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Spectral-Collocation Method for Volterra Delay Integro-Differential Equations with Weakly Singular Kernels

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Abstract. A spectral Jacobi-collocation approximation is proposed for Volterra delay integro-differential equations with weakly singular kernels. In this paper, we consider the special case that the underlying solutions of equations are sufficiently smooth. We provide a rigorous error analysis for the proposed method, which shows that both the errors of approximate solutions and the errors of approximate derivatives decay exponentially in L^∞ norm and weighted L^2 norm. Finally, two numerical examples are presented to demonstrate our error analysis.

AMS subject classifications: 65R20, 65M70, 45D05

Key words: Volterra integro-differential equations, spectral Jacobi-collocation method, pantograph delay, weakly singular kernel.

1 Introduction

In this paper, we consider the following weakly singular Volterra integro-differential equations with pantograph delays

$$y'(t)=a(t)y(t)+b(t)y(qt)+f(t)+\int_0^t (t-s)^{-\mu}K_1(t,s)y(s)ds + \int_0^{qt}(qt-\tau)^{-\mu}K_2(t,\tau)y(\tau)d\tau, \quad t\in I:=[0,T], \tag{1.1a}$$

$$y(0)=y_0, \tag{1.1b}$$

where $0<\mu<1$, $0<q<1$, $y(t)$ is the unknown function, the functions $a,b,f\in C(I)$ and $K_1,K_2\in C(I\times I)$ with $K_1(t,t),K_2(t,qt)\neq 0$ for $t\in I$.

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It is well known that spectral methods have excellent error properties and provide exponential rates of convergence for smooth problems. In the last decades, they have become a class of important methods to numerically solve certain partial differential equations. And there are many existing works on this subject (see e.g., books [3–5, 17, 18] and references therein). Recently, several authors have applied spectral methods to solve Volterra-type integral or integro-differential equations. In [28], spectral and pseudo-spectral Jacobi-Galerkin approaches were introduced to solve the second kind Volterra integral equations. In [20], spectral and pseudo-spectral Jacobi-Petrov-Galerkin approaches were developed to solve the second kind Volterra integro-differential equations. Wan, Chen and Huang [21] used the spectral Galerkin method to solve the nonlinear Volterra integral equations of the second kind. The authors in [1, 9, 10, 19, 22, 23, 26, 31] proposed the spectral Legendre-collocation method for Volterra integral or integro-differential equations with smooth kernels. In [6–8, 12, 24, 25, 29, 30, 32] the spectral Jacobi-collocation method was successfully applied to solve Volterra integral or integro-differential equations with weakly kernels and fractional integro-differential equations. And the spectral accuracy of the approaches is verified both theoretically and numerically in [1, 6–10, 12, 19–26, 28–32].

The numerical treatment of (1.1a)-(1.1b) is not simple, mainly due to the fact that the solutions of (1.1a)-(1.1b) usually have a weak singularity at $t=0$. In this work, we will consider the special case that the exact solutions of (1.1a)-(1.1b) are smooth (see [2]). The main purpose of this paper is to develop a spectral Jacobi-collocation method for Volterra integro-differential equations with weakly singular kernels and pantograph delays based on the works of [23] and [32]. This paper is arranged as follows. In Section 2, we describe the spectral methods for (1.1a)-(1.1b). Some useful lemmas for derivation of the error estimates are given in Section 3. The error estimates in L^∞ norm and weighted L^2 norm of the approximation scheme (2.12a)-(2.12c) are provided in Section 4. Section 5 contains two numerical experiments which are used to verify the theoretical results obtained in Section 4. Finally, in Section 6, we end with the conclusion and the future work.

2 Jacobi-collocation method

Let $\omega^{\alpha,\beta}(x)=(1-x)^\alpha(1+x)^\beta$ be a weight function in the usual sense, for $\alpha,\beta>-1$. As defined in [3–5, 17, 18], the set of Jacobi polynomials $\{J_n^{\alpha,\beta}(x)\}_{n=0}^\infty$ forms a complete $L^2_{\omega^{\alpha,\beta}}(-1,1)$-orthogonal system, where $L^2_{\omega^{\alpha,\beta}}(-1,1)$ is a weighted space defined by

$$L^2_{\omega^{\alpha,\beta}}(-1,1)=\{v: v \text{ is measurable and } \|v\|_{\omega^{\alpha,\beta}}<\infty\},$$

equipped with the norm

$$\|v\|_{\omega^{\alpha,\beta}}=\Big(\int_{-1}^{1}|v(x)|^2\omega^{\alpha,\beta}(x)dx\Big)^{\frac{1}{2}},$$

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and the inner product

$$(u,v)_{\omega^{\alpha,\beta}}=\int_{-1}^{1}u(x)v(x)\omega^{\alpha,\beta}(x)dx,\quad \forall u,v\in L^2_{\omega^{\alpha,\beta}}(-1,1).$$

When $\omega^{\alpha,\beta}(x)=1$, $L^2_{\omega^{\alpha,\beta}}(-1,1)$ and $\|\cdot\|_{\omega^{\alpha,\beta}}$ are denoted simply by $L^2(-1,1)$ and $\|\cdot\|$ respectively.

For a given positive integer N, we denote the collocation points by $\{x_i\}_{i=0}^N$, which is the set of $(N+1)$ Jacobi Gauss points, corresponding to the weight $\omega^{\alpha,\beta}(x)$. Let $\mathcal{P}_N$ denote the space of all polynomials of degree not exceeding N. For any $v\in C[-1,1]$, we can difine the Lagrange interpolation polynomial $I_N^{\alpha,\beta}v\in\mathcal{P}_N$, satisfying

$$I_N^{\alpha,\beta}v(x_i)=v(x_i),\quad 0\le i\le N,$$

see, e.g., [4,17]. The Lagrange interpolation polynomial can be written in the form

$$I_N^{\alpha,\beta}v(x)=\sum_{i=0}^{N}v(x_i)F_i(x),$$

where $F_i(x)$ is the Lagrange interpolation basis function associated with $\{x_i\}_{i=0}^N$.

Using the variable transformation $\tau=qs$, Eqs. (1.1a)-(1.1b) can be rewritten as follows:

$$\begin{aligned}y'(t)=&a(t)y(t)+b(t)y(qt)+f(t)+\int_0^t(t-s)^{-\mu}K_1(t,s)y(s)ds\\&+q\int_0^t(qt-qs)^{-\mu}K_2(t,qs)y(qs)ds,\end{aligned}\tag{2.1a}$$

$$y(0)=y_0.\tag{2.1b}$$

It follows by applying the transformation $t=\frac{T}{2}(1+x)$ that

$$\begin{aligned}u'(x)=&\tilde{a}(x)u(x)+\tilde{b}(x)u(qx+q-1)+\tilde{f}(x)\\&+\frac{T}{2}\int_0^{\frac{T}{2}(1+x)}\Big(\frac{T}{2}(1+x)-s\Big)^{-\mu}K_1\Big(\frac{T}{2}(1+x),s\Big)y(s)ds\\&+\frac{qT}{2}\int_0^{\frac{T}{2}(1+x)}\Big(\frac{qT}{2}(1+x)-qs\Big)^{-\mu}K_2\Big(\frac{T}{2}(1+x),qs\Big)y(qs)ds,\quad x\in[-1,1],\end{aligned}\tag{2.2a}$$

$$u(-1)=u_{-1}=y_0,\tag{2.2b}$$

where

$$\begin{aligned}u(x)&=y\Big(\frac{T}{2}(1+x)\Big),&\tilde{a}(x)&=\frac{T}{2}a\Big(\frac{T}{2}(1+x)\Big),\\\tilde{b}(x)&=\frac{T}{2}b\Big(\frac{T}{2}(1+x)\Big),&\tilde{f}(x)&=\frac{T}{2}f\Big(\frac{T}{2}(1+x)\Big).\end{aligned}$$

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Furthermore, we make a linear transformation

$$s=\frac{T}{2}(1+\theta),\quad \theta\in[-1,x],$$

then Eq. (2.2a) becomes

$$\begin{aligned}u'(x)=&\tilde{a}(x)u(x)+\tilde{b}(x)u(qx+q-1)+\tilde{f}(x)+\int_{-1}^{x}(x-\theta)^{-\mu}\tilde{K}_1(x,\theta)u(\theta)d\theta\\&+\int_{-1}^{x}(x-\theta)^{-\mu}\tilde{K}_2(x,\theta)u(q\theta+q-1)d\theta,\end{aligned}\tag{2.3}$$

where

$$\begin{aligned}\tilde{K}_1(x,\theta)&=\Big(\frac{T}{2}\Big)^{2-\mu}K_1\Big(\frac{T}{2}(1+x),\frac{T}{2}(1+\theta)\Big),\\\tilde{K}_2(x,\theta)&=q^{1-\mu}\Big(\frac{T}{2}\Big)^{2-\mu}K_2\Big(\frac{T}{2}(1+x),\frac{qT}{2}(1+\theta)\Big).\end{aligned}$$

For the sake of naturally applying the spectral collocation method, we restate (2.2b) as

$$u(x)=u_{-1}+\int_{-1}^{x}u'(\theta)d\theta.\tag{2.4}$$

Assume that Eqs. (2.3)-(2.4) hold at the collocation points $\{x_i\}_{i=0}^{N}$ on $[-1,1]$, i.e.,

$$\begin{aligned}u'(x_i)=&\tilde{a}(x_i)u(x_i)+\tilde{b}(x_i)u(qx_i+q-1)+\tilde{f}(x_i)+\int_{-1}^{x_i}(x_i-\theta)^{-\mu}\tilde{K}_1(x_i,\theta)u(\theta)d\theta\\&+\int_{-1}^{x_i}(x_i-\theta)^{-\mu}\tilde{K}_2(x_i,\theta)u(q\theta+q-1)d\theta,\end{aligned}\tag{2.5a}$$

$$u(x_i)=u_{-1}+\int_{-1}^{x_i}u'(\theta)d\theta.\tag{2.5b}$$

Let $v(x_i)=u(qx_i+q-1)$, we have

$$v(x_i)=u_{-1}+q\int_{-1}^{x_i}u'(q\theta+q-1)d\theta.\tag{2.6}$$

In order to obtain high order accuracy of the approximate solutions, we make use of some appropriate quadrature rule to compute the integral terms in (2.5a)-(2.6). Based on this

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idea, we rewrite the integral terms as

$$\int_{-1}^{x_i}(x_i-\theta)^{-\mu}\tilde{K}_1(x_i,\theta)u(\theta)d\theta$$
$$=\Big(\frac{1+x_i}{2}\Big)^{1-\mu}\int_{-1}^{1}(1-\eta)^{-\mu}\tilde{K}_1(x_i,\theta_i(\eta))u(\theta_i(\eta))d\eta, \tag{2.7a}$$
$$\int_{-1}^{x_i}(x_i-\theta)^{-\mu}\tilde{K}_2(x_i,\theta)u(q\theta+q-1)d\theta$$
$$=\Big(\frac{1+x_i}{2}\Big)^{1-\mu}\int_{-1}^{1}(1-\eta)^{-\mu}\tilde{K}_2(x_i,\theta_i(\eta))u(q\theta_i(\eta)+q-1)d\eta, \tag{2.7b}$$
$$\int_{-1}^{x_i}u'(\theta)d\theta=\frac{1+x_i}{2}\int_{-1}^{1}u'(\theta_i(\eta))d\eta, \tag{2.7c}$$
$$\int_{-1}^{x_i}u'(q\theta+q-1)d\theta=\frac{1+x_i}{2}\int_{-1}^{1}u'(q\theta_i(\eta)+q-1)d\eta, \tag{2.7d}$$

by making the following variable change

$$\theta=\theta_i(\eta)=\frac{1+x_i}{2}\eta+\frac{x_i-1}{2},\quad -1\le\eta\le 1. \tag{2.8}$$

Next, using $(N+1)$-points Jacobi Gauss quadrature formula, two integral terms in the right-hand sides of (2.7a) and (2.7b) can be respectively approximated by

$$\int_{-1}^{1}(1-\eta)^{-\mu}\tilde{K}_1(x_i,\theta_i(\eta))u(\theta_i(\eta))d\eta\approx\sum_{k=0}^{N}\tilde{K}_1(x_i,\theta_i(\bar{\eta}_k))u(\theta_i(\bar{\eta}_k))\bar{\omega}_k, \tag{2.9a}$$
$$\int_{-1}^{1}(1-\eta)^{-\mu}\tilde{K}_2(x_i,\theta_i(\eta))u(q\theta_i(\eta)+q-1)d\eta\approx\sum_{k=0}^{N}\tilde{K}_2(x_i,\theta_i(\bar{\eta}_k))u(q\theta_i(\bar{\eta}_k)+q-1)\bar{\omega}_k, \tag{2.9b}$$

where the set $\{\bar{\eta}_k\}_{k=0}^{N}$ is the set of $(N+1)$ Jacobi Gauss points, corresponding to the weight $\omega^{-\mu,0}(x)$. Similarly, two integral terms in the right-hand sides of (2.7c) and (2.7d) can be respectively approximated by

$$\int_{-1}^{1}u'(\theta_i(\eta))d\eta=\sum_{k=0}^{N}u'(\theta_i(\hat{\eta}_k))\hat{\omega}_k, \tag{2.10a}$$
$$\int_{-1}^{1}u'(q\theta_i(\eta)+q-1)d\eta=\sum_{k=0}^{N}u'(q\theta_i(\hat{\eta}_k)+q-1)\hat{\omega}_k, \tag{2.10b}$$

where the set $\{\hat{\eta}_k\}_{k=0}^{N}$ is the set of $(N+1)$ Jacobi Gauss points, corresponding to the weight $\omega^{0,0}(x)$, i.e., $\{\hat{\eta}_k\}_{k=0}^{N}$ is $(N+1)$ Legendre Gauss points.

We denote u_i', v_i and u_i as the approximations of $u'(x_i)$, $v(x_i)$ and $u(x_i)$ respectively, and expand $u'(x)$, $u(x)$ by using Lagrange interpolation polynomials, i.e.,

$$u'(x)\approx u_N'(x)=\sum_{j=0}^{N}u_j'F_j(x),\quad u(x)\approx u_N(x)=\sum_{j=0}^{N}u_jF_j(x), \tag{2.11}$$

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where $F_j(x)$ is the Lagrange interpolation basis function associated with the Jacobi collocation points $\{x_i\}_{i=0}^N$. Then, the Jacobi-collocation method is to seek $\{u_i'\}_{i=0}^N$, $\{v_i\}_{i=0}^N$ and $\{u_i\}_{i=0}^N$ to satisfy the following collocation equations:

$$u_i' = \tilde{a}(x_i)u_i + \tilde{b}(x_i)v_i + \tilde{f}(x_i) + \Big(\frac{1+x_i}{2}\Big)^{1-\mu}\sum_{j=0}^N u_j\Big(\sum_{k=0}^N \tilde{K}_1(x_i,\theta_i(\bar{\eta}_k))F_j(\theta_i(\bar{\eta}_k))\bar{\omega}_k\Big)$$
$$+\Big(\frac{1+x_i}{2}\Big)^{1-\mu}\sum_{j=0}^N u_j\Big(\sum_{k=0}^N \tilde{K}_2(x_i,\theta_i(\bar{\eta}_k))F_j(q\theta_i(\bar{\eta}_k)+q-1)\bar{\omega}_k\Big), \tag{2.12a}$$

$$u_i = u_{-1} + \frac{1+x_i}{2}\sum_{j=0}^N u_j'\Big(\sum_{k=0}^N F_j(\theta_i(\hat{\eta}_k))\hat{\omega}_k\Big), \tag{2.12b}$$

$$v_i = u_{-1} + \frac{q(1+x_i)}{2}\sum_{j=0}^N u_j'\Big(\sum_{k=0}^N F_j(q\theta_i(\hat{\eta}_k)+q-1)\hat{\omega}_k\Big). \tag{2.12c}$$

By writing $U_N' = (u_0', u_1', \cdots, u_N')^T$, $U_N = (u_0, u_1, \cdots, u_N)^T$ and $V_N = (v_0, v_1, \cdots, v_N)^T$, we obtain the matrix statement of (2.12a)-(2.12c):

$$U_N' = (A + C + D)U_N + BV_N + F, \tag{2.13}$$
$$U_N = U_{-1} + GU_N', \tag{2.14}$$
$$V_N = U_{-1} + LU_N', \tag{2.15}$$

where $A = \mathrm{diag}(\tilde{a}(x_0), \tilde{a}(x_1), \cdots, \tilde{a}(x_N))$, $B = \mathrm{diag}(\tilde{b}(x_0), \tilde{b}(x_1), \cdots, \tilde{b}(x_N))$, $F = (\tilde{f}(x_0), \tilde{f}(x_1), \cdots, \tilde{f}(x_N))^T$ and $U_{-1} = (u_{-1}, u_{-1}, \cdots, u_{-1})^T$. The entries of the matrices C, D, G and L are given by

$$C_{ij} = \Big(\frac{1+x_i}{2}\Big)^{1-\mu}\sum_{k=0}^N \tilde{K}_1(x_i,\theta_i(\bar{\eta}_k))F_j(\theta_i(\bar{\eta}_k))\bar{\omega}_k,$$
$$D_{ij} = \Big(\frac{1+x_i}{2}\Big)^{1-\mu}\sum_{k=0}^N \tilde{K}_2(x_i,\theta_i(\bar{\eta}_k))F_j(q\theta_i(\bar{\eta}_k)+q-1)\bar{\omega}_k,$$
$$G_{ij} = \frac{1+x_i}{2}\sum_{k=0}^N F_j(\theta_i(\hat{\eta}_k))\hat{\omega}_k, \quad L_{ij} = \frac{q(1+x_i)}{2}\sum_{k=0}^N F_j(q\theta_i(\hat{\eta}_k)+q-1)\hat{\omega}_k.$$

Remark 2.1. Since F_j, $j = 0,1,\cdots,N$ are polynomials of degree not exceeding N, we have

$$\int_{-1}^{x_i}\sum_{j=0}^N u_j'F_j(\theta)d\theta = \frac{1+x_i}{2}\int_{-1}^{1}\sum_{j=0}^N u_j'F_j(\theta_i(\eta))d\eta = \frac{1+x_i}{2}\sum_{j=0}^N u_j'\sum_{k=0}^N F_j(\theta_i(\hat{\eta}_k))\hat{\omega}_k,$$
$$\int_{-1}^{x_i}\sum_{j=0}^N u_j'F_j(q\theta+q-1)d\theta = \frac{1+x_i}{2}\int_{-1}^{1}\sum_{j=0}^N u_j'F_j(q\theta_i(\eta)+q-1)d\eta$$
$$= \frac{1+x_i}{2}\sum_{j=0}^N u_j'\sum_{k=0}^N F_j(q\theta_i(\hat{\eta}_k)+q-1)\hat{\omega}_k.$$

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3 Some useful lemmas

Throughout the paper, C will denote a generic positive constant that is independent of N but depends on the interval length T, bounds of the given functions a, b, f, K_1, K_2 and the index μ.

We first introduce a weighted Hilbert space. For a nonnegative integer m, define

$$H^m_{\omega^{\alpha,\beta}}(-1,1):=\{v(x):v^{(k)}(x)\in L^2_{\omega^{\alpha,\beta}}(-1,1),\ 0\le k\le m\},$$

with the norm

$$\|v\|_{m,\omega^{\alpha,\beta}}=\Big(\sum_{k=0}^{m}\|v^{(k)}(x)\|^2_{\omega^{\alpha,\beta}}\Big)^{\frac{1}{2}}.$$

To bound approximation errors of Jacobi polynomials, only some of the L^2 norms appearing on the right-hand side of above norm enter into play. Thus, it is convenient to introduce the seminorm

$$|v|^{m;N}_{\omega^{\alpha,\beta}}:=|v|_{H^{m;N}_{\omega^{\alpha,\beta}}(-1,1)}=\Big(\sum_{k=\min(m,N+1)}^{m}\|v^{(k)}(x)\|^2_{\omega^{\alpha,\beta}}\Big)^{\frac{1}{2}}.$$

Next, we introduce a Höld space. For $r\ge 0$ and $\kappa\in[0,1]$, $C^{r,\kappa}([-1,1])$ denotes the space of functions whose r-th derivatives are Hölder continuous with exponent κ, endowed with the usual norm

$$\|v\|_{r,\kappa}=\max_{0\le k\le r}\max_{x\in[-1,1]}|v^{(k)}(x)|+\max_{0\le k\le r}\sup_{-1\le x\ne y\le 1}\frac{|v^{(k)}(x)-v^{(k)}(y)|}{|x-y|^{\kappa}}.$$

We now give some elementary lemmas, which are important for the derivation of the error estimates in the subsequent section.

Lemma 3.1 (see [4]). *Assume that Gauss quadrature formula relative to the Jacobi weight is used to integrate the product $v\phi$, where $v\in H^m_{\omega^{\alpha,\beta}}(-1,1)$ for some $m\ge 1$ and $\phi\in\mathcal{P}_N$, then*

$$|(v,\phi)_{\omega^{\alpha,\beta}}-(v,\phi)_N|\le CN^{-m}|v|^{m;N}_{\omega^{\alpha,\beta}}\|\phi\|_{\omega^{\alpha,\beta}},\tag{3.1}$$

where

$$(v,\phi)_N=\sum_{k=0}^{N}v(x_k)\phi(x_k)\omega_k.\tag{3.2}$$

Lemma 3.2 (see [13]). *Assume that $\{F_j(x)\}_{j=0}^N$ is the set of N-th Lagrange interpolation polynomials associated with the Jacobi Gauss points $\{x_i\}_{i=0}^N$, then*

$$\|I^{\alpha,\beta}_N\|_\infty:=\max_{x\in[-1,1]}\sum_{j=0}^{N}|F_j(x)|=\begin{cases}\mathcal{O}(\log N), & -1<\alpha,\beta\le-\dfrac{1}{2},\\ \mathcal{O}(N^{\gamma+\frac{1}{2}}), & \gamma=\max\{\alpha,\beta\},\ \textit{otherwise}.\end{cases}\tag{3.3}$$

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Lemma 3.3. *For any function v satisfying $v\in H^m_{\omega^{\alpha,\beta}}(-1,1)$ $(m\geq 1)$, we have*

$$\|v-I_N^{\alpha,\beta}v\|_{\omega^{\alpha,\beta}}\leq CN^{-m}|v|^{m;N}_{\omega^{\alpha,\beta}}, \tag{3.4a}$$

$$\|v-I_N^{\alpha,\beta}v\|_{\infty}\leq\begin{cases} CN^{\frac{1}{2}-m}\log N|v|^{m;N}_{\omega^c}, & -1<\alpha,\beta\leq-\dfrac{1}{2},\\ CN^{1+\gamma-m}|v|^{m;N}_{\omega^c}, & \gamma=\max\{\alpha,\beta\},\ \textit{otherwise}, \end{cases} \tag{3.4b}$$

where $\omega^c=\omega^{-1/2,-1/2}$.

Proof. The inequality (3.4a) can be found in [4]. We now prove (3.4b). Let $I_N^c v\in\mathcal{P}_N$ denote the interpolant of v at the Chebyshev Gauss points. From (5.5.28) in [4], the interpolation error estimate in the maximum norm is given by

$$\|v-I_N^c v\|_{\infty}\leq CN^{\frac{1}{2}-m}|v|^{m;N}_{\omega^c}. \tag{3.5}$$

By using (3.4) and Lemma 3.2, and noting that

$$I_N^{\alpha,\beta}p(x)=p(x),\quad \forall p(x)\in\mathcal{P}_N,$$

we get that

$$\begin{aligned}\|v-I_N^{\alpha,\beta}v\|_{\infty}&=\|(I-I_N^{\alpha,\beta})(v-I_N^c v)\|_{\infty}\leq\big(1+\|I_N^{\alpha,\beta}\|_{\infty}\big)\|v-I_N^c v\|_{\infty}\\ &\leq\begin{cases} CN^{\frac{1}{2}-m}\log N|v|^{m;N}_{\omega^c}, & -1<\alpha,\beta\leq-\dfrac{1}{2},\\ CN^{1+\gamma-m}|v|^{m;N}_{\omega^c}, & \gamma=\max\{\alpha,\beta\},\ \text{otherwise}. \end{cases}\end{aligned}$$

So, we complete the proof. □

The following result follows directly from Theorem 0 in [27].

Lemma 3.4. *Suppose that*

$$v(t)\leq a(t)+C\int_0^t v(s)ds,\quad 0\leq t\leq T, \tag{3.6}$$

where $a(t)$, $v(t)$ are nonnegative and locally integrable functions defined on $[0,T]$, and C is a positive constant. Then

$$v(t)\leq a(t)+C\int_0^t a(s)ds,\quad 0\leq t\leq T. \tag{3.7}$$

Lemma 3.5. *Assume that a nonnegative function $E(x)\in L^2_{\omega^{\alpha,\beta}}(-1,1)\bigcap L^{\infty}(-1,1)$ satisfies*

$$E(x)\leq H(x)+L\int_{-1}^x E(\theta)d\theta,\quad -1\leq x\leq 1, \tag{3.8}$$

where $H(x)$ is a nonnegative function satisfying $H(x)\in L^2_{\omega^{\alpha,\beta}}(-1,1)\bigcap L^{\infty}(-1,1)$, and L is a positive constant. Then

$$\|E(x)\|_{\infty}\leq C\|H(x)\|_{\infty}, \tag{3.9a}$$

$$\|E(x)\|_{\omega^{\alpha,\beta}}\leq C\|H(x)\|_{\omega^{\alpha,\beta}}. \tag{3.9b}$$

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Proof. It follows from Lemma 3.4 and (3.8) that

$$E(x)\leq H(x)+L\int_{-1}^{x}H(\theta)d\theta,\quad -1\leq x\leq 1, \tag{3.10}$$

which gives (3.9a) and

$$\|E(x)\|\leq L\Big\|\int_{-1}^{x}H(\theta)d\theta\Big\|+\|H(x)\|. \tag{3.11}$$

Clearly,

$$\Big\|\int_{-1}^{x}H(\theta)d\theta\Big\|\leq\|H(x)\|. \tag{3.12}$$

Then, we have

$$\|E(x)\|\leq C\|H(x)\|. \tag{3.13}$$

Consequently, the desired inequality (3.9b) follows from (3.13). □

Lemma 3.6 (see [15,16]). *For nonnegative integer r and $\kappa\in(0,1)$, there exists a constant $C_{r,\kappa}>0$ such that for any function $v\in C^{r,\kappa}([-1,1])$, there exists a polynomial function $\mathcal{F}_N v\in\mathcal{P}_N$ such that*

$$\|v-\mathcal{F}_N v\|_\infty\leq C_{r,\kappa}N^{-(r+\kappa)}\|v\|_{r,\kappa}. \tag{3.14}$$

Actually, as stated in [15] and [16], $\mathcal{F}_N$ is a linear operator from $C^{r,\kappa}([-1,1])$ into $\mathcal{P}_N$.

Lemma 3.7 (see [8]). *Let $\kappa\in(0,1)$ and $\mathcal{M}_1$, $\mathcal{M}_2$ be respectively defined by*

$$\mathcal{M}_1 v=\int_{-1}^{x}(x-\theta)^{-\mu}\tilde{K}_1(x,\theta)v(\theta)d\theta, \tag{3.15a}$$

$$\mathcal{M}_2 v=\int_{-1}^{x}(x-\theta)^{-\mu}\tilde{K}_2(x,\theta)v(q\theta+q-1)d\theta. \tag{3.15b}$$

Then, for any function $v\in C([-1,1])$, there exists a positive constant C such that

$$\frac{|\mathcal{M}_i v(x')-\mathcal{M}_i v(x'')|}{|x'-x''|^\kappa}\leq C\max_{x\in[-1,1]}|v(x)|,\quad i=1,2, \tag{3.16}$$

under the assumption that $0<\kappa<1-\mu$, for any $x',x''\in[-1,1]$ and $x'\neq x''$. This implies that

$$\|\mathcal{M}_i v\|_{0,\kappa}\leq C\max_{x\in[-1,1]}|v(x)|,\quad i=1,2,\quad 0<\kappa<1-\mu. \tag{3.17}$$

Lemma 3.8 (see [11]). *For any measurable function $f\geq 0$, the following generalized Hardy's inequality*

$$\Big(\int_a^b|(\mathcal{T}f)(x)|^q u(x)dx\Big)^{\frac{1}{q}}\leq C\Big(\int_a^b|f(x)|^p v(x)dx\Big)^{\frac{1}{p}} \tag{3.18}$$

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holds if and only if

$$\sup_{a<x<b}\left(\int_x^b u(t)dt\right)^{\frac{1}{q}}\left(\int_a^x v^{1-p'}(t)dt\right)^{\frac{1}{p'}}<\infty,\quad p'=\frac{p}{p-1},\tag{3.19}$$

for the case $1<p\leq q<\infty$. *Here,* $\mathcal{T}$ *is an operator of the form*

$$(\mathcal{T}f)(x)=\int_a^x K(x,t)f(t)dt,$$

with $K(x,t)$ *a given kernel,* u,v *weight functions, and* $-\infty\leq a<b\leq\infty$.

Lemma 3.9 (see [14]). *For every bounded function* $v(x)$, *there exists a constant* C *independent of* v *such that*

$$\left\|\sum_{j=0}^N v(x_j)F_j(x)\right\|_{\omega^{\alpha,\beta}}\leq C\max_{x\in[-1,1]}|v(x)|,\tag{3.20}$$

where $F_i(x)$ *is the Lagrange interpolation basis function associated with the Jacobi collocation points* $\{x_i\}_{i=0}^N$.

4 Convergence analysis

Firstly, we derive the error estimates in the L^∞ norm.

Theorem 4.1. *Let* $u(x)$ *be the exact solution of Eqs.* (2.3)-(2.4), *which is assumed to be sufficiently smooth. Assume that*

$$u_N(x)=\sum_{j=0}^N u_jF_j(x),\quad u_N'(x)=\sum_{i=0}^N u_j'F_j(x),\tag{4.1}$$

where u_j, u_j' *are given by* (2.12a)-(2.12c). *If*

$$\gamma=\max\{\alpha,\beta\}<\min\left\{\frac{1}{2}-\mu,0\right\}\quad\text{and}\quad u\in H^{m+1}_{\omega^{\alpha,\beta}}(-1,1)\cap H^{m+1}_{\omega^c}(-1,1),\quad(m\geq1).$$

Then

$$\|u(x)-u_N(x)\|_\infty\leq\begin{cases}CN^{-m}\log N\big(\mathcal{K}^*\|u(x)\|_\infty+N^{\frac{1}{2}}\mathcal{U}\big), & -1<\alpha,\beta\leq-\frac{1}{2},\\ CN^{\frac{1}{2}+\gamma-m}\big(\mathcal{K}^*\|u(x)\|_\infty+N^{\frac{1}{2}}\mathcal{U}\big), & \text{otherwise},\end{cases}\tag{4.2}$$

and

$$\|u'(x)-u_N'(x)\|_\infty\leq\begin{cases}CN^{-m}\log N\big(\mathcal{K}^*\|u(x)\|_\infty+N^{\frac{1}{2}}\mathcal{U}\big), & -1<\alpha,\beta\leq-\frac{1}{2},\\ CN^{\frac{1}{2}+\gamma-m}\big(\mathcal{K}^*\|u(x)\|_\infty+N^{\frac{1}{2}}\mathcal{U}\big), & \text{otherwise},\end{cases}\tag{4.3}$$

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provided that N is sufficiently large, where

$$\mathcal{K}^*=\max_{0\le i\le N}|\tilde{K}_1(x_i,\theta_i(\cdot))|^{m;N}_{\omega^{-\mu,0}}+\max_{0\le i\le N}|\tilde{K}_2(x_i,\theta_i(\cdot))|^{m;N}_{\omega^{-\mu,0}}, \tag{4.4a}$$

$$\mathcal{U}=|u'(x)|^{m;N}_{\omega^c}+|u(x)|^{m;N}_{\omega^c}. \tag{4.4b}$$

Proof. Following the notation of (3.2), we let

$$(\tilde{K}(x_i,\theta_i(\cdot)),\phi(\theta_i(\cdot)))_N=\sum_{k=0}^{N}\tilde{K}(x_i,\theta_i(\bar{\eta}_k))\phi(\theta_i(\bar{\eta}_k))\bar{\omega}_k. \tag{4.5}$$

Then the numerical scheme (2.12a)-(2.12c) can be written as

$$\begin{aligned}u_i'=&\tilde{a}(x_i)u_i+\tilde{b}(x_i)v_i+\tilde{f}(x_i)+\Big(\frac{1+x_i}{2}\Big)^{1-\mu}(\tilde{K}_1(x_i,\theta_i(\cdot)),u_N(\theta_i(\cdot)))_N\\&+\Big(\frac{1+x_i}{2}\Big)^{1-\mu}(\tilde{K}_2(x_i,\theta_i(\cdot)),u_N(q\theta_i(\cdot)+q-1))_N,\end{aligned} \tag{4.6a}$$

$$u_i=u_{-1}+\int_{-1}^{x_i}u_N'(\theta)d\theta, \tag{4.6b}$$

$$v_i=u_{-1}+q\int_{-1}^{x_i}u_N'(q\theta+q-1)d\theta, \tag{4.6c}$$

where (4.6b) and (4.6c) are obtained by Remark 2.1. By (2.8), (4.5) and (4.6a), we get

$$\begin{aligned}u_i'=&\tilde{a}(x_i)u_i+\tilde{b}(x_i)v_i+\tilde{f}(x_i)+\int_{-1}^{x_i}(x_i-\theta)^{-\mu}\tilde{K}_1(x_i,\theta)u_N(\theta)d\theta\\&+\int_{-1}^{x_i}(x_i-\theta)^{-\mu}\tilde{K}_2(x_i,\theta)u_N(q\theta+q-1)d\theta-I_{i,1}-I_{i,2},\end{aligned} \tag{4.7}$$

where

$$\begin{aligned}I_{i,1}=&\Big(\frac{1+x_i}{2}\Big)^{1-\mu}(\tilde{K}_1(x_i,\theta_i(\cdot)),u_N(\theta_i(\cdot)))_{\omega^{-\mu,0}}\\&-\Big(\frac{1+x_i}{2}\Big)^{1-\mu}(\tilde{K}_1(x_i,\theta_i(\cdot)),u_N(\theta_i(\cdot)))_N,\end{aligned} \tag{4.8a}$$

$$\begin{aligned}I_{i,2}=&\Big(\frac{1+x_i}{2}\Big)^{1-\mu}(\tilde{K}_2(x_i,\theta_i(\cdot)),u_N(q\theta_i(\cdot)+q-1))_{\omega^{-\mu,0}}\\&-\Big(\frac{1+x_i}{2}\Big)^{1-\mu}(\tilde{K}_2(x_i,\theta_i(\cdot)),u_N(q\theta_i(\cdot)+q-1))_N.\end{aligned} \tag{4.8b}$$

Using Lemma 3.1 gives

$$|I_{i,1}|\le CN^{-m}|\tilde{K}_1(x_i,\theta_i(\cdot))|^{m;N}_{\omega^{-\mu,0}}\|u_N(\theta_i(\cdot))\|_{\omega^{-\mu,0}} \tag{4.9}$$

and

$$|I_{i,2}|\le CN^{-m}|\tilde{K}_2(x_i,\theta_i(\cdot))|^{m;N}_{\omega^{-\mu,0}}\|u_N(q\theta_i(\cdot)+q-1)\|_{\omega^{-\mu,0}}. \tag{4.10}$$

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Subtracting (4.7) from (2.5a), subtracting (4.6b) from (2.5b) and subtracting (4.6c) from (2.6) lead to the error equations:

$$u'(x_i)-u_i'=\tilde{a}(x_i)(u(x_i)-u_i)+\tilde{b}(x_i)(v(x_i)-v_i)+\int_{-1}^{x_i}(x_i-\theta)^{-\mu}\tilde{K}_1(x_i,\theta)e(\theta)d\theta$$
$$+\int_{-1}^{x_i}(x_i-\theta)^{-\mu}\tilde{K}_2(x_i,\theta)e(q\theta+q-1)d\theta+I_{i,1}+I_{i,2}, \tag{4.11a}$$
$$u(x_i)-u_i=\int_{-1}^{x_i}\bar{e}(\theta)d\theta, \tag{4.11b}$$
$$v(x_i)-u_i=q\int_{-1}^{x_i}\bar{e}(q\theta+q-1)d\theta, \tag{4.11c}$$

where $e(x)=u(x)-u_N(x)$, $\bar{e}(x)=u'(x)-u_N'(x)$. Substituting (4.11b), (4.11c) into (4.11a) gives

$$u'(x_i)-u_i'=\tilde{a}(x_i)\int_{-1}^{x_i}\bar{e}(\theta)d\theta+q\tilde{b}(x_i)\int_{-1}^{x_i}\bar{e}(q\theta+q-1)d\theta+\int_{-1}^{x_i}(x_i-\theta)^{-\mu}\tilde{K}_1(x_i,\theta)e(\theta)d\theta$$
$$+\int_{-1}^{x_i}(x_i-\theta)^{-\mu}\tilde{K}_2(x_i,\theta)e(q\theta+q-1)d\theta+I_{i,1}+I_{i,2}. \tag{4.12}$$

Multiplying $F_i(x)$ on both sides of (4.12) and (4.11b), and summing up from $i=0$ to N yield

$$I_N^{\alpha,\beta}u'(x)-u_N'(x)$$
$$=I_N^{\alpha,\beta}\Big(\tilde{a}(x)\int_{-1}^{x}\bar{e}(\theta)d\theta\Big)+qI_N^{\alpha,\beta}\Big(\tilde{b}(x)\int_{-1}^{x}\bar{e}(q\theta+q-1)d\theta\Big)+I_N^{\alpha,\beta}\Big(\int_{-1}^{x}(x-\theta)^{-\mu}\tilde{K}_1(x,\theta)e(\theta)d\theta\Big)$$
$$+I_N^{\alpha,\beta}\Big(\int_{-1}^{x}(x-\theta)^{-\mu}\tilde{K}_2(x,\theta)e(q\theta+q-1)d\theta\Big)+\sum_{i=0}^{N}I_{i,1}F_i(x)+\sum_{i=0}^{N}I_{i,2}F_i(x), \tag{4.13a}$$
$$I_N^{\alpha,\beta}u(x)-u_N(x)=I_N^{\alpha,\beta}\Big(\int_{-1}^{x}\bar{e}(\theta)d\theta\Big). \tag{4.13b}$$

Consequently,

$$\bar{e}(x)=\tilde{a}(x)\int_{-1}^{x}\bar{e}(\theta)d\theta+q\tilde{b}(x)\int_{-1}^{x}\bar{e}(q\theta+q-1)d\theta+\int_{-1}^{x}(x-\theta)^{-\mu}\tilde{K}_1(x,\theta)e(\theta)d\theta$$
$$+\int_{-1}^{x}(x-\theta)^{-\mu}\tilde{K}_2(x,\theta)e(q\theta+q-1)d\theta+\sum_{l=1}^{7}J_l(x), \tag{4.14a}$$
$$e(x)=\int_{-1}^{x}\bar{e}(\theta)d\theta+J_8(x)+J_9(x), \tag{4.14b}$$

where

$$J_1(x)=\sum_{i=0}^{N}I_{i,1}F_i(x),\quad J_2(x)=\sum_{i=0}^{N}I_{i,2}F_i(x),\quad J_3(x)=u'(x)-I_N^{\alpha,\beta}u'(x),$$
$$J_4(x)=I_N^{\alpha,\beta}\Big(\tilde{a}(x)\int_{-1}^{x}\bar{e}(\theta)d\theta\Big)-\tilde{a}(x)\int_{-1}^{x}\bar{e}(\theta)d\theta,$$

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$$J_5(x)=qI_N^{\alpha,\beta}\Big(\tilde{b}(x)\int_{-1}^{x}\bar{e}(q\theta+q-1)d\theta\Big)-q\tilde{b}(x)\int_{-1}^{x}\bar{e}(q\theta+q-1)d\theta,$$
$$J_6(x)=I_N^{\alpha,\beta}\Big(\int_{-1}^{x}(x-\theta)^{-\mu}\tilde{K}_1(x,\theta)e(\theta)d\theta\Big)-\int_{-1}^{x}(x-\theta)^{-\mu}\tilde{K}_1(x,\theta)e(\theta)d\theta,$$
$$J_7(x)=I_N^{\alpha,\beta}\Big(\int_{-1}^{x}(x-\theta)^{-\mu}\tilde{K}_2(x,\theta)e(q\theta+q-1)d\theta\Big)-\int_{-1}^{x}(x-\theta)^{-\mu}\tilde{K}_2(x,\theta)e(q\theta+q-1)d\theta,$$
$$J_8(x)=u(x)-I_N^{\alpha,\beta}u(x),\quad J_9(x)=I_N^{\alpha,\beta}\Big(\int_{-1}^{x}\bar{e}(\theta)d\theta\Big)-\int_{-1}^{x}\bar{e}(\theta)d\theta.$$

By (4.14a) and (4.14b), we get

$$\begin{aligned}\bar{e}(x)=&\tilde{a}(x)\int_{-1}^{x}\bar{e}(\xi)d\xi+q\tilde{b}(x)\int_{-1}^{x}\bar{e}(q\xi+q-1)d\xi+\int_{-1}^{x}\int_{-1}^{\theta}(x-\theta)^{-\mu}\tilde{K}_1(x,\theta)\bar{e}(\xi)d\xi d\theta\\&+\int_{-1}^{x}\int_{-1}^{q\theta+q-1}(x-\theta)^{-\mu}\tilde{K}_2(x,\theta)\bar{e}(\xi)d\xi d\theta+H(x)\\=&\int_{-1}^{x}\Big(\int_{\xi}^{x}(x-\theta)^{-\mu}\tilde{K}_1(x,\theta)d\theta+\tilde{a}(x)\Big)\bar{e}(\xi)d\xi\\&+\int_{-1}^{qx+q-1}\Big(\int_{\frac{\xi}{q}+\frac{1}{q}-1}^{x}(x-\theta)^{-\mu}\tilde{K}_2(x,\theta)d\theta+\tilde{b}(x)\Big)\bar{e}(\xi)d\xi+H(x),\end{aligned}\tag{4.15}$$

where

$$\begin{aligned}H(x)=&\int_{-1}^{x}(x-\theta)^{-\mu}\tilde{K}_1(x,\theta)(J_8(\theta)+J_9(\theta))d\theta\\&+\int_{-1}^{x}(x-\theta)^{-\mu}\tilde{K}_2(x,\theta)(J_8(q\theta+q-1)+J_9(q\theta+q-1))d\theta+\sum_{l=1}^{7}J_l(x).\end{aligned}\tag{4.16}$$

Denote $D:=\{(x,\theta):-1\le\theta\le x,\ x\in[-1,1]\}$, we have

$$\begin{aligned}&\Big|\int_{\xi}^{x}(x-\theta)^{-\mu}\tilde{K}_1(x,\theta)d\theta+\tilde{a}(x)\Big|\\&\le\max_{(x,\theta)\in D}|\tilde{K}_1(x,\theta)|\int_{\xi}^{x}(x-\theta)^{-\mu}d\theta+\max_{x\in[-1,1]}|\tilde{a}(x)|\\&\le\frac{2^{1-\mu}}{1-\mu}\max_{(x,\theta)\in D}|\tilde{K}_1(x,\theta)|+\max_{x\in[-1,1]}|\tilde{a}(x)|\\&\equiv C_1,\end{aligned}\tag{4.17a}$$

$$\begin{aligned}&\Big|\int_{\frac{\xi}{q}+\frac{1}{q}-1}^{x}(x-\theta)^{-\mu}\tilde{K}_2(x,\theta)d\theta+\tilde{b}(x)\Big|\\&\le\max_{(x,\theta)\in D}|\tilde{K}_2(x,\theta)|\int_{\frac{\xi}{q}+\frac{1}{q}-1}^{x}(x-\theta)^{-\mu}d\theta+\max_{x\in[-1,1]}|\tilde{b}(x)|\\&\le\frac{2^{1-\mu}}{1-\mu}\max_{(x,\theta)\in D}|\tilde{K}_2(x,\theta)|+\max_{x\in[-1,1]}|\tilde{b}(x)|\\&\equiv C_2.\end{aligned}\tag{4.17b}$$

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Then Eq. (4.15) gives

$$\begin{aligned}|\bar{e}(x)| \leq & C_1 \int_{-1}^{x} |\bar{e}(\xi)| d\xi + C_2 \int_{-1}^{qx+q-1} |\bar{e}(\xi)| d\xi + |H(x)| \\ \leq & (C_1+C_2) \int_{-1}^{x} |\bar{e}(\xi)| d\xi + |H(x)|. \end{aligned} \tag{4.18}$$

It follows from Lemma 3.5 that

$$\|\bar{e}(x)\|_\infty \leq C \|H(x)\|_\infty. \tag{4.19}$$

Eq. (4.14b) gives

$$\|e(x)\|_\infty \leq 2\|\bar{e}(x)\|_\infty + \|J_8(x)\|_\infty + \|J_9(x)\|_\infty. \tag{4.20}$$

By virtue of (4.16), we have

$$\|H(x)\|_\infty \leq C \sum_{l=1}^{9} \|J_l(x)\|_\infty. \tag{4.21}$$

Applying (4.9), (4.10), (4.20) and Lemma 3.2 yields

$$\|J_1(x)\|_\infty = \Big\| \sum_{i=0}^{N} I_{i,1} F_i(x) \Big\|_\infty \leq C \max_{1\leq i\leq N} |I_{i,1}| \max_{x\in[-1,1]} \sum_{j=0}^{N} |F_j(x)|$$

$$\leq \begin{cases} CN^{-m} \log N \max\limits_{1\leq i\leq N} |\tilde{K}_1(x_i,\theta_i(\cdot))|^{m;N}_{\omega^{-\mu},0} (\|e(x)\|_\infty + \|u(x)\|_\infty), & -1<\alpha,\beta\leq -\dfrac{1}{2}, \\ CN^{\frac{1}{2}+\gamma-m} \max\limits_{1\leq i\leq N} |\tilde{K}_1(x_i,\theta_i(\cdot))|^{m;N}_{\omega^{-\mu},0} (\|e(x)\|_\infty + \|u(x)\|_\infty), & \text{otherwise}, \end{cases}$$

$$\leq \begin{cases} CN^{-m} \log N \max\limits_{1\leq i\leq N} |\tilde{K}_1(x_i,\theta_i(\cdot))|^{m;N}_{\omega^{-\mu},0} (2\|\bar{e}(x)\|_\infty + \|J_8(x)\|_\infty \\ \quad + \|J_9(x)\|_\infty + \|u(x)\|_\infty), & -1<\alpha,\beta\leq -\dfrac{1}{2}, \\ CN^{\frac{1}{2}+\gamma-m} \max\limits_{1\leq i\leq N} |\tilde{K}_1(x_i,\theta_i(\cdot))|^{m;N}_{\omega^{-\mu},0} (2\|\bar{e}(x)\|_\infty + \|J_8(x)\|_\infty \\ \quad + \|J_9(x)\|_\infty + \|u(x)\|_\infty), & \text{otherwise}, \end{cases}$$

$$\|J_2(x)\|_\infty = \Big\| \sum_{i=0}^{N} I_{i,2} F_i(x) \Big\|_\infty \leq C \max_{1\leq i\leq N} |I_{i,2}| \max_{x\in[-1,1]} \sum_{j=0}^{N} |F_j(x)|$$

$$\leq \begin{cases} CN^{-m} \log N \max\limits_{1\leq i\leq N} |\tilde{K}_2(x_i,\theta_i(\cdot))|^{m;N}_{\omega^{-\mu},0} (\|e(x)\|_\infty + \|u(x)\|_\infty), & -1<\alpha,\beta\leq -\dfrac{1}{2}, \\ CN^{\frac{1}{2}+\gamma-m} \max\limits_{1\leq i\leq N} |\tilde{K}_2(x_i,\theta_i(\cdot))|^{m;N}_{\omega^{-\mu},0} (\|e(x)\|_\infty + \|u(x)\|_\infty), & \text{otherwise}, \end{cases}$$

$$\leq \begin{cases} CN^{-m} \log N \max\limits_{1\leq i\leq N} |\tilde{K}_2(x_i,\theta_i(\cdot))|^{m;N}_{\omega^{-\mu},0} (2\|\bar{e}(x)\|_\infty + \|J_8(x)\|_\infty \\ \quad + \|J_9(x)\|_\infty + \|u(x)\|_\infty), & -1<\alpha,\beta\leq -\dfrac{1}{2}, \\ CN^{\frac{1}{2}+\gamma-m} \max\limits_{1\leq i\leq N} |\tilde{K}_2(x_i,\theta_i(\cdot))|^{m;N}_{\omega^{-\mu},0} (2\|\bar{e}(x)\|_\infty + \|J_8(x)\|_\infty \\ \quad + \|J_9(x)\|_\infty + \|u(x)\|_\infty), & \text{otherwise}. \end{cases}$$

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Employing Lemma 3.3 gives

$$\|J_3(x)\|_\infty \le \begin{cases} CN^{\frac{1}{2}-m}\log N|u'(x)|^{m;N}_{\omega^c}, & -1<\alpha,\beta\le -\frac{1}{2}, \\ CN^{1+\gamma-m}|u'(x)|^{m;N}_{\omega^c}, & \text{otherwise}, \end{cases}$$

$$\|J_8(x)\|_\infty \le \begin{cases} CN^{\frac{1}{2}-m}\log N|u(x)|^{m;N}_{\omega^c}, & -1<\alpha,\beta\le -\frac{1}{2}, \\ CN^{1+\gamma-m}|u(x)|^{m;N}_{\omega^c}, & \text{otherwise}. \end{cases}$$

By letting $m=1$ in (3.4b) yields the following estimates

$$\|J_4(x)\|_\infty \le \begin{cases} CN^{-\frac{1}{2}}\log N\|\bar{e}(x)\|_\infty, & -1<\alpha,\beta\le -\frac{1}{2}, \\ CN^{\gamma}\|\bar{e}(x)\|_\infty, & \text{otherwise}, \end{cases}$$

$$\|J_5(x)\|_\infty \le \begin{cases} CN^{-\frac{1}{2}}\log N\|\bar{e}(x)\|_\infty, & -1<\alpha,\beta\le -\frac{1}{2}, \\ CN^{\gamma}\|\bar{e}(x)\|_\infty, & \text{otherwise}, \end{cases}$$

$$\|J_9(x)\|_\infty \le \begin{cases} CN^{-\frac{1}{2}}\log N\|\bar{e}(x)\|_\infty, & -1<\alpha,\beta\le -\frac{1}{2}, \\ CN^{\gamma}\|\bar{e}(x)\|_\infty, & \text{otherwise}. \end{cases}$$

We now estimate $J_6(x)$ and $J_7(x)$. Due to Lemma 3.2, Lemma 3.6 and Lemma 3.7, we obtain

$$\begin{aligned}
&\|J_6(x)\|_\infty = \|(I_N^{\alpha,\beta}-I)\mathcal{M}_1 e\|_\infty = \|(I_N^{\alpha,\beta}-I)(\mathcal{M}_1 e-\mathcal{F}_N\mathcal{M}_1 e)\|_\infty \\
&\le (1+\|I_N^{\alpha,\beta}\|_\infty)\|\mathcal{M}_1 e-\mathcal{F}_N\mathcal{M}_1 e\|_\infty \le (1+\|I_N^{\alpha,\beta}\|_\infty)CN^{-\kappa}\|\mathcal{M}_1 e\|_{0,\kappa} \\
&\le \begin{cases} CN^{-\kappa}\log N\|e(x)\|_\infty, & -1<\alpha,\beta\le -\frac{1}{2}, \\ CN^{\frac{1}{2}+\gamma-\kappa}\|e(x)\|_\infty, & \text{otherwise}, \end{cases} \\
&\le \begin{cases} CN^{-\kappa}\log N(2\|\bar{e}(x)\|_\infty+\|J_8(x)\|_\infty+\|J_9(x)\|_\infty), & -1<\alpha,\beta\le -\frac{1}{2}, \\ CN^{\frac{1}{2}+\gamma-\kappa}(2\|\bar{e}(x)\|_\infty+\|J_8(x)\|_\infty+\|J_9(x)\|_\infty), & \text{otherwise}, \end{cases} \\
&\|J_7(x)\|_\infty = \|(I_N^{\alpha,\beta}-I)\mathcal{M}_2 e\|_\infty = \|(I_N^{\alpha,\beta}-I)(\mathcal{M}_2 e-\mathcal{F}_N\mathcal{M}_2 e)\|_\infty \\
&\le (1+\|I_N^{\alpha,\beta}\|_\infty)\|\mathcal{M}_2 e-\mathcal{F}_N\mathcal{M}_2 e\|_\infty \le (1+\|I_N^{\alpha,\beta}\|_\infty)CN^{-k}\|\mathcal{M}_2 e\|_{0,\kappa} \\
&\le \begin{cases} CN^{-\kappa}\log N\|e(x)\|_\infty, & -1<\alpha,\beta\le -\frac{1}{2}, \\ CN^{\frac{1}{2}+\gamma-\kappa}\|e(x)\|_\infty, & \text{otherwise}, \end{cases} \\
&\le \begin{cases} CN^{-\kappa}\log N(2\|\bar{e}(x)\|_\infty+\|J_8(x)\|_\infty+\|J_9(x)\|_\infty), & -1<\alpha,\beta\le -\frac{1}{2}, \\ CN^{\frac{1}{2}+\gamma-\kappa}(2\|\bar{e}(x)\|_\infty+\|J_8(x)\|_\infty+\|J_9(x)\|_\infty), & \text{otherwise}, \end{cases}
\end{aligned}$$

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where, in the fifth step of the estimates of $J_6(x)$ and $J_7(x)$, we have both used Lemma 3.7 under the following assumption:

$$\begin{cases} 0<\kappa<1-\mu, & -1<\alpha,\beta\leq-\dfrac{1}{2}, \\ \max\left\{\dfrac{1}{2}+\gamma,0\right\}<\kappa<1-\mu, & \text{otherwise.} \end{cases}$$

Thus, we obtain the desired estimates (4.2) and (4.3) by combing the above estimates and (4.19)-(4.21). □

Next, we prove the error estimates in weighted L^2 norm.

Theorem 4.2. *If the hypotheses given in Theorem 4.1 hold, then*

$$\begin{aligned} &\|u(x)-u_N(x)\|_{\omega^{\alpha,\beta}} \\ &\leq\begin{cases} CN^{-m}\Big(\mathcal{V}_1+\log N\mathcal{V}_2+N^{\frac{1}{2}}\log N\mathcal{U}+N^{-\kappa}\log N\mathcal{V}_2\Big), & -1<\alpha,\beta\leq-\dfrac{1}{2}, \\ CN^{-m}\Big(\mathcal{V}_1+N^{\frac{1}{2}+\gamma}\mathcal{V}_2+N^{1+\gamma}\mathcal{U}+N^{\frac{1}{2}+\gamma-\kappa}\mathcal{V}_2\Big), & \textit{otherwise}, \end{cases} \end{aligned} \tag{4.22}$$

and

$$\begin{aligned} &\|u'(x)-u'_N(x)\|_{\omega^{\alpha,\beta}} \\ &\leq\begin{cases} CN^{-m}\Big(\mathcal{V}_1+\log N\mathcal{V}_2+N^{\frac{1}{2}}\log N\mathcal{U}+N^{-\kappa}\log N\mathcal{V}_2\Big), & -1<\alpha,\beta\leq-\dfrac{1}{2}, \\ CN^{-m}\Big(\mathcal{V}_1+N^{\frac{1}{2}+\gamma}\mathcal{V}_2+N^{1+\gamma}\mathcal{U}+N^{\frac{1}{2}+\gamma-\kappa}\mathcal{V}_2\Big), & \textit{otherwise}, \end{cases} \end{aligned} \tag{4.23}$$

for any $\kappa\in(0,1-\mu)$ provided that N is sufficiently large, where

$$\begin{aligned} &\mathcal{V}_1=|u(x)|^{m;N}_{\omega^{\alpha,\beta}}+|u'(x)|^{m;N}_{\omega^{\alpha,\beta}}+\mathcal{K}^*\|u(x)\|_{2,\omega^c}, \\ &\mathcal{V}_2=\mathcal{K}^*\|u(x)\|_\infty. \end{aligned}$$

Proof. Using (4.14b), (4.16), (4.18), Lemma 3.5 and Lemma 3.8 leads to the estimates:

$$\begin{aligned} \|\bar{e}(x)\|_{\omega^{\alpha,\beta}}&\leq C\Big(\sum_{l=1}^{9}\|J_l(x)\|_{\omega^{\alpha,\beta}}+\|J_8(qx+q-1)\|_{\omega^{\alpha,\beta}}+\|J_9(qx+q-1)\|_{\omega^{\alpha,\beta}}\Big) \\ &\leq C\Big(\sum_{l=1}^{9}\|J_l(x)\|_{\omega^{\alpha,\beta}}+\|J_8(x)\|_\infty+\|J_9(x)\|_\infty\Big) \end{aligned} \tag{4.24}$$

and

$$\|e(x)\|_{\omega^{\alpha,\beta}}\leq\|\bar{e}(x)\|_{\omega^{\alpha,\beta}}+\|J_8(x)\|_{\omega^{\alpha,\beta}}+\|J_9(x)\|_{\omega^{\alpha,\beta}}. \tag{4.25}$$

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It follows from (4.9), (4.10) and Lemma 3.9 that

$$\|J_1(x)\|_{\omega^{\alpha,\beta}} = \|\sum_{i=0}^{N} I_{i,1}F_i(x)\|_{\omega^{\alpha,\beta}} \le C \max_{1\le i\le N}|I_{i,1}|$$
$$\le CN^{-m} \max_{1\le i\le N}|\tilde{K}_1(x_i,\theta_i(\cdot))|^{m;N}_{\omega^{-\mu},0}(\|e(x)\|_\infty + \|u(x)\|_\infty),$$
$$\|J_2(x)\|_{\omega^{\alpha,\beta}} = \|\sum_{i=0}^{N} I_{i,2}F_i(x)\|_{\omega^{\alpha,\beta}} \le C \max_{1\le i\le N}|I_{i,2}|$$
$$\le CN^{-m} \max_{1\le i\le N}|\tilde{K}_2(x_i,\theta_i(\cdot))|^{m;N}_{\omega^{-\mu},0}(\|e(x)\|_\infty + \|u(x)\|_\infty).$$

By virtue of the estimate in Theorem 4.1, $(m=1)$, we get

$$\|e(x)\|_\infty \le C(\|u(x)\|_\infty + \|u(x)\|_{2,\omega^c}).$$

Then

$$\sum_{l=1}^{2}\|J_l(x)\|_{\omega^{\alpha,\beta}} \le CN^{-m}\mathcal{K}^*(\|u(x)\|_\infty + \|u(x)\|_{2,\omega^c}).$$

Employing Lemma 3.3 gives

$$\|J_3(x)\|_{\omega^{\alpha,\beta}} \le CN^{-m}|u'(x)|^{m;N}_{\omega^{\alpha,\beta}},$$
$$\|J_8(x)\|_{\omega^{\alpha,\beta}} \le CN^{-m}|u(x)|^{m;N}_{\omega^{\alpha,\beta}}.$$

Letting $m=1$ in (3.4a) yields the following estimates

$$\|J_4(x)\|_{\omega^{\alpha,\beta}} \le CN^{-1}\|\bar{e}(x)\|_\infty,$$
$$\|J_5(x)\|_{\omega^{\alpha,\beta}} \le CN^{-1}\|\bar{e}(qx+q-1)\|_\infty \le CN^{-1}\|\bar{e}(x)\|_\infty,$$
$$\|J_9(x)\|_{\omega^{\alpha,\beta}} \le CN^{-1}\|\bar{e}(x)\|_\infty.$$

Finally, due to Lemma 3.6, Lemma 3.7 and Lemma 3.9, we have

$$\|J_6(x)\|_{\omega^{\alpha,\beta}} = \|(I_N^{\alpha,\beta}-I)\mathcal{M}_1 e\|_{\omega^{\alpha,\beta}} = \|(I_N^{\alpha,\beta}-I)(\mathcal{M}_1 e-\mathcal{F}_N\mathcal{M}_1 e)\|_{\omega^{\alpha,\beta}}$$
$$\le \|I_N^{\alpha,\beta}(\mathcal{M}_1 e-\mathcal{F}_N\mathcal{M}_1 e)\|_{\omega^{\alpha,\beta}} + \|\mathcal{M}_1 e-\mathcal{F}_N\mathcal{M}_1 e\|_{\omega^{\alpha,\beta}}$$
$$\le C\|\mathcal{M}_1 e-\mathcal{F}_N\mathcal{M}_1 e\|_\infty \le CN^{-\kappa}\|\mathcal{M}_1 e\|_{0,\kappa} \le CN^{-\kappa}\|e(x)\|_\infty,$$
$$\|J_7(x)\|_{\omega^{\alpha,\beta}} = \|(I_N^{\alpha,\beta}-I)\mathcal{M}_2 e\|_{\omega^{\alpha,\beta}} = \|(I_N^{\alpha,\beta}-I)(\mathcal{M}_2 e-\mathcal{F}_N\mathcal{M}_2 e)\|_{\omega^{\alpha,\beta}}$$
$$\le \|I_N^{\alpha,\beta}(\mathcal{M}_2 e-\mathcal{F}_N\mathcal{M}_2 e)\|_{\omega^{\alpha,\beta}} + \|\mathcal{M}_2 e-\mathcal{F}_N\mathcal{M}_2 e\|_{\omega^{\alpha,\beta}}$$
$$\le C\|\mathcal{M}_2 e-\mathcal{F}_N\mathcal{M}_2 e\|_\infty \le CN^{-\kappa}\|\mathcal{M}_2 e\|_{0,\kappa} \le CN^{-\kappa}\|e(x)\|_\infty,$$

where, in the last step of the estimates of $J_6(x)$ and $J_7(x)$, we have both used lemma 3.7 for any $\kappa\in(0,1-\mu)$. Using the above estimates, the estimates of $J_8(x)$ and $J_9(x)$ in L^∞ norm in the proof of Theorem 4.1, together with (4.2), (4.3), (4.24), (4.25), the desired estimates (4.22) and (4.23) are obtained. □

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5 Numerical experiments

The error estimates attained in last section indicate that the proposed Jacobi-collocation method possess exponential rate of convergence if the exact solutions of the equations are smooth. To confirm the theoretical prediction, two numerical experiments are carried out by considering the following examples.

Example 5.1. Consider the following initial-value problem

$$u'(x)=\sin(x)u(x)+e^x u(qx+q-1)+f(x)+\int_{-1}^{x}(x-\theta)^{-\frac{2}{3}}u(\theta)d\theta + \int_{-1}^{x}(x-\theta)^{-\frac{2}{3}}(q\theta+q-1)^{-3}u(q\theta+q-1)d\theta, \quad x\in[-1,1], \tag{5.1a}$$

$$u(-1)=-1, \tag{5.1b}$$

with $f(x)$ chosen so that $u(x)=x^3$. By calculation, we get

$$f(x)=3x^2-x^3\sin(x)-e^x(qx+q-1)^3-\frac{27}{4}(1+x)^{\frac{4}{3}}+\frac{81}{14}(1+x)^{\frac{7}{3}}-\frac{243}{140}(1+x)^{\frac{10}{3}}.$$

We use the Jacobi-collocation method (2.12a)-(2.12c) to solve (5.1a)-(5.1b) with q=0.05. The numerical errors in L^∞ and $L^2_{\omega^{\alpha,\beta}}$ norms are showed in Table 1 and Table 2. In Fig. 1, we plot the error curves of approximate solution and approximate derivative in semi-log scale. As expected, the errors decay exponentially. This indicates that the convergence of the numerical scheme described above is exponential.

Table 1: Example 5.1: the errors of approximate solution $u_N(x)$ $(\alpha=-1/3,\ \beta=-1/4)$.

N	2	4	6	8	10
L^∞ error	2.02624	4.88787e-12	8.158807e-12	8.82894e-12	8.94218e-12
$L^2_{\omega^{\alpha,\beta}}$ error	1.19466	2.95107e-12	4.92681e-12	5.32993e-12	5.39833e-12

Table 2: Example 5.1: the errors of approximate derivative $u'_N(x)$ $(\alpha=-1/3,\ \beta=-1/4)$.

N	2	4	6	8	10
L^∞ error	4.47857	1.45124e-11	2.35425e-11	2.54299e-11	2.57581e-11
$L^2_{\omega^{\alpha,\beta}}$ error	3.16340	8.73818e-12	1.40734e-11	1.51852e-11	1.53847e-11

Example 5.2. Consider the following initial-value problem

$$u'(x)=\sin(x)u(x)+x^2u(qx+q-1)+f(x)+\int_{-1}^{x}(x-\theta)^{-\frac{1}{2}}e^{-\theta}u(\theta)d\theta + \int_{-1}^{x}(x-\theta)^{-\frac{1}{2}}e^{-q\theta-q+1}u(q\theta+q-1)d\theta, \quad x\in[-1,1], \tag{5.2a}$$

$$u(-1)=e^{-1}, \tag{5.2b}$$

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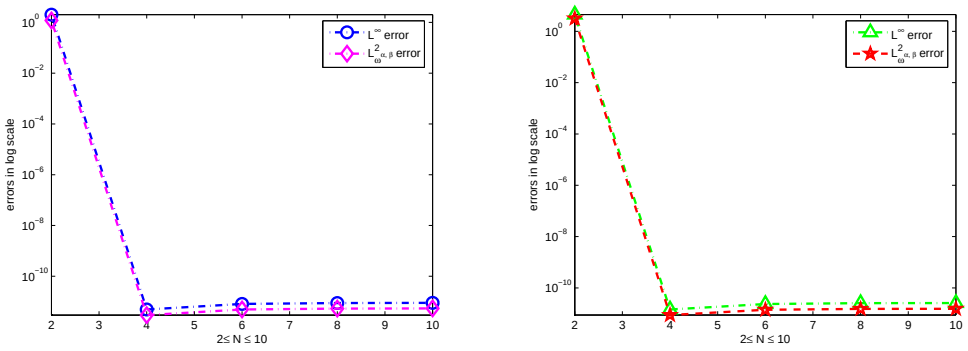


Figure 1: Example 5.1: The errors of approximate solution $u_N(x)$ (left) and the errors of approximate derivative $u'_N(x)$ (right) versus the number of collocation points in L^∞ and $L^2_{\omega^{\alpha,\beta}}$ norms $(\alpha=-1/3,\ \beta=-1/4)$.

with $f(x)$ chosen so that $u(x)=e^x$. By calculation, we get

$$f(x)=(1-\sin x)e^x-x^2e^{qx+q-1}-4(1+x)^{\frac{1}{2}}.$$

The numerical errors of approximate solution and approximate derivative of (5.2a)-(5.2b) with $q=0.95$ are plotted in Fig. 2 in L^∞ and $L^2_{\omega^{\alpha,\beta}}$ norms. In Table 3 and Table 4, we present the corresponding errors with several values of N. These results show that the desired exponential rate of convergence is obtained.

Table 3: Example 5.2: the errors of approximate solution $u_N(x)$ $(\alpha=-2/5,\ \beta=-1/5)$.

N	6	8	10	12	14
L^∞ error	4.64599e+1	4.82756e-1	1.84044e-3	7.58855e-5	8.66808e-7
$L^2_{\omega^{\alpha,\beta}}$ error	3.53149e+1	3.67170e-1	1.39361e-3	5.76008e-5	6.58017e-7
N	16	18	20	22	24
L^∞ error	6.41397e-9	6.97575e-11	2.00018e-11	3.05249e-11	4.36096e-11
$L^2_{\omega^{\alpha,\beta}}$ error	4.86830e-9	5.27833e-11	1.50652e-11	2.30240e-11	3.29558e-11

Table 4: Example 5.2: the errors of approximate derivative $u'_N(x)$ $(\alpha=-2/5,\ \beta=-1/5)$.

N	6	8	10	12	14
L^∞ error	9.65516e+1	1.00539e+0	3.83683e-3	1.58093e-4	1.80574e-6
$L^2_{\omega^{\alpha,\beta}}$ error	7.07444e+1	7.35570e-1	2.79215e-3	1.15396e-4	1.31825e-6
N	16	18	20	22	24
L^∞ error	1.33615e-8	1.44880e-10	4.13607e-11	6.32738e-11	9.05871e-11
$L^2_{\omega^{\alpha,\beta}}$ error	9.75284e-9	1.05506e-10	2.99888e-11	4.59490e-11	6.58629e-11

6 Conclusions and future work

Under the smoothness assumption of the underlying solutions, this paper examined the

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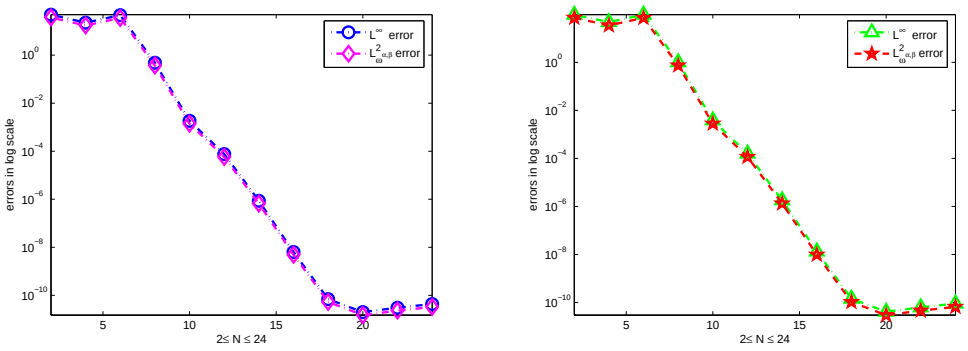


Figure 2: Example 5.2: The errors of approximate solution $u_N(x)$ (left) and the errors of approximate derivative $u'_N(x)$ (right) versus the number of collocation points in L^∞ and $L^2_{\omega^{\alpha,\beta}}$ norms $(\alpha=-2/5,\ \beta=-1/5)$.

spectral Jacobi-collocation method for Volterra delay integro-differential equations with weakly singular kernels, and established the error estimates in L^∞ norm and weighted L^2 norm. The numerical experiments indicate that spectral approximation of the problems presented in this paper is effective and very accurate. In the next work, we will consider the spectral collocation method for the case when the underlying solutions are not smooth.

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