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Dimension subgroups modulo n

BY SIEGFRIED MORAN
University of Kent at Canterbury

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Let G be an arbitrary group and $Z_n(G)$ denote the group algebra of G over the integers modulo n. If $\Delta^i(G)$ denotes ith power of the augmentation ideal $\Delta(G)$ of $Z_n(G)$, then

$$G \cap (1+\Delta^i(G))$$

is easily seen to be a normal subgroup of G. It is denoted by $D_{i,n}(G)$ and is called ith dimension subgroup of G modulo n. It can be shown that these dimension subgroups are determined by the dimension subgroups modulo a power of a prime p. Hence we shall restrict our attention to these dimension subgroups.

In the case when G is a free group F, all $D_{i,p^h}(F)$ have been determined by Lazard (3) (the case $h = 1$ is due to Zassenhaus (4)). He showed that

$$D_{i,p^h}(F) = F_{i,p^h}$$

which consists of all those elements of F which have a representation as an infinite product of the form

$$\prod_{j=1}^{\infty} z_j^{p^{k_j}}$$

satisfying $$jp^{(k_j-h+1)^+} \geqslant i$$

for all j. Here $m^+ = \sup(m, 0)$. Also every k_j is a non-negative integer or ∞, where we use the convention that $p^\infty = 0$, and z_j is an element of ${}^jF$ such that z_j does not belong to $({}^jF)^p$ modulo ${}^{j+1}F$ for $j = 1, 2, \ldots$.

It was shown by Jennings (2) and Lazard (3) that if an arbitrary group G is represented as a factor group F/N of a free group F, then

$$D_{i,p}(F/N) = F_{i,p}.N/N \quad \text{for} \quad i = 1, 2, \ldots.$$

In Chapter I, § 8 of (3) Lazard has raised the problem as to whether

$$D_{i,p^h}(F/N) = F_{i,p^h}.N/N$$

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for arbitrary positive integers i and h. It is easy to see that in order to decide this question it is sufficient to consider the case when F/N is an arbitrary finite p-group. Below we show

(1) For every prime p and integer $h > 1$ (with the possible exception of $p = h = 2$) there exists a finite p-group F/N of order p^{h+3} with the property that

$$D_{p+1,\,p^h}(F/N) \neq F_{p+1,\,p^h}.N/N.$$

(2) If F/N is a p-group, then

$$D_{c+1,\,p^h}(F/N) = F_{c+1,\,p^h}.N/N = {}^{c+1}F.F^{p^h}.N/N$$

for $c+1 = 1, 2, \ldots, p$.

Proof of (1). Let G be the group with generators x and y and defining relations

$$x^{p^{h+1}} = y^{p^2} = x^{p^{h-1}}(y,x) = 1, \quad \text{with} \quad h \geqslant 2.$$

As pointed out to me by David Cappitt, this group can be realized as an extension of the cyclic group of order p^{h+1} generated by x by the cyclic group of order p^2 generated by y. Also G is nilpotent of class 3 or 2 according as $h = 2$ or $h > 2$. The order of G is p^{h+3}. Now in the group algebra $Z_{p^h}(G)$ we have that

$$\begin{aligned} x^{p^h} &= (1+(x-1))^{p^h} \\ &= 1 + \binom{p^h}{p}(x-1)^p + \binom{p^h}{p+1}(x-1)^{p+1} + \ldots \end{aligned}$$

which belongs to $1+\Delta^{p+1}(G)$, since

$$1 = x^{p^{h-1}}(y,x) = (1+(x-1))^{p^{h-1}}.(1+y^{-1}x^{-1}[y-1, x-1]),$$

which gives that $\binom{p^h}{p}(x-1)^p$ belongs to $\Delta^{p+1}(G)$. Thus we can conclude that x^{p^h} is a non-trivial element of $D_{p+1,\,p^h}(G)$.

On the other hand $x^{p^h}.N$ does not belong to $F_{p+1,\,p^h}.N/N$. This follows from Lazard's formula for elements of this subgroup and the fact that

$$G^{p^{h+1}} = ({}^2G)^{p^2} = ({}^3G)^{p^2} = 1.$$

There is an exceptional case. It occurs when $h = p = 2$. Then

$$x^4 = (x,y)^2 = (x^2, y) = ((x,y), y)$$

in G and thus $x^4.N$ belongs to $F_{3,\,4}.N/N$.

Proof of (2). It is easy to verify that

$$D_{c+1,\,p^h}(F/N) \geqslant F_{c+1,\,p^h}.N/N = {}^{c+1}F.F^{p^h}.N/N$$

for $c < p$, and

$$D_{c+1,\,p^h}(F/N)/({}^{c+1}F.F^{p^h}.N/N) \leqslant D_{c+1,\,p^h}((F/N)/({}^{c+1}F.F^{p^h}.N/N)).$$

Hence we may assume that G is a p-group which is nilpotent of class c, where $c < p$, and $G^{p^h} = 1$. It is now sufficient to show that $D_{c+1,\,p^h}(G) = 1$.

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According to Lazard (3) groups of this sort are Lie rings under appropriately defined operations. We now give some of the essential details of this theory. Let x and y be elements of G. Then by $z_1, z_2, \ldots, z_i, \ldots$ we denote the finite sequence of basic commutators of weight less than p on x and y. These basic commutators are ordered according to weight. Lazard (3), Chapter II, Formula (2·8) defines

$$x+y = \prod_i z_i^{h_i} \quad \text{and} \quad [x,y] = \prod_i z_i^{k_i},$$

where h_i and k_i $(i = 1, 2, \ldots)$ are certain rational numbers whose denominators in their irreducible representations have prime factors not exceeding the weight of z_i, for all i. These operations define the structure of a Lie ring G_L on G. The sequences $h_1, h_2, \ldots$ and $k_1, k_2, \ldots$ of rational numbers are fixed and start off as follows

$$x+y = xy(x,y)^{-\frac{1}{2}} \ldots \quad \text{and} \quad [x,y] = (x,y)\,(x,y,y)^{\frac{1}{2}} \ldots.$$

Using the latter of these formulae it can now be verified, by the usual laws of commutator calculus, that G_L is nilpotent of class $\leqslant c$. It follows from Lazard (3), Chapter II, Théorème 4·3 that G_L under the operation of Campbell–Hausdorff multiplication $\circ$ is a group $(G_L, \circ)$ naturally isomorphic to G. It can now be verified as above that if G is nilpotent of class c, then so is G_L. It is worth noting that the unit element 1 of G is the O-element of G_L. We will also need the fact that $p^h.(G_L) = 0$, that is, G_L can be considered to be a Z_{p^h}-module.

Consider the following single-valued mapping σ of the group algebra $Z_{p^h}((G_L, \circ))$ onto G_L defined by $\sigma(g) = g$ for all g belonging to $(G_L, \circ)$. This defines a homomorphism of the group algebra considered as a Z_{p^h}-module onto the Lie ring G_L considered as a Z_{p^h}-module. It is important to note that σ is not a ring homomorphism, since $Z_{p^h}((G_L, \circ))$ and G_L are different types of rings. Also $\circ$ and $+$ are not related by the distributive law in $(G_L, \circ)$. This is shown, for example, by the fact that

$$(1+1)\circ g = g \quad \text{while} \quad 1\circ g + 1\circ g = 2g$$

for all g in G_L.

Now let g belong to $D_{c+1,\,p^h}((G_L, \circ))$. Then

$$g = 1 + \Sigma\lambda(g_1 - 1) \ldots (g_{c+1} - 1),$$

where $g_1, g_2, \ldots, g_{c+1}$ belong to $(G_L, \circ)$ and λ is an integer modulo p^h. We multiply out and this gives that

$$g = 1 + \Sigma\lambda((g_1 \circ \ldots \circ g_{c+1}) - \ldots (-1)^c (g_1 + \ldots + g_{c+1}) + (-1)^{c+1}).$$

Applying σ to both sides of this expression gives that

$$g = \Sigma\lambda((g_1 \circ \ldots \circ g_{c+1}) - \ldots (-1)^c (g_1 + \ldots + g_{c+1}))$$

in G_L. More specifically we note that it suffices to show that $(g_1 - 1) \ldots (g_{c+1} - 1)$ is mapped onto zero by σ. Now $(g_1 - 1) \ldots (g_{c+1} - 1)$ equals

$$\Sigma(-1)^{c+1-s} g_{\alpha_1} \circ g_{\alpha_2} \circ \ldots \circ g_{\alpha_s} \qquad (*)$$

where the summation is taken over all distinct subsets $\{\alpha_1, \ldots, \alpha_s\}$ of $\{1, \ldots, c+1\}$ with $1 \leqslant \alpha_1 < \alpha_2 < \ldots < \alpha_s \leqslant c+1$. The image γ of $(g_1 - 1) \ldots (g_{c+1} - 1)$ under σ has the

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same expression as (*), but this is now considered as an element of G_L. We now use the formula of Dynkin which asserts that (see, for instance, Jacobson (1), p. 173)†

$$g_1 \circ g_2 \circ \ldots \circ g_{c+1} = \sum_m \sum_{p_i^{(j)}} \frac{(-1)^{m-1}}{m \sum_{i,j} p_i^{(j)}} \cdot \frac{[g_1^{p_1(1)}, \ldots, g_{c+1}^{p_{c+1}(1)}, \ldots, g_1^{p_1(m)}, \ldots, g_{c+1}^{p_{c+1}(m)}]}{p_1^{(1)}! \ldots p_{c+1}^{(1)}! \ldots p_1^{(m)}! \ldots p_{c+1}^{(m)}!},$$

where $p_1^{(r)} + \ldots + p_{c+1}^{(r)} > 0$ for $r = 1, 2, \ldots, m$ is the condition determining the second Σ and we are using the usual convention for repeated Lie products. Take the term exhibited behind the double summation signs in Dynkin's formula and suppose that it has t distinct g_i genuinely appearing. Then, in the above expression (*) for γ holding in G_L, we have that the coefficient of this term is a scalar multiple of

$$1 - {}^{c+1-t}C_{c-t} + {}^{c+1-t}C_{c-t-1} - \ldots + (-1)^{c+1-t} {}^{c+1-t}C_0 = 0,$$

for every $t \leqslant c$. Since G_L is nilpotent of class c, we have that

$$\gamma = 0 \quad \text{in} \quad G_L,$$

which establishes the required result.

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† It is better to use a shortened form of this formula which involves only Lie products of weight $\leqslant p$. This avoids ambiguities of the form o/p.

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