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Almost minimal systems and periodicity in hyperspaces

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Abstract. Given a self-map of a compact metric space X, we study periodic points of the map induced on the hyperspace of closed non-empty subsets of X. We give some necessary conditions on admissible sets of periods for these maps. Seemingly unrelated to this, we construct an almost totally minimal homeomorphism of the Cantor set. We also apply our theory to give a full description of admissible period sets for induced maps of the interval maps. The description of admissible periods is also given for maps induced on symmetric products.

1. *Introduction*

A continuous function $f\colon X \to X$ from a compact metric space X to itself induces a continuous function $2^f\colon 2^X \to 2^X$ on the (compact metric) space of closed non-empty subsets of X with Hausdorff metric by defining $2^f(C) = f(C) = \{f(x) : x \in C\}$ for all closed $C \in 2^X$. The study of such systems was initiated by Bauer and Sigmund [**5**], where they can been seen as a (simplified) topological version of the induced dynamics on the space of measures over X and their evolution under f. Edalat [**11**] has argued that such systems are the natural approach to dynamical systems, from a computational and domain theoretic point of view.

Clearly, one is interested in which properties of the dynamical system (X, f) transfer to the system $(2^X, 2^f)$ and there have been a number of results in this direction. For example, Banks [**4**] proves that f is (weakly) mixing if and only if 2^f is (weakly) mixing and that 2^f is transitive if and only if f is weakly mixing. In [**5**] and also [**19**], it is shown that if f has positive entropy, then the entropy of 2^f is infinite. Kwietniak and Oprocha go on to show [**19**] that if f does not have a dense set of recurrent points, then the entropy of 2^f is at least $\log 2$. Guirao *et al* in [**16**] show that all the usual notions of chaos except for

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the classical Devaney's definition are transferred from T to 2^T, whereas the converse does not hold for any of them. Recently, the authors jointly with Ramírez in [**13**] have proved that 2^f is chain transitive if and only if f is chain weakly mixing if and only if f is chain transitive and has a fixed point. The first two authors [**12**] also show that 2^f has shadowing if and only if f does.

However, little work has been done on the periodic points of 2^f other than some results on periodicity in symmetric products in [**15**], which we extend in §9. It turns out that the theory here is of some interest. Moreover, the techniques that seem to be necessary to construct hyperspaces with various periods are of independent interest.

Example 3.1 shows that the set of periods of 2^f, denoted by $\mathrm{Per}(2^f)$, can be $\mathbb{N}$ even when f has no periodic points at all and, in Theorem 3.2, we show quite easily that if f is an interval map, then the set of periods of 2^f is either $\{1\}$ or $\{1, 2\}$ or $\mathbb{N}$.

THEOREM 3.2. *Let f be a continuous map of a compact interval to itself. Then* $\mathrm{Per}(2^f)$ *is either* $\{1\}$ *or* $\{1, 2\}$ *or* $\mathbb{N}$.

Trivially, if x is a point of period n under f, then $\{x\}$ is a point of period n under 2^f. Since finite unions of points are again closed sets, it follows, for example, that if f has a period four point and a period six point, then 2^f has points of period one, two, three, four, six and twelve. It is possible to give a complete description of admissible periods in 2^f that arise in this fashion, i.e., those that are formed exclusively of points that were periodic in the original system (X, f), and this is done in §9. More generally, if 2^f has points of period four and six consisting of pairwise disjoint subsets of X, then it would again have points of period one, two, three, four, six and twelve. It is reasonable to ask then, whether this is a general property of induced maps on hyperspaces.

A positive result in this direction is the following theorem.

THEOREM 5.1. *Given a continuous map $f: X \to X$, the set of periods* $\mathrm{Per}(2^f)$ *of the induced map on* 2^X *contains* $\mathrm{Per}(f) \cup \{1\}$ *and is closed under taking prime power divisors.*

Whether $\mathrm{Per}(2^f)$ is closed under taking arbitrary divisors remains open. We are, however, able to show that $\mathrm{Per}(2^f)$ needs not to be closed under taking least common multiples. In §7, we construct a dynamical system (Z, g) on the Cantor set such that the induced system $(2^Z, 2^g)$ only has periodic points of periods one, two and three. In particular, it does not have a period six point. In Theorem 7.4, we generalize this construction to an induced system which admits only those periods that are divisors† of two positive integers p and q.

THEOREM 7.4. *Let $p, q \in \mathbb{N}$. There exists a continuous onto map of the Cantor set $g: Z \to Z$ for which the periods appearing in the induced map are exact divisors of either p or q, i.e.*

$$\mathrm{Per}(2^g) = \{m \mid m|p \text{ or } m|q\}.$$

To obtain those results, we first construct an *almost totally minimal system* over the Cantor set which is of interest in its own right (see Theorem 6.1). Recall that a system

† Here, and elsewhere in the text, by divisors we actually mean *positive* divisors.

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(X, f) having a fixed point $x^0 \in X$ is almost totally minimal if, after removing its (unique) fixed point, the remaining non-compact system $(X^* = X \setminus \{x^0\}, f)$ is totally minimal, meaning that the full orbit with respect to any iterate of f of any point is dense in X^* (see §6 for a precise definition).

THEOREM 6.1. *There exists an almost totally minimal homeomorphism $T : \mathcal{C} \to \mathcal{C}$ of the Cantor set $\mathcal{C}$. Such T has exactly one fixed point $x^0 \in \mathcal{C}$ and the full T^m-orbit of every other point is dense in $\mathcal{C}$ for every $m \in \mathbb{N}$:*

(i) $(\exists!\, x^0 \in \mathcal{C})\ T(x^0) = x^0$*; and*

(ii) $(\forall y \in \mathcal{C} \setminus \{x^0\})\ (\forall m \in \mathbb{N})\ \overline{\{T^{mk}(y) \mid k \in \mathbb{Z}\}} = \mathcal{C}$.

This theorem is proved using *graph covers*, a tool first devised by Gambaudo and Martens in [**14**] to give a combinatorial description of minimal systems over the Cantor set. This theory has proved useful not just for describing algebraic structure of such systems but also for constructing maps on the Cantor set with particular properties. Shimomura [**24**], for example, uses it to construct a transitive, completely scrambled zero-dimensional system. Essentially the same method but formalized in a slightly different way was used by Akin, Glasner and Weiss in [**1**] to give a generic self homeomorphism of the Cantor set. Bernardes and Darji extend some of their results in [**6**] and again make use of the same method. There is a close link between these graph covers, Bratteli–Vershik diagrams and Kakutani–Rokhlin towers and it would be possible to pass from one representation to another. As an example, in §8, we give a Bratteli–Vershik representation of the almost totally minimal system from Theorem 6.1.

The rest of the paper is organized as follows. In §2, we briefly introduce the notation. A complete description of the situation for interval maps is given in §3. In §4, we introduce minimal building blocks of periodic sets in hyperspaces and derive some fundamental properties related to these. In §5, we seek to characterize admissible periods for induced maps. In §6, an almost totally minimal homeomorphism of the Cantor set is constructed, which we then use in §7 to obtain a map whose induced system admits only periods one, two and three. In §8, we give a Bratteli–Vershik representation of the system constructed in §6. Section 9 contains results on periods in symmetric products. At the very end, in §10, we list some problems naturally arising from our considerations that are still without a satisfying resolution.

2. *Preliminaries*

All the spaces under consideration will be compact and metric unless specified otherwise. Given a continuous self-map $f : X \to X$ of such a space, the inverse limit $\varprojlim(X, f)$ is defined as the set of full orbits of this system

$$\varprojlim(X, f) = \left\{ \bar{x} = (x_i)_{i\in\mathbb{Z}} = (\dots, x_{-1}, x_0, x_1, x_2, \dots) \in \prod_{i\in\mathbb{Z}} X \mid f(x_i) = x_{i+1} \right\}.$$

Note the slightly unconventional enumeration of indices above. We shall say that any $\bar{x} \in \varprojlim(X, f)$ with $x_0 = x$ is a *full orbit* of the point $x \in X$. There exists a natural homeomorphism of $\varprojlim(X, f)$ called (right) shift $\sigma : \varprojlim(X, f) \to \varprojlim(X, f)$, given by $(\sigma(\bar{x}))_i = x_{i+1}$ for all $i \in \mathbb{Z}$. For more on inverse limits see [**18**].

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Recall that to a full orbit $\bar{x} \in \varprojlim(X, f)$ one can associate two limit sets, ω- and α-limit sets, which are the accumulation sets of the forward and backward orbit of x_0, respectively.

$$\omega(\bar{x}) = \bigcap_{m=0}^{\infty} \overline{\bigcup_{n=m}^{\infty} \{x_n\}},$$
$$\alpha(\bar{x}) = \bigcap_{m=0}^{\infty} \overline{\bigcup_{n=m}^{\infty} \{x_{-n}\}}.$$

Both of these are closed and strongly f-invariant meaning that $f(\omega(\bar{x})) = \omega(\bar{x})$ and $f(\alpha(\bar{x})) = \alpha(\bar{x})$ (see, e.g., [**3**, **7**]). Note that we could equivalently say that they are fixed points of the induced map 2^f (see below). As a shorthand, we denote their union $\omega(\bar{x}) \cup \alpha(\bar{x})$ by $\lambda(\bar{x})$, which is again a closed and strongly f-invariant set.

The set of all non-empty closed (and compact) subsets of X is denoted by 2^X and can be metrized using the *Hausdorff metric*

$$H(A, B) = \inf\{\varepsilon > 0 \mid A \subseteq N_X(B, \varepsilon) \text{ and } B \subseteq N_X(A, \varepsilon)\},$$
$$\text{where } N_X(S, \varepsilon) = \{x \in X \mid (\exists y \in S)\, d(x, y) < \epsilon\},$$

with the corresponding induced topology coinciding with *Vietoris' topology* (see, e.g., [**21**] for more details).

The induced map $2^f : 2^X \to 2^X$ on the hyperspace 2^X is defined by

$$2^f(S) = f(S) = \{f(s) \mid s \in S\}$$

and is continuous with respect to Hausdorff metric.

As we shall constantly be dealing with the sets of periods of different functions, it is convenient to introduce a symbol $\mathrm{Per}(f)$ for the subset of natural numbers such that $k \in \mathrm{Per}(f)$ if and only if there exists a point $x \in X$ with the fundamental period k. $\mathrm{Per}(2^f)$ will consequentially be the set of all the fundamental periods of points in 2^X.

We shall also be interested in restrictions of the map 2^f to a few 2^f-invariant subsets of 2^X. We introduce a special symbol for each of these restrictions, namely,

$$f_n = 2^f|_{F_n(X)},$$
$$f^{<\omega} = 2^f|_{F(X)},$$

where $F_n(X) = \{A \in 2^X \mid A \text{ has at most } n \text{ points}\}$ is the *n-fold symmetric product of X* and $F(X) = \bigcup_{n=1}^{\infty} F_n(X)$ is the collection of all finite subsets of X. Occasionally, we shall write just f for any of the above maps (including 2^f itself) as this does not lead to any confusion, and it is useful to keep the notation simple, especially when we need to refer to the nth iterate of the map 2^f, which we simply denote by f^n.

The usual *n-fold Cartesian product* will also be of interest as the n-fold symmetric product can be seen as a quotient of this space. Somewhat unconventionally, we denote the product space $\underbrace{X \times X \times \cdots \times X}_{n-times}$ by $X^{(n)}$ and the induced map by $f^{(n)}$. This is not to be confused with f^n which is simply the nth iterate of f.

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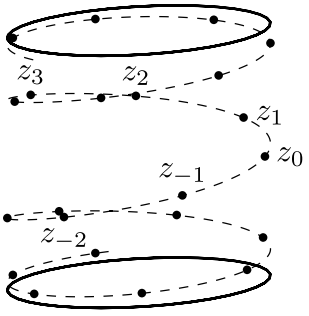


FIGURE 1. The set X from Example 3.1.

3. *Two simple results*

To put our results into perspective, we start with two related results.

Example 3.1. Let $C = \mathbb{S}^1 \times [0, 1]$ be a cylinder where $\mathbb{S}^1 = [0, 1]/\sim$ denotes the unit circle obtained from the interval $[0, 1]$ with its endpoints identified. Let $g\colon C \to C$ be an irrational rotation by $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ about the central axis combined with an upward displacement that preserves the bases, e.g. $(\theta, z) \overset{g}{\mapsto} (\theta + \alpha \bmod 1, 2z - z^2)$. Note that this is a homeomorphism of C. Let $X \subset C$ be the set consisting of the two bases $\mathbb{S}^1 \times \{0, 1\}$ and a full orbit $\{\dots, z_{-1}, z_0, z_1, z_2, \dots\}$ of a point z_0 on a generating line, say, $z_0 = (0, 1/2)$, where we set $z_k = g^k(z_0)$ for all $k \in \mathbb{Z} \setminus \{0\}$ (see Figure 1). Then, clearly, $f = g|_X$ has no periodic points and, at the same time, for any $k \in \mathbb{N}$ the set $\mathbb{S}^1 \times \{0, 1\} \cup \{\dots, z_{-k}, z_0, z_k, \dots\}$ is periodic under 2^f with period k.

Block and Coven [**8**] prove that if f is be a continuous map from a compact interval to itself and every point is chain recurrent, then either f^2 is the identity map or f is turbulent. From this, one can easily deduce the following theorem.

THEOREM 3.2. *Let f be a continuous map of a compact interval to itself. Then* $\mathrm{Per}(2^f)$ *is either* $\{1\}$ *or* $\{1, 2\}$ *or* $\mathbb{N}$.

Proof. If f^2 is the identity, then $\mathrm{Per}(2^f)$ is either $\{1\}$ or $\{1, 2\}$. If f^2 turbulent, then f must have a periodic point with period which is not a power of two (see [**7**, p. 33]). But if f has a point of period $2^n(2k + 1)$, then, by Šarkovs′kiĭ's theorem, it has points of period $2^{n+1}m$ for any positive integer m, from which it follows that 2^f has points of period m, i.e. that $\mathrm{Per}(2^f) = \mathbb{N}$.

So suppose that there exists a point x_0 that is not chain recurrent.† Clearly, x_0 is not periodic. Also, we may assume that f is onto, as otherwise one can take a restriction to the surjective core of the map f and repeat the same reasoning. We can therefore take a full orbit $\bar{x} \in \varprojlim(X, f)$ of x_0 and consider the limit set

$$\lambda(\bar{x}) = \alpha(\bar{x}) \cup \omega(\bar{x}) = \bigcap_{m=0}^{\infty} \overline{\bigcup_{n=m}^{\infty} \{x_{-n}, x_n\}}.$$

† Compare this with Proposition 4.2. Also note that whether a point is *(chain) recurrent* depends only on its forward orbit, as is common in the literature; but, for convenience, our notion of *non-recurrent* point involves the full λ-limit set of the point. As a results, it is possible for a point to be neither recurrent nor non-recurrent.

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We claim that x_0 is non-recurrent, i.e. $x_0 \notin \lambda(\bar{x})$. Firstly note that $x_0 \notin \omega(x_0)$ as otherwise it would be recurrent and hence chain recurrent. Another possibility is that there exists an increasing subsequence $(p_k)_{k\in\mathbb{N}}$ such that $x_{-p_k} \to x_0$ as $k \to \infty$. But then $x_{-p_k+1} \to x_1$ and, for any $\epsilon > 0$, one can choose $k_0 \in \mathbb{N}$ large enough so that $x_{-p_{k_0}+1}$ is ϵ close to x_1. Thus $\{x_0, x_{-p_{k_0}+1}, x_{-p_{k_0}+2}, \dots, x_0\}$ is an ϵ-chain from x_0 to x_0 making x_0 a chain recurrent point. This is a contradiction.

As x_0 is non-recurrent, by Proposition 4.2, we immediately obtain all periods in $\mathrm{Per}(2^f)$, which proves our corollary. □

4. *Elementary periodic points*

We shall first describe the most basic type of periodic points that appear in the induced dynamics on 2^X. It captures both the periods arising from cycles in (X, f) via $x \mapsto \{x\}$ embedding as well as the periods as in Example 3.1.

Given a point $\bar{x} = (\dots, x_{-2}, x_{-1}, x_0, x_1, x_2, \dots) \in \varprojlim(X, f)$, we define the *set of periods of* $\bar{x}$ by

$$\mathrm{Per}(\bar{x}) = \{k \in \mathbb{N} \mid \overline{\{x_{mk} \mid m \in \mathbb{Z}\}} \in 2^X \text{ is periodic with period } k\}.$$

We use the same symbol as before but the meaning will be clear from the context. Note that this set does not depend on the choice of the starting point, i.e. $\mathrm{Per}(\bar{x}) = \mathrm{Per}(\sigma(\bar{x})) = \mathrm{Per}(\sigma^{-1}(\bar{x}))$. It may, however, depend on the chosen backward orbit of x_0. For example, if x_0 is a fixed point which also has a history of infinitely many isolated points, then $\mathrm{Per}((\dots, x_{-2}, x_{-1}, x_0, x_0, \dots)) = \mathbb{N}$ but $\mathrm{Per}((\dots, x_0, x_0, x_0, \dots)) = \{1\}$.

Remark 4.1. If $\{x_0, x_1, \dots, x_{p-1}\}$ is a p-cycle in (X, f), then it is not hard to check that $\mathrm{Per}((\dots, x_{p-1}, x_0, \dots, x_{p-1}, x_0, \dots))$ is the set of all divisors of p.

The situation akin to that in Example 3.1 occurs whenever there exists a *non-recurrent point*, i.e. a point whose full orbit does not accumulate at the point itself. Formally, $x \in X$ is *non-recurrent* if there exists a full orbit $\bar{x} \in \varprojlim(X, f)$ of $x = x_0$ such that $x \notin \lambda(\bar{x})$.

PROPOSITION 4.2. *Let $f : X \to X$ be a continuous map and assume that there exists a non-recurrent point x_0 with a full orbit $\bar{x} = (\dots, x_{-1}, x_0, x_1, \dots) \in \varprojlim(X, f)$. Then* $\mathrm{Per}(\bar{x}) = \mathbb{N}$ *and therefore also* $\mathrm{Per}(2^f) = \mathbb{N}$.

Proof. Let $k \in \mathbb{N}$ be arbitrary and set $S = \overline{\{x_{mk} \mid m \in \mathbb{Z}\}} \in 2^X$. S is clearly mapped to itself under k iterates of 2^f. It is therefore periodic with a fundamental period that divides k. To show that this is in fact k, it suffices to see that $x_0 \notin (2^f)^n(S)$ for any $0 < n < k$.

To that end, take any such n and note that $(2^f)^n(S) = \overline{\{x_{mk+n} \mid m \in \mathbb{Z}\}}$. Recall that $x_0 \notin \lambda(\bar{x})$ and so, if x_0 was in $(2^f)^n(S)$, it could only be equal to x_{m_0k+n} for some $m_0 \in \mathbb{Z}$ and thus x_0 would have to be periodic. This would then imply that $x_0 \in \omega(\bar{x}) \subseteq \lambda(\bar{x})$, which gives a contradiction. □

Remark 4.3. It could, however, happen that x itself is recurrent but if it has a point x_{-k} for some $k > 0$ in the backward part of its orbit that is non-recurrent, then the result still holds as $\mathrm{Per}(\bar{x}) = \mathrm{Per}(\sigma^{-k}(\bar{x})) = \mathbb{N}$.

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We have already used the obvious fact that $\mathrm{Per}(\bar{x}) \subseteq \mathrm{Per}(2^f)$ and one might suspect that $\bigcup_{\bar{x}\in\varprojlim(X,f)} \mathrm{Per}(\bar{x}) = \mathrm{Per}(2^f)$, but this is not the case. It suffices to take a system that consists of five points, two of which form a 2-cycle and the other three a 3-cycle. Then, clearly, $6 \in \mathrm{Per}(2^f)$ but no point in the inverse limit has six in its set of periods.

$\mathrm{Per}(\bar{x})$, however, has a nice structure. Below we show that it is closed under taking divisors and least common multiples. This, in particular, implies that when it is finite, $\mathrm{Per}(\bar{x})$ is simply the set of divisors of its largest element. But what if it is infinite? Does it have to be $\mathbb{N}$? A negative answer to this provides the odometer, a classical example of a transitive system over the Cantor set.

Example 4.4. Let $X = \Sigma_2 = \{0, 1\}^{\mathbb{N}}$. The 2-adic odometer $f\colon X \to X$ is defined recursively by

$$f(\xi_0, \xi_1, \dots) = \begin{cases} (1, \xi_1, \dots) & \text{if } \xi_0 = 0, \\ (0, f(\xi_1, \xi_2, \dots)) & \text{otherwise.} \end{cases}$$

It is not hard to see that $\mathrm{Per}(\bar{x}) = \{1, 2, 2^2, 2^3, \dots\}$ is the set of all powers of 2 for any $\bar{x} \in \varprojlim(X, f)$, and that $\mathrm{Per}(2^f)$ is the same set.

We note, in passing, that this is also an example of a system where 2^f-periodic points are dense in 2^X. This is simply because any closed set in Σ_2 can be approximated by a finite union of clopen cylinders to an arbitrary precision and these cylinders are clearly periodic under 2^f and so is their finite union. We remind the reader that a cylinder in Σ_2 is a set of the form

$$\{(\xi_0, \xi_1, \dots) \in \Sigma_2 \mid \xi_i = a_i \text{ for } 0 \le i \le n\},$$

for some $n \in \mathbb{N}$ and a choice of $a_i \in \{0, 1\}$, and that all such sets form a clopen basis of the Cantor topology on Σ_2.

To prove the aforementioned structural result for elementary sets of periods, we shall need two lemmata.

LEMMA 4.5. *If $S \in 2^X$ is a k-periodic, then none of its $k-1$ iterates under 2^f can be a subset of S.*

Proof. Otherwise, let us assume that $(2^f)^j(S) \subseteq S$ for some $0 < j < k$. Using the usual notation, one would write $f^j(S) \subseteq S$ and from there

$$S = f^{kj}(S) \subseteq f^{(k-1)j}(S) \subseteq \cdots \subseteq f^j(S) \subseteq S.$$

This makes S periodic with a period strictly less than k, which is a contradiction. □

LEMMA 4.6. *Let $\bar{x} \in \varprojlim(X, f)$. Then*

$$\mathrm{Per}(\bar{x}) = \{k \in \mathbb{N} \mid \exists N_0 \in \mathbb{N} \text{ s.t. } x_{-N_0 k} \notin \overline{\{x_l \mid l \in \mathbb{Z} \text{ and } k \not| l\}}\}. \tag{1}$$

Also

$$\mathrm{Per}(\bar{x}) = \{k \in \mathbb{N} \mid \exists N_0 \in \mathbb{N} \text{ s.t. } \forall N \ge N_0 \quad x_{-Nk} \notin \overline{\{x_l \mid l \in \mathbb{Z} \text{ and } k \not| l\}}\} \tag{2}$$

and, in particular,

$$\mathrm{Per}(\bar{x}) = \bigcup_{N=0}^{\infty} \{k \in \mathbb{N} \mid x_{-Nk} \notin \overline{\{x_l \mid l \in \mathbb{Z} \text{ and } k \not| l\}}\}. \tag{3}$$

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Proof. (2) follows easily from (1) if one recalls that $x_{-Nk} \notin \overline{\{x_l \mid l \in \mathbb{Z} \text{ and } k \not| l\}}$ implies that $x_{-(N+1)k} \notin \overline{\{x_l \mid l \in \mathbb{Z} \text{ and } k \not| l\}}$. And from there it is clear that the sets under the union sign in (3) form a monotonically increasing family of sets converging to the set in (1).

We now turn to proving that the first claim holds. We first show that any k satisfying the defining statement of the set in (1) is necessarily in $\mathrm{Per}(\bar{x})$. Let such a $k \in \mathbb{N}$ be fixed and let N_0 be as in the definition of the set. It is clear that $x_{-N_0 k}$ is a 'distinguishing feature': an element contained in $\overline{\{x_{mk} \mid m \in \mathbb{Z}\}}$ but which cannot be in any of its $k-1$ forward iterates under 2^f. The set $\overline{\{x_{mk} \mid m \in \mathbb{Z}\}}$ is thus truly a k-period point.

Conversely, take a $k \in \mathbb{N}$ which does not satisfy the statement in (1). Hence, there are infinitely many points in the k-step backward orbit of x_0 that are also in $\overline{\{x_l \mid l \in \mathbb{Z} \text{ and } k \not| l\}}$. In particular, infinitely many of them are in $\overline{\{x_{km+j} \mid m \in \mathbb{Z}\}}$ for some $0 < j < k$. Similar reasoning as in the first paragraph of this proof allows us to conclude that, not just infinitely many, but all of the elements of the k-step orbit $\{x_{mk} \mid m \in \mathbb{Z}\}$ are contained in $\overline{\{x_{km+j} \mid m \in \mathbb{Z}\}}$ and hence their closure as well. But this implies that $\overline{\{x_{km} \mid m \in \mathbb{Z}\}}$ is not a k-periodic point by Lemma 4.5. □

PROPOSITION 4.7. *Let* $\bar{x} \in \varprojlim(X, f)$. *Then* $\mathrm{Per}(\bar{x})$ *is non-empty (always contains at least* 1*) and closed under taking divisors and least common multiples.*

Proof. Clearly $1 \in \mathrm{Per}(\bar{x})$. The rest is proved by invoking Lemma 4.6. Let $k \in \mathrm{Per}(\bar{x})$, $d|k$ and let N_0 be chosen as in (1), i.e. $x_{-N_0 k} \notin \overline{\{x_l \mid l \in \mathbb{Z} \text{ and } k \not| l\}}$ and hence $x_{-(N_0 k/d)d} \notin \overline{\{x_l \mid l \in \mathbb{Z} \text{ and } k \not| l\}} \supseteq \overline{\{x_l \mid l \in \mathbb{Z} \text{ and } d \not| l\}}$. Therefore $d \in \mathrm{Per}(\bar{x})$.

As we have already seen that this set is closed under taking divisors, it will suffice to show that for any two co-prime $m, n \in \mathrm{Per}(\bar{x})$ their product mn is in there as well. Chose N_0 to be the greater of the two integers associated to m and n in the context of (1). Then $x_{-(N_0 m)n} \notin \overline{\{x_l \mid l \in \mathbb{Z} \text{ and } n \not| l\}}$ and $x_{-(N_0 n)m} \notin \overline{\{x_l \mid l \in \mathbb{Z} \text{ and } m \not| l\}}$. As m and n are co-prime, $nm \not| l$ if and only if $n \not| l$ or $m \not| l$, and hence $x_{-N_0(mn)} \notin \overline{\{x_l \mid l \in \mathbb{Z} \text{ and } n \not| l\}} \cup \overline{\{x_l \mid l \in \mathbb{Z} \text{ and } m \not| l\}} = \overline{\{x_l \mid l \in \mathbb{Z} \text{ and } mn \not| l\}}$. Therefore $mn \in \mathrm{Per}(\bar{x})$. □

5. *Admissible sets of periods for* 2^f

A few facts about $\mathrm{Per}(2^f)$ are obvious. By considering singleton sets, we see that $\mathrm{Per}(2^f) \supseteq \mathrm{Per}(f)$ and also $1 \in \mathrm{Per}(2^f)$ as $X \in 2^X$ is a fixed point. In fact, $\mathrm{Per}(f^{<\omega}) = [\mathcal{D}(\mathrm{Per}(f))]^{\dagger} \subseteq \mathrm{Per}(2^f)$ since $F(X) \subseteq 2^X$. It turns out that $\mathrm{Per}(2^f)$ itself is closed under taking prime power divisors. The question of whether it must be closed under taking any divisor remains open. Surprisingly, it needs not to be closed under taking least common multiples. The construction of such an example occupies §7. Intuitively, one might think that if $A_0, A_1, \ldots, A_{n-1}$ is an n-cycle and $B_0, B_1, \ldots, B_{m-1}$ is an m-cycle in 2^X, and if $d|n$, then $A_0 \cup A_d \cup \cdots \cup A_{n-d}$ should be a d-periodic point and $A_0 \cup B_0$ should be a $[m, n]$-periodic point. But this does not hold in general as it could happen that their fundamental period is smaller than expected. As a trivial example, consider a map over $X = \{0, 1, 2, 3\}$ given by $f(x) = x + 1 \bmod 4$. Then $A_0 = \{0, 1, 2\} \in 2^X$ is 2^f-periodic with period four but $A_0 \cup f^2(A_0) = X$ is a fixed rather than a period two point in 2^X.

† See Remark 9.3 below.

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THEOREM 5.1. *Given a continuous map $f: X \to X$, the set of periods* $\mathrm{Per}(2^f)$ *of the induced map on* 2^X *contains* $\mathrm{Per}(f) \cup \{1\}$ *and is closed under taking prime power divisors.*

Proof. We have already seen that $\mathrm{Per}(f) \cup \{1\} \subseteq \mathrm{Per}(2^f)$. Given an $n \in \mathrm{Per}(2^f)$ and its prime factorization $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$, where $r \geq 1$, for the second part of the claim, it will suffice to find a full orbit $\bar{x} \in \varprojlim(X, f)$ for which $\mathrm{Per}(\bar{x})$ contains $p_1^{\alpha_1}$, as the result will then follow from Proposition 4.7.

To that end, we set $k = p_1^{\alpha_1}$ and $l = n/p_1 = p_1^{\alpha_1 - 1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$, and let $A_0, \ldots, A_{n-1}$ be a periodic orbit for 2^f of period n. By Lemma 4.5, there exists $x_0 \in A_0 \setminus A_l$ and, as the A_i are mapped surjectively onto each other, it is possible to find a full orbit $\bar{x} = (\ldots, x_{-1}, x_0, x_1, x_2, \ldots)$ of x_0 such that $x_{mn+i} \in A_i$ for all $m \in \mathbb{Z}$. We claim that $k \in \mathrm{Per}(\bar{x})$, i.e. that $\overline{\{x_{mk} \mid m \in \mathbb{Z}\}}$ is k-periodic under 2^f.

Firstly, note that $x_{-mn+i} \in A_i \setminus A_{i+k \bmod n}$ for $m \in \mathbb{N}$ as, otherwise, if $x_{-mn+i} \in A_{i+k \bmod n}$, then after mapping it forward by f^{mn-i} we would have a contradiction with $x_0 \in A_0 \setminus A_k$. Similar reasoning allows us to conclude that, as we go backwards along the orbit, the preimages $x_0, x_{-n}, x_{-2n}, x_{-3n}, \ldots$ belong to, possibly, more and more complements of different A_i and, as there are only finitely many of those, this number must stabilize. The backward iterate at which this happens is then taken to be x_0 and all the other indices are shifted accordingly. Note that this does not affect the claim we wish to prove as $\mathrm{Per}(\bar{x}) = \mathrm{Per}(\sigma^t(\bar{x}))$ for any $t \in \mathbb{Z}$. Also note that, for this new x_0, we still have $x_0 \in A_0 \setminus A_l$. This modification will, however, allow that, from $x_0 \in A_j$ for some j, we can infer that $x_{mn+i} \in A_{i+j \bmod n}$ for all $m \in \mathbb{Z}$, whereas before we could conclude this only for positive m. Essentially the same trick was used previously in the proof of Lemma 4.6.

After we have altered the enumeration in $\bar{x}$, we are ready to conclude the proof by showing that (now modified) $x_0 \notin \overline{\{x_t \mid t \in \mathbb{Z} \text{ and } k \not| t\}}$. Note that

$$\overline{\{x_t \mid t \in \mathbb{Z} \text{ and } k \not| t\}} = \bigcup_{i=1}^{k-1} \overline{\{x_{mk+i} \mid m \in \mathbb{Z}\}}$$

and, for the sake of getting a contradiction, we assume that $x_0 \in \overline{\{x_{mk+i} \mid m \in \mathbb{Z}\}}$ for some $0 < i < k$. This means that either $x_0 = x_{mk+i}$ for some $m \in \mathbb{Z}$ and therefore $x_0 \in A_{\tilde{i}}$, where

$$\tilde{i} \equiv mk + i \pmod{n} \quad \text{and hence } \tilde{i} \equiv i \pmod{k},$$

or x_0 is a limit of such points (x_{mk+i}) out of which infinitely many must fall within the same congruence class with respect to n, say, $\tilde{i}$, and thus are also in the same $A_{\tilde{i}}$ for some $0 \leq \tilde{i} < n$, which must also satisfy $\tilde{i} \equiv i \pmod{k}$. As $A_{\tilde{i}}$ is closed, in both cases we get $x_0 \in A_0 \cap A_{\tilde{i}}$.

From here we can conclude that, in the periodic case, $x_0 \in A_{\tilde{i}} \cap A_{2\tilde{i} \bmod n}$ as $x_0 = x_{mk+i} = x_{2mk+2i}$. Also, if x_0 is a limit point of some subsequence in $\{x_{mn+\tilde{i}} \mid m \in \mathbb{Z}\} \subseteq A_{\tilde{i}}$, using the reasoning described above and in light of the fact that $x_0 \in A_{\tilde{i}}$, this subsequence is also in $A_{2\tilde{i} \bmod n}$ and so must be x_0 as its limit. In both cases, we henceforth conclude that $x_0 \in A_0 \cap A_{\tilde{i}} \cap A_{2\tilde{i} \bmod n}$. Continuing in this fashion, we see that $x_0 \in A_j$ for any j that satisfies

$$j \equiv s\tilde{i} \pmod{n} \text{ for some } s \in \mathbb{N}_0.$$

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To obtain a contradiction, with $x_0 \notin A_l$ it now only remains to show that there exist $s \in \mathbb{N}_0$ for which $s\tilde{i} \equiv l \pmod{n}$. It is an elementary fact from number theory that there exists $\tilde{s} \in \mathbb{N}_0$ such that $\tilde{s}\tilde{i} \equiv (\tilde{i}, n) \pmod{n}$, where (a, b) stands for the greatest common divisor of integers a and b. Note that $\tilde{i} \not\equiv 0 \pmod{k}$ and hence $p_1^{\alpha_1} = k \not| \tilde{i}$. Thus $(\tilde{i}, n) = (\tilde{i}, n/p_1) = (\tilde{i}, l) \mid l$ and setting $s = l/(\tilde{i}, n)\tilde{s}$ gives the desired conclusion. □

We now turn to showing that $\mathrm{Per}(2^f)$ is not always closed under taking least common multiples. But for this, we need first to revisit the notion of *almost minimal systems*.

6. *Almost totally minimal Cantor system*

Recall that any two non-compact, locally compact, totally disconnected, metrizable spaces with no isolated points are homeomorphic (see, e.g., [**9**, Proposition 1.1]). These are essentially equal to the Cantor set without a point, which is, in turn, homeomorphic to a countable union of Cantor sets. We shall denote such a set by $\mathcal{C}^*$. In [**9**], Danilenko gives a direct limit construction of a class of minimal systems on $\mathcal{C}^*$. The system $(\mathcal{C}^*, T)$ is said to be *minimal* if the only non-empty, closed, strongly T-invariant subset of $\mathcal{C}^*$ is $\mathcal{C}^*$ itself, or, equivalently, if any full orbit† of any point is dense in $\mathcal{C}^*$. In the compact case, this is equivalent to asking that the ω-limit set of any point is the whole of the state space, but here one has to be careful with the definition of minimality as some points can have their ω-limit sets empty due to non-compactness even if their full orbits are dense. In fact, Danilenko proved that, for any invertible minimal map on $\mathcal{C}^*$, the set of points with their forward orbit dense in $\mathcal{C}$ (those points x for which $\omega(x) = \mathcal{C}^*$) will be a dense, G_δ set with empty interior (see [**9**, Theorem 1.2]).

The class of maps constructed there conveniently extends to a class of continuous homeomorphisms of the Cantor set $\mathcal{C}$, which is obtained as a one point compactification of $\mathcal{C}^*$ with a point at infinity being mapped onto itself. Such maps, with one fixed point and all the other points having dense full orbits, are referred to as *almost minimal systems* and, in particular, they are examples of *essentially minimal systems*, which are defined as those systems that have one unique minimal subsystem (see [**17**]). To make our proofs in §7 work, we need a map that is *almost totally minimal*, meaning that the system $(\mathcal{C}, T^m)$ is almost minimal for any iterate T^m of the original map T, where $m \in \mathbb{N}$: in other words, that the system $(\mathcal{C}^*, T|_{\mathcal{C}^*})$ with the fixed point removed is totally minimal in the usual sense.

THEOREM 6.1. *There exists an almost totally minimal homeomorphism $T : \mathcal{C} \to \mathcal{C}$ of the Cantor set $\mathcal{C}$. Such T has exactly one fixed point $x^0 \in \mathcal{C}$ and all the other points have their full orbit with respect to T^m dense in $\mathcal{C}$, for any $m \in \mathbb{N}$, i.e.*

$$(\forall x \in \mathcal{C} \setminus \{x^0\})\ (\forall m \in \mathbb{N}) \quad \overline{\mathcal{O}}_{T^m}(x) = \mathcal{C},$$

where $\overline{\mathcal{O}}_{T^m}(x) = \overline{\mathcal{O}}_{T^m}(\bar{x}) = \overline{\{x_{mk} \mid k \in \mathbb{Z}\}}$ and $\bar{x} = (x_i)_{i\in\mathbb{Z}} \in \varprojlim(\mathcal{C}, T)$ is the full orbit of $x = x_0$.

Remark 6.2. We remark, in passing that, as $\mathcal{C}$ is a Baire space without isolated points, $\overline{\mathcal{O}}_T(\bar{x}) = \mathcal{C}$ is equivalent to $\lambda(\bar{x}) = \mathcal{C}$ for any full orbit $\bar{x} \in \varprojlim(\mathcal{C}, T)$.

† If T is non-invertible, then there will points with more than one distinct full orbits.

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To prove this theorem, we revisit Gambaudo and Martens' combinatorial approach for constructing self-maps of the Cantor set (or indeed any zero-dimensional compact metric space) via *graph covers* (see [**14**]).

6.1. *Graph covers.* It is a well-known fact that any totally disconnected, compact and Hausdorff space (these are sometimes called *Stone spaces*) can be obtained as an inverse limit of a (countable) system of discrete finite spaces. This property actually characterizes Stone spaces via Stone duality (see [**22**, Proposition 8-5]), which is why these are also occasionally called *profinite spaces*.

It turns out, using similar ideas, that it is possible to give a complete description not just for Stone spaces, but also for the self-maps on them by adding arrows to the discrete spaces forming the inverse system and thus creating an inverse limit of directed graphs. We briefly recall the main results of this theory following Shimomura's treatment in [**23**].

A *graph*† is a pair $G = (V, E)$, where V is a finite set of vertices and $E \subseteq V \times V$ is a set of directed edges. We additionally require that each vertex has at least one outgoing and one incoming edge. A *graph homomorphism* between graphs (V, E) and (V', E') is a vertex map $\phi\colon V \to V'$ that respects the edges, i.e. for any pair $(v, w) \in E$, it must be $(\phi(v), \phi(w)) \in E'$. A graph homomorphism is said to be *+-directional* if $\phi(w_1) = \phi(w_2)$ whenever both $(v, w_1) \in E$ and $(v, w_2) \in E$. If, additionally, $(v_1, w) \in E$ and $(v_2, w) \in E$ implies that $\phi(v_1) = \phi(v_2)$, it is said that ϕ is *bidirectional*. A *graph cover* is a +-directional homomorphism of graphs that is also *edge-surjective*, meaning that the map that ϕ naturally induces on the set of edges $\phi\colon E \to E'$ is surjective.

Given a sequence of graph covers $G_0 \xleftarrow{\phi_0} G_1 \xleftarrow{\phi_1} G_2 \xleftarrow{\phi_2} \cdots$, one forms a Stone space as the inverse limit

$$G_\infty = \varprojlim G_i = \left\{ (v^{(i)})_{i\geq 0} \in \prod_{i=0}^{\infty} V_i \mid v^{(i)} = \phi_i(v^{(i+1)}) \text{ for all } i \in \mathbb{N}_0 \right\}.$$

It is possible to define a self-map $\phi_\infty\colon G_\infty \to G_\infty$ by setting

$$\phi_\infty((v^{(i)})_{i\geq 0}) = (\phi_i(w^{(i+1)}))_{i\geq 0},$$

where $w^{(i)}$ is any vertex for which the edge $(v^{(i)}, w^{(i)})$ is in E_i. Intuitively, the map is given by the rule 'follow the arrows' except that if there is more than one outgoing arrow from the chosen vertex at the any given level, we might need to peek at one level below, the finer resolution of which will help us decide which of the arrows to follow. +-directionality of the covers ensures that this process, or indeed ϕ_∞, is a well-defined continuous self-map of the Stone space G_∞. If, additionally, each of the bounding maps in the sequence is bidirectional, then the map ϕ_∞ is a homeomorphism (see [**23**, Lemma 3.5]).

The full correspondence is given by the following theorem.

THEOREM 6.3. [**23**, Theorem 3.9] *A topological (compact, metric, surjective) dynamical system is zero-dimensional if and only if it is topologically conjugate to G_∞ for some sequence of covers $G_0 \xleftarrow{\phi_0} G_1 \xleftarrow{\phi_1} G_2 \xleftarrow{\phi_2} \cdots$.*

† All graphs that we consider are directed.

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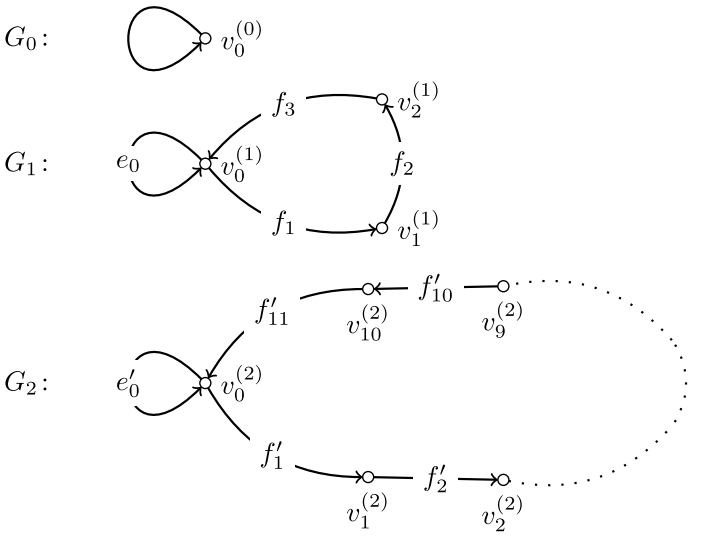


FIGURE 2. The first three steps of the construction, with $C_0 = 1$ and $C_1 = 2$.

We remark that these results have a flavour of those by Mioduszewski in [**20**] but are more elaborate as here we are dealing only with zero-dimensional spaces.

We are now ready to construct the system described in Theorem 6.1. The inspiration for this came from the construction in [**24**]. Very recently, similar techniques appeared in [**25**].

Proof of Theorem 6.1. Let $G_0 = (\{v_0^{(0)}\}, \{(v_0^{(0)}, v_0^{(0)})\})$ be one vertex with a self-loop. Given G_i, we shall inductively define G_{i+1}. The graph G_{i+1} will consist of $|V_{i+1}| = (|V_i| + C_i) \cdot (i+1)! + 1$ vertices† $V_{i+1} = \{v_0^{(i+1)}, \ldots, v_{|V_{i+1}|-1}^{(i+1)}\}$ which make a cycle in that order and, additionally, E_{i+1} also contains a self-loop at $v_0^{(i+1)}$ (see Figure 2). The constants $C_i \in \mathbb{N}$ are inductively chosen so that $|V_i| + C_i$ and $(i+1)!$ are co-prime. It remains to specify the cover $\phi_i : V_{i+1} \to V_i$ by the formula

$$\phi_i(v_k^{(i+1)}) = v_{I(i,k)}^{(i)},$$

where

$$I(i,k) = \begin{cases} 0 & \text{if } k \equiv 0, 1, 2, \ldots \text{ or } C_i \pmod{|V_i| + C_i}, \\ l & \text{otherwise, where } l \equiv k - C_i \pmod{|V_i| + C_i}. \end{cases}$$

It is perhaps easier to see what is going if we look at how ϕ_i maps the edges of G_{i+1} onto the edges of G_i. Let us denote the self loops in G_i and G_{i+1} by e_0 and e_0', respectively, and let $\{f_1, \ldots, f_{|V_i|}\}$ and $\{f_1', \ldots, f_{|V_{i+1}|}'\}$ be successive edges in their bigger cycles, respectively. The formulae above capture the fact that the cycle $\{f_1', \ldots, f_{|V_{i+1}|}'\}$ on the $(i+1)$st level is wound $(i+1)!$ times over the full circuit (with the first edge being

† Thus $|V_n|$ grows faster than superfactorials $n!(n-1)! \cdots 2!1!$.

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repeated C_i times) in the graph G_i in the succession

$$\underbrace{\underbrace{e_0 \dots e_0}_{C_i \text{ times}} f_1 \dots f_{|V_i|} \ \underbrace{e_0 \dots e_0}_{C_i \text{ times}} f_1 \dots f_{|V_i|} \quad \cdots \quad \underbrace{e_0 \dots e_0}_{C_i \text{ times}} f_1 \dots f_{|V_i|}}_{(i+1)! \text{ times}} \ e_0.$$

In particular, looking at Figure 2, $\phi_1 : G_2 \to G_1$ maps denoted edges as

$$\begin{array}{cccccccccccc} f_1', & f_2', & f_3', & f_4', & f_5', & f_6', & f_7', & f_8', & f_9', & f_{10}', & f_{11}' \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ e_0, & e_0, & f_1, & f_2, & f_3, & e_0, & e_0, & f_1, & f_2, & f_3, & e_0. \end{array}$$

Also note the additional twist over e_0 at the end, which is needed to ensure that these are bidirectional covers.

Let G_∞ be the inverse limit of the just constructed inverse system $G_0 \xleftarrow{\phi_0} G_1 \xleftarrow{\phi_1} G_2 \xleftarrow{\phi_2} \cdots$. As each vertex in G_i is covered with at least two vertices of G_{i+1}, one easily checks that G_∞ has no isolated points and is therefore homeomorphic to the Cantor set. As we mentioned earlier, the sequence consists of bidirectional covers and $\phi_\infty : G_\infty \to G_\infty$ is thus a homeomorphism of the Cantor set.

It remains to be seen that (G_∞, ϕ_∞) is almost totally minimal. One fixed point of the system is clearly $(v_0^{(i)})_{i\geq 0}$. Any other point $(v_{p_i}^{(i)})_{i\geq 0} \in G_\infty$ will have $p_i > 0$ for all i large enough. Let $x_\infty = (v_{p_i}^{(i)})_{i\geq 0}$ be one such a point. We wish to show that, for any given $m \in \mathbb{N}$, the full orbit under ϕ_∞^m of this point is dense in G_∞.

We shall denote by $[v_k^{(i)}]$ the set of all the points in G_∞ with ith coordinate equal to $v_k^{(i)}$. Recall that these sets $[v_k^{(i)}]$, where $i \in \mathbb{N}_0$ and $0 \leq k \leq |V_i| - 1$, form a clopen basis for the topology on G_∞. They are neatly nested within each other and those on the level $i+1$ refine all those on levels less than i. In fact, one has the relation $[v_k^{(i+1)}] \subseteq [\phi_i(v_k^{(i+1)})] = [v_{I(i,k)}^{(i)}]$. It will thus suffice to see that, for any level $L \in \mathbb{N}_0$, one can find ϕ_∞^m-iterates of x_∞ that hit each of the sets $[v_k^{(L)}]$, for $0 \leq k \leq |V_L| - 1$.

Let L be one such level and choose $M \geq \max\{L, m-1\}$ large enough so that $p_{M+1} > 0$. We claim that $\phi_\infty^{(M+1)!}$-iterates of x_∞ hit each of the sets $[v_k^{(M)}]$, for $0 \leq k \leq |V_M| - 1$, which will suffice as $m|(M+1)!$ and $M \geq L$.

Note that, as $p_{M+1} > 0$, it is possible to infer where a large portion of iterates of x_∞ are mapped within level $M+1$ without actually having to look at what happens at the level below†. In particular, we know that $\phi_\infty^{-p_{M+1}+k}(x_\infty) \in [v_k^{(M+1)}]$ for each $0 \leq k \leq |V_{M+1}| - 1$. This means that those same iterates hit each of the sets $[v_k^{(M)}]$, for $0 \leq k \leq |V_M| - 1$, at least $(M+1)!$ times, as each of the vertices in G_{M+1} covers those of G_M with at least multiplicity $(M+1)!$ This, along with the choice of C_M (the number of successive repetitions of $v_0^{(M)}$ in the covering map), creates an apt offset implying that $\phi_\infty^{(M+1)!}$-iterates of x_∞ end up in all of the sets $[v_k^{(M)}]$.

To put it precisely, let $p_{M+1} = q(M+1)! + r$ with $0 \leq r < (M+1)!$ as in Euclidean division. Then if $k = s(M+1)! + r$ with $0 \leq s \leq |V_M| + C_M - 1$, $\phi_\infty^{(s-q)(M+1)!}(x_\infty) =$

† By the level below, we mean level $M+2$, i.e. a level with a finer structure.

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$\phi_\infty^{-q(M+1)!-r+k}(x_\infty) = \phi_\infty^{-p_{M+1}+k}(x_\infty) \in [v_k^{(M+1)}] \subseteq [v_{I(M,k)}^{(M)}]$. This is justified by noting that k falls within the required range

$$0 \le r \le k \le (M+1)!(|V_M| + C_M) = |V_{M+1}| - 1.$$

As $|V_M| + C_M$ and $(M+1)!$ are co-prime, it is an elementary number theoretic fact that $s(M+1)!$, and hence also $k = s(M+1)! + r$ will run through all the residue classes modulo $|V_M| + C_M$ as s runs through $\{0, 1, \ldots, |V_M| + C_M - 1\}$. Inspecting the definition of $I(M, k) = I(M, s(M+1)! + r)$, we see that this attains all the values in $\{0, 1, \ldots, |V_M| - 1\}$ when s goes through $\{0, 1, \ldots, |V_M| + C_M - 1\}$. This completes our proof. □

7. $2, 3 \in \mathrm{Per}(2^f) \implies 6 \in \mathrm{Per}(2^f)$

Using Theorem 6.1, we can now construct a map on the Cantor set $\mathcal{C}$ for which the induced map on the hyperspace $2^{\mathcal{C}}$ has points of period two and three but no cycle of length six.

Let $A \sqcup B = \mathcal{C}$ be a partition† of the Cantor set in two clopen set such that the fixed point x^0 is in A. Let $\hat{X} = A \times \{0, 1\} \sqcup B \times \{0\}$. We define a map $f\colon \hat{X} \to \hat{X}$ by

$$f(x, i) = \begin{cases} (T(x), 1) & \text{if } i = 0 \text{ and } x \in T^{-1}(A), \\ (T(x), 0) & \text{otherwise.} \end{cases}$$

But this map is not surjective as, for example, $T(B) \cap A \times \{0\}$ is not contained in the range of f, and it is also two-to-one on $A \cap T^{-1}(B) \times \{0, 1\}$ as $f(x, 0) = f(x, 1) = (T(x), 0)$ for all $x \in A \cap T^{-1}(B)$. It is possible to restrict the dynamics to the surjective core of the map $f|_{\tilde{X}}\colon \tilde{X} \to \tilde{X}$, where $\tilde{X} = \bigcap_{k=0}^{\infty} f^k(\hat{X})$ is the set of all points in $\hat{X}$ that have preimages going infinitely back. This is still not the map we are after as $\tilde{X}$ is the largest closed strongly f-invariant set, and we would like our map to be defined on a minimal closed, strongly f-invariant set that contains $B \times \{0\}$. Such a set exists by Zorn's lemma, but it turns out that it is unique and can be constructed as follows.

Let

$$N(x, B) = \min\{k \in \mathbb{N}_0 \mid T^{-k}(x) \in B\}$$

be the time elapsed since x last visited B. If the minimum above does not exist, we set $N(x, B) = \infty$. Clearly, $N(x, B) = 0$ for all $x \in B$, but it is also finite for all other $x \in \mathcal{C}$ which have their backward orbit dense in $\mathcal{C}$, which we know is a G_δ dense set in $\mathcal{C}$ with empty interior. In fact, $N(x, B)$ is finite on a much larger (dense and open) set $U = \{x \in \mathcal{C} \mid N(x, B) < \infty\}$. To see that U is open, it suffices to notice that $N(x, B)$ is a locally constant function. What we mean by this is that, for any point $x \in U$, there exists a (cl)open neighbourhood containing x on which the function $N(\,\cdot\,, B)$ is constant $(= N(x, B))$. This is easy to see as one just needs to ensure that the neighbourhood is small enough so that the $N(x, B)$ backward iterates of the points in it follow tightly those of x. We now set

$$X = \overline{\{(x, N(x, B) \bmod 2) \mid x \in U\}},$$

where the closure is taken in $\hat{X}$. Note that X is still homeomorphic to the Cantor set since it is defined as the closure of a subset of the Cantor set with no isolated points. It is also

† We use symbol ⊔ to denote a disjoint union.

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immediately clear that X can be written as

$$X = \overline{\bigcup_{k=0}^{\infty} f^k(B \times \{0\})}.$$

To prove that this is the unique minimal closed strongly f-invariant set containing $B \times \{0\}$, it only remains to be seen that $X \subseteq f(X)$. Actually, $B \times \{0\} \subseteq f(X)$ will suffice. For this we recall the natural factor map, the projection to the first coordinate $\pi : \hat{X} \to \mathcal{C}$, given by $\pi(x, i) = x$. It is, in fact, a semi-conjugacy as one can verify that $\pi \circ f = T \circ \pi$. Using this, $T(\pi(X)) = \pi(f(X)) \subseteq \pi(X)$ and so $\pi(X)$ is a T-invariant set clearly containing B. Remembering that the set of points with their forward orbit dense under T is itself dense in $\mathcal{C}$, we conclude that $\pi(X) = \mathcal{C}$. But then, given any $(b, 0) \in B \times \{0\}$, set $b_{-1} = T^{-1}(b)$ and then, since $\pi(X) = \mathcal{C}$, either $(b_{-1}, 0)$ or $(b_{-1}, 0)$ or both are in X. In any case, one of these will map to $(b, 0)$ under f, which finishes our proof.

Another important thing to note is that the map π restricted to X is almost a homeomorphism as it is injective everywhere except, possibly, at some points of $\pi^{-1}(\mathcal{C} \setminus U) \cap X$, where it can be two-to-one. To see that π is injective on $\pi^{-1}(U) \cap X$, recall that, for any $u \in U$, one can find a clopen neighbourhood V_u where $N(\cdot, B)$ is constant and hence $V_u \times \{N(u, B) \bmod 2\}$ is contained in X but $V_u \times \{1 - N(u, B) \bmod 2\}$ has an empty intersection with X.

This observation allows us to prove the following lemma.

LEMMA 7.1. *For any fixed natural number m, X is the least closed f^m-invariant set that contains $B \times \{0\}$.*

Proof. Let $m \in \mathbb{N}$ be fixed and let $S \subseteq X$ be a closed, f^m-invariant subset of X containing $B \times \{0\}$. We need to see that $S = X$. Reasoning similarly as before, we see that $\pi(S)$ is a closed T^m-invariant subset of $\mathcal{C}$ containing B. Thus, by Theorem 6.1, $\pi(S) = \mathcal{C}$. If we now remember that π is injective on $\pi^{-1}(U) \cap X = \{(x, N(x, B) \bmod 2) \mid x \in U\}$, we get $\{(x, N(x, B) \bmod 2) \mid x \in U\} \subseteq S$ and, from there, immediately $S = X$. □

Remark 7.2. The statement of Lemma 7.1 could equivalently be written as

$$X = \overline{\bigcup_{k=0}^{\infty} f^{km}(B \times \{0\})},$$

for any $m \in \mathbb{N}$.

LEMMA 7.3. *For any $(x, i) \in X \setminus \pi^{-1}(x^0)$ and any $m \in \mathbb{N}$,*

$$\overline{\mathcal{O}}_{f^m}((x, i)) = X$$

for any chosen full orbit of (x, i).

Proof. Using the fact that π is a semi-conjugacy and Theorem 6.1,

$$\mathcal{C} = \overline{\mathcal{O}}_{T^m}(x) = \overline{\mathcal{O}}_{T^m}(\pi(x, i)) \subseteq \pi(\overline{\mathcal{O}}_{f^m}((x, i))),$$

and hence $\pi(\overline{\mathcal{O}}_{f^m}((x, i))) = \mathcal{C}$. But π is injective on $B \times \{0\}$ and hence $B \times \{0\} \subset \overline{\mathcal{O}}_{f^m}((x, i))$ and, as $\overline{\mathcal{O}}_{f^m}((x, i))$ is f^m-invariant and closed, by Lemma 7.1 it must be that $\overline{\mathcal{O}}_{f^m}((x, i)) = X$. □

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We are now ready to finish the construction of our example. To avoid confusion, we abstain from denoting points in X as ordered pairs, and we let the alternating 2-cycle be made of points $x^0, x^1 \in X$. Let $Z = X \times \{0, 1, 2\}/\sim$ be a quotient space obtained by gluing the points in $\{(x^0, i) \mid i = 0, 1, 2\}$ together and, likewise, the points in $\{(x^1, i) \mid i = 0, 1, 2\}$. Let us call those two points z^0 and z^1. We define the map $g\colon Z \to Z$ by

$$g(x, i) = (f(x), i + 1 \bmod 3).$$

Note that this rule respects the quotient relation and therefore accounts for a well-defined map on Z. Let us inspect closer the induced map 2^g. It is clear that $\{z^0\}$ gives a 2-cycle in 2^Z as z^0 is 2-periodic for g. It is also clear that the natural embedding of X in Z as $X \times \{0\}$ produces a 3-cycle in 2^Z. It remains to show that no 6-cycle in 2^Z exists. For if it did, it would have to contain at least one point $(x, i) \in Z$ other than z^0 or z^1. Then, clearly, $\overline{\mathcal{O}}_{g^6}((x, i))$ would also have to be contained in this set for some choice of the full orbit of (x, i). But $\overline{\mathcal{O}}_{g^6}((x, i)) = \overline{\mathcal{O}}_{f^6}(x) \times \{i\} = X \times \{i\}$ and therefore our initial set must have been of the form $X \times F/\sim$ for some $F \subseteq \{0, 1, 2\}$ and hence was not periodic with fundamental period six.

7.1. *General p and q.* Using the idea of the time elapsed since the last visit to B, it becomes obvious how to generalize this construction for p and q other than two and three. We set

$$X = \overline{\{(x, N(x, B) \bmod p) \mid x \in U\}},$$

and define a map $f\colon X \to X$ by

$$f(x, i) = \begin{cases} (T(x), i + 1 \bmod p) & \text{if } x \in T^{-1}(A), \\ (T(x), 0) & \text{otherwise.} \end{cases}$$

This is a continuous map on the Cantor set X with a p-cycle $(x^0, 0), (x^0, 1), \ldots, (x^0, p-1)$. As before, using the projection map $\pi\colon X \to \mathcal{C}$, one checks that f is onto, and that $\pi(X) = \mathcal{C}$. Analogously one sees that π is injective on $\pi^{-1}(U) \cap X = \{(x, N(x, B) \bmod p) \mid x \in U\}$, and hence any closed subset of X that projects onto the whole of $\mathcal{C}$ must be X itself. This enables one to prove that Lemmata 7.1 and 7.3 still hold even with this, more general, definition of X and f. In fact, their statements along with the proofs (up to changing two to p) still hold word for word.

One can now set $Z = X \times \{0, 1, \ldots, q-1\}/\sim$, where $\sim$ identifies q distinct p-cycles into one p-periodic orbit $\{z^0, \ldots, z^{p-1}\} \subset Z$. The map $g\colon Z \to Z$ given by

$$g(x, i) = (f(x), i + 1 \bmod q)$$

is once again well defined and clearly satisfies $p, q \in \mathrm{Per}(2^g)$. We claim that any other $m \in \mathrm{Per}(2^g)$ must be a divisor of either p or q.

Firstly, if we assume that the periodic point $S \in 2^Z$ is completely contained in $\{z^0, \ldots, z^{p-1}\}$, then, by our previous results, we know that $m|p$. If otherwise, there exists a point $(x, i) \in S$ other than any of $z^0, \ldots z^{p-1}$, then $\overline{\mathcal{O}}_{g^{mq}}((x, i))$ is contained in S for some choice of the full orbit of (x, i). By Lemma 7.3, $\overline{\mathcal{O}}_{g^{mq}}((x, i)) = \overline{\mathcal{O}}_{f^{mq}}((x)) \times \{i\} = X \times \{i\}$ and so $S = X \times F/\sim$ for some $F \subseteq \{0, 1, \ldots, q-1\}$. From here, we immediately get that $m|q$. Thus we have proved the following theorem.

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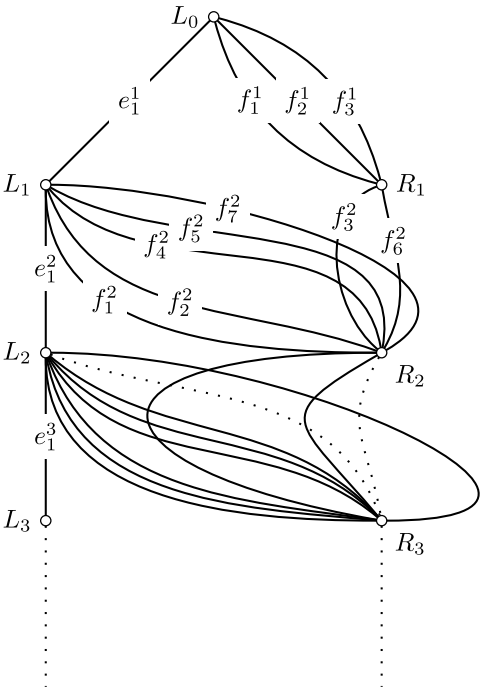


FIGURE 3. Bratteli–Vershik representation of the system from Theorem 6.1.

THEOREM 7.4. *Let $p, q \in \mathbb{N}$. There exists a continuous onto map of the Cantor set $g\colon Z \to Z$ for which the periods appearing in the induced map are exact divisors of either p or q, i.e.*

$$\mathrm{Per}(2^g) = \{m \mid m|p \text{ or } m|q\}.$$

8. *Bratteli–Vershik representation*

We have mentioned before that it would be possible to give rewritten rules for translating a description of a system given by graph covers to those using Bratteli–Vershik diagrams or Kakutani–Rokhlin towers. In this section, we give an example of how this is done by providing another, equivalent construction of the system $(\mathcal{C}, T)$ from Theorem 6.1 using the former.

There are many references explaining the theory behind Bratteli–Vershik diagrams, e.g. [**10**, **17**], but to us the most relevant was a recent preprint [**2**] by Amini, Elliott and Golestani in which they give a category theory treatment of the matter highlighting the connection between these diagrams and essentially minimal systems.

Consider the infinite graph in Figure 3, where all the edges are oriented downwards even if this is not depicted there. Additionally, for each node other than the root L_0 which has no incoming edges, an ordering is given on the set of incoming edges into that node. This order is depicted in the figure as the order in which the edges connect to that node going from left to right.

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We have yet to explain how this infinite graph is constructed. Recall that, in the proof of Theorem 6.1, we obtained a sequence of numbers $(C_i)_{i\in\mathbb{N}_0}$, which was of some significance in the construction of the map T. Here, any node L_{i+1} will have only one incoming edge coming from the node L_i for any $i \in \mathbb{N}_0$. The incoming edges into the node R_{i+1} will be: C_{i+1} edges coming form the node L_i; then one edge from the node R_i; then another C_{i+1} edges from L_i; and another one from R_i; and this sequence of $C_{i+1}+1$ edges is to be repeated in total $(i+1)!$ times, after which the last one in is the edge from L_i. Note that we could simply say that there are $(i+1)!\cdot(C_{i+1})+1$ edges coming from L_i and $(i+1)!$ from R_i into R_{i+1}, but, by listing them as above, we implicitly specified the order on those edges incoming into the node R_{i+1} for any $i \in \mathbb{N}$.

The space we shall be considering is the space of all infinite paths starting from L_0 and then following the sequence of edges down through the vertices L_i/R_i. Encoding these paths as sequences of edges allows one to see this space as the Cantor set of symbolic sequences. It remains to specify the map on this space of infinite paths.

Recall that for each node other than the root, there exists the minimal and the maximal ingoing edge into that node. A path is said to be minimal (respectively, maximal) if it consists entirely of minimal (respectively, maximal) edges. It is not hard to check that in our graph the path $e_1^1 e_1^2 e_1^3 \dots$ that is always staying on the left side is both minimal and maximal and no other minimal nor maximal path exits. This path will be the fixed point of our map. For any other path, $p^1 p^2 p^3 \dots$ there must exist a $k \in \mathbb{N}_0$ such that the edge p^k is not maximal. Choose this k to be the smallest possible and then we let our function map this path to the path $q^1 q^2 \dots q^k p^{k+1} p^{k+2} p^{k+3} \dots$, where q^k is the successor of the edge p^k and the q^is for $i < k$ are chosen in such a way that $q^1 q^2 \dots q^{k-1}$ forms the (unique) minimal path connecting L_0 and the source vertex of the edge q^k. Equivalently, $q^1 \dots q^k$ is chosen to be the successor of the path $p^1 \dots p^k$ with respect to the natural induced order on the sequence of paths (of length k) terminating at the source of p^{k+1}.

We leave to the reader to check that this does give the same (or formally, conjugated) system as the method given in §6. For this, it is helpful to see how this representation was derived from Figure 2.

Each vertex above stands for a cycle in the graph covers representation. To be precise, the self-loop of the graph G_0 is represented by the root L_0, the self-loop of any G_i is represented by the vertex L_i, and the bigger cycle $v_0^{(i)} v_1^{(i)} \dots v_{|V_i|}^{(i)}$ in G_i corresponds to the right side vertex R_i. Finally, the edges and the order in which they connect vertices L_i and R_i to the vertices L_{i+1} and R_{i+1} are to be inferred from the way the cycles in the graph G_{i+1} wrap around through the cycles of G_i. In particular, the loop e_0' in G_2 just goes once through the loop e_0 in G_1 and hence only one edge from L_1 to L_2. On the other hand, the cycle $f_1' f_2' \dots f_{11}'$ in G_2 wraps firstly two times around e_0, then once around the bigger cycle in G_1, then repeats this, and then finally winds the last time over e_0. This means that the edges coming into R_2 are: two edges from L_1; followed by one from R_1; then again two from L_1 and one from R_1; and lastly one from L_1.

At the very end, we mention that the corresponding representation using Kakutani–Rokhlin towers is obtained by associating the cycles in the graphs G_i with the eponymous towers and the vertices within those cycles correspond to the levels of the towers.

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9. *Periodicity in symmetric products*

In this section, we shall be concerned only with periodic points in 2^X made up entirely of periodic points for (X, f). We shall see that, in this setting, the anomalies such as those in Theorem 7.4 are not possible.

Firstly, note that one can reduce the problem to studying periodic points of $f^{<\omega}$. Namely, let $S \in 2^X$ be a periodic point with fundamental period $k = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, where each point of S is periodic under f. Then, as in the proof of Theorem 5.1, we can find points $x_1, \ldots, x_r$ in S for which $p_i^{\alpha_i} \in \mathrm{Per}(\bar{x}_i)$, where each $\bar{x}_i$ is simply a full periodic orbit of x_i. It could happen that some of these x_i are represented by the same point or by points that are members of the same full orbit, which, in this case, means the same cycle. For a moment, let us assume that this is not the case and that all of these r full orbits are mutually disjoint when considered as subsets of X. Then, for each $1 \le i \le r$, let $S_i \in 2^X$ be the canonical $k_i = p_i^{\alpha_i}$-periodic set given by $S_i = \{x_i, f^{k_i}(x_i), f^{2k_i}(x_i), \ldots\}$. Note that the S_i are finite and mutually disjoint, and thus $\bigcup_{i=1}^r S_i$ is a k-periodic point for 2^f but also for $f^{<\omega}$.

Suppose now that x_1 and x_2 belong to the same orbit. Then, using Proposition 4.7, we can actually conclude that $l_1 = k_1 k_2 \in \mathrm{Per}(\bar{x}_1) = \mathrm{Per}(\bar{x}_2)$. Grouping the elements of the same orbits in this way and discarding all but one representative for each of the orbits, we obtain a factorization of $k = l_1 \cdots l_s$ in $s \le r$ co-prime factors, where each l_i is an element of some $\mathrm{Per}(\bar{x}_{t(i)})$ with $x_{t(1)}, \ldots, x_{t(s)}$ all belonging to distinct orbits. After setting $S_i = \{x_{t(i)}, f^{l_i}(x_{t(i)}), f^{2l_i}(x_{t(i)}), \ldots\}$, the union $\bigcup_{i=1}^s S_i$ will again be a k-periodic point for both 2^f and $f^{<\omega}$.

Below we give explicit formulae for $\mathrm{Per}(f^{(n)})$, $\mathrm{Per}(f_n)$ and $\mathrm{Per}(f^{<\omega})$ in terms of $\mathrm{Per}(f)$, where by $\mathrm{Per}(f)$ we denote the *set of periods* of the periodic points of f.

PROPOSITION 9.1. *We have the following identities.*

$$\mathrm{Per}(f^{(n)}) = \{[m_1, \ldots, m_n] \mid m_i \in \mathrm{Per}(f) \text{ for all } 1 \le i \le n\}, \tag{4}$$

$$\mathrm{Per}(f_n) = \bigcup_{l=1}^{n} \left\{ [d_1, \ldots, d_l] \mid d_i | m_i \in \mathrm{Per}(f) \text{ for all } 1 \le i \le l \text{ and } \frac{m_1}{d_1} + \cdots + \frac{m_l}{d_l} \le n \right\}, \tag{5}$$

$$\mathrm{Per}(f^{<\omega}) = \bigcup_{l=1}^{\infty} \{[d_1, \ldots, d_l] \mid d_i | m_i \in \mathrm{Per}(f) \text{ for all } 1 \le i \le l\}, \tag{6}$$

where $[k_1, \ldots, k_m]$ *stands for the least common multiple of* $k_1, \ldots, k_m$.

Proof. Statement (4) is easy. Given points $q_1, \ldots, q_n$ with fundamental periods $m_1, \ldots, m_n$, it is clear that the point $(q_1, \ldots, q_n) \in X^{(n)}$ has fundamental period $[m_1, \ldots, m_n]$. And, conversely, any periodic point of $f^{(n)}$ must arise in this fashion.

We shall now prove (5). Let $Q = \{q_1, \ldots, q_n\}$ be a periodic point in $F_n(X)$. Clearly, each q_i must be a periodic point of f. Then Q can be naturally partitioned into sets $Q_1, \ldots Q_l$, where each Q_i contains points belonging to the same cycle under f, and no two points from distinct Q_i belong to the same cycle. All the points in Q_i have the same period m_i under f. Let us denote the periods of $Q, Q_1, \ldots, Q_l \in F_n(X)$

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by $d, d_1, \ldots, d_l$, respectively. Clearly, $d = [d_1, \ldots, d_l]$ and, since $f_n^{m_i}(Q_i) = Q_i$, also $d_i | m_i$. By the construction, $l \leq n$ as every Q_i must contain at least one point from Q.

Lastly, we claim that $m_i/d_i \leq |Q_i|$, where $|\cdot|$ denotes the cardinality of a set. This is because $f_n^{d_i}$ acts on Q_i as a permutation over set of cardinality $|Q_i|$ of order at most m_i/d_i. But a permutation cannot have order larger than the cardinality of the set it acts upon, and thus we get the claim. From this it follows $\sum_{i=1}^{l} m_i/d_i \leq |\bigcup_{i=1}^{l} Q_i| = |Q| = n$, which finishes the proof of the inclusion $\subseteq$ in (5). The other inclusion follows by noting that any period of the form $[d_1, \ldots, d_l]$, where $1 \leq l \leq n$, $d_i | m_i$ for some $m_i \in \mathrm{Per}(f)$ and $\sum_{i=1}^{l} m_i/d_i \leq n$, can be realized by taking points $r_1, \ldots, r_l$ of periods $m_1, \ldots, m_l$, respectively, and then setting $Q = \bigcup_{i=1}^{l} Q_i$, where, for each $1 \leq i \leq l$, the set Q_i is formed of m_i/d_i equispaced iterates of r_i. More precisely, $Q_i = \{r_i, f^{d_i}(r_i), f^{2d_i}(r_i), \ldots, f^{(m_i/d_i - 1)d_i}(r_i)\}$.

Finally, note that $\mathrm{Per}(f^{<\omega}) = \bigcup_{n \in \mathbb{N}} \mathrm{Per}(f_n)$ and therefore (6) follows immediately from (5). □

Remark 9.2. An important observation that implicitly drives the proof of the second statement above was that no periods are lost if one only considers points in $F_n(X)$ formed of equispaced† iterates of points in X. To take a simple example, consider a point x which is periodic with fundamental period six in some system (X, f). Even though $\{x, f(x)\}$, $\{x, f(x), f^2(x)\}$ and many others are period six points for f_3, there is a simple, equispaced point of period six for f_3, namely, $\{x\}$.

Remark 9.3. Note that statement (6) could equally be written as

$$\mathrm{Per}(f^{<\omega}) = [\mathcal{D}(\mathrm{Per}(f))],$$

where $\mathcal{D}(S) = \{d \in \mathbb{N} \mid d | n \text{ for some } n \in S\}$ is the set of divisors of S and $[S] = \{[k_1, \ldots, k_l] \mid l \in \mathbb{N}, k_1, \ldots k_l \in S\}$ is the set of least common multiples of a set $S \subseteq \mathbb{N}$. In particular, $\mathrm{Per}(f^{<\omega})$ is the smallest set containing $\mathrm{Per}(f)$ that is closed under taking least common multiples and divisors.

Let us show how Proposition 9.1 can be used to compute the periods of induced maps on hyperspaces.

Example 9.4. Let $f\colon [0, 1] \to [0, 1]$ be a 2^r interval map ($r > 1$), i.e. a map for which $\mathrm{Per}(f) = \{1, 2, 2^2, \ldots 2^{r-1}, 2^r\}$. One possible construction of such a map is described in [**7**, Example I.13]. For any positive integer n, all the sets $\mathrm{Per}(f_n)$, $\mathrm{Per}(f^{(n)})$ and $\mathrm{Per}(f^{<\omega})$ clearly contain $\mathrm{Per}(f)$. Note that, in our special case, if $d_i | m_i$ and $m_i \in \mathrm{Per}(f)$, then $d_i \in \mathrm{Per}(f)$ and hence also $[d_1, \ldots, d_l] = \max\{d_1, \ldots, d_l\} \in \mathrm{Per}(f)$. Therefore $\mathrm{Per}(f_n) = \mathrm{Per}(f^{(n)}) = \mathrm{Per}(f^{<\omega}) = \mathrm{Per}(f)$. Note that, using Theorem 3.2 and the fact $\mathrm{Per}(f) \subseteq \mathrm{Per}(2^f)$, we can also conclude that $\mathrm{Per}(2^f) = \mathbb{N}$.

† Formally, we say that $Q = \{q_1, \ldots, q_m\}$ is an equispaced point in $F_n(X)$ if $\min\{k \geq 1 \mid f^k(q_i) \in Q\}$ does not depend on $1 \leq i \leq m$.

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10. *Concluding remarks*

Nearly all the results contained in here were driven by the problem of finding a complete characterization of the set $\mathrm{Per}(2^f)$ in terms of $\mathrm{Per}(f)$. The first step would be to find constraints on $\mathrm{Per}(2^f)$ and this already seems involved. We have shown that this set must be closed under taking prime divisors, but the question remains whether it is closed under taking divisors in general. Whatever turns out to be the truth, one will still be faced with the laborious task of constructing maps attaining all the admissible periodicity sets in between, if one seeks such a full characterization. Some of these will be easy, but we have already gone through much trouble in order to construct examples having two co-prime periods in $\mathrm{Per}(2^f)$ but not their product. We conjecture that, more generally, a map with $\mathrm{Per}(2^f) = \{m \mid m|p \text{ for } p \in P\}$ exists for any finite $P \subset \mathbb{N}$.

One plausible strategy for obtaining such results easily would be to find a way to embed any Cantor set dynamics (X, f) within another ('bigger') Cantor set dynamic $(\hat{X}, \hat{f})$, where X would be a closed, nowhere dense set in $\hat{X}$. Additionally, one would require that $(\hat{X} \setminus X, \hat{f})$ is totally minimal. More precisely, we formulate this as the following question.

Question 10.1. Let (X, f) be a dynamical system on the Cantor set X. Does there always exist a system on the Cantor set $(\hat{X}, \hat{f})$ and an embedding (continuous injection) $i : X \to \hat{X}$ which:

(a) intertwines the dynamics of f and $\hat{f}$, i.e. $\hat{f} \circ i = i \circ f$;

(b) embeds X as an $\hat{f}$-invariant set whose complement in $\hat{X}$ is also invariant and non-empty; and

(c) is such that, for any $m \in \mathbb{N}$ and any full orbit $\bar{x} \in \varprojlim(\hat{X}, \hat{f})$ of a point $x = x_0$ in $\hat{X} \setminus i(X)$, $\overline{\mathcal{O}}_{\hat{f}^m}(\bar{x}) = \hat{X}$?

Note that these imply that the embedding $i(X)$ is nowhere dense in $\hat{X}$.

Using this one could, starting with our p, q example as (X, f), construct a system with $\mathrm{Per}(2^f)$ consisting solely of divisors of p, q and r by taking the set $\hat{X} \times \{0, 1, \ldots, r-1\}/\sim$ (where $\sim$ identifies r copies of X into just one copy) and the map $(x, i) \mapsto (\hat{f}(x), i + 1 \bmod r)$. This procedure can then be repeated with the system just obtained as a starting point. Continuing in this way would enable one to construct systems with an increasing hierarchy of subsystems nested within each other. It is not hard to check that $\mathrm{Per}(2^f)$ of such a map would be exactly of the form above.

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