = -H_n, \quad n = 1, 3, 5, \cdots, \qquad \tilde{H}_n = H_n, \quad n = 0, 2, 4, \cdots,$$

the leading coefficients of the polynomial with odd subscript are negative and the new coefficients are $\tilde{\alpha}_0 = 0$ and $\tilde{\alpha}_1 = -1$ such that

$$\tilde{A} = \tilde{\alpha}_0 A_0 + \tilde{\alpha}_1 A_1 = -A.$$

When k obeys Poisson distribution, associated with Charlier polynomial chaos, we have $w(\xi) = \lambda^\xi/\xi$, $\xi = 0, 1, 2, \cdots$, $\lambda > 0$. The Charlier orthogonal polynomials satisfy the three-term recurrence relation

$$\lambda C_{n+1}(\xi) = (n + \lambda - \xi)C_n(\xi) - nC_{n-1}(\xi), \quad n = 0, 1, \cdots \tag{3.12}$$

together with $C_{-1}(\xi) = 0$ and $C_0(\xi) = 1$, so we obtain

$$C_0(\xi) = 1, \quad C_1(\xi) = (\lambda - \xi)/\lambda, \quad C_2(\xi) = [(\lambda - \xi)^2 + (\lambda - \xi) - \lambda]/\lambda^2, \tag{3.13}$$

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and $\alpha_0 = 1,\ \alpha_1 = -1$. If $P = 4$, we use symbolic integration to obtain

$$A_1 = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ -1 & 1 & -2 & 0 & 0 \\ 0 & -1 & 2 & -3 & 0 \\ 0 & 0 & -1 & 3 & -4 \\ 0 & 0 & 0 & -1 & 4 \end{pmatrix}, \quad A = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 1 & -2 & 2 & 0 & 0 \\ 0 & 1 & -3 & 3 & 0 \\ 0 & 0 & 1 & -4 & 4 \\ 0 & 0 & 0 & 1 & -5 \end{pmatrix}. \tag{3.14}$$

The leading coefficients of the Charlier orthogonal polynomial are negative when the subscript is an odd number, so there is no need to deal with the polynomials as before and it follows that

$$\tilde{A} = A\,.$$

We note that the matrix $\tilde{A}$ is three-diagonal, all the elements are positive except on the diagonal, and $\tilde{A}$ is a column diagonal dominant matrix for a discrete distribution. (For other distributions, the matrix $\tilde{A}$ has similar properties.) Omitting the inhomogeneous term, we have the difference scheme

$$Y^{n+1} - Y^n = \tau \tilde{A} Y^{n+1}, \tag{3.15}$$

or

$$(I - \tau\tilde{A})\, Y^{n+1} = Y^n. \tag{3.16}$$

If we take D_1=diag($\tilde{A}$) and $D_2 = \tilde{A} - D_1$, then Eq. (3.16) can be rewritten as

$$(I - \tau D_1 - \tau D_2) Y^{n+1} = Y^n, \tag{3.17}$$

or

$$\left[I - \tau D_2 (I - \tau D_1)^{-1}\right](I - \tau D_1) Y^{n+1} = Y^n. \tag{3.18}$$

From the properties of matrix $\tilde{A}$, the elements of the matrix $I - \tau D_1$ and D_2 are all non-negative, and setting $D = D_2(I - \tau D_1)^{-1}$ all elements of the matrix D are non-negative. From Eq. (3.18), we obtain

$$Y^{n+1} = (I - \tau D_1)^{-1}(I - \tau D)^{-1} Y^n. \tag{3.19}$$

When the time step size τ is small, we can always guarantee $\rho(\tau D) < 1$, so

$$(I - \tau D)^{-1} = \sum_{m=0}^{\infty} (\tau D)^m \approx I + \tau D + \tau^2 D^2 + \tau^3 D^3 \tag{3.20}$$

such that the elements of $(I - \tau\tilde{A})^{-1} = (I - \tau D_1)^{-1}(I - \tau D)^{-1}$ are all non-negative. If the initial value is non-negative, we can guarantee that all numerical solutions of the resulting ordinary differential equations are non-negative, as verified in Section 4 below.

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3.3. Error estimates

In this subsection, we define following two errors for the mean and variance of the solution:

$$\varepsilon_{mean}(t) = \left| \frac{\bar{y}_{Number}(t) - \bar{y}(t)}{\bar{y}(t)} \right|, \qquad \varepsilon_{var}(t) = \left| \frac{\sigma_{Number}(t) - \sigma(t)}{\sigma(t)} \right|, \tag{3.21}$$

where $\bar{y}(t) = E[y(t)]$ is the mean and $\sigma(t) = E[(y(t) - \bar{y}(t))^2]$ is the variance of the solution. In general, the relation of probability density function (PDF) $f(\xi)$ to the weighting function $w(\xi)$ is $f(\xi) = cw(\xi)$, where c is a positive constant. In the case of a continuous distribution,

$$\bar{y}(t) = E[y(t)] = \int_S \left(c_0 e^{-\xi t} + \int_0^t e^{\xi(s-t)} g(s) \mathrm{d}s \right) f(\xi) \mathrm{d}\xi \,, \tag{3.22}$$

$$\sigma(t) = E\left[(y(t) - \bar{y}(t))^2 \right] = \int_S \Big(y(t) - \bar{y}(t) \Big)^2 f(\xi) \mathrm{d}\xi \,; \tag{3.23}$$

and in the case of a discrete distribution,

$$\bar{y}(t) = E[y(t)] = \sum_{\xi=0}^{N} \left(c_0 e^{-\xi t} + \int_0^t e^{\xi(s-t)} g(s) \mathrm{d}s \right) f(\xi) \,, \tag{3.24}$$

$$\sigma(t) = E\left[(y(t) - \bar{y}(t))^2 \right] = \sum_{\xi=0}^{N} \Big(y(t) - \bar{y}(t) \Big)^2 f(\xi) \,. \tag{3.25}$$

In both cases we also obtain

$$y_{Number}(t) = \sum_{i=0}^{P} y_i(t) H_i \,, \tag{3.26}$$

$$\bar{y}_{Number}(t) = E\left[\sum_{i=0}^{P} y_i(t) H_i \right] = c \langle H_0 H_0 \rangle y_0(t) \,, \tag{3.27}$$

$$\sigma_{Number}(t) = c \sum_{i=0}^{P} \langle H_i H_i \rangle y_i(t)^2 - c^2 \langle H_0 H_0 \rangle^2 y_0(t)^2 \,. \tag{3.28}$$

From the above, we get the error of mean associated with $y_0(t)$, but the error of the variance for every $y_i(t)$, $i = 0, 1, \cdots, P$.

4. Numerical Experiments

We present numerical solutions for Eq. (1.1) using polynomial chaos expansions. We consider three continuous distributions — viz. the Gaussian distribution, Gamma distribution and Beta distribution; and then three discrete distributions — viz. the Poisson distribution, Binomial distribution and Negative Binomial distribution. Here we do not present

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numerical results using the Runge-Kutta method, as some relevant results can be found in Ref. [12]. Without loss of generality, we take $T = 1$ and $\hat{y} = 1$. Since $c_0 = 1$, it follows that the solution y and deterministic solution y_D are

$$y(t;\xi) = e^{-\xi t} + \int_0^t e^{\xi(s-t)} g(s)\mathrm{d}s\,, \quad y_D(t) = e^{-\bar{\xi} t} + \int_0^t e^{\bar{\xi}(s-t)} g(s)\mathrm{d}s\,. \tag{4.1}$$

4.1. Continuous distributions

(I) Gaussian distribution and Hermite chaos.
Assume that ξ obeys the Gauss distribution with PDF

$$f(\xi) = \frac{1}{\sqrt{2\pi}} e^{-\xi^2/2}\,, \quad -\infty < \xi < \infty\,, \tag{4.2}$$

and has zero mean value $\bar{\xi} = 0$. Eq. (1.1) can be turned into a series of ordinary differential equations using the Hermite polynomial chaos, when the FD and FE methods can be used. If we take $g(t) = \sin(t)$, then the deterministic solution is

$$y_D(t) = 2 - \cos(t). \tag{4.3}$$

In this case, we have $g(t) \geq 0$ for all $t \in [0, T]$, so the inhomogeneous term does not affect the positive or negative feature of the solution, and we obtain the non-negative numerical solutions of y_i $(i = 0, 1, \cdots, P)$. Fig. 1 shows the solution solved by the FE method. The solution solved by the FD method is almost the same. Fig. 2 shows error convergence of the mean and variance by the FE and FD methods, respectively. We see that if the mesh size is fixed, the errors tend to be stable when the degree of orthogonal polynomial reaches a certain value. On the other hand, when the highest order of orthogonal polynomial is fixed the errors decrease with decreasing mesh size. In addition, the convergence of the FE method is better than that of the FD method.

(II) Gamma distribution and Laguerre chaos.
Assume that ξ is the Gamma distribution with PDF

$$f(\xi;\alpha) = \frac{e^{-\xi}\xi^{\alpha}}{\Gamma(\alpha+1)}\,, \quad 0 \leq \xi < \infty, \quad \alpha > -1\,, \tag{4.4}$$

and the mean of ξ is $\bar{\xi} = \alpha + 1$. If we take $g(t) = 0$ and $\alpha = 0$, the mean exact and deterministic solutions are

$$\bar{y}(t) = \int_0^{\infty} y(t;\xi) f(\xi)\mathrm{d}\xi = \frac{1}{1+t}\,, \qquad y_D(t) = e^{-t}\,. \tag{4.5}$$

Here we use Laguerre chaos to solve Eq. (1.1). Fig. 3 shows the solution using the FE method, which is almost the same as that obtained by the FD method, and the y_i non-negative. Fig. 4 shows that the mean and variance errors for the FE and FD methods differ for different values of α.

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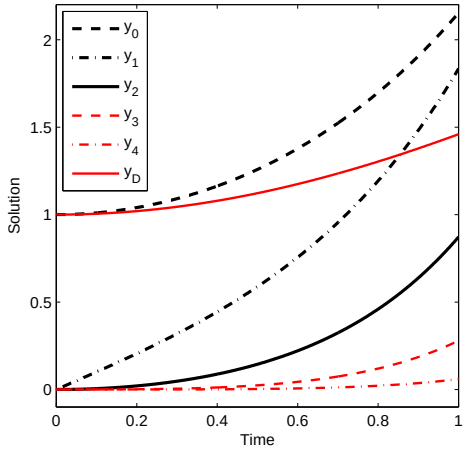
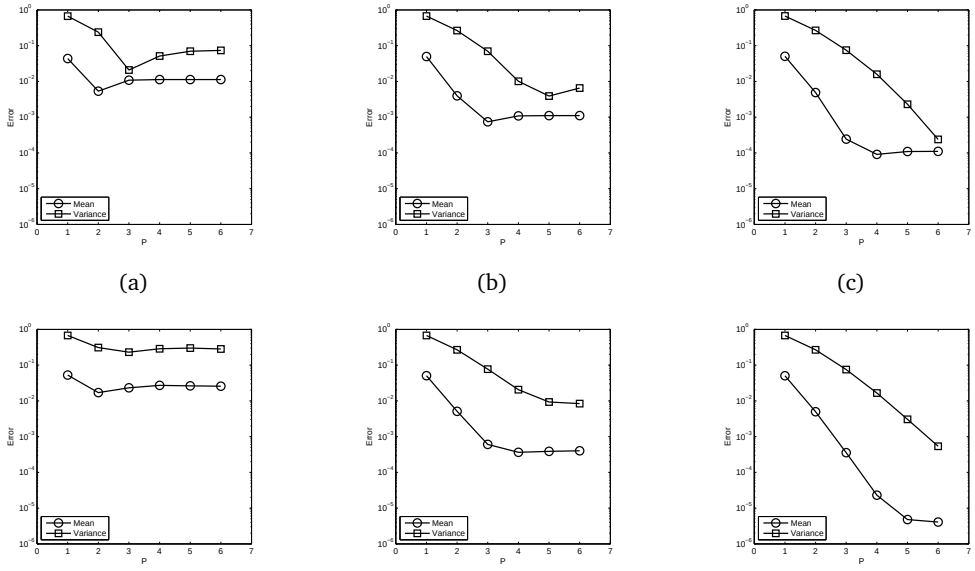


Figure 1: Solution for Gaussian random input by the FE method when $P = 4$ and $\tau = 10^{-3}$.



(a) (b) (c)

(d) (e) (f)

Figure 2: For the Gauss distribution when $g(t) = \sin(t)$, the error in the mean and the variance for the FD method: (a) $\tau = 10^{-2}$, (b) $\tau = 10^{-3}$, (c) $\tau = 10^{-4}$ — and the error in the mean and the variance for the FE method: (d) $\tau = 10^{-1}$, (e) $\tau = 10^{-2}$, (f) $\tau = 10^{-3}$.

(III) Beta distribution and Jacobi chaos.

Assume that ξ is the Beta distribution with PDF

$$f(\xi;\alpha,\beta) = \frac{(1-\xi)^{\alpha}(1+\xi)^{\alpha}}{2^{\alpha+\beta+1}B(\alpha+1,\beta+1)}\,, \quad -1 < \xi < 1\,,\ \alpha,\beta > -1\,, \tag{4.6}$$

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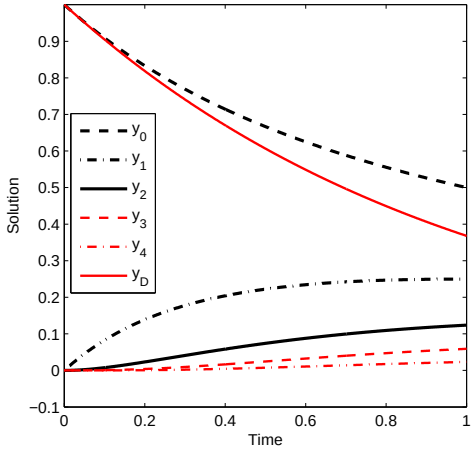


Figure 3: Solution for Gamma random input by the FE method when $P=4$, $\alpha=0$ and $\tau=10^{-3}$.

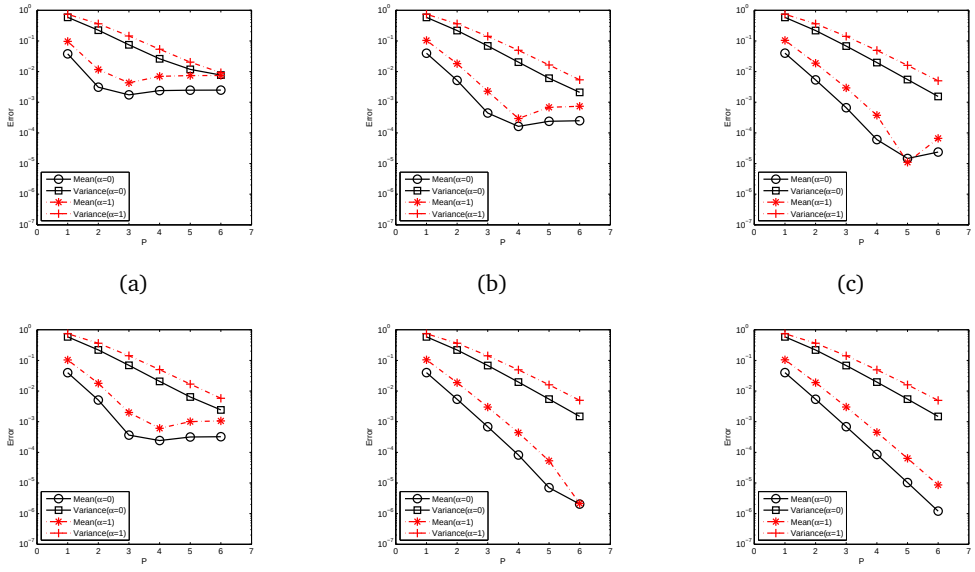


Figure 4: For the Gamma distribution when $g(t)=0$, the error in the mean and the variance for the FD method: (a) $\tau=10^{-2}$, (b) $\tau=10^{-3}$, (c) $\tau=10^{-4}$ — and the error in the mean and the variance for the FE method: (d) $\tau=10^{-1}$, (e) $\tau=10^{-2}$, (f) $\tau=10^{-3}$.

where $B(\alpha,\beta)=\Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha+\beta)$ is the Beta-function. If $g(t)=0$, $\alpha=0$ and $\beta=0$, the mean of k is $\bar{\xi}=0$, and the mean exact and deterministic solutions are

$$\bar{y}(t)=\int_{-1}^{1} y(t;\xi)f(\xi)\mathrm{d}\xi=\frac{e^{t}-e^{-t}}{t},\quad y_D(t)=1. \tag{4.7}$$

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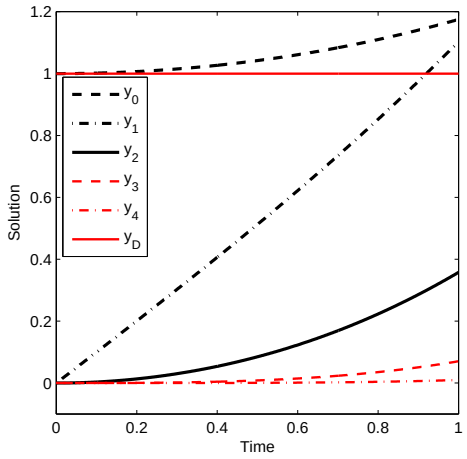
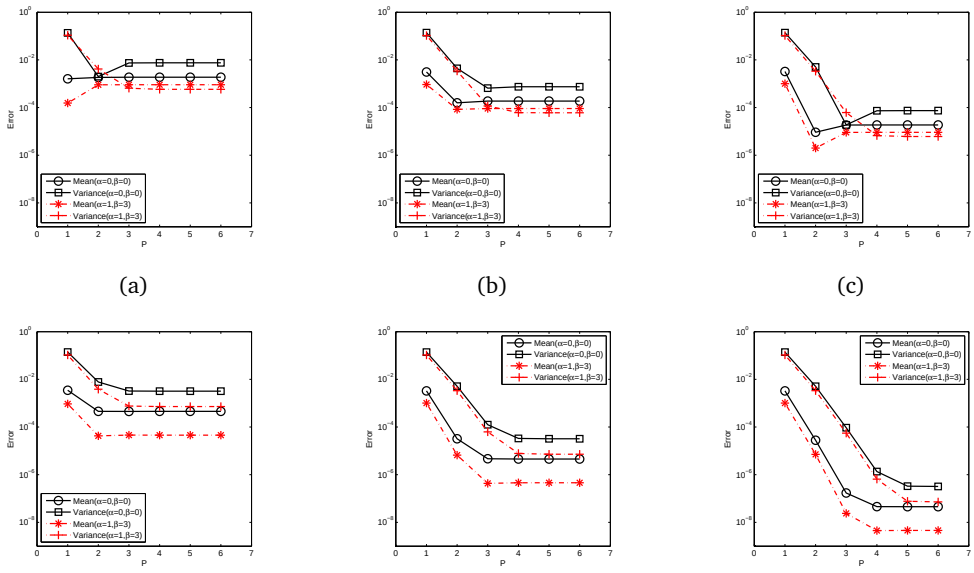


Figure 5: Solution with Beta random input by the FE method when $P=4$, $\alpha=0$, $\beta=0$ and $\tau=10^{-3}$.



(a) (b) (c)

(d) (e) (f)

Figure 6: For the Beta distribution when $g(t)=0$, the error in the mean and the variance by FD method: (a) $\tau=10^{-2}$, (b) $\tau=10^{-3}$, (c) $\tau=10^{-4}$ — and the error in the mean and the variance by FE method: (d) $\tau=10^{-1}$, (e) $\tau=10^{-2}$, (f) $\tau=10^{-3}$.

Here the Jacobi chaos is used to solve Eq. (1.1). Fig. 5 shows the solution by the FE method, and again the solution by the FD method is almost the same. For different values of α and β, Fig. 6 shows the error of the mean and variance solutions by the FD and FE methods are also different, and as in the case above the numerical solutions for y_i are non-negative. Further, the errors change with different α and β.

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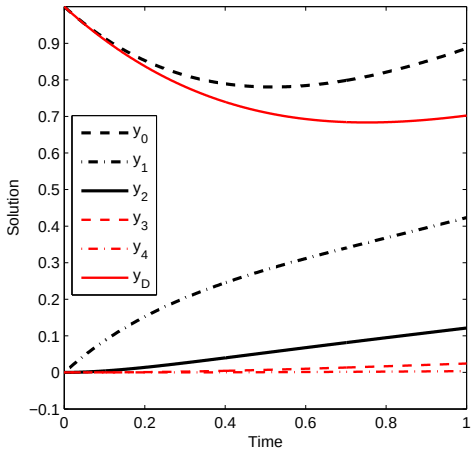


Figure 7: Solution with Poisson random input by FE method when $P=4$, $\lambda=1$ and $\tau=10^{-3}$.

4.2. Discrete distributions

(IV) Poisson distribution and Charlier chaos.
Assume that ξ is the Poisson distribution with PDF

$$f(\xi;\lambda)=e^{-\lambda}\frac{\lambda^{\xi}}{\xi!}, \quad \xi=0,1,\cdots, \ \lambda>0, \tag{4.8}$$

and the mean of ξ is $\bar{\xi}=\lambda$. We employ the Charlier chaos expansion in Eq. (1.1). If $g(t)=\sin(t)$ and $\lambda=1$, then the deterministic solution is

$$y_D(t)=\frac{1}{2}\left(\sin(t)-\cos(t)+3e^{-t}\right). \tag{4.9}$$

Fig. 7 shows the solution solved the FE method, and Fig. 8 shows errors of the mean and the variance for the FD and FE methods with different values of λ. As in the continuous case, if the mesh size is fixed the errors tend to be stable when the degree of orthogonal polynomial reaches a certain value, but when the highest order of orthogonal polynomial is fixed they decrease with decreasing of mesh size and change with different λ. The numerical solutions of y_i are again all non-negative.

(V) Binomial distribution and Krawtchouk chaos.
We assume ξ to be a Binomial distribution with PDF

$$f(\xi;q,N)=\begin{pmatrix} N \\ \xi \end{pmatrix} q^{\xi}(1-q)^{N-\xi}, \quad \xi=0,1,\cdots,N, \ 0\le q\le 1, \tag{4.10}$$

and the mean of ξ is $\bar{\xi}=Nq$. Here we employ Krawtchouk chaos expansion to Eq. (1.1). If $g(t)=0$, $q=0.5$ and $N=5$, the mean exact solution and deterministic solutions are

$$\bar{y}(t)=\sum_{\xi=0}^{N} y(t;\xi)f(\xi)=\left(\frac{1}{2}+\frac{e^{-t}}{2}\right)^5, \quad y_D(t)=e^{-\frac{5}{2}t}. \tag{4.11}$$

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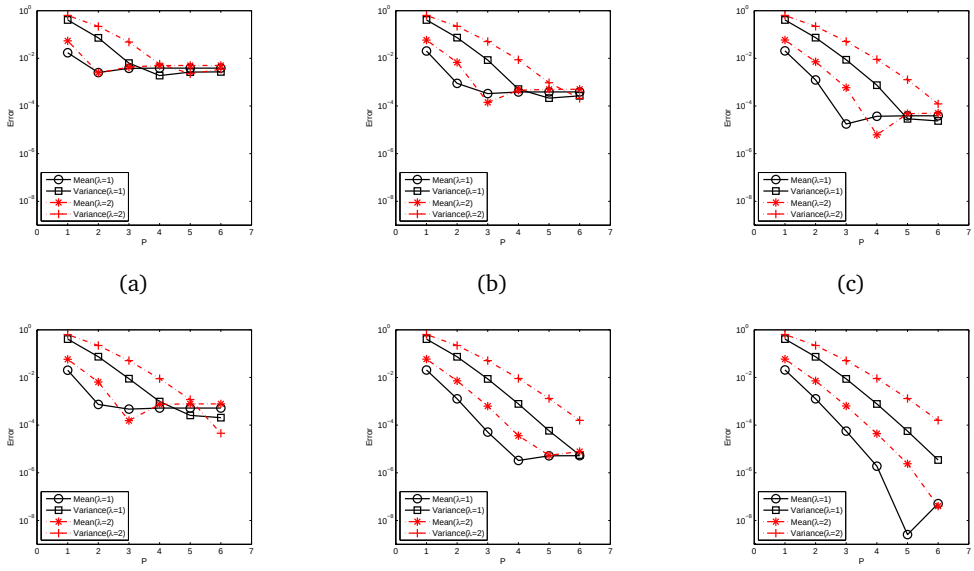


Figure 8: For the Poisson distribution when $g(t) = \sin(t)$, the error in the mean and the variance for the FD method: (a) $\tau = 10^{-2}$, (b) $\tau = 10^{-3}$, (c) $\tau = 10^{-4}$ — and the error in the mean and the variance for the FE method: (d) $\tau = 10^{-1}$, (e) $\tau = 10^{-2}$, (f) $\tau = 10^{-3}$.

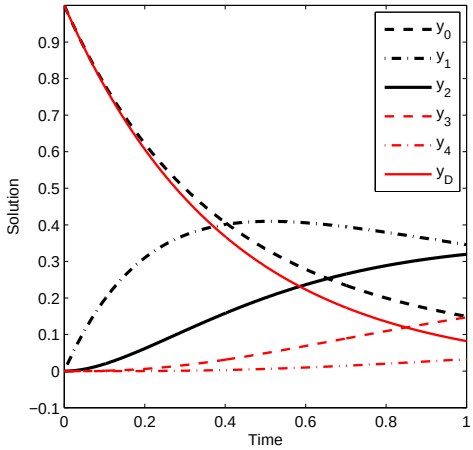


Figure 9: Solution with Binomial random input by the FE method when $P = 4$, $q = 0.5$, $N = 5$ and $\tau = 10^{-3}$.

Fig. 9 shows the numerical solution by the FE method, and Fig. 10 shows errors of the mean and variance for the FE and FD methods with different values of q and N. The errors change with different q and N, and the numerical solutions y_i are all non-negative once again.

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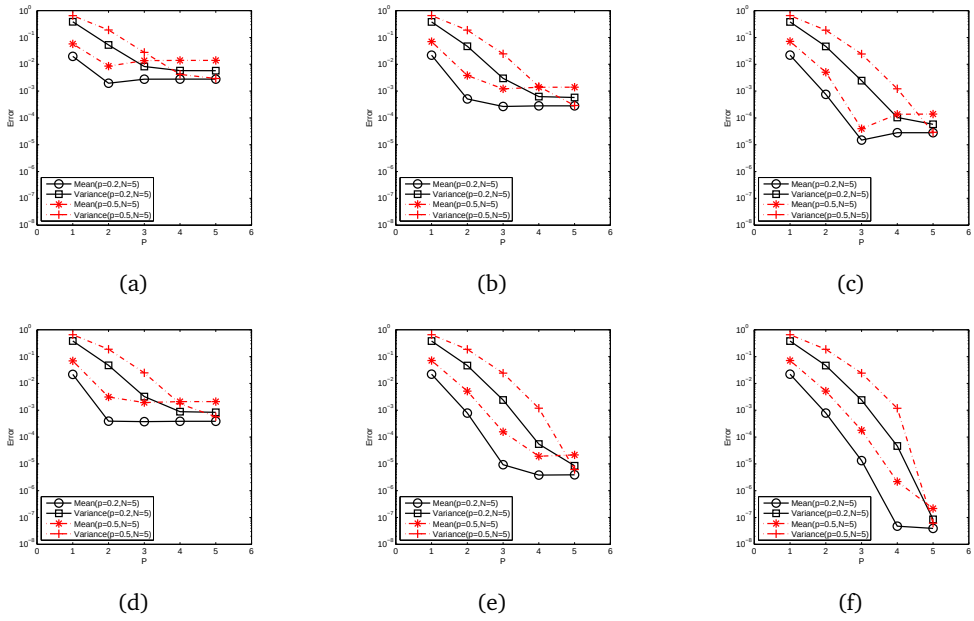


Figure 10: For the Binomial distribution when $g(t)=0$, the error in the mean and the variance for the FD method: (a) $\tau=10^{-2}$, (b) $\tau=10^{-3}$, (c) $\tau=10^{-4}$ — and the error in the mean and the variance for the FE method: (d) $\tau=10^{-1}$, (e) $\tau=10^{-2}$, (f) $\tau=10^{-3}$.

(VI) Negative binomial distribution and Meixner chaos.
Assume ξ to be the negative Binomial distribution with PDF

$$f(\xi;\beta,c)=\frac{(\beta)_\xi}{\xi!}(1-c)^\beta c^\xi\,,\quad \xi=0,1,\cdots,\ 0\le c\le 1\,,\ \beta>0\,, \tag{4.12}$$

where

$$(\beta)_\xi=\begin{cases}1\,, & \xi=0\,,\\ \beta(\beta+1)\cdots(\beta+\xi-1)\,, & \xi=1,2,\cdots\,.\end{cases} \tag{4.13}$$

and the mean of $k(\omega)$ is $\bar{k}=\beta(1-c)/c$. Here we employ Meixner chaos in Eq. (1.1). If $g(t)=0$, $\beta=1$ and $c=0.5$, the mean exact solution and deterministic solutions are

$$\bar{y}(t)=\sum_{\xi=0}^{\infty}y(t;\xi)f(\xi)=\frac{1}{2-e^{-t}},\qquad y_D(t)=e^{-t}. \tag{4.14}$$

Fig. 11 shows the solution by the FE method, and Fig. 12 shows the error of the mean and variance for the FE and FD methods for different values of β and c. Again, the errors change with different β and c, and the y_i are all non-negative.

5. Conclusions

For different distributions, a stochastic differential equation turns into a set of ordinary differential equations for the corresponding polynomial chaos. For this set of ordinary dif-

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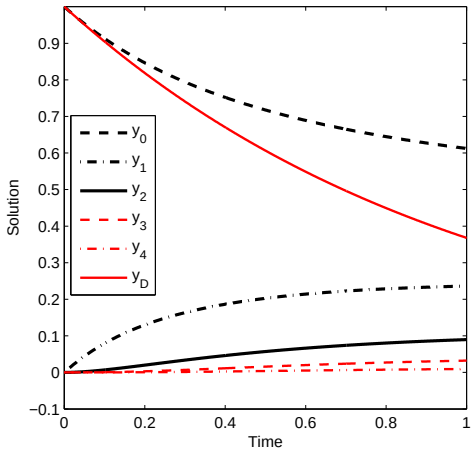
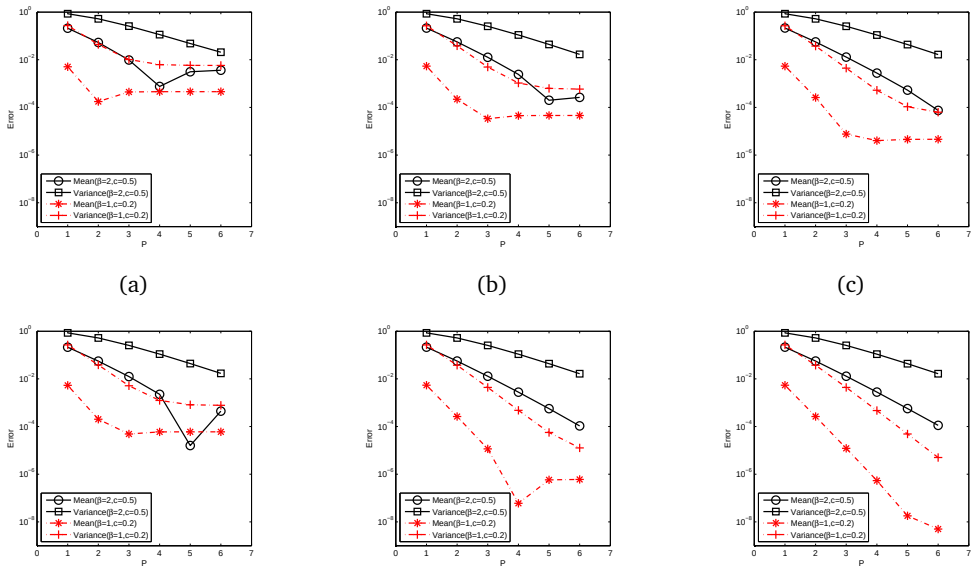


Figure 11: Solution with negative Binomial random input by the FE method when $P = 4$, $\beta = 1$, $c = 0.5$, $\tau = 10^{-3}$.



(a) (b) (c)

(d) (e) (f)

Figure 12: For the negative Binomial distribution, the error in the mean and variance for the FD method: (a) $\tau = 10^{-2}$, (b) $\tau = 10^{-3}$, (c) $\tau = 10^{-4}$ — and the error in the mean and the variance for the FE method: (d) $\tau = 10^{-1}$, (e) $\tau = 10^{-2}$, (f) $\tau = 10^{-3}$.

ferential equations, the resulting solutions are all non-negative. When the highest order of orthogonal polynomial is fixed, the error of first-order implicit FD and FE methods decrease with decreasing mesh size. On the other hand, when the mesh size is fixed, these errors tend to be stable if the degree of the orthogonal polynomial reaches a certain value. In

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addition, the convergence of the FE method is generally better than the FD method in the same circumstances.

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