Math. Proc. Camb. Phil. Soc. (1983), **93**, 135
Printed in Great Britain

Eigen-values of Lipschitz kernels

BY J. B. READE
University of Manchester

(*Received* 7 *May* 1982; *revised* 11 *June* 1982)

1. *Abstract*

Asymptotic estimates are obtained for the eigen-values of symmetric kernels which satisfy a Lipschitz condition of order α, where $0 < \alpha \leqslant 1$, on a bounded region. For $0 < \alpha < 1$, the eigen-values are shown to be $O(1/n^{\alpha+\frac{1}{2}})$ for general kernels, and $O(1/n^{\alpha+1})$ for positive definite kernels. For $\alpha = 1$, the eigen-values are shown to be $o(1/n^{\frac{3}{2}})$ in general, but only $O(1/n^2)$ when the kernel is positive definite. Examples are given to show these estimates are best possible for powers of n.

The estimates for general kernels are known. In 1931, Hille and Tamarkin used infinite determinants to obtain weaker estimates in the case $0 < \alpha < 1$, but the same estimates as ours for $\alpha = 1$. In 1937, Smithies sharpened the estimates for $0 < \alpha < 1$ to the ones we give by a variant of the method used by Weyl in 1912 for smooth kernels. Cochran obtained the same estimates in 1972 using Fredholm determinants. Our proof uses Weyl's method in its original form for $0 < \alpha < 1$, and in Smithies' variant for $\alpha = 1$, so in all cases simplifies previous proofs.

The estimates for positive definite kernels are, as far as we know, new. The proofs use the method of our recent paper on smooth positive definite kernels.

2. *General kernels*

We say the symmetric kernel $K(x,t) = \overline{K(t,x)} \in \mathrm{Lip}_\alpha [a,b]^2$ if there is a constant C such that

$$|K(x,t) - K(y,u)| \leqslant C(|x-y|^\alpha + |t-u|^\alpha),$$

for all $x, y, t, u \in [a,b]$. Let the eigen-values of $K(x,t)$ be

$$|\lambda_1| \geqslant |\lambda_2| \geqslant \ldots \geqslant |\lambda_n| \geqslant \ldots .$$

LEMMA 1. *For symmetric* $K(x,t) \in \mathrm{Lip}_\alpha [0,1]^2$, *where* $0 < \alpha < 1$, *we have*

$$\lambda_n = O(1/n^{\alpha+\frac{1}{2}}).$$

Proof. (See (8), p. 449.) For each N, subdivide the unit square $[0,1]^2$ into N^2 equal subsquares, and let $R_N(x,t)$ be constant on each subsquare, agreeing with $K(x,t)$ at the centre. $R_N(x,t)$ is a symmetric kernel of rank N such that

$$|R_N(x,t) - K(x,t)| = O\left(\frac{1}{N^\alpha}\right),$$

uniformly in $x, t \in [0,1]$. Therefore, by E. Schmidt's lemma (see (8), p. 447),

$$\sum_{N+1}^{\infty} \lambda_n^2 \leqslant \int_0^1 \int_0^1 |R_N(x,t) - K(x,t)|^2 \, dx \, dt$$
$$= O\left(\frac{1}{N^{2\alpha}}\right).$$

https://doi.org/10.1017/S0305004100060412 Published online by Cambridge University Press

Hence
$$\lambda_n^2 = O\left(\frac{1}{n^{2\alpha+1}}\right),$$
and the lemma follows.

To construct an example to show the estimate of Lemma 1 is best possible, we make use of the Rudin–Shapiro polynomials which are defined as follows. (See (6).)
$$P_0(z) = Q_0(z) = 1,$$
$$P_{n+1}(z) = P_n(z) + z^{2^n} Q_n(z),$$
$$Q_{n+1}(z) = P_n(z) - z^{2^n} Q_n(z).$$

If ϵ_n is the coefficient of z^n in $P_k(z)$ for $2^k > n$, then $\epsilon_n = \pm 1$, and
$$\sum_0^N \epsilon_n e^{int} = O(N^{\frac{1}{2}}),$$
uniformly in t. (See (6), p. 855.)

LEMMA 2. $\sum_1^\infty \frac{\epsilon_n e^{int}}{n^{\alpha+\frac{1}{2}}} \in \mathrm{Lip}_\alpha[-\pi, \pi]$ *for* $0 < \alpha < 1$.

Proof (See (10), p. 118.) Let $s_N(t) = \sum_1^N \epsilon_n e^{int}$.
Then, for any $\beta > 0$, we have
$$\begin{aligned}\sum_1^N \frac{\epsilon_n e^{int}}{n^{\beta+\frac{1}{2}}} &= s_1(t) + \sum_2^N \frac{s_n(t) - s_{n-1}(t)}{n^{\beta+\frac{1}{2}}} \\ &= \sum_1^{N-1} s_n(t)\left(\frac{1}{n^{\beta+\frac{1}{2}}} - \frac{1}{(n+1)^{\beta+\frac{1}{2}}}\right) + \frac{s_N(t)}{N^{\beta+\frac{1}{2}}} \\ &= \sum_1^{N-1} O\left(\frac{1}{n^{\beta+1}}\right) + O\left(\frac{1}{N^\beta}\right) \\ &= O(1).\end{aligned}$$

Similarly, $\sum_1^N \frac{\epsilon_n e^{int}}{n^{\frac{1}{2}}} = O(\log N)$, and $\sum_1^N \frac{\epsilon_n e^{int}}{n^{\frac{1}{2}-\beta}} = O(N^\beta)$,

where, again, $\beta > 0$. Therefore $\sum_1^\infty (\epsilon_n e^{int}/n^{\alpha+\frac{1}{2}})$ is uniformly convergent by Dirichlet's test. (See (9) pp. 17 and 50.) Also
$$\begin{aligned}\sum_{N+1}^\infty \frac{\epsilon_n e^{int}}{n^{\alpha+\frac{1}{2}}} &= \sum_{N+1}^\infty \frac{s_n(t) - s_{n-1}(t)}{n^{\alpha+\frac{1}{2}}} \\ &= -\frac{s_N(t)}{(N+1)^{\alpha+\frac{1}{2}}} + \sum_{N+1}^\infty s_n(t)\left(\frac{1}{n^{\alpha+\frac{1}{2}}} - \frac{1}{(n+1)^{\alpha+\frac{1}{2}}}\right) \\ &= O\left(\frac{1}{N^\alpha}\right) + \sum_{N+1}^\infty O\left(\frac{1}{n^{\alpha+1}}\right) \\ &= O\left(\frac{1}{N^\alpha}\right).\end{aligned}$$

To show $\sum_1^\infty (\epsilon_n e^{int}/n^{\alpha+\frac{1}{2}})$ is Lip_α, let $h > 0$ and consider
$$\begin{aligned}\sum_1^\infty \frac{\epsilon_n e^{in(t+h)}}{n^{\alpha+\frac{1}{2}}} - \sum_1^\infty \frac{\epsilon_n e^{int}}{n^{\alpha+\frac{1}{2}}} &= \sum_1^\infty \frac{\epsilon_n (e^{in(t+h)} - e^{int})}{n^{\alpha+\frac{1}{2}}} \\ &= \sum_1^N + \sum_{N+1}^\infty,\end{aligned}$$

https://doi.org/10.1017/S0305004100060412 Published online by Cambridge University Press

where $1/(N+1) < h \leqslant 1/N$.

$$\begin{aligned}\operatorname{Re}\sum_1^N &= \sum_1^N \frac{\epsilon_n(\cos n(t+h)-\cos nt)}{n^{\alpha+\frac{1}{2}}}\\ &= -h\sum_1^N \frac{\epsilon_n \sin n(t+\theta h)}{n^{\alpha-\frac{1}{2}}},\end{aligned}$$

for some $0 < \theta < 1$,

$$\begin{aligned}&= -h \operatorname{Im}\sum_1^N \frac{\epsilon_n e^{in(t+\theta h)}}{n^{\alpha-\frac{1}{2}}}\\ &= h\,O(N^{1-\alpha})\\ &= O(h^\alpha).\end{aligned}$$

Similarly,

$$\operatorname{Im}\sum_1^N = O(h^\alpha).$$

$$\begin{aligned}\left|\sum_{N+1}^\infty\right| &\leqslant \left|\sum_{N+1}^\infty \frac{\epsilon_n e^{in(t+h)}}{n^{\alpha+\frac{1}{2}}}\right| + \left|\sum_{N+1}^\infty \frac{\epsilon_n e^{int}}{n^{\alpha+\frac{1}{2}}}\right|\\ &= O\left(\frac{1}{N^\alpha}\right)\\ &= O(h^\alpha).\end{aligned}$$

COROLLARY. *For $0 < \alpha < 1$, there exist symmetric $K(x,t) \in \operatorname{Lip}_\alpha[-\pi,\pi]^2$ whose eigen-values $\lambda_n \sim 1/n^{\alpha+\frac{1}{2}}$.*

Proof. Take $K(x,t) = \sum_1^\infty (\epsilon_n e^{in(x-t)}/n^{\alpha+\frac{1}{2}})$.

3. *Positive definite kernels*

Let $PD\operatorname{Lip}_\alpha[a,b]^2$ denote the set of all symmetric kernels $K(x,t) \in \operatorname{Lip}_\alpha[a,b]^2$ which are positive (semi) definite, i.e. $\lambda_n \geqslant 0$ for all n.

LEMMA 3. *For $K(x,t) \in PD\operatorname{Lip}_\alpha[0,1]^2$, where $0 < \alpha < 1$, we have*

$$\lambda_n = O\left(\frac{1}{n^{\alpha+1}}\right).$$

Proof (See (4).) For each N, let

$$R_N(x,t) = N\sum_1^N \psi_n(x)\,\psi_n(t),$$

where

$$\begin{aligned}\psi_n(x) &= 1 \quad \text{if} \quad \frac{n-1}{N} \leqslant x \leqslant \frac{n}{N},\\ &= 0 \quad \text{otherwise.}\end{aligned}$$

Let $J(x,t)$ be the unique positive definite square root of $K(x,t)$, and let

$$S_N(x,t) = \int_0^1\int_0^1 J(x,u)\,R_N(u,v)\,J(v,t)\,du\,dv.$$

$S_N(x,t)$ is symmetric of rank N and $K(x,t) - S_N(x,t)$ is positive definite and continuous. Also

$$\begin{aligned}\int_0^1 (K(x,x)-S_N(x,x))\,dx &= \int_0^1\int_0^1 R_N(x,t)\left[\frac{K(x,x)+K(t,t)}{2} - K(x,t)\right]dx\,dt\\ &\leqslant \int_0^1\int_0^1 R_N(x,t)\,C|x-t|^\alpha\,dx\,dt\\ &\leqslant \int_0^1\int_0^1 R_N(x,t)\,\frac{C}{N^\alpha}\,dx\,dt\\ &= \frac{C}{N^\alpha}.\end{aligned}$$

https://doi.org/10.1017/S0305004100060412 Published online by Cambridge University Press

Therefore
$$\sum_{N+1}^{\infty} \lambda_n \leqslant \frac{C}{N^\alpha},$$

which gives
$$\lambda_n = O\left(\frac{1}{n^{\alpha+1}}\right),$$

as required.

LEMMA 4. $\sum_1^\infty (e^{int}/n^{\alpha+1}) \in \mathrm{Lip}_\alpha [-\pi, \pi]$ *for* $0 < \alpha < 1$.

Proof. Clearly, $\sum_1^\infty (e^{int}/n^{\alpha+1})$ is uniformly convergent and its sum is C^1 except at $t = 0$. For any $h > 0$, we have

$$\sum_1^\infty \frac{e^{inh}}{n^{\alpha+1}} - \sum_1^\infty \frac{1}{n^{\alpha+1}} = \sum_1^\infty \frac{e^{inh}-1}{n^{\alpha+1}}$$
$$= \sum_1^N + \sum_{N+1}^\infty,$$

where $1/(N+1) < h \leqslant 1/N$.

$$\left|\sum_1^N\right| \leqslant \sum_1^N \frac{|e^{inh}-1|}{n^{\alpha+1}}$$
$$\leqslant \sum_1^N \frac{nh}{n^{\alpha+1}}$$
$$= h \sum_1^N \frac{1}{n^\alpha}$$
$$= h\, O(N^{1-\alpha})$$
$$= O(h^\alpha).$$
$$\sum_{N+1}^\infty = O\left(\sum_{N+1}^\infty \frac{1}{n^{\alpha+1}}\right)$$
$$= O\left(\frac{1}{N^\alpha}\right)$$
$$= O(h^\alpha).$$

COROLLARY. *For* $0 < \alpha < 1$, *there exist* $K(x,t) \in PD\,\mathrm{Lip}_\alpha [-\pi, \pi]^2$ *with eigen-values* $\lambda_n = 1/n^{\alpha+1}$.

Proof. Take $K(x,t) = \sum_1^\infty (e^{in(x-t)}/n^{\alpha+1})$.

4. *The case* $\alpha = 1$

If $K(x,t) \in \mathrm{Lip}_1 [a,b]^2$, we can say
$$\lambda_n = o\left(\frac{1}{n^{\frac{3}{2}}}\right),$$

however, if $K(x,t) \in PD\,\mathrm{Lip}_1 [a,b]^2$, we can only say
$$\lambda_n = O\left(\frac{1}{n^2}\right).$$

LEMMA 5. *If* $f(t) \in \mathrm{Lip}_1 [-\pi, \pi]$ *and* $f(-\pi) = f(\pi)$, *and the nth Fourier coefficient of* $f(t)$ *is*
$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, e^{-int}\, dt,$$

then
$$\Sigma n^2 |c_n|^2 < \infty.$$

https://doi.org/10.1017/S0305004100060412 Published online by Cambridge University Press

Proof.
$$c_n = -\frac{1}{2\pi i n}\int_{-\pi}^{\pi} f'(t)\, e^{-int}\, dt$$
$$= -\frac{c_n'}{in},$$

where c_n' is the nth Fourier coefficient of $f'(t)$. The integration by parts is justified by the fact that $f(t)$ is absolutely continuous with $f'(t) \in L^\infty[-\pi, \pi]$. (See (1), p. 58.) Therefore

$$\Sigma n^2 |c_n|^2 = \Sigma |c_n'|^2$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} |f'(t)|^2\, dt$$
$$< \infty.$$

COROLLARY. *If*
$$s_N(t) = \sum_{n=-N}^{N} c_n e^{int},$$

is the Nth partial sum of the Fourier series of $f(t)$, then

$$\int_{-\pi}^{\pi} |s_N(t) - f(t)|^2\, dt = o\left(\frac{1}{N^2}\right).$$

Proof.
$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |s_N(t) - f(t)|^2\, dt = \sum_{|n| \geqslant N+1} |c_n|^2$$
$$\leqslant \frac{1}{N^2} \sum_{|n| \geqslant N+1} n^2 |c_n|^2$$
$$= o\left(\frac{1}{N^2}\right).$$

LEMMA 6. *If $K(x,t) \in \mathrm{Lip}_1[-\pi, \pi]^2$ and $K(-\pi, t) = K(\pi, t)$ for all t, $K(x, -\pi) = K(x, \pi)$ for all x, and if $s_N(x,t)$ is the Nth partial sum of the double Fourier series of $K(x,t)$, i.e.*

$$s_N(x,t) = \sum_{m=-N}^{N} \sum_{n=-N}^{N} c_{mn}\, e^{i(mx - nt)},$$

where
$$c_{mn} = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} K(x,t)\, e^{i(nt - mx)}\, dx\, dt,$$

then
$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |s_N(x,t) - K(x,t)|^2\, dx\, dt = o\left(\frac{1}{N^2}\right).$$

Proof. Similar to Lemma 5.

LEMMA 7. *For symmetric $K(x,t) \in \mathrm{Lip}_1[-\pi, \pi]^2$,*

$$\lambda_n = o\left(\frac{1}{n^{\frac{3}{2}}}\right).$$

Proof. $s_N(x,t)$ is symmetric of rank $2N+1$, therefore, by Lemma 6 and E. Schmidt's lemma,

$$\sum_{2N+2}^{\infty} \lambda_n^2 = o\left(\frac{1}{N^2}\right)$$

which gives the required result.

If $K(x,t)$ does not satisfy the condition $K(-\pi, t) = K(\pi, t)$ all t, $K(x, -\pi) = K(x, \pi)$ all x, then it differs from a kernel which does by a Lip_1 kernel of finite rank, so the asymptotic behaviour of the eigen-values is not affected. (See e.g. (5) p. 239.)

https://doi.org/10.1017/S0305004100060412 Published online by Cambridge University Press

LEMMA 8. $\sum_1^\infty (\epsilon_n e^{int}/n^{\frac{1}{2}+\alpha}) \in C^1[-\pi,\pi]$ *for* $\alpha > 0$, *where* (ϵ_n) *are the Rudin–Shapiro coefficients.*

Proof. The derived series is $i\sum_1^\infty (\epsilon_n e^{int}/n^{\frac{1}{2}+\alpha})$ which is uniformly convergent. (See proof of Lemma 2.)

COROLLARY. *The estimate of Lemma 7 cannot be improved to* $1/n^{\frac{3}{2}+\alpha}$ *for any* $\alpha > 0$.

Proof. The kernel $K(x,t) = \sum_1^\infty (\epsilon_n e^{in(x-t)}/n^{\frac{3}{2}+\alpha})$ is C^1, a fortiori Lip_1, and its eigenvalues $\sim 1/n^{\frac{3}{2}+\alpha}$.

LEMMA 9. *For* $K(x,t) \in PD\,\mathrm{Lip}_1[0,1]^2$,

$$\lambda_n = O\left(\frac{1}{n^2}\right).$$

Proof. As Lemma 3.

LEMMA 10. $\sum_1^\infty (\cos nt/n^2) \in \mathrm{Lip}_1[-\pi,\pi]$.

Proof. Its sum is $\dfrac{(\pi-|t|)^2}{4} - \dfrac{\pi^2}{12}$,

for $|t| \leqslant \pi$.

COROLLARY. *Lemma 9 is best possible.*

Proof. Consider $K(x,t) = \sum_1^\infty \dfrac{\cos n(x-t)}{n^2}$.

REFERENCES

(1) BURKILL, J. C. *The Lebesgue integral* (Cambridge, 1958).
(2) COCHRAN, J. A. *Analysis of linear integral equations* (Bell Telephone Laboratories, 1972.)
(3) HILLE, E. and TAMARKIN, J. E. On the characteristic values of linear integral equations. *Acta Math.* **57** (1931), 1–76.
(4) READE, J. B. Eigen-values of positive definite kernels. *S.I.A.M. Journal in Analysis.* (To appear.)
(5) RIESZ, F. and NAGY, B. S. *Functional analysis* (Budapest and Szeged, 1952.)
(6) RUDIN, W. Some theorems on Fourier coefficients. *Proc. Amer. Math. Soc.* **10** (1959), 855–859.
(7) SMITHIES, F. The eigen-values and singular values of integral equations. *Proc. London Math. Soc.* (2) **43** (1937), 255–279.
(8) WEYL, H. Das Asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen. *Math. Ann.* **71** (1912), 441–479.
(9) WHITTAKER, E. T. and WATSON, G. N. *A course of modern analysis*, 4th ed. (Cambridge, 1927.)
(10) ZYGMUND, A. *Trigonometrical series* (Wilno, 1935.)

https://doi.org/10.1017/S0305004100060412 Published online by Cambridge University Press