SUBGROUPS OF ALGEBRAIC GROUPS CONTAINING REGULAR UNIPOTENT ELEMENTS

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To the memory of Brian Hartley

Let G be a simple algebraic group over an algebraically closed field K of characteristic p with $p \geqslant 0$. Then G contains finitely many conjugacy classes of unipotent elements. There is a unique class of unipotent elements of largest dimension – the class of *regular unipotent elements*; the centralizer of any such element is a unipotent group of dimension equal to the rank of G. For example, if G is the linear group $\mathrm{SL}(V)$ on the finite dimensional vector space V over K, then the regular unipotent elements are precisely the unipotent matrices consisting of a single Jordan block.

In this paper we determine the maximal closed subgroups of G which contain regular unipotent elements. Maximal subgroups are either reductive or parabolic. Since parabolic subgroups contain representatives of all unipotent classes, we concentrate on reductive subgroups.

The main results give a complete classification of maximal closed (not necessarily connected) reductive subgroups containing regular unipotent elements. The results are stated separately for classical and exceptional groups. The most interesting examples occur for G of exceptional type and we state this result first. Throughout the paper, T_k will denote a torus of dimension k and h will denote the Coxeter number of G.

THEOREM A. *Let X be a maximal closed reductive subgroup of the exceptional group G with X of positive dimension. Then X contains a regular unipotent element of G if and only if the containment $X < G$ is one of the following:*

(a) $A_1 < G$ (*for* $p = 0$ *or* $p > h$);
(b) $F_4 < E_6$;
(c) $A_2.2 < G_2$ (*for* $p = 2$);
(d) $D_4.S_3 < F_4$ (*for* $p = 3$);
(e) $(D_4\, T_2).S_3 < E_6$ (*for* $p = 3$);
(f) $(E_6\, T_1).2 < E_7$ (*for* $p = 2$);
(g) $A_1^7.L_3(2) < E_7$ (*for* $p = 7$).

To state the result for classical groups, it is convenient to take $G = \mathrm{I}(W)$, the isometry group of the vector space W. Here we regard G as stabilizing a form on its natural module W; we obtain that $\mathrm{I}(W)$ is equal to $\mathrm{SL}(W)$, $\mathrm{Sp}(W)$ or $\mathrm{SO}(W)$, according to the form being, respectively, the zero form, a non-degenerate alternating form or a non-degenerate quadratic form. We shall sometimes write $\mathrm{I}(d)$ to indicate a classical group over a space of dimension d. For orthogonal groups G we take

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dim $W \geqslant 7$. When $p = 2$, we omit orthogonal groups in odd dimension, since these are (abstractly) isomorphic to the corresponding symplectic groups.

THEOREM B. *Let X be a maximal closed reductive subgroup of $G = \mathrm{I}(W)$ of positive dimension. Then X contains a regular unipotent element of G if and only if the containment $X < G$ satisfies one of the following:*

(i) *X is reducible on W:*
  (a) $B_{l-1} < D_l$;
  (b) $(D_{l-1} T_1)2 < D_l$ (*for* $p = 2$);

(ii) *X is irreducible and imprimitive on W:*
  (a) $\mathrm{stab}_{\mathrm{I}(W)}\{W_1, \ldots, W_k\} < \mathrm{I}(W)$, *where* $W = W_1 \perp \cdots \perp W_k$ *and* $\dim W_i = \dim W/k$ *for all i; further*,

$$k = p^s \text{ if } G \neq D_l \quad (\text{for } p > 0);$$
$$k = l = 2^s + 1 \text{ if } G = D_l \quad (\text{for } p = 2);$$

  (b) $A_{l-1} T_1.2 < D_l$ *with l even* (*for* $p = 2$);

(iii) *X^0 is irreducible, but tensor decomposable on W:*
  (a) $A_1^2.2 < C_2$ (*for* $p = 2$);
  (b) $A_1^3.S_3 < C_4$ (*for* $p = 3$);

(iv) *X^0 is irreducible and tensor indecomposable on W:*
  (a) $A_1 < \mathrm{I}(W)$ (*for* $p = 0$ *or* $p > h$);
  (b) $\mathrm{Sp}(W) < \mathrm{SL}(W)$ (*for* $\dim W$ *even*);
  (c) $\mathrm{SO}(W) < \mathrm{SL}(W)$ (*for* $\dim W$ *odd*, $p \neq 2$);
  (d) $\mathrm{SO}(W).2 < \mathrm{Sp}(W)$ (*for* $p = 2$);
  (e) $G_2 < \mathrm{SO}_7$ (*respectively* Sp_6 *if* $p = 2$);
  (f) $A_2.2 < \mathrm{Sp}_8$ (*for* $p = 2$);
  (g) $B_3 < \mathrm{SO}_8$.

We remark that in even characteristic the group C_3 has the spin representation of dimension 8; this however factors through B_3, yielding the configuration in (g).

We note that there are very few connected examples.

A special case of Theorem B has also been proved by I. D. Suprunenko [**15**, 1.9].

The proof of Theorem A is in Section 3 while Section 2 contains the proof of Theorem B. In Section 4 we establish the following consequences of the above theorems.

THEOREM C. *Assume that $p = 0$ or that p is a good prime. Let G act morphically and primitively on an affine algebraic variety Ω. Then some unipotent element of G is fix-point-free on Ω, unless $p = 2$ and (G, X) is one of $(B_l$ or C_l, $D_l.2)$, $(G_2, A_2.2)$ or $(C_2, A_1^2.2)$, where X is the stabilizer of a point in Ω.*

We remark that if X is a parabolic subgroup then every element of G fixes points in Ω, so the hypothesis that Ω is affine is necessary.

Finally, we obtain some information on primitive actions where a unipotent element fixes a finite non-empty subset.

THEOREM D. *Assume that $p = 0$ or that p is odd and a good prime. Let G act morphically and primitively on a variety Ω. Suppose that a unipotent element u of G*

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fixes f points of Ω, *where* $0 < f < \infty$. *Then the point stabilizer is a maximal parabolic subgroup of* G, u *is a regular unipotent element and* $f = 1$.

1. *Preliminaries*

Throughout the paper G will be a simple algebraic group over a field K in characteristic p (possibly 0). Write Φ for the corresponding root system, with a fixed subsystem Φ^+ of positive roots, and a system Δ of fundamental roots. Fixing a maximal torus T of G, we let U_α denote the T-root subgroup of G corresponding to the root α.

There is a convenient survey of the basic properties of regular unipotent elements in connected reductive algebraic groups in [**2**, Section 5.1]. In particular, the existence, as well as the uniqueness, of the conjugacy class is established in [**2**, 5.1.2]. We shall use freely the following well-known criterion.

LEMMA 1.1. [**2**, 5.1.3]. *Let* u *be a unipotent element in the connected reductive algebraic group* G. *The following conditions are equivalent:*

(i) u *is regular;*

(ii) u *lies in a unique Borel subgroup of* G;

(iii) u *is conjugate to an element* $\prod U_\alpha(\lambda_\alpha)$, *where the product is over the positive roots* $\alpha \in \Phi^+$ *and* $\lambda_\alpha \neq 0$ *for all the fundamental roots* $\alpha \in \Delta$.

Next we present a few preliminary results concerning the case where G is a classical algebraic group. We write W for the underlying module of G, and put $d = \dim W$. If G is of type B_l in even characteristic, W is taken to be the irreducible symplectic module of dimension $2l$. Information on the conjugacy classes in G is contained in the paper of Wall [**17**] (see also [**2**, Sections 5.11 and 13.1]). Our next lemma contains all the information we shall need.

LEMMA 1.2. (i) *The regular unipotent elements in* A_l, B_l *and* C_l *have a unique Jordan block on* W. *Indeed,* B_l *and* C_l *contain a regular unipotent of the corresponding linear group on* W.

(ii) *The regular unipotent element in* D_l *has two Jordan blocks, of size* 1 *and* $2l-1$ *if* $p \neq 2$, *and size* 2 *and* $2l-2$ *if* $p = 2$. *If* $p = 2$ *then* $D_l.2$ *contains a regular unipotent of* A_{2l-1}.

We remark that in each case, the class of regular unipotent elements is determined uniquely by their Jordan normal form.

Proof of Lemma 1.2. (i) The assertion is clear for A_l. Indeed, identifying A_l with SL_{l+1} in the usual way, it is easy to check that $\prod U_{\alpha_i}(1)$ is a single Jordan block, where the product is taken over all fundamental roots. By Lemma 1.1 this is a regular unipotent element. For B_l, C_l note that these groups occur as fixed points under an involutory graphs automorphism of A_{2l}, A_{2l-1}, respectively. The graph automorphism, say τ, permutes root groups for roots in Δ. Let α be a fundamental root. There are three possibilities: $\alpha^\tau = \alpha$; $\alpha^\tau = \beta$ with β orthogonal to α; and $\alpha^\tau = \beta$ with α, β adjacent in the Dynkin diagram. Correspondingly, we obtain fixed points $U_\alpha(1)$, $U_\alpha(1)\,U_\beta(1)$, $U_\alpha(1)\,U_\beta(1)\,U_{\alpha+\beta}(-1/2)$, except that the last case does not occur for $p = 2$. Multiplying these fixed points over all orbits and rearranging, we see from Lemma 1.1

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that the result is a regular unipotent element in the appropriate classical group. If $p = 2$ the fixed point group of A_{2l} is B_l and does not contain a regular element. However, our convention is that we take W as the symplectic module on which B_l induces the full symplectic group. So the result follows, as we have seen that C_l contains a regular unipotent element of A_{2l-1}.

(ii) First note that B_{l-1} is the centralizer of the usual involutory graph automorphism τ of D_l and the argument of (i) shows that this group contains a regular unipotent element of D_l.

If the defining characteristic is not 2, the assertion about the Jordan form follows from this observation and (i), noting that the representation of B_{l-1} is semisimple. So assume that $p = 2$. We claim that $D_l.2$ contains a regular unipotent of B_l. To see this, consider the usual fundamental root elements $U_{\alpha_1}(1), \ldots, U_{\alpha_l}(1)$ of B_l. Then D_l contains $U_{\alpha_1}(1), \ldots, U_{\alpha_{l-1}}(1)$, and $D_l.2$ contains also s_l, the lth reflection (as $D_l.2$ contains a maximal torus and the corresponding Weyl group of B_l). Now

$$s_l = U_{\alpha_l}(1)\, U_{-\alpha_l}(1)\, U_{\alpha_l}(1) = U_{-\alpha_l}(1)\, U_{\alpha_l}(1)\, U_{-\alpha_l}(1) = U_{\alpha_l}(1)^{U_{-\alpha_l}(1)}.$$

Hence $D_l.2$ contains

$$U_{\alpha_1}(1) \cdots U_{\alpha_{l-1}}(1)\, U_{\alpha_l}(1)^{U_{-\alpha_l}(1)} = (U_{\alpha_1}(1) \cdots U_{\alpha_l}(1))^{U_{-\alpha_l}(1)}.$$

It follows that the subgroup $D_{l-1}.2$ of D_l contains a regular unipotent u of D_l.

Now $D_{l-1}.2$ acts on a nondegenerate space, say E, of dimension $2(l-1)$. Viewing this group within B_{l-1}, it follows that E can be identified within a module on which B_{l-1} induces the symplectic group, C_{l-1}. It therefore follows from (i) that u induces a single Jordan block on E. But u also induces a Jordan block on the orthogonal 2-space, so this completes the argument.

Next we state two elementary observations on powers of Jordan blocks.

LEMMA 1.3. (i) *If u is an indecomposable unipotent Jordan block matrix of size k over a field of positive characteristic p, and if p^a is a power of p satisfying $k = cp^a + b$ with $0 \leqslant b < p^a$, then u^{p^a} has p^a Jordan blocks, of which b have size $c+1$ and the remaining $p^a - b$ have size c.*

(ii) *If u is transitive on the set $\{W_1, \ldots, W_k\}$ of summands of a direct decomposition of V and if u^k induces a single Jordan block on each of the W_i, then u is a single Jordan block on V.*

Next we collect together some useful information on the orders of regular unipotent elements. For classical groups, this is immediate from the above two results; for exceptional groups, we refer to [**16**] or [**5**].

LEMMA 1.4. *Let $p > 0$ and let p^a be the order of a regular unipotent of G. If G is classical then a is minimal such that p^a is greater or equal to d (respectively $d-2$ for G of type D_l). If G is exceptional, then a is minimal with $p^a > 29, 17, 11, 11, 5$, according to $G = E_8, E_7, E_6, F_4$, or G_2.*

Finally we give an easy result on tensor products of matrices.

LEMMA 1.5. *Let the matrix c be the tensor product of the $r \times r$ unipotent matrix a and the $s \times s$ matrix b, with $1 < r \leqslant s$. Then c has at least two Jordan blocks. Moreover, if*

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c has precisely two blocks, then a and b are indecomposable Jordan blocks, $r = 2$ and the blocks of c have sizes $s-1$ and $s+1$ (respectively both have size s in the case where p divides s).

Proof. First observe that if J_i, J_j and J_k are Jordan blocks, then $(J_i \oplus J_j) \otimes J_k = (J_i \otimes J_k) \oplus (J_j \otimes J_k)$.

Assume now that a and b are indecomposable Jordan blocks, with respect to bases $v_1, \ldots, v_r$ and $w_1, \ldots, w_s$, respectively. Then the vectors $v_r \otimes w_s$ and $v_{r-1} \otimes w_s - v_r \otimes w_{s-1}$ are fixed by c. Furthermore, if both r and s are greater than 2, then the vector

$$v_{r-2} \otimes w_s + v_r \otimes w_{s-2} + v_r \otimes w_{s-1} - v_{r-1} \otimes w_{s-1}$$

is also fixed. Finally, to prove the last assertion, take a to be the 2×2 Jordan block. One checks that the fixed space has dimension 2. By induction on n, we have

$$(c-I)^n = \begin{pmatrix} b-I & b \\ 0 & b-I \end{pmatrix}^n = \begin{pmatrix} (b-I)^n & nb(b-I)^{n-1} \\ 0 & (b-I)^n \end{pmatrix},$$

so $(c-I)^{s+1} = 0$ (and in fact $(c-I)^s = 0$ if p divides s).

2. *Classical groups*

Throughout this section G is a classical simple algebraic group over an algebraically closed field in characteristic $p \geqslant 0$. Our goal here is to prove Theorem B. We assume that X is a closed reductive subgroup of positive dimension containing a regular unipotent element. In the first part we assume also that X is maximal amongst closed reductive subgroups of G. We continue to write W for the natural module for G and d for the dimension of W.

The first proposition deals with the case where X^o is not a simple group acting irreducibly and tensor-indecomposably on W. The main tool here is the recent reduction theorem of Liebeck and Seitz [**8**].

PROPOSITION 2.1. *Let X be a maximal closed reductive subgroup of positive dimension of the classical simple group G acting naturally on the space W, and assume that X contains a regular unipotent element of G. Assume also that X^o is not a simple group acting on W irreducibly and tensor-indecomposably. Then* (i), (ii), *or* (iii) *of Theorem B holds. Conversely, each of these subgroups X contains a regular unipotent element of G.*

Proof. We are assuming that it is not the case that X^o is a simple group acting irreducibly and tensor indecomposably on W. Consequently, the main result in [**8**] determines the structure of X and the action of X on W. In what follows, we consider the various possibilities given by that result.

(a) First assume that X is reducible on W. If X stabilizes a singular subspace, then X is contained in a parabolic subgroup P of G. The maximality of X forces $X = P$, contrary to hypothesis. Therefore, X stabilizes a proper nonsingular subspace of W and the assertion now follows from the Jordan normal form of the regular unipotent elements given in Lemma 1.2.

(b) Next assume that X preserves a tensor decomposition $W = W_1 \otimes W_2$, with $\dim W_1 \neq \dim W_2$. Then Lemmas 1.2 and 1.5 force $d \leqslant 6$ and G to be of type D. This is not so as we are assuming that $\dim(W) \geqslant 7$ when G is orthogonal.

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(c) Next assume that X^o stabilizes the tensor decomposition $W = W_1 \otimes \cdots \otimes W_k$ with $\dim \mathrm{W_i} = \mathrm{r}$ for all i, for some r and k. Then $d = r^k$. If $p = 0$ then any power of u is again a regular unipotent (cf. [**2**, 5.1.4]); but here $u^{k!}$ is a tensor product, contrary to Lemma 1.5. So $p > 0$. Let p^a be the largest power of p with $p^a \leqslant k$, and let p^b be the smallest power with $p^b \geqslant r$. Then u^{p^a} is a tensor product of matrices, each of order dividing p^b. It follows that $|u^{p^a}|$ divides p^b, and so $|u|$ divides p^{a+b}. On the other hand, by Lemma 1.4, $|u| \geqslant d-2 = r^k - 2 \geqslant r^{p^a} - 2$. Since $r > p^{b-1}$, we have $(b-1)p^a \leqslant a+b$. We deduce that either (i) $b = a = 1$, $r = 2$, and $p \leqslant 3$ or (ii) $a = 1$ and $b = p = 2$. In the first case, maximality of X implies that X is $A_1^2 2$ with $p = 2$ or $A_1^3 S_3$ with $p = 3$, and G is C_2 or C_4, respectively. In the second case $r = 3$ and the only possibility is $k = 2$. But then $G = \mathrm{SL}_9$ (recall that for $p = 2$ we do not consider $\mathrm{SO}(W)$ when $\dim(W)$ is odd) and u has a single Jordan block. However, here $|u| = 8$, a contradiction.

(d) Next suppose that X preserves a partition $W = W_1 \oplus \cdots \oplus W_k$ with all $\dim \mathrm{W}_i$ equal to m, say, so $d = km$, and with the W_i nondegenerate if G is not of type A. If $p = 0$, then all powers of u are regular. But then some regular element acts on each of the W_i, contradicting Lemma 1.2. Hence $p > 0$.

Assume first that $G \neq D_l$. Since u has just one Jordan block, of size d, Lemma 1.4 forces k to be a power of p, say $k = p^a$, and u acts on $\Delta = \{W_1, \ldots, W_k\}$ as a cycle. We show that examples arise here and work with $G = A_l$, as the other cases are similar. So let $X = stab(W_1 \oplus \cdots \oplus W_k) = (\mathrm{SL}(d/p^a)^{p^a} \cdot T_{p^a-1}) \cdot S_{p^a}$. Let $y \in X^o$ be the block diagonal matrix $(J, I, \ldots, I)$ with all blocks of size m and J an indecomposable Jordan block. Let z be an m-block matrix with entries 0 and I such that z^Δ is a p^a-cycle. Put $u = yz$. Then u^{p^a} is the Jordan matrix with p^a blocks of size m, whence Lemma 1.3 implies that u is just one Jordan block.

Now let $G = D_l$. It follows from Lemmas 1.2 and 1.3 that each of the summands has size at most 2. Then $p = 2$, $m = 2$, and $k = l$, forcing $X = N(T)$. Another application of Lemma 1.2 shows that either $\langle u \rangle$ is transitive on $\{W_1, \ldots W_l\}$ or else $\langle u \rangle$ on $\{W_1, \ldots W_l\}$ fixes one of the W_i and its transitive on the remaining ones. The transitive case is out, since here u^l would have the same action on each 2-space W_i, contrary to Lemma 1.2. In the latter case, $l-1$ must be a power of 2 and we claim that this provides an example. Indeed, if $B = \{e_1, f_1, \ldots, e_l, f_l\}$ is a standard basis, we can take u to be the orthogonal transformation which acts on B as a disjoint product of a $(2l-2)$-cycle and a 2-cycle. For example, one can take

$$u = (e_1 f_2 e_3 f_4 \cdots e_{l-2} f_{l-1} f_1 e_2 \cdots f_{l-2} e_{l-1}) \cdot (e_l f_l).$$

(e) Finally assume that G is C_l or D_l and $X = A_{l-1} T_1.2$ preserves a partition $W = W_1 \oplus W_2$ with $\dim \mathrm{W}_i = l$ and W_i totally isotropic. Here consideration of the Jordan normal form forces $p = 2$. Also, $G = D_l$ by maximality of X, since $X < D_l.2 < C_l$. And l must be even, since otherwise X contains outer elements of $D_l.2$. Now the regular unipotent of D_l stabilizes a partition $W = W_1 \perp W_2$, with $\dim W_1 = 2l-2$.

Since l is even, D_l contains $A_{l-1}.2$, where the A_{l-1} corresponds to the fundamental roots $\{\alpha_1, \ldots, \alpha_{l-1}\}$ of Δ. Let E be the A_{l-2} subgroup of A_{l-1} corresponding to $\{\alpha_1, \ldots, \alpha_{l-2}\}$. If t is a graph automorphism of A_{l-1}, w_o is the long word in A_{l-1} and w_1 is the long word in E, then $j = t \cdot w_o \cdot w_1$ acts as a graph automorphism of E.

Let $m = (l-2)/2$ and set $u = U_1(1) \cdots U_m(1) \cdot j$. Then

$$u^2 = (U_1(1) \cdots U_m(1))(U_{l-2}(1) \cdots U_{m+1}(1)),$$

is a regular element of E. So u^2 has Jordan form with two blocks of size $l-1$ and two trivial blocks. It follows that u is a regular unipotent element of G and is contained in $A_{l-1}.2$.

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This completes the proof of Proposition 2.1.

To complete the proof of Theorem B we must now consider the case where X^o acts irreducibly and tensor indecomposably on W. In the following result we do not require the assumption of maximality.

PROPOSITION 2.2. *Suppose that X is a closed subgroup of the classical group G with X^o simple, irreducible and tensor-indecomposable on the natural module of G. Assume that X contains a regular unipotent element of G. Then one of the following holds:*

(i) *$X = A_1$ and $p = 0$ or $p > h$, where h is the Coxeter number of G;*
(ii) *X is a classical subgroup B_l or C_l when G is A_{2l} or A_{2l-1}, respectively;*
(iii) *$X = D_l.2$ in the orthogonal representation with $p = 2$ and $G = C_l$;*
(iv) *$X = G_2$ in dimension 7 with G either A_6 or B_3 (respectively dimension 6 if $p = 2$ with G either A_5 or C_3);*
(v) *$X = A_2.2$ in dimension 8, $p = 2$ and G is A_7 or C_4;*
(vi) *$X = B_3$ and $G = D_4$.*

Proof. If $X = A_1$ with $p > 0$, then regular unipotent elements must have order p. The hypothesis implies that X has an irreducible restricted representation of dimension $\dim W$. Using Lemma 1.2 we find that $p > h$. From now on we assume that $X \neq A_1$.

I. First assume that $u \in X^o$ and it is not the case that $G = D_n$ with $p = 2$.

Let U be a maximal unipotent subgroup of X containing u. Consider the commutator sequence $W > [W, U] > [W, U, U] = [W, U^2] > \ldots$. Except possibly for the D_n case, we see from the action of the regular unipotent element u that each term has codimension precisely 1 in the preceding term. In the D_n case the codimensions are all 1, with at most 1 exception, and if there is an exception the codimension at this point is 2.

Let λ be the high weight of $W|X$. As the representation is tensor indecomposable, λ is a twist of a restricted weight. Ignoring the twist, we may take λ to be restricted. Let $\{\alpha_1, \ldots, \alpha_k\}$ be a base of $\Sigma(X)$. For the moment exclude the cases where $X^o = B_l$, C_l, F_4, or G_2 with $p = 2$ ($p = 2$ or 3 in the G_2 case). Then [**10**, 2.3] applies and shows that the term $[W, U^s]$ in the above commutator series is the sum of all weight spaces of W for weights $\lambda - \sum c_i \alpha_i$, where $\sum c_i \geqslant s$. So the quotient of successive terms in the series can be regarded as the direct sum of all weight spaces for weights of a given level.

Suppose that $G = D_n$ and let w_o denote the long word in the Weyl group of X. Either $\gamma = w_o$ or $\gamma = w_o \cdot \tau$ induces -1 on the root lattice of X, where τ is a graph automorphism of X. Since X fixes a form on W, it follows that γ acts on the weights of W, sending each weight to its negative, and $\gamma(\lambda) = \mu$, the low weight. Then $\gamma(\lambda - \sum c_i \alpha_i) = \mu + \sum c_i \alpha_i$, so that a weight and its image add to 0.

Thus γ induces a symmetry of levels. It follows that if some term in the commutator series has codimension 2 in the preceding one, then the set of weights at this level is fixed by γ. So if a quotient has dimension at least two, then it is in the middle of the sequence of quotients. Notice also that if the 0-weight occurs, then it occurs within this quotient.

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Suppose there exist i, j with $i \neq j$ such that $\langle \lambda, \alpha_i \rangle \neq 0 \neq \langle \lambda, \alpha_j \rangle$. Then [**10**, 2.3] implies that there are exactly two such nodes and that $[W, U]/[W, U, U]$ has dimension 2. So $G = D_n$ and all other quotients have dimension 1. Moreover, the earlier remarks force one of $\lambda - \alpha_i$, $\lambda - \alpha_j$ and $2\lambda - (\alpha_i + \alpha_j)$ to be the 0-weight. But this is impossible as λ is dominant.

Hence, $\lambda = c\omega_i$ for some i. Let α_j be a node adjoining α_i. Then $\lambda - \alpha_i - \alpha_j$ is a weight of level 2. If α_i is not an end node, let α_k be a second node adjoining α_i. Then $\lambda - \alpha_i - \alpha_j$ and $\lambda - \alpha_i - \alpha_k$ are both weights of level 2, and we again have a contradiction unless $X = A_3$, $\lambda = \lambda_2$, and (iii) holds. So α_i is an end node. Also, $c = 1$, since otherwise, $\lambda - 2\alpha_i$ is a second weight of level two.

Suppose that X has rank l. If $X = A_l$, then W is a natural module for X and $X = G$, a contradiction. Suppose that $X = C_l$, D_l, or B_l. If $i = 1$, then $W \downarrow X$ is also the natural module and (ii) holds (since D_l does not contain regular unipotent elements of A_{2l-1}). Now suppose that $\lambda = \lambda_l$ (similarly for $\lambda = \lambda_{l-1}$ if $X = D_l$). If $X = B_l$ or C_l we may assume that $l > 2$, as otherwise we again have that (ii) holds. Then $\lambda - \alpha_{l-2} - \alpha_{l-1} - \alpha_l$ is a weight of level 3. In the B_l case, $\lambda - \alpha_{l-1} - 2\alpha_l$ is another such, whereas in the C_l case we have $\lambda - 2\alpha_{l-1} - \alpha_l$ appearing at level 3. None of these is the 0 weight. Moreover, they do not sum to 0 unless $X = B_3$, in which case (vi) holds. If $X = D_l$, use the weights $\lambda - \alpha_{l-2} - \alpha_{l-1} - \alpha_l$ and $\lambda - \alpha_{l-3} - \alpha_{l-2} - \alpha_l$. This gives a contradiction unless $l = 4$, in which case $X = G$, which is impossible.

Similar arguments give contradictions if X is of exceptional type, except for the case when $X = G_2$ and W is the restricted representation of dimension 7, where (iv) holds.

We are left with the previously excluded cases where $p = 2$ (or 3). We illustrate the argument for $X = B_l$ or C_l, the other cases being similar. Here $|u| = 2^a$ where a is minimal subject to $2^a \geqslant 2l$. Then $2^{a-1} < 2l$, so $2^a < 4l$. But then $\dim \mathrm{W} - 2 < 4l$. Counting conjugates of dominant weights we see that W is a natural module or a spin module for B_3 or C_3. For the natural modules, the result holds. In the rank 3 cases, comparing Lemma 1.4 for X and G we see that $G = D_4$, which has been excluded in I.

II. Now assume that $u \notin X^o$. If $p = 0$, then replacing u by an appropriate power, we reduce to the previous case. So assume that $p > 0$. Now u induces an automorphism of $X^o = A_l$, D_l or E_6, which is in the coset of a graph automorphism (so the characteristic is 2, or possibly 3 if $X^o = D_4$). The high weight λ of $W \downarrow X^o$ must be symmetric with respect to the graph automorphism, since u acts on W. We temporarily exclude the case where $p = 3$ and $X = D_4$.

First assume that $X = A_l$ with $l \geqslant 6$. Counting conjugates of dominant weights we see easily that the dimension d satisfies $d \geqslant 5l$. If $G \neq D_n$, then $|u| = 2^a$, where a is minimal subject to $2^a \geqslant \dim \mathrm{W}$. Thus $2^a \geqslant 5l$ and $2^{a-2} > l$. Now Lemma 1.4 applied to X implies that $|u^2| \leqslant 2^{a-2}$, a contradiction. If $G = D_n$, the order formula gives only $2^a \geqslant \dim \mathrm{W} - 2$. But here we still have $2^{a-2} > l$, a contradiction.

Now consider the small values of l. If $l = 2$, then we have $|u| \leqslant 8$, whence $\dim \mathrm{W} \leqslant 10$, with equality possible only if $G = D_5$. So here $W \downarrow X^0$ is the adjoint representation and (v) holds. If $l = 3$, then $|u| \leqslant 8$, and, as above, the only possibility is $\lambda = \omega_2$, which gives (iii). If $l = 4$ or 5, then $|u| \leqslant 16$ and again this is inconsistent with Lemma 1.4.

Now suppose that $X = D_l$. First assume that $d \geqslant 8l - 10$. Then $|u| = 2^a$, where a satisfies $2^a \geqslant 8l - 12$ (the bound is slightly better if G is not of type D). Then $|u^4| \geqslant 2l - 3$ and u^4 has order at least that of a regular unipotent element of D_l. This is a contradiction, since regular unipotent elements have maximal order among

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unipotents and $u^2 \in X^o$. Now suppose that $d \leqslant 8l-10$. Since $W|X$ is invariant under u, the high weight is fixed by the graph automorphism. Hence, $\langle \lambda, \alpha_l \rangle = \langle \lambda, \alpha_{l-1} \rangle$. Counting conjugates of the high weight and, if necessary, of a subdominant weight, we see that $W|X$ is the usual module for D_l, so that (iii) holds.

If $X = E_6$, then the information on the orders of regular unipotent elements of E_6 in Lemma 1.4 implies that $|u^2| \leqslant 16$, so $|u| \leqslant 32$. However, counting conjugates one easily sees that E_6 has no nontrivial self-dual module of dimension less than 72. This is a contradiction.

Now assume that $X = D_4$ with u inducing an element in the coset of the triality automorphism, then $|u| = 27$. The only module for which this is consistent with the order of u viewed as an element of G is when $W|X$ is the adjoint module of dimension 28 and $G = D_{14}$. Take $u \in N(U)$. Then $u^3 \in U$. Lemmas 1.2 and 1.3 together imply that u^3 has three Jordan blocks on W of size 9 and a trivial block. Hence the fixed point space has dimension 4.

On the other hand, one checks that if one expresses u^3 as a product of root elements, there is no contribution from U_{α_2}. Consequently, u^3 cannot be a regular unipotent element of D_4. It follows that $C_{D_4}(u^3)$ has dimension strictly greater than 4, and so u^3 must have fixed space on $L(D_4)$ of dimension strictly greater than 4. This is a contradiction.

III. Finally, assume that $u \in X^o$, $G = D_n$, and $p = 2$. Here $|u| = 2^a$, where a is minimal subject to $2^a \geqslant 2n-2$. Also, u has maximal order among unipotent elements of G, so this holds for X as well. Suppose that X has classical type. Then Lemma 1.4 applied to X implies that $2^{a-1} < l+1, 2l-2, 2l, 2l$, according to $X = A_l$, D_l, B_l, or C_l. Hence, $2^a \leqslant 2l+2, 4l-4, 4l, 4l$, respectively. Combining with the above inequality we have $2n \leqslant 2l+4,\ 4l-2,\ 4l+2,\ 4l+2$, respectively. Counting conjugates of high weights and subdominant weights we find that the only possibilities are when $(X, G) = (B_3, D_4)$, or $(X, G) = (C_3, D_4)$ with $p = 2$, where the embedding is given by a spin module. (Note that B_4, C_4 and D_5 in dimension 16 are also out by the first inequality.) Now, the spin representation for C_3 factors through B_3, so the only configuration is (vi).

For the remaining cases, where X is an exceptional group, we use Lemma 1.4 to find $|u|$ and count conjugates of the high weight and subdominant weights to get lower bounds for dim W. A comparison easily shows that the only possibility is $X = G_2$ with W of dimension 6. This is impossible, since G_2 is not a subgroup of $D_3 = A_3$. This is a contradiction, completing the proof of Proposition 2.2.

LEMMA 2.3. *Each of the possibilities* (i) *to* (vi) *in Proposition* 2.2. *gives examples.*

Proof. (i) The irreducible tensor indecomposable representations of A_1 arise from the action of SL_2 on homogeneous polynomials in variables corresponding to a basis of the usual 2-dimensional module. A direct check shows that unipotent elements are represented by a single Jordan block.

(ii) and (iii) are well known – see Lemma 1.2.

To see (iv) consider G_2 as the group of fixed points of a triality automorphism of D_4. The element $U_{\alpha_2}(1)\, U_{\alpha_1}(1)\, U_{\alpha_3}(1)\, U_{\alpha_4}(1)$ is fixed by triality and is regular by Lemma 1.1. Since $G_2 < B_3$ (respectively $G_2 < C_3$ if $p = 2$), it follows that G_2 contains a regular unipotent element of B_3 (respectively C_3 if $p = 2$).

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(v) Take $u = U_{\alpha_1}(1)\,t$ in X, where t is a graph automorphism of $X^o = A_2$. Then $u^2 = U_{\alpha_1}(1)\,U_{\alpha_2}(1)$ is a regular unipotent element of X^o of order 4. One checks that u^4 has four Jordan blocks of size 2 on $L(X^o)$. Hence u has a single Jordan block on W.

(vi) follows from (i) of Proposition 2.1, using triality.

3. *Exceptional groups*

Our goal in this section is to establish Theorem A, which we do in two steps. We first reduce to the cases of Theorem A together with a few extra possibilities, and then show that only the configurations in Theorem A actually occur.

As before, X is maximal among closed reductive subgroups of G, and X contains a regular unipotent element of the exceptional group G.

LEMMA 3.1. *Either Theorem* A *holds or G and X satisfy one of the following:*

(i) $G = E_7$, $X = (D_4 \times 2^2).S_3$, $p = 3$;

(ii) $G = E_7$, $X = (D_4 A_1^3).S_3$, $p = 3$;

(iii) $G = E_8$, $X = (E_6 A_2).2$, $p = 2$.

Proof. Take G to be an exceptional group of adjoint type and apply the results of Liebeck–Seitz [**7**], which give the maximal reductive subgroups of G, subject to certain characteristic restrictions. These restrictions occur only when X^o is simple with trivial centralizer. Our strategy is to use Lemma 1.4 when the characteristic is small, and otherwise to make full use of the results in [**7**].

Write $X^o = X_1 \cdots X_k$, a commuting product of simple groups. The restrictions on the characteristic occur only when $k = 1$ with $C_G(X^o) = 1$.

1. Suppose X^o contains a maximal torus of G. We claim that $u \notin X^o$. For applications later, for this particular claim we shall not use maximality and shall assume only that X^o is a connected reductive subgroup of maximal rank. Suppose that $u \in X^o$, and let $u \in B_X = U_X T$, a Borel subgroup of X, where T is a maximal torus of G. Embed this in a Borel subgroup $B_G = U_G T$ of G. Order roots so that B_G is a standard Borel subgroup. The root subgroups of B_X are also root subgroups of G, since they are T-invariant. Express u as a product of root elements of U_X with respect to a fixed ordering consistent with height. Regarding u as a regular element of U_G we see that the expression for u must involve nonidentity elements from each root group corresponding to a fundamental root of $\Sigma(G)$. But then uniqueness of expressions implies that each of these root groups is in U_X. It follows that $X^o = G$, a contradiction. This establishes the claim.

If $p = 0$, then all powers of u are regular, so this contradicts the above paragraph. Hence, $p \neq 0$ and is bounded above by the largest prime dividing the order of the Weyl group.

Note that $C(X)^o \leqslant C(T)^o = T$. If $X^o = T$, a maximal torus of G, then we can regard u as an element of the Weyl group; this contradicts the information on orders in Lemma 1.4 (cf. [**3**]). So X^o must contain simple factors. Maximality of X implies that the centralizer of $(X^o)'$ is a torus. From [**1**, Proposition 28] and the table of subsystems at the end of that paper we see that the possibilities are as follows.

If $p = 7$, the only possibilities are $X^o = A_1^7$ or A_1^8, with $G = E_7$ or E_8, respectively. In the first case (g) of Theorem A holds. In the second case, $X^o\langle u\rangle \leqslant E_7 A_1$, which contains no regular unipotent element of E_8 by the second paragraph. If $p = 5$, then

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X^o must contain a normal subgroup of type A_1^5 with $\langle u \rangle$ permuting the factors transitively. However, [**1**] shows that there is no such subgroup.

If $p = 3$, then either u fixes all simple components of X^o, in which case some component must have type D_4, or there is an orbit of size 3. Suppose that $X_1 = D_4$, with u inducing a triality automorphism. If $G = E_8$, then $X^o = D_4 \circ D_4$, $u^3 \in X^o$, and $|u| \leqslant 27$, contradicting Lemma 1.4. If $G = F_4$ or E_6, then (d) or (e) of Theorem A holds. And if $G = E_7$, we have (ii) of this lemma. So now assume that there is a nontrivial orbit of u on the simple components of X^0. Without loss, assume that $X_1^u \neq X_1$.

If $X_1 \neq A_1$, then a rank consideration implies that $X_1 = A_2$ and (X^o, G) is one of the following: (A_2^3, E_6), $(A_2^3 T_1, E_7)$, (A_2^4, E_8). In each case, $u^3 \in X^0$ so $|u| \leqslant 9$, contradicting Lemma 1.4. Now assume that $X_1 = A_1$. Then $X^0\langle u \rangle \leqslant (A_1^3)\, C_G(A_1^3)\langle u \rangle$. The structure of the second factor is given by [**1**]. If $G = F_4$ or E_6, then $C_G(A_1^3)' = A_1$ and so u is contained in the centralizer of an involution, contradicting the fact that u is regular. Similarly, if $G = E_8$, then $C_G(A_1^3) = A_1 D_4$ and we have the same contradiction. If $G = E_7$ there are two classes of subsystem groups of type A_1^3, with centralizers of type D_4 and A_1^4, respectively. In the latter case, $|u| \leqslant 9$, contradicting Lemma 1.4. In the former case $X^o \leqslant A_1^3 D_4$ and of maximal rank. As $|u| = 27$ we must have $X^o = A_1^3 D_4$ or $A_1^3 A_3$. The former is (ii) of this lemma, while in the latter case $u \in C(Z(A_3))$, contradicting the fact that u is regular.

Finally, assume that $p = 2$. First assume that u fixes X_1. If $X_1 = A_1$, then with one exception $A_1 N_G(A_1)$ is connected, contradicting the first paragraph. The exceptional case is $G = F_4$ with the A_1 corresponding to a short root. Here $u \in A_1 A_3.2$, so $|u| \leqslant 8$, contradicting Lemma 1.4. Suppose that $X_1 = A_2$. If $X_1 \lhd X$, then $X^o = A_2 C_G(A_2)^o$. The cases $G = G_2$ and E_8 are accounted for in (c) of Theorem A and (iii) of this lemma, respectively. The other cases contradict Lemma 1.4. Similarly, if X_1 is not normal in X, then $X^o\langle u \rangle \leqslant A_2 A_2 C_G(A_2 A_2)\langle u \rangle$ and considering possibilities we again contradict Lemma 1.4.

At this point one checks the remaining possibilities for X_1 in the same way. In each case, $X^o \leqslant X_1 C_G(X_1)^o$. If $X_1 \lhd X$, then $X^o = X_1 C_G(X_1)^o$ so that X^o is given by [**1**]. Otherwise, by looking at an orbit product one obtains a contradiction to Lemma 1.4. Returning to the first case we see that in several instances u normalizes an A_1 or A_2 component of X^o. In view of what has been done so far, we are done if any of these occur. The only cases that do not succumb to this analysis have (X^o, G) one of (D_4, F_4), $(D_4 T_2, E_6)$, $(D_5 T_1, E_6)$, $(E_6 T_1, E_7)$, $(D_6 A_1 A_1, E_8)$ or $(A_3 D_5, E_8)$. In the first case, we see that u is contained in a subgroup of F_4 of type B_4 or C_4, according to whether the D_4 system consists of long or short roots. In either case we contradict the second paragraph. In the second case, we see that u is contained in a subgroup of $B_4 < F_4$, so it is not even regular in F_4. The third case is impossible, since it follows from [**1**] that $u \in X^o = X$. The fourth case is (f) of Theorem A and in the last two cases X is contained in a maximal rank subgroup of type D_8, contradicting the second paragraph.

So we may now assume that u stabilizes no simple factor of X^o. Reorder if necessary so that X_1 has rank as large as possible. If $X_1 = A_1$, then all simple factors of X^o have rank 1 and we contradict the order information, as usual. It is easy to go through the remaining possibilities, in each case getting a contradiction.

2. Now assume that X^o does not contain a maximal torus of G. We must consider configurations from [**7**, Table II] and cases where the main result of that paper cannot

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be applied due to the characteristic restrictions. First consider the latter. As mentioned earlier there are characteristic restrictions only when X^o is simple with trivial centralizer. These are given in the table of prime restrictions in [**7**]. Here the rank of X^o is small and in all cases $p \leqslant 7$. Comparing $|u|$ as given by Lemma 1.4 with the order of u considered as an element of X, we obtain a contradiction.

We now assume that X^o is one of the groups given in [**7**]. Since u is a regular unipotent element, $C_G(u)$ contains no nonidentity semisimple elements. This rules out certain configurations. In particular, consider the cases $X = (D_4 \times 2^2) S_3 < E_7$, with $p \neq 2$, and $A_1 \times S_5 < E_8$, with $p \neq 2$, 3, or 5. The only way u could lie in X without centralizing a semisimple element is when (i) of this lemma holds.

The remaining cases are where X^o is one of the groups in [**7**, Table II]. Then $C_G(X^o) = 1$ and either $u \in X^o$ or $p = 2$. Suppose that $|u| = p$. Then p is odd, so $u \in X^o$. Using the result of Testerman [**16**] we see that u is contained in a subgroup of X^o of type A_1. On the other hand, a result of [**6**] implies that there is a unique class of elements A_1 containing u, and the main result of [**12**] shows that these groups are maximal among closed connected subgroups of G. So this is a contradiction unless $X = A_1$, in which case the theorem holds.

Now suppose that $|u| > p$. Then Lemma 1.4 shows that p is small. If p is odd then $u \in X^o$, and if $p = 2$ then $u^2 \in X^o$. We can thus compare the information from Lemma 1.4 for the groups G and X^o. The only possibilities for (G, X^o) are (E_6, F_4), (E_6, C_4), $(E_7, A_1 F_4)$ and $(E_8, G_2 F_4)$. The first case occurs in Theorem A.

For the second case note that p is odd (otherwise $C_4 < F_4$). Here C_4 occurs as the group of fixed points of an outer automorphism of $G = E_6$. Set $C_G(u) = Q$, a unipotent group. There is an involutory automorphism, say t, of G such that $u \in C_G(t) = F_4$. Then $C_{\mathrm{Aut}(G)}(u) = Q\langle t \rangle$. However, $Q\langle t \rangle$ has just one class of involutions. It follows that u cannot be centralized by an involution in $\mathrm{Aut}(G)$ having fixed point group C_4.

Suppose that $G = E_7$, with $X = A_1 F_4$. Let $B = VT_1$ be a Borel subgroup of the A_1 factor, with $V = R_u(B)$. Let F be the F_4 factor of X and embed $B \times F$ in a parabolic subgroup P with V contained in the unipotent radical. Write $P = QL$, where $Q = R_u(P)$ and L is a Levi factor containing T_1. This parabolic subgroup must be of type E_6. As T_1 centralizes F_4, we see that $T_1 \leqslant Z(L)$. Hence, T_1 acts without fixed points on Q and it follows that $C_G(T_1) = L$. In particular, F is contained in $L' = E_6$.

We may take a system of root subgroups for G such that P is a standard parabolic subgroup, L' is a subsystem subgroup, and the F_4 is embedded in the usual way.

Let U be a maximal unipotent subgroup of F and embed U in a maximal unipotent subgroup of L'. The latter is the product of all root subgroups corresponding to positive roots in the E_6 subsystem.

The elements of V can be expressed as products of root elements. But V is centralized by the 1-dimensional torus of F consisting of all elements of F of the form $h_{\alpha_1}(c)\, h_{\alpha_6}(c)$. It follows that V is contained in the product of all root subgroups of Q for roots other than α_7. Hence, a maximal unipotent subgroup of X, namely VU, is contained in the product of all root subgroups for positive roots other than α_7. This product is the unipotent radical of a minimal parabolic subgroup of G and hence is contained in infinitely many Borel subgroups of G. It follows that X cannot contain a regular unipotent element of G.

Essentially the same argument settles the remaining case, where $G = E_8$ and $X = G_2 F_4$. Let D be a parabolic subgroup of G_2 of the form $VA_1 T_1$, where the A_1 factor is generated by opposite root subgroups corresponding to a long root and its

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negative. Embed $VA_1\,T_1 \times F_4$ in a parabolic subgroup $P < G$ such that $V < R_u(P)$. The Levi factor of P contains a group S of type $A_1 \times F_4$ and so must be of type E_7 or $E_6\,A_1$. The former is impossible since here $R_u(P)'$ has dimension 1, whereas $\dim \mathrm{V}' = 3$. As above, $C_G(T_1)$ is a Levi factor of P, so the F_4 factor is contained in the E_6 component of the Levi factor, while the A_1 factor of S is normal in the Levi factor. At this point we can assume the parabolic is standard and that the F_4 embeds in the usual way in the E_6 factor of this Levi subgroup. We now argue as in the preceding paragraph that V is contained in the product of all root subgroups of Q for compound positive roots involving α_7. Again we conclude that a maximal unipotent subgroup of X is contained in the unipotent radical of a minimal parabolic subgroup of G. Hence, X cannot contain a regular unipotent element, a contradiction. This completes the proof of the lemma.

The proof of Theorem A is completed in the next lemma. In a number of places we refer to Lawther's paper [**5**] in which the Jordan normal forms of unipotent elements of exceptional groups on modules of relatively small dimension are given. Our original proof was independent of this – indeed [**5**] was partly motivated by our work. However, reference to [**5**] has shortened the proof significantly.

LEMMA 3.2. *The subgroups in Lemma* 3.1 *do not contain regular unipotent elements, whereas the examples in Theorem* A *do contain regular unipotent elements.*

Proof. If $p > h$, then Lemma 1.4 ensures that regular unipotent elements have order p. Then the main result of [**16**] shows that regular unipotent elements lie in a group of type A_1. Also, it is clear from the embedding of F_4 in E_6, that the former contains regular unipotent elements of the latter. Next consider the group $A_2.2 < G_2$ with $p = 2$. The A_2 is generated by all long root subgroups in a fixed root system and $A_2.2$ contains the Weyl group of G_2. Say the root system of G_2 has base $\{\alpha, \beta\}$, with α long. Then $u = U_\alpha(1)\, s_\beta \in A_2.2$. We claim this is a regular unipotent element of G_2. We can write $s_\beta = U_{-\beta}(1)\, U_\beta(1)\, U_{-\beta}(1)$, so, conjugating u by $\mathrm{U}_{-\beta}(1)$, we get the regular element $U_\alpha(1)\, U_\beta(1)$.

A similar, but more complicated, argument works for $D_4\,S_3 < F_4$, when $p = 3$. However, we use a shortcut instead. By [**5**], the regular unipotent elements are the only elements in F_4 of order 27. On the other hand, the sporadic simple group Fi_{23} contains elements of order 27, and has a maximal subgroup $O_8^+(3)\,S_3$ of index prime to 3 [**3**, pp. 177–179]. It follows that $D_4\,S_3$ contains a regular unipotent element of F_4.

Now consider the case $D_4\,T_2\,S_3 < E_6$ with $p = 3$. We can take the $D_4\,S_3$ to lie in F_4. We know that regular unipotent elements of F_4 are regular in E_6 and we have just seen that $D_4\,S_3$ contains regular unipotent elements of F_4. So we have regular elements in this case also.

We are left with five cases (two for Theorem A and three for 3.1).

1. If $X = (D_4 \times 2^2)S_3$ with $G = E_7$ and $p = 3$, then X does not contain a regular unipotent element of G.

This D_4 is a natural subgroup of a subsystem group of type A_7. The Lie algebra $L(E_7)$ restricted to A_7 has $L(A_7)$ as a direct summand. If the regular unipotent element x of G lies in X, then x^3 lies in the D_4 in A_7. In view of Lemma (1.2)(ii) x^3 has more

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than one Jordan block on the natural 8-dimensional module and hence lies in a Levi subgroup of A_7. It follows that $L(E_7)$ restricted to x^3 has a trivial direct summand. This is contrary to [**5**].

2. If $X = (E_6 T_1).2$ with $G = E_7$ and $p = 2$, then X contains a regular unipotent element of G.

By [**5**], it is enough to show that $E_6.2$ contains an element of order 32. This is true, since the group ${}^2E_6(2).2$, which is a subgroup of $E_6.2$, contains an element of order 32 by [**3**]. Alternatively, it suffices to show that the subgroup $2_+^{1+20} L_6(2).2$ of $E_6(2).2$ contains an element of order 32.

3. If $X = (E_6 \circ A_2).2$ with $G = E_8$ and $p = 2$, then X does not contain a regular unipotent element of G.

The square of the regular unipotent element of E_8 has Jordan blocks $16^{14}\,12^2$ on $L(E_8)$ by [**5**]. On the other hand, the restriction of $L(E_8)$ to X^o has $L(A_2)$ as a direct summand of dimension 8.

4. If $X = (D_4 \circ A_1 \circ A_1 \circ A_1).S_3$ and $G = E_7$ with $p = 3$, then X does not contain a regular unipotent element of G.

Assume that a regular unipotent element x of G lies in X. Since $x \notin X^o$ by orders, we have $x = x_1 x_2 t$ with $x_1 \in D_4, x_2 \in A_1 \circ A_1 \circ A_1$ where x_1 and x_2 commute and t is a triality element of order 3. Then x^3 is a product of 3-elements, one from D_4 and the other from $A_1 \circ A_1 \circ A_1$. On the other hand,

$$x^3 = x_1 x_2 (x_1^{t^2} x_2^{t^2})(x_1^t x_2^t) = (x_1 x_1^{t^2} x_1^t)(x_2 x_2^{t^2} x_2^t).$$

Since $(x_2 x_2^{t^2} x_2^t)^3 = 1$ and the two factors above commute, we have $x^9 = (x_1 x_1^{t^2} x_1^t)^3 = (x_1 t)^9$. Thus x and $x_1 t$ have the same 9th powers. By [**5**] it follows that $(x_1 t)$ is also a regular unipotent element of G. This is impossible since, for example, it lies in the subgroup F_4.

5. If $X = (A_1^7) L_3(2)$ and $G = E_7$ with $p = 7$, then X contains a regular unipotent element of G.

Let $U_\alpha(1)$ be a unipotent element in one of the components A_1 of X^o, and let w be an element of order 7 in $X \setminus X^o$. We claim that $x = wU_\alpha(1)$ is a regular unipotent element of G. Consider x^7 which is a regular unipotent element of $X^o = A_1^7$. Now $X^o < A_1 D_6$. Let W be the 56-dimensional module for G. This restricts to $A_1 D_6$ as direct sum of $2 \otimes 12$ and a spin module for D_6. Now the 24-dimensional summand restricted to X^o is the sum of three 8-dimensional representations, each the tensor product of natural representations for three of the factors. We deduce that W restricts to X^o as $(2 \otimes 2 \otimes 2)^7$. Hence x^7 has Jordan form $2^{14}\,4^7$ on W. It follows from [**5**] that x is a regular unipotent of G.

This completes the proof of Lemma 3.2 and hence the proof of Theorem A.

4. *Proofs of Theorems* C *and* D

In this section we prove Theorems C and D. Both results are concerned with the primitive action of G on a variety Ω. Let X denote a point stabilizer in this action.

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We begin with Theorem C. Here Ω is an affine algebraic variety, so by [**9**] X is reductive. Thus X is a reductive maximal subgroup of G. Suppose that the result is false. Then X contains representatives of all unipotent classes of G. In particular, X is one of the groups in Theorem A or Theorem B. Using the description of the unipotent conjugacy classes in simple algebraic groups given in [**2**, Sections 5.11 and 13.1] one quickly disposes of most of the possibilities. For example, consider first the examples in Theorem A.

Clearly $X = A_1$ is not possible, since there is only one unipotent class. Next, F_4 has 16 unipotent classes (respectively 20 if $p = 2$), whereas E_6 has 21. The next possibility, $A_2.2 < G_2$ with $p = 2$, gives an example: there are 4 unipotent classes in G_2, of which two consist of involutions and the other two of elements of order 8 and their squares; and $A_2.2$ contains involutions with two different Jordan normal forms on the 6-dimensional module, as well as elements of order 8. Next we dispose of $D_4.S_3 < F_4$ with $p = 3$ by considering classes of elements of order 3: in F_4 there are seven such classes (cf. [**5**]); D_4 has six such classes, three of which are fused by the triality automorphism (cf. [**2**, p. 396]), and there are two classes of triality automorphisms (cf. [**4**, 9.1]), giving six such classes in $D_4.S_3$. It follows that $(D_4\, T_2).S_3 < E_6$ with $p = 3$ is also not an example here.

To see that $(E_6\, T_1).2 < E_7$ with $p = 2$ is impossible, we use the results of Lawther [**5**]. In [**5**, Table D], it is shown that a unipotent element of E_7 of type $E_7(a_1)$ has square of type A_6, a class not represented in E_6. So such a unipotent element cannot lie in a group of type $(E_6\, T_1).2$. Finally, in $A_1^7.L_3(2)$ all elements of order 49 have conjugate 7th powers, whereas that is not so in E_7 (cf. [**5**]).

The examples in Theorem B(iv) are ruled out, using the information in [**2**, Chapters 5 and 13], except for $D_l.2 < C_l$ with $p = 2$. This leads to an example as is shown in Lemma 4.1 below. The configuration in Theorem B(iii)(a) occurs, while (iii)(b) is disposed of as above. The reducible candidates in Theorem B(i) are also easily eliminated: an element of Jordan form l^2 in A_{l-1} in D_l fixes neither a nonsingular hyperplane nor a nonsingular 2-space.

Finally, consider the imprimitive examples in Theorem B(ii). Here we argue as follows. Let e be the largest nontrivial Jordan block size of an element of $I(W_1)$. Choose a unipotent element t of G having a single nontrivial Jordan block of size c, where we take $c > e$ minimal (respectively $c = 3$ for the orthogonal case with $\dim W_1 = 2$).

Assume that t permutes the set $\{W_1, \ldots, W_k\}$; we shall show that $\dim W_i = 2 = k$. First note that by construction t must have a nontrivial orbit. Next use a dimension argument to check that t fixes vectors in $W_i \oplus W_j$ for all $i \neq j$. An easy argument with orbits now gives a contradiction unless $p = 2$. By choice of t there is a unique fixed point in $[W, t]$. This forces t^2 to act as a regular unipotent element on W_1 and W_2. The choice of t forces $\dim \mathrm{W} = 4$ with $G = \mathrm{Sp}_4$. This configuration appears in Theorem C.

LEMMA 4.1. *Let $X = D_l.2 < C_l = G$ with $p = 2$. Then each conjugacy class of G meets X.*

Proof. We are grateful to Dr N. F. J. Inglis for the proof given here. Let g be an element preserving our symplectic form $(,)$. We construct a g-invariant quadratic form Q supported by $(,)$. Note that Q is g-invariant if and only if $Q(v(g+1)) = (v, vg)$ for all v, since $Q(v(g+1)) = Q(v) + Q(vg) + (v, vg)$. We therefore define Q on

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$W(g+1)$ by $Q(x) = (v, vg)$ if $x = v(g+1)$. This is well defined, since if $u(g+1) = v(g+1)$ then $(u+v)g = u+v$, so

$$\begin{aligned}(u, ug) &= (v+u+v, vg+(u+v)g)\\ &= (v, vg)+(v, (u+v)g)+(u+v, vg)+(u+v, (u+v)g) = (v, vg).\end{aligned}$$

Also, Q, as defined so far, gives a quadratic form on $W(g+1)$ supported by $(\ ,\ )$, since

$$\begin{aligned}Q(\lambda(v+vg)) &= (\lambda v, \lambda vg) = \lambda^2 Q(v+vg),\\ Q(v+vg+w+wg) &= (v+w, (v+w)g) = (v, vg)+(w, wg)+(v, wg)+(w, vg)\\ &= Q(v+vg)+Q(w+wg)+(v+vg, w+wg).\end{aligned}$$

Now extend Q by defining it to be 0 on a fixed basis of a fixed complement to $W(g+1)$ and extending the definition to all of W using $(\ ,\ )$. This gives the desired g-invariant non-singular quadratic form.

We now come to the proof of Theorem D. Let u be a unipotent element fixing a finite number of points in the action on the variety Ω, with $u \in X$. Since $C_G(u)$ permutes the conjugates of X containing u, we conclude that $C = C_G(u)^o \leqslant X$. By [**14**, III.3.15], all unipotent elements of $C_G(u)$ are contained in C. Let V be a maximal unipotent subgroup of C with $u \in V$. Another application of [**14**, III.3.15] shows that V contains all unipotent elements of its centralizer. Hence [**11**, Theorem 1] implies that X is a maximal parabolic subgroup P of G. Write $P = QL$, where $Q = R_u(X)$ and L is a Levi subgroup.

We claim next that u is a regular unipotent of G. Suppose that this is not so. Clearly G has rank at least two. Each Borel subgroup of G containing u is contained in a unique conjugate of P. By [**14**, III.1.12], u must be contained in infinitely many Borel subgroups of G. But by hypothesis, u is contained in only finitely many conjugates of P. Replacing P by a conjugate, we may assume that u is contained in infinitely many Borel subgroups of P.

It follows from the above remarks that uQ is not a regular unipotent element of $P/Q \simeq L$. Hence there is a minimal parabolic $P_0 \leqslant P$ with $u \in R_u(P_0)$. Take such a P and take P_0 as a standard parabolic subgroup with respect to some maximal torus of L. Then P is the maximal parabolic subgroup corresponding to some fundamental root α_k and P_0 is the minimal parabolic subgroup corresponding to a fundamental root α_i. Among all the possible P-conjugates of u and all the i for which the above condition holds, choose one for which there are as few nodes as possible between α_i and α_k. Write $Q_i = R_u(P_0)$.

Suppose that there is a fundamental root α_j adjoining α_i with α_j between α_i and α_k. Express u as a product of elements from root groups. As $u \in Q_i$, the expression does not involve a nonidentity element of U_{α_i}. On the other hand, the expression must involve an element of U_{α_j}, as otherwise $u \in Q_j$, contradicting the choice of i. Consider the element $u^{s_i} \in Q_i$. If this element does not involve U_{α_j}, then the conjugate is in Q_j, contrary to the choice of i. If the conjugate does involve U_{α_j}, then u must involve $U_{(\alpha_j)s_i}$. Now $(\alpha_j)s_i = \alpha_j + c\alpha_i$ with c equal to 1, 2 or 3. In any case, let $s = U_{-\alpha_i}(d)$ and consider u^s. Using the commutator relations one checks that for some choice of d, the element u^s will not involve a nonidentity element of U_{α_j}. Once again this contradicts the choice of i. We conclude that α_i must adjoin α_k.

If no element of U_{α_k} is involved in the expression for u, then $u \in Q_k$. As Q_k is normalized by Q and s_k, we see that $u^{qs_k} \in Q_k$ for all $q \in Q$. Therefore, $u \in P^{s_k q}$ for all

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$q \in Q$. However, we are assuming that u is in only finitely many conjugates of P. It follows that $Q = Q^o \leqslant N_G(P^{s_k}) = P^{s_k}$. Hence $Q^{s_k} \leqslant P$. But this is a contradiction, as $U_{-\alpha_k} \leqslant Q^{s_k}$, whereas this group is not contained in P. So u involves nonidentity elements of U_{α_k}.

Similarly this also holds for u^{s_i}. So u involves nonidentity elements of U_{α_k} and $U_{\alpha_k + c\alpha_i}$. Conjugating by an element of $U_{-\alpha_i}$ results in an element with no factor from U_{α_k} and replacing u by this conjugate we contradict the last paragraph.

We have now proved that u is a regular unipotent element. Hence u is contained in a unique Borel subgroup of G, which in turn is contained in a unique conjugate of P.

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