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Edge-colourings

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The focus of this chapter is on two cornerstones in modern edge-colouring: the famous Goldberg–Seymour conjecture, and ideas culminating in the method of Tashkinov trees. We also discuss extreme examples, the classification problem, and the computational complexity of edge-colouring.

1. Introduction

The central question concerning edge-colourings is: given a graph G, what is its *chromatic index* $\chi'(G)$ – that is, the minimum number of colours needed to edge-colour G? We explore this question in detail in this chapter.

The most obvious lower bound on the chromatic index of a graph G is $\chi'(G) \geq \Delta$ (where Δ is the maximum degree in G), and in 1916 König [39] proved that all bipartite graphs satisfy this bound with equality.

Theorem 1.1 *If G is a bipartite graph, then $\chi'(G) = \Delta$.*

The most famous upper bounds on the chromatic index are the following two results. The first of these was proved by Shannon [75] in 1949, and the second was proved by Vizing [81], and independently by Gupta [27], in the 1960s. The

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second result is widely known as *Vizing's theorem*, and it makes use of the maximum edge-multiplicity of a graph G, denoted by $\mu = \mu(G)$.

Theorem 1.2 *For every graph G, $\chi'(G) \leq \frac{3}{2}\Delta$.*

Theorem 1.3 *For every graph G, $\chi'(G) \leq \Delta + \mu$.*

In terms of exactly determining the chromatic index of a graph G, Theorem 1.2 gives a range of about $\frac{1}{2}\Delta$ possible values. When $\mu < \frac{1}{2}\Delta$, Vizing's theorem further narrows this window. In particular, when G is a simple graph ($\mu = 1$), Vizing's theorem says that there are only two possible values for the chromatic index: Δ and $\Delta + 1$. However, even in the case of simple graphs (and in fact even in the case of 3-regular simple graphs), it is NP-hard to determine the chromatic index exactly, as shown by Holyer [35].

Each colour class in an edge-colouring of a graph is a matching in that graph. Hence, given a set S of vertices in a graph G, where $|S|$ is odd and at least 3, the maximum size of a colour class in $G[S]$ is $\frac{1}{2}(|S| - 1)$. Thus the following is a natural lower bound for the chromatic index:

$$\rho = \rho(G) = \max\left\{ \frac{2|E(G[S])|}{|S| - 1} : S \subseteq V(G), |S| \geq 3 \text{ and odd} \right\}.$$

In fact, we also get $\chi'(G) \geq \lceil \rho \rceil$, since the chromatic index is an integer. The famous *Goldberg–Seymour conjecture*, posed independently by Goldberg [18] and Seymour [73] in the 1970s, says that any graph with chromatic index higher than $\Delta + 1$ must have chromatic index $\lceil \rho \rceil$.

Conjecture A *For any graph G, $\chi'(G)$ is Δ, $\Delta + 1$ or $\lceil \rho \rceil$.*

Conjecture **A** is an amazing generalization of Vizing's theorem for simple graphs. Moreover, it suggests a dichotomy within edge-colouring. While Holyer's result tells us that it is NP-hard to distinguish between chromatic index Δ and $\Delta + 1$, the truth of the Goldberg–Seymour conjecture would imply the following for any graph G:

Determining whether or not $\chi'(G) > \Delta + 1$ can be done in polynomial time. Furthermore, the exact value of $\chi'(G)$ can be computed in polynomial time whenever $\chi'(G) > \Delta + 1$.

This implication is not immediate, but it can be deduced from well-known results in combinatorial optimization. In Section 7 we explain this in detail.

The focus of this chapter is on the Goldberg–Seymour conjecture and the techniques used in edge-colouring, particularly the 'method of Tashkinov trees'. In Section 2 we explain the core concepts of this method, taking as an example a proof of König's theorem. In Section 3 we discuss Tashkinov trees, and present easy proofs of Shannon and Vizing's theorems. We also present other important upper bounds for the chromatic index, all of which can be obtained in a similar manner. In Section 4 we survey the current status of work towards the Goldberg–Seymour conjecture, and show more advanced proof ideas that lead to approximations of the conjecture.

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Given a known bound for the chromatic index – say, $\chi'(G) \leq k$ for all graphs G – can we characterize those G for which $\chi'(G) = k$? Our discussion of such extreme graphs begins in Section 5, where we also touch on graphs at the other end of the scale – those with chromatic index Δ. When we restrict the characterization of extreme graphs to simple graphs, we meet the so-called *classification problem*, the NP-hard problem of distinguishing simple graphs with chromatic index Δ or $\Delta + 1$. This is a very rich area, and we provide a brief survey in Section 6. Section 7 contains the explanation mentioned above, concerning the complexity implications of the Goldberg–Seymour conjecture and the dichotomy between low and high chromatic index. Our final section contains some final thoughts about this dichotomy, Conjecture **A** and the method of Tashkinov trees.

There are many well-studied variations and problems concerning edge-colourings that we do not attempt to cover here. The curious reader is referred to the thorough and thoughtful new book, *Graph Edge-colouring: Vizing's Theorem and Goldberg's Conjecture*, by Stiebitz, Scheide, Toft and Favrholt [77]. This book also contains more information on nearly everything we discuss here and we refer to it henceforth as Stiebitz *et al.* [77].

2. Elementary sets and Kempe changes

A *partial edge-colouring* φ of a graph G is an edge-colouring of some subgraph of G. The edges of this subgraph are referred to as the *domain* of φ and denoted by $\mathrm{dom}(\varphi)$. Suppose we want to find a k-edge-colouring of a graph G, or at least show that one exists. It is trivial to find a partial k-edge-colouring φ of G with $|\mathrm{dom}(\varphi)| = 1$, and we can view this as a starting point. The next step is to find a k-edge-colouring φ' of G for which $|\mathrm{dom}(\varphi')| = |\mathrm{dom}(\varphi)| + 1$, and then iterate this process. We can phrase such an argument in an algorithmic way (given any partial k-edge-colouring whose domain is not $E(G)$, we find one with a larger domain) or in an extremal way (given a partial k-edge-colouring with maximum domain, we show that its domain is $E(G)$). Each phrasing has its own merit and will suit us at different times in this chapter.

Given a graph G and a partial edge-colouring φ of G, let $\overline{\varphi}(v)$ be the set of colours in φ that are not used on any edge incident to v – that is, the set of colours *missing* at v. A set of vertices $W \subseteq V(G)$ is said to be *φ-elementary* if no pair of vertices in W have a common missing colour – that is, $\overline{\varphi}(w_1) \cap \overline{\varphi}(w_2) = \varnothing$ for each pair of distinct vertices $w_1, w_2 \in W$. If the ends of an uncoloured edge are *not* φ-elementary, then there is some colour missing at both ends, and that colour could be used on the edge in question. It is worth stating this formally, as follows.

Theorem 2.1 *Let G be a graph and let φ be a partial k-edge-colouring of G that leaves $e = vw$ uncoloured. If $\{v, w\}$ is not φ-elementary, then there is a k-edge-colouring of $dom(\varphi) \cup \{e\}$.*

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In the next section we discuss Tashkinov's Theorem (Theorem 3.1), which is a generalization of Theorem 2.1. A prerequisite to this topic is the fundamental concept of Kempe changes, which we describe now and use, along with Theorem 2.1, to prove König's theorem.

Given a partial k-edge-colouring of a graph G, any pair of colours $a, b \in \{1, 2, \ldots, k\}$ induces a subgraph $G(a, b)$ of G, where every component is a path or an even cycle. Note that switching a and b on any such component of $G(a, b)$ results in another k-edge-colouring of G. We call any such switch a *Kempe change*. First introduced by Kempe in his erroneous proof of the four-colour theorem, Kempe changes have been widely used in edge-colouring.

Given a partial k-edge-colouring φ of a graph G and a vertex v in G with $a \in \overline{\varphi}(v)$, $b \notin \overline{\varphi}(v)$, one can consider a maximal (a, b)-*alternating path* starting at v – that is, a path whose edges alternate in the colours a and b. Performing a Kempe change along this path, we obtain a k-edge-colouring with domain $\mathrm{dom}(\varphi)$, where $b \in \overline{\varphi}(v)$ and $a \notin \overline{\varphi}(v)$. This simple procedure is a staple in edge-colouring.

Let us now give a quick proof of König's theorem, that all bipartite graphs are Δ-edge-colourable. Let φ be a partial Δ-edge-colouring of a bipartite graph G with maximum domain, and suppose (for a contradiction) that some edge $e = vw$ is uncoloured. Then there exist some colour $a \in \overline{\varphi}(v)$ and some colour $b \in \overline{\varphi}(w)$. By Theorem 2.1, $\{v, w\}$ must be φ-elementary, so $a \neq b$ and we can make a Kempe change along the maximal (a, b)-alternating path beginning at w. The alternating path cannot end at v, since this would create an odd cycle, and so in the resulting colouring φ', $b \in \overline{\varphi'}(v) \cap \overline{\varphi'}(w)$. However, Theorem 2.1 then guarantees a Δ-edge-colouring of $\mathrm{dom}(\varphi') \cup \{e\}$, contradicting the fact that φ has maximum domain. König's theorem is thus proved.

The above proof of König's theorem can be thought of as a blueprint for the method of Tashkinov trees that we see next. We have already mentioned that the statement of Tashkinov's theorem (Theorem 3.1) is a generalization of Theorem 2.1. Moreover, the proof of Tashkinov's theorem is made up entirely of sophisticated sequences of Kempe changes.

3. Tashkinov trees and upper bounds

Let G be a graph and let φ be a partial edge-colouring of G in which edge e_1 is uncoloured. A φ-*Tashkinov tree* is a sequence $T = (v_1, e_1, v_2, e_2, \ldots, v_t)$ of distinct vertices $v_1, v_2, \ldots, v_t$ and edges $e_1, e_2, \ldots, e_{t-1}$ of G such that $e_1 = v_1v_2$ and, for each e_i with $i \geq 2$,

(T1) e_i has endpoints v_l, v_{i+1}, for some $l \leq i$
(T2) $\varphi(e_i) \in \overline{\varphi}(v_k)$, for some $k \leq i$.

We say that such a Tashkinov tree *starts* at its uncoloured edge e_1, and that it is φ-*elementary* if $V(T)$ is φ-elementary.

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Given a partial edge-colouring φ of a graph G and an edge e not coloured by φ, e and its endpoints constitute a φ-Tashkinov tree. Hence the following result, proved by Tashkinov [78] in 2000 and known as *Tashkinov's theorem*, generalizes Theorem 2.1 when $k \geq \Delta + 1$.

Theorem 3.1 *Let G be a graph, let φ be a partial k-edge-colouring of G with $k \geq \Delta + 1$, and assume that e is uncoloured by φ. If there is a φ-Tashkinov tree starting at e that is not φ-elementary, then there is a k-edge-colouring of* $\mathrm{dom}(\varphi) \cup \{e\}$.

Theorem 3.1 actually generalizes Theorem 2.1, not only in the sense that its outcome is to guarantee the existence of a 'better' colouring, but in that it actually provides an algorithm to find such a colouring. This algorithm – that is, the proof of Theorem 3.1 – consists of a sophisticated sequence of Kempe changes. It provides the promised new colouring using at most a polynomial number (in n and Δ) of Kempe changes (see [45] or [77]).

There are two special cases of Tashkinov's theorem that are much easier to prove – that is, they require fewer Kempe changes. In (T1), if we can replace 'l' with 'i' then T is also called a *φ-Kierstead path*; if we can replace 'l' with '1' then T may be called a *φ-Vizing fan*.

Theorem 3.2 *Let G be a graph, let φ be a partial k-edge-colouring of G, and assume that e is uncoloured by φ. Assume that*
either
(1) *$k \geq \Delta + 1$ and there is a φ-Kierstead path T starting at e;*
or that
(2) *$k \geq \Delta$ and there is a φ-Vizing fan T starting at e.*
If T is not φ-elementary, then there is a k-edge-colouring of $dom(\varphi) \cup \{e\}$.

Case 2 of the above theorem was essentially proved by Vizing himself in his original argument for Vizing's theorem. In 1984 Kierstead [38] gave a new proof of Vizing's theorem, and on the way he proved case 1 of Theorem 3.2, a result which directly paved the way for Theorem 3.1. In fact, Tashkinov's proof of Theorem 3.1 is inductive and reduces to the case where the tree in question is a path – that is, a Kierstead path.

To apply Tashkinov's theorem, we typically want to consider a *maximal φ-Tashkinov tree* starting at e, where 'maximal' refers to the number of vertices in the tree. There may be a variety of such trees, with edge- and vertex-orderings that are distinct. However, it is easy to see that all maximal φ-Tashkinov trees starting at e must have the same vertex-set. We denote the vertex-set of any maximal φ-Tashkinov tree starting at e by $\mathcal{T}(e, \varphi)$. It is often helpful to consider a *maximal φ-Kierstead path* starting at e, or a *maximal φ-Vizing fan* starting at e, although the vertex-sets of such trees can vary considerably.

We now show how Theorem 3.1 implies both Shannon's and Vizing's theorems – in fact, we will see that these results follow from Case 1 of Theorem 3.2. To this end, let φ be a partial k-edge-colouring of a graph G, where $k \geq \Delta + 1$. (The value

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of k is either $\lfloor \frac{3}{2}\Delta \rfloor$ or $\Delta + \mu$ here, and we note that the case of $\lfloor \frac{3}{2}\Delta \rfloor < \Delta + 1$ is trivially resolved.) Suppose, for a contradiction, that φ has maximum domain but some edge e is left uncoloured. By Tashkinov's theorem, $\mathcal{T}(e, \varphi)$ is φ-elementary, so setting $|\mathcal{T}(e, \varphi)| = t$ we get

$$k \geq |\cup_{v \in \mathcal{T}(e,\varphi)} \overline{\varphi}(v)| \geq 2 + \sum_{v \in \mathcal{T}(e,\varphi)} (k - \deg(v)) \geq 2 + t(k - \Delta), \tag{1}$$

where the '2' accounts for the edge e being uncoloured. Rearranging gives

$$k \leq \Delta + \frac{\Delta - 2}{t - 1}. \tag{2}$$

We know that $t \geq 3$ since if $e = vw$, each colour in $\overline{\varphi}(v)$ appears on an edge incident with w, and we may pick any such edge as a second edge for a φ-Tashkinov tree. However, using $t \geq 3$ and $k = \lfloor \frac{3}{2}\Delta \rfloor$ in inequality (2) yields a contradiction. Thus we have proved Shannon's theorem.

To get Vizing's theorem, we go in a slightly different direction within the same argument, and pick some vertex $x \in \mathcal{T}(e, \varphi)$ that is not an endpoint of e. Since $\mathcal{T}(e, \varphi)$ is φ-elementary, x must be incident to an edge of every colour in $\bigcup_{v \in \mathcal{T}(e,\varphi), v \neq x} \overline{\varphi}(v)$, and by maximality, all these edges must be between x and $\mathcal{T}(e, \varphi) - x$. As there can only be at most $\mu(t - 1)$ such edges, we get

$$\mu(t - 1) \geq |\cup_{v \in \mathcal{T}(e,\varphi), v \neq x} \overline{\varphi}(v)| \geq 2 + (t - 1)(k - \Delta), \tag{3}$$

where the second inequality comes from inequality (1). Rearranging, we get

$$(t - 1)(\Delta + \mu - k) \geq 2. \tag{4}$$

Setting $k = \Delta + \mu$ in (4) yields a contradiction, and this proves Vizing's theorem.

The above proofs of Vizing's and Shannon's theorems are stated in the language of Tashkinov trees, but can be rephrased in terms of Kierstead paths, so that only Case 2 of Theorem 3.2 need be applied. To do so, simply consider a maximal φ-Kierstead path beginning at e and proceed as before, choosing x as the last vertex in this path in the proof of Vizing's theorem.

We discuss results that require the full strength of Tashkinov's theorem in the next section. First, however, we note that we can get even more from Theorem 3.2.

We can actually improve the condition $t \geq 3$ in the above argument to $t \geq g_o$, where g_o is the odd girth of the underlying simple graph. We see this by taking colours $a \in \overline{\varphi}(v)$ and $b \in \overline{\varphi}(w)$ and considering a maximal (a, b)-alternating path from w. (This path must end at v, since otherwise a single Kempe change causes $\{v, w\}$ to be non-elementary and hence contradict Theorem 2.1.) Putting $t \geq g_o$ in equation (2) yields a contradiction when $k = \Delta + 1 + \lfloor (\Delta - 2)/(g_o - 1) \rfloor$, and thus leads to a proof that of the following result, originally proved by Goldberg [21] in 1984.

Theorem 3.3 *For every non-bipartite graph G, $\chi'(G) \leq \Delta + 1 + \frac{\Delta - 2}{g_o - 1}$.*

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Theorem 3.3 generalizes Shannon's theorem. A more careful analysis of the above argument for Vizing's theorem also yields a generalization. This generalization was proved by Steffen [76] in 2000 and uses the girth g of the underlying graph.

Theorem 3.4 *For every graph G containing a cycle, $\chi'(G) \leq \Delta + 1 + \frac{(\mu-1)}{\lfloor g/2 \rfloor}$.*

There is another well-known way of generalizing Vizing's theorem. Ore [53] proved the following local refinement in the 1960s, using the parameter $\mu(v)$ to denote the maximum multiplicity of an edge incident with v; we also use $\mu(v, w)$ to denote the number of edges between a specific pair of vertices v and w.

Theorem 3.5 *For every graph G, $\chi'(G) \leq \max\{\deg(v) + \mu(v) : v \in V(G)\}$.*

Another recent local refinement is due to Stiebitz *et al.* [77].

Theorem 3.6 *For every graph G with at least two vertices,*

$$\chi'(G) \leq \Delta + \min\{\mu(G - v) : v \in V(G)\}.$$

Theorem 3.6 generalizes an earlier result of Chetwynd and Hilton [4]. Theorem 3.5 is best proved using Case 2 of Theorem 3.2, as follows. We observe that if φ is a partial edge-colouring of a graph G, and $T = (v_1, e_1, v_2, \ldots, v_t)$ is a maximal φ-Vizing fan that is φ-elementary, then

$$\left| \bigcup_{i=2}^{t} \overline{\varphi}(v_i) \right| = \sum_{i=2}^{t} |\overline{\varphi}(v_i)| \leq \left(\sum_{i=2}^{t} \mu(v_1, v_i) \right) - 1, \tag{5}$$

since every colour that is missing at one of $v_2, v_3, \ldots, v_t$ must be present on an edge between v_1 and $v_2, v_3, \ldots, v_t$ (since otherwise the fan is not maximal), and e is uncoloured. On the other hand, as in (1), we get

$$\sum_{i=2}^{t} |\overline{\varphi}(v_i)| \geq 1 + \sum_{i=2}^{t} (k - \deg(v_i)). \tag{6}$$

Combining (5) and (6) and using Case 1 of Theorem 3.2, we get the following.

Theorem 3.7 *Let G be a graph and let φ be a partial k-edge-colouring of G with $k \geq \Delta$. Assume that φ has maximum domain but that $e = vw$ is uncoloured. Then there is a vertex-set $X \subseteq N(v)$ such that $|X| \geq 2$, $w \in X$, and*

$$\sum_{x \in X} (\deg(x) + \mu(v, x) - k) \geq 2.$$

Theorem 3.7 is commonly referred to as the *fan inequality*. If we set $k = \max\{\deg(v) + \mu(v) : v \in V(G)\}$ in Theorem 3.7 we see that for this value it is not possible to have a partial k-edge colouring of maximum domain that does not colour all of G, hence proving Ore's theorem. The proof of Theorem 3.6 also relies on the fan inequality, although we omit those details here.

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The bounds mentioned in this section are arguably the most important in edge-colouring, and we discuss graphs that achieve them in Section 5. First however, we present more advanced results using Tashkinov trees, requiring the full generality of Theorem 3.1 – namely, we discuss approximations towards the Goldberg–Seymour conjecture.

4. Towards the Goldberg–Seymour conjecture

An equivalent way of stating the Goldberg–Seymour conjecture (Conjecture **A**), as given in our introductory section, is that for any graph G,

$$\chi'(G) \leq \max\{\lceil \rho \rceil, \Delta + 1\}.$$

This conjecture is far from being established, but there is mounting evidence towards it, as we discuss in this chapter.

The Goldberg–Seymour conjecture is true asymptotically: Kahn [37] has shown that, for any graph G,

$$\chi'(G) \sim \max\{\rho, \Delta\} \quad \text{as} \quad \max\{\rho, \Delta\} \to \infty.$$

The conjecture was shown by Marcotte [43] to be true for all graphs that do not contain a K_5^--minor, where K_5^- is K_5 with an edge removed. Most other results towards Conjecture **A** take the form of an approximation – namely, that $\chi'(G) \leq \max\{\lceil \rho \rceil, \Delta + s\}$ for some specific $s > 1$.

There is a family of results, starting in the 1970s, which assert that

$$\chi' \leq \max\left\{\lceil \rho \rceil, \quad \Delta + 1 + \frac{\Delta - 2}{m - 1}\right\}, \tag{7}$$

for increasing values of m. Such results have been obtained by Goldberg [21], [20] for $m = 9$, Nishizeki and Kashiwagi [51] and (independently) Tashkinov [78] for $m = 11$, Favrholt, Stiebitz and Toft [16] for $m = 13$, Scheide [69], [66] for $m = 15$, and Kurt [41] for $m = 25$. Since $(\Delta - 2)/24 < 1$ when $\Delta \leq 25$, the best of these results show that the Goldberg–Seymour conjecture holds when the maximum degree is at most 25.

Scheide [69] (see also [67]) and Chen, Yu and Zang [3] independently proved a different sort of approximation to the Goldberg–Seymour conjecture. Building on earlier work by Favrholdt, Stiebitz and Toft [16], they proved the following.

Theorem 4.1 *For every graph G, $\chi'(G) \leq \max\left\{\lceil \rho \rceil, \quad \Delta + \sqrt{\Delta/2}\right\}$.*

Theorem 4.1 greatly improves the above-mentioned approximations when Δ is large. This result can be tightened slightly by replacing $\sqrt{\Delta/2}$ by $\sqrt{(\Delta - 1)/2}$ (see [68], [45]), and Kurt [41] has announced that he can replace it with $\sqrt[3]{\Delta/2}$. We would like to replace this term by $\log \Delta$ as a next step, but this remains out of reach at this point. On the other hand, by allowing the parameter μ, Haxell and McDonald [28] proved the following.

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Theorem 4.2 *For every graph G, $\chi'(G) \leq \max\left\{\lceil\rho\rceil, \quad \Delta + 2\sqrt{\mu \log \Delta}\right\}$.*

Another way to obtain approximations for the Goldberg–Seymour conjecture is to use the number of vertices n as a parameter. Plantholt [56] proved that every graph with even order $n \geq 572$ satisfies

$$\chi'(G) \leq \max\{\Delta, \lceil\rho\rceil\} + 1 + \sqrt{n \log(n/6)},$$

and he recently made a further improvement that applies to all values of n (see [58]). In fact, the Goldberg–Seymour conjecture is known to be true for graphs for which n is very small. In 1997 Plantholt and Tipnis [59] completely determined the chromatic index of all graphs with at most 10 vertices, and showed that all graphs of this size satisfied the Goldberg–Seymour conjecture. Scheide's result of (7) (or rather, its proof) also implies the truth of the conjecture for graphs with at most 15 vertices. We explain this in more detail shortly.

The aforementioned work of Kahn, Marcotte and Plantholt *et al.* does not use the method of Tashkinov trees, but all the other results do, even those that predate Theorem 3.1. Tashkinov's association with (7) refers to his 2000 paper where Tashkinov trees were first introduced. However, as we saw in the last section, specialized versions of this method have long been used in edge-colouring.

In Section 3 we saw how to use Tashkinov trees (or even Kierstead paths) to k-edge-colour a graph G when $k = \lfloor \frac{3}{2}\Delta \rfloor$ (Shannon's theorem), $k = \Delta + \mu$ (Vizing's theorem) and $k = \Delta + 1 + \lfloor (\Delta - 2)/(g_0 - 1) \rfloor$ (Goldberg's theorem). But how can we use this technique to prove k-edge-colourability when $k = \max\{\lceil\rho\rceil, \Delta + s\}$ for some $s \geq 1$? We now explore this question.

Let φ be a partial k-edge-colouring of a graph G with maximum domain, and let e be an edge uncoloured by φ. Then $\mathcal{T}(e, \varphi)$ is φ-elementary by Tashkinov's theorem. This means that if $|\mathcal{T}(e, \varphi)| = t$, then every colour that is missing at a vertex in $\mathcal{T}(e, \varphi)$ must occur on exactly $\frac{1}{2}(t-1)$ edges in $G[\mathcal{T}(e, \varphi)]$ (and hence t must be odd), as such a colour cannot be missing at any other vertex in $\mathcal{T}(e, \varphi)$ and cannot occur on any edge in $\delta(\mathcal{T}(e, \varphi))$ (since then this edge could be added to $\mathcal{T}(e, \varphi)$). By (1), the number of such missing colours is at least $t(k - \Delta) + 2$. It follows from the definition of ρ that

$$\chi'(G) \geq \rho \geq \frac{2|E(G[\mathcal{T}(e, \varphi)])|}{t - 1} > t(k - \Delta) + 2, \tag{8}$$

where the strict inequality arises from the fact that the uncoloured edge e also lies in $G[\mathcal{T}(e, \varphi)]$. *If* we knew instead that *all* k colours occurred on exactly $\frac{1}{2}(t-1)$ edges of $G[\mathcal{T}(e, \varphi)]$, we could deduce (by using $k = \max\{\lceil\rho\rceil, \Delta + s\}$ in (8)) that

$$\chi'(G) \geq \rho > k \geq \rho,$$

which is a contradiction and hence proves the k-colourability of G. Thus, we may assume that there is some colour in $\{1, 2, \ldots, k\}$ that appears on fewer than $\frac{1}{2}(t-1)$ edges of $G[\mathcal{T}(e, \varphi)]$. Since this colour cannot be missing at any vertex in $G[\mathcal{T}(e, \varphi)]$,

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it must appear on multiple (in fact, at least 3) edges of $\delta(\mathcal{T}(e,\varphi))$; such a colour is called a *defective colour*. The existence of a defective colour yields further information about G: in particular, Favrholdt, Stiebitz and Toft [16] (see also [77]) have proved several such facts using Kempe changes, allowing us to improve the bound in inequality (2), and obtain the following result; similar statements are given in [77] and [45].

Theorem 4.3 *Let G be a graph, s be a positive integer, and $k = \max\{\Delta + s, \lceil \rho \rceil\}$. If there is a partial k-edge-colouring φ of G with maximum domain where e is uncoloured and $|\mathcal{T}(e,\varphi)| \geq t$, then*

$$s \leq \frac{\Delta - 3}{t+1}.$$

In the light of this result, if we wish to prove that

$$\chi'(G) \leq \max\left\{\lceil \rho \rceil, \Delta + 1 + \frac{\Delta - 3}{t+1}\right\}$$

for a given value of t, we need 'only' build a large enough Tashkinov tree ($|\mathcal{T}(e,\varphi)| \geq t$) and use Theorem 4.3 to get a contradiction. The results of form (7) are roughly obtained in this way by guaranteeing $t \geq m-2$, but as m increases this tends to require substantial case analysis. The bound of Theorem 4.1 requires showing that t is large compared to the number of colours k $(= \Delta + s)$ – in particular, $t \geq 2s+3$; this can be done easily by analyzing the number of times that a colour on an edge of a Tashkinov tree must appear in $\mathcal{T}(e,\varphi)$. Taking this idea further, and considering the parameter μ as well, one can show that $t > 1 + (1 + \frac{1}{2}s\mu)^s$, from which Theorem 4.2 follows.

If we set $s = 1$ in the above arguments and are able to show the existence of a Tashkinov tree of size t, then we can deduce that all graphs with fewer than t vertices satisfy the Goldberg–Seymour conjecture. In fact, using results from [16] we can tighten this further to prove the conjecture whenever $n < t + 3$; in particular, this yields the aforementioned result due to Scheide [67] for graphs with up to 15 vertices.

5. Extreme graphs

Which graphs have the highest (and lowest) chromatic indices? For example, if we know that all graphs G satisfy $\chi'(G) \leq k$, can we characterize those G for which $\chi'(G) = k$?

If H is a simple graph, then we denote by $H^{(\mu)}$ the *μ-multiple* of H – that is, the graph obtained from H by replacing each edge with μ parallel edges. Multiples of even cliques and multiples of even cycles are known to have chromatic index equal to Δ. On the other hand, by considering the vertex-set of an odd clique or an odd cycle, we can see that

$$\rho(K^{(\mu)}_{d+1}) \geq \Delta + \mu \quad \text{and} \quad \rho(C^{(\mu)}_{g_o}) \geq \Delta + 1 + \lfloor (\Delta - 2)/(g_o - 1) \rfloor,$$

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for all even d and odd g_o. Since $\lceil \rho \rceil$ is a lower bound on the chromatic index and we can use the theorems of Vizing and Goldberg, this tells us that

$$\chi'(K_{d+1}^{(\mu)}) = \Delta + \mu \quad \text{and} \quad \chi'(C_k^{(\mu)}) = \Delta + 1 + \lfloor (\Delta - 2)/(g_o - 1) \rfloor$$

for all d even and k odd. We thus obtain a class of examples (multiples of odd cliques) that achieve Vizing's bound and a class of examples (multiples of odd cycles) that achieve Goldberg's bound; the latter can be specified to yield a class of examples (multiples of triangles) that achieve Shannon's bound. These actually turn out to be the only extreme graphs for Shannon's and Goldberg's bounds – namely, Vizing [83] proved that

$$\chi'(G) = \tfrac{3}{2}\Delta \quad \text{if and only if} \quad G = K_3^{(\mu)},$$

and McDonald [46] generalized this to

$$\chi'(G) = \Delta + 1 + \frac{\Delta - 2}{g_o - 1} \text{ if and only if } \quad G = C_{g_o}^{(\mu)} \text{ and } (g_o - 1) \text{ divides } 2(\mu - 1).$$

These proofs can be deduced from the proof of Goldberg's bound presented in Section 3 by letting $k = \chi'(G) - 1$; instead of seeking a contradiction, we seek to understand the necessity of the last colour. This value of k in (2) implies that $t \leq g_o$, and hence inequalities (1) and (2) are satisfied with equality, leading to our desired characterization.

While all multiples of odd cliques achieve Vizing's bound of $\chi'(G) = \Delta + \mu$, these are not the only graphs to do so; for example, we can easily check that $\chi'(K_7^{(2)} - e) = \Delta + \mu$. Scheide and Stiebitz [70] listed pairs (Δ, μ) for which there exists a graph with maximum degree Δ and multiplicity μ achieving the bound, and proved that their list is complete provided that Conjecture **A** holds. Necessary structural conditions for graphs achieving Vizing's bound are also known (see [38], [46]), and if we restrict ourselves to multiples of simple graphs, a descriptive characterization of sorts is attainable; here it turns out that odd cliques do tell the full story (see [47]).

In the absence of a general descriptive characterization, however, simply characterizing non-simple graphs achieving Vizing's bound as having chromatic index $\lceil \rho \rceil$ would be highly desirable. This would mean proving that $\chi'(G) \leq \max\{\lceil \rho \rceil, \Delta + \mu - 1\}$ when $\mu \geq 2$, and partial results towards this have been obtained (see [28] and [16]). Without the condition on μ Holyer's result says that there would be no hope here, and perhaps this helps to explain why the problem is so difficult. In contrast, one can extend the characterization of Goldberg's bound even further to deduce that $\chi'(G) \leq \max\{\lceil \rho \rceil, \Delta + 1 + (\Delta - 3)/(g_o + 3)\}$ (see [48]).

We have already encountered two families of graphs that have the lowest possible chromatic index (Δ): bipartite graphs and multiples of even cliques. Because of Holyer's work we know that we cannot hope for a characterization of such graphs. However, Seymour [72] made the following conjecture, characterizing regular planar graphs of chromatic index Δ.

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Conjecture B *Let G be a regular planar graph. Then $\chi'(G) = \Delta$ if and only if $\rho(G) \leq \Delta$.*

Note that this conjecture does not follow from the Goldberg–Seymour conjecture, which says that $\chi'(G) = \Delta$ or $\Delta + 1$ if $\rho(G) \leq \Delta$. The truth of Conjecture **B** is well known to be equivalent to the four-colour theorem when $\Delta = 3$, and has been established for all $\Delta \leq 8$ with a string of deep papers by Guenin [26], Dvořák, Kawarabayashi and Král [12], Kawarabayashi and Edwards (see [14], [9]) and Chudnovsky, Edwards and Seymour [10]. There are also significant results towards two generalizations of Conjecture **B**, one asserting that it holds even if 'planar graph' is replaced by 'graph containing no Petersen minor', and the other asserting that the chromatic index of any planar graph is equal to $\max\{\Delta, \lceil\rho\rceil\}$ (see [74], [44], [61], [63] and [62]). The methods in these papers vary greatly from what we have been discussing so far – in particular, we see induction, discharging and reducible configurations, as introduced in Chapter 1.

When we restrict our discussion of extreme graphs to simple graphs, the scope of our problem paradoxically seems to increase. Since simple graphs can take only one of two values for the chromatic index, every simple graph can be considered extreme, in the sense that it meets either the upper bound $\Delta + 1$ of Vizing's theorem or the lower bound Δ. Work on edge-colouring simple graphs has been prolific; Fiorini and Wilson's 1977 book *Edge-colourings of Graphs* [17] (see also the later survey by Hilton and Wilson [30]) focuses almost exclusively on simple graphs. We devote the next section to a short survey of this important topic.

6. The classification problem and critical graphs

Vizing's theorem divides all simple graphs into one of two classes: those with chromatic index Δ and those with chromatic index $\Delta + 1$. The former simple graphs are called *class 1* and the latter are called *class 2*, according to the convention introduced by Wilson. The problem of distinguishing between them, which we have already seen is NP-hard, is commonly called the *classification problem*.

To follow up on results from the previous section, the classification problem has been well studied for planar graphs. Vizing [82] conjectured that all simple planar graphs with $\Delta \geq 6$ are of class 1, and this has now been established save for the case $\Delta = 6$; the case $\Delta = 7$ was recently proved independently by Grünewald [23], Sanders and Zhao [64] and Zhang [88]. The classification problem has also been studied for graphs embedded on other surfaces by Sanders and Zhao [65] and Luo and Zhao [42].

Even when we step away from talking about planar (or embedded) graphs, graphs of class 1 are much more abundant than those of class 2. In fact, Erdős and Wilson [15] proved in 1977 that almost all simple graphs are of class 1.

One way to prove that a simple graph is of class 2 is to show that $\lceil\rho\rceil \geq \Delta + 1$; the easiest case is when we need only analyze the entire vertex-set $V(G)$ (as opposed

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to various odd subsets S of $V(G)$). With this in mind, Hilton defined a simple graph G to be *overfull* if $m \geq \lfloor \frac{1}{2}n \rfloor \Delta + 1$. Such an inequality is not possible unless n is odd and $n \geq 3$, so we see that every overfull graph has too many edges to be of class 1 – that is, for any overfull graph G,

$$\chi'(G) \geq \lceil \rho \rceil \geq \left\lceil \frac{2m}{n-1} \right\rceil \geq \left\lceil \frac{\Delta(n-1)+2}{n-1} \right\rceil = \Delta + 1.$$

Every simple regular graph of odd order is easily seen to be overfull, and is thus of class 2; additional examples can be found in the early paper of Beineke and Wilson [2]. Hoffman and Rodger [34] proved that a complete multipartite graph is of class 2 if and only if it is overfull. *Hilton's overfull conjecture*, posed by Hilton in 1985 and first appearing in [5] and [6], states that when $\Delta > \frac{1}{3}|V(G)|$, overfullness is the only possible reason for a simple graph G to be of class 2.

Conjecture C *Let G be a simple graph with $\Delta > \frac{1}{3}n$. If G is of class 2, then G contains an overfull subgraph H with $\Delta(H) = \Delta(G)$.*

Note that the $\frac{1}{3}n$ condition is best possible: consider $P - v$, where P is the Petersen graph. An earlier and weaker version of Conjecture **C**, where G is a regular simple graph with $2n$ vertices and $\Delta \geq n$, appeared in Chetwynd and Hilton [4] and is known as the 1-*factorization conjecture*. This conjecture is known to be true asymptotically, as shown by Perković and Reed [55], and there are other supporting results (see, for example, [7] and [52]). Little progress has been made on Conjecture **C** in general, save for work by Plantholt [57], although Hilton and Johnson explored consequences of the conjecture in [29]. Generalizations of the 1-factorization conjecture and the overfull conjecture for all (not necessarily simple) graphs have been proposed by Plantholt and Tipnis [60] and Stiebitz *et al.* [77], respectively; the first of these was proved asymptotically by Plantholt and Tipnis [60] and Vaughan [80].

Many results on the classification problem give structural properties of class 2 graphs. Thomassen [79] proved that every class 2 graph contains two vertices of degree Δ that are joined by Δ edge-disjoint paths. A number of authors (see [33], [32], [4] and [7]) have studied the subgraph induced by all vertices of maximum degree, called the *core* of a graph, and found conditions for a graph to be of class 1 or class 2 depending on its core. Of special interest are the *Hilton graphs*, connected simple graphs of class 2 with a core of $\Delta = 1$ or 2 (see [31]).

Results about class 2 graphs (in particular, those mentioned in the above paragraph) very often make use of the following result, known as *Vizing's adjacency lemma*. Here we encounter for the first time in this chapter the important notion of *critical* as it relates to edge-colouring: an edge e in a graph G is *critical* if $\chi'(G - e) < \chi'(G)$, and G is *critical* if every edge of G is a critical edge.

Theorem 6.1 *Let G be a simple graph of class 2 and let $e = vw$ be a critical edge of G. Then v is adjacent to at least $\Delta - \deg(w) + 1$ vertices of degree Δ.*

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This lemma, originally proved by Vizing [82], can be seen as a consequence of the fan inequality (Theorem 3.7). Namely, if we apply Theorem 3.7 with $k = \Delta$ and $\mu = 1$ and let $Z = X \setminus w$, where X is the set of vertices guaranteed by Theorem 3.7, we deduce that $w \notin Z \subseteq N(v)$ and

$$\sum_{z \in Z} (\deg(z) + 1 - \Delta) \geq \Delta - \deg(w) + 1.$$

Hence the vertices of Z give the outcome of Vizing's adjacency lemma. With considerably more care, one can also prove a number of generalizations of Vizing's adjacency lemma (which provide additional structure), including those due to Goldberg [19] and Andersen [1], Choudum and Kayarthi [8] and Kostochka and Stiebitz [40].

Every simple graph G of class 2 contains as a subgraph a critical simple graph G' of class 2 with the same maximum degree (see Vizing [82]). Thus, a study of simple graphs of class 2 amounts to a study of *critical* simple graphs of class 2. Further, the only critical simple graphs that are of class 1 are stars (as these are the only graphs where deleting any edge causes the maximum degree to drop). Therefore, the classification problem can be seen as the problem of describing critical simple graphs of class 2.

There are constructions for critical simple graphs of class 2 by authors such as Jacobsen [36] (using Hajós's construction), Grünewald and Steffen [24] and Yap [87]. Vizing [83], [84] also made the following conjectures about critical simple graphs of class 2.

Conjecture D *If G is a simple critical n-vertex graph of class 2, then:*

- *the independence number of G is at most $\frac{1}{2}n$*
- *G contains a 2-regular spanning subgraph*
- *the number of edges in G is at least $\frac{1}{2}((\Delta - 1)n + 3)$.*

Note that the first conjecture above, commonly known as *Vizing's independence number conjecture*, is implied by the second conjecture, referred to as *Vizing's 2-factor conjecture*. The latter is known to be true for all overfull graphs (see Grünewald and Steffen [25]) and the former is additionally known to be true when when $\frac{1}{2}$ is replaced by $\frac{3}{5}$ (Woodall [86]). The last conjecture above, *Vizing's average degree conjecture*, is known to be true when $\frac{1}{2}((\Delta-1)n+3)$ is replaced by $\frac{1}{3}(\Delta+1)n$ (Woodall [85]), and there are a number of other supporting results; see [77] for a survey.

It is worth noting that nearly all of our earlier discussions can be phrased in terms of critical graphs. In particular, both of the following conjectures are equivalent to the Goldberg–Seymour conjecture; the first is known as the *critical multigraph conjecture* and the second is known as *Andersen's conjecture*. Of interest is the fact that Andersen proposed his conjecture in 1975 while an M.Sc. student at the University of Aarhus; it was later published in [1].

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Conjecture E *Every critical graph G with $\chi'(G) \geq \Delta + 2$ has n odd and $m = \frac{1}{2}(\chi'(G) - 1)(n - 1) + 2$.*

Conjecture F *If e is an edge in a critical graph G with $\chi'(G) \geq \Delta + 2$, then there is a $(\chi'(G) - 1)$-edge-colouring of $G - e$ such that no colour is missing at more than one vertex.*

Unlike the situation for simple graphs, we have reason to hope that there may be a polynomial-time algorithm for $\chi'(G)$-edge-colouring *any* graph G with $\chi'(G) \geq \Delta + 2$. This is why we chose not to restrict our earlier discussions to critical graphs. We explain this hope in the following section.

7. The dichotomy of edge-colouring

Let G be a graph, and let $\mathcal{M}(G)$ be the set of matchings in G. As previously mentioned, each colour class in an edge-colouring is a matching. Hence, we can express the chromatic index of G as

$$\chi'(G) = \min\left\{ \sum_{M \in \mathcal{M}(G)} x_M : \sum_{M : e \in M} x_M = 1 \text{ for all } e \in E, \quad x \in \mathbb{Z}^{\mathcal{M}(G)} \text{ and } x \geq 0 \right\},$$

where $x_M \in \{0, 1\}$ indicates whether or not the matching M is a colour class in the edge-colouring $x \in \{0, 1\}^{\mathcal{M}(G)}$. The linear programming relaxation gives rise to the *fractional chromatic index* of G, $\chi'_f(G)$ – namely,

$$\chi'_f(G) = \min\left\{ \sum_{M \in \mathcal{M}(G)} x_M : \sum_{M : e \in M} x_M = 1 \text{ for all } e \in E, \quad x \in \mathbb{R}^{\mathcal{M}(G)} \text{ and } x \geq 0 \right\},$$

where we view $x_M \in [0, 1]$ as the weight with which we pick the matching M. The *matching polytope* $\mathcal{MP}(G)$ of G is defined to be the convex hull of the characteristic vectors of matchings in G, and this definition leads immediately to the following:

$$\chi'_f(G) = \min\{\beta : (1/\beta, 1/\beta, \ldots, 1/\beta) \in \mathcal{MP}(G)\}. \tag{9}$$

Edmonds [13] gave a combinatorial description of $\mathcal{MP}(G)$ that is now known as *Edmonds's matching polytope theorem*, a short proof of which appears in [71, pp. 109–111]. This theorem and equation (9) yield the following useful interpretation of the fractional chromatic index:

$$\chi'_f(G) = \max\{\rho, \Delta\}. \tag{10}$$

We can now explain why the Goldberg–Seymour conjecture implies what we claimed it did in Section 1 – namely the following.

Determining whether or not $\chi'(G) > \Delta + 1$ can be done in polynomial time. Moreover, the exact value of $\chi'(G)$ can be computed in polynomial time whenever $\chi'(G) > \Delta + 1$.

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To this end, note that the Goldberg–Seymour conjecture and (10) imply that

$$\chi'(G) > \Delta + 1 \iff \chi'(G) = \lceil \rho \rceil > \Delta + 1 \iff \chi'(G) = \lceil \chi'_f(G) \rceil > \Delta + 1.$$

Hence our claim amounts to being able to compute $\chi'_f(G)$ in polynomial time.

Linear programs are known to be solvable in time that is polynomial in their number of constraints. The linear program describing the fractional chromatic index has an exponential number of constraints, however, as is the case with the linear program describing the fractional chromatic number. Despite this, although computing the fractional chromatic number is known to be NP-hard, there is a polynomial-time algorithm to compute the fractional chromatic index because of (9) and the work of Edmonds.

Equation (9) reduces the computation of $\chi'_f(G)$ to the following separation problem over the matching polytope.

Given a weighting of the edges in G, either identify a constraint of the matching polytope that is violated, or prove that no such constraint exists.

This is because we know, for example, that $(\frac{1}{\Delta-1}, \frac{1}{\Delta-1}, \ldots, \frac{1}{\Delta-1}) \notin \mathcal{MP}(G)$ and $(\frac{3}{2\Delta}, \frac{3}{2\Delta}, \ldots, \frac{3}{2\Delta}) \in \mathcal{MP}(G)$ (by Shannon's theorem), and we can use bisection to find the minimum as in (9).

There are two ways to show that we can solve this separation problem in polynomial time, both relying on the work of Edmonds. First, Padberg and Rao [54] solved this directly by making use of Edmonds's matching polytope theorem. Alternatively, we now know that for the matching polytope (or even more generally), a separation problem can be solved in polynomial time if and only if an optimization problem can be solved in polynomial time (see [22] and [11, p.238]). A polynomial-time algorithm for optimizing over the matching polytope was found by Edmonds in the 1960s; it is commonly called *Edmonds's blossom algorithm*.

8. Final thoughts

Our discussions have centred on the Goldberg–Seymour conjecture (Conjecture **A**) and the method of Tashkinov trees. This method consists of modifying a given partial edge-colouring with sequences of Kempe changes and resulting *extensions* (colouring an edge e with a colour a, provided that a is missing at both ends of e). Therefore, if we want to know whether the method of Tashkinov trees can be used to prove the Goldberg–Seymour conjecture, we need to ask the following question.

Can every partial $\chi'(G)$-edge-colouring of G be turned into a $\chi'(G)$-edge-colouring of G by a sequence of Kempe changes and extensions?

An affirmative answer to this question would settle positively the following conjecture of Vizing from 1965, now known as *Vizing's interchange conjecture*. Observe

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(for what follows) that a Kempe change may reduce the frequency of a colour, and when this change involves a single edge, the frequency may decrease to 0.

Conjecture G *Every edge-colouring of G can be turned into a $\chi'(G)$-edge-colouring of G by a sequence of Kempe changes.*

Two colourings are said to be *Kempe equivalent* if one can be obtained from the other by a sequence of Kempe changes. Vizing [83] noted that not all edge-colourings of a given graph are Kempe equivalent, but that Conjecture **G** holds when $\chi'(G) = \Delta + \mu$ or $\chi'(G) = \frac{3}{2}\Delta$, and we see that the same is true for all the bounds proved using the method of Tashkinov trees. More results on Kempe equivalence of edge-colourings have been found by Mohar [50] and McDonald, Mohar and Scheide [49].

In the previous section we saw that the Goldberg–Seymour conjecture implies that there exists a polynomial-time algorithm for determining $\chi'(G)$, whenever $\chi'(G) > \Delta + 1$. Is it possible that when $\chi'(G) > \Delta + 1$, every partial $\chi'(G)$-edge-colouring of G can be turned into a $\chi'(G)$-edge-colouring of G by a polynomial number of Kempe changes and extensions? We know that such a polynomial-time algorithm cannot exist for graphs with lower chromatic index (see Holyer [35]), but there may well be such a dichotomy in edge-colouring. In fact, graphs that have extremely high chromatic index appear to be among the easiest to edge-colour efficiently, perhaps because they are 'far away' from the classification problem. Accordingly, while one might hope to prove the Goldberg–Seymour conjecture first for graphs with $\mu \leq 2$, or even just for 2-multiples of simple graphs, these may be among the most difficult cases to settle.

References

1. L. Andersen, On edge-colourings of graphs, *Math. Scand.* **40** (1977), 161–175.
2. L. W. Beineke and R. J. Wilson, On the edge-chromatic number of a graph, *Discrete Math.* **5** (1973), 15–20.
3. G. Chen, X. Yu and W. Zang, Approximating the chromatic index of multigraphs, *J. Combin. Optim.* **21** (2011), 219–246.
4. A. G. Chetwynd and A. J. W. Hilton, Regular graphs of high degree are 1-factorizable, *Proc. London Math. Soc. (2)* **50** (1985), 193–206.
5. A. G. Chetwynd and A. J. W. Hilton, Star multigraphs with three vertices of maximum degree, *Math. Proc. Cambridge Philos. Soc.* **100** (1986), 303–317.
6. A. G. Chetwynd and A. J. W. Hilton, The edge chromatic class of graphs with maximum degree at least $|V| - 3$, *Graph Theory in Memory of G. A. Dirac* (eds. L. Andersen, I. Jakobsen, C. Thomassen, B. Toft and P. Vestergaard), Vol. 1, Ann. Discrete Math. **41**, North-Holland (1989), 91–110.
7. A. G. Chetwynd and A. J. W. Hilton, 1-factorizing regular graphs of high degree – an improved bound, *Discrete Math.* **75** (1989), 103–112.
8. S. Choudum and K. Kayathri, An extension of Vizing's adjacency lemma on edge chromatic critical graphs, *Discrete Math.* **206** (1999), 97–103.
9. M. Chudnovsky, K. Edwards, K. Kawarabayashi and P. Seymour, Edge-colouring seven-regular planar graphs, to appear.

https://doi.org/10.1017/CBO9781139519793.008 Published online by Cambridge University Press

10. M. Chudnovsky, K. Edwards and P. Seymour, Edge-colouring eight-regular planar graphs, to appear.
11. W. J. Cook, W. H. Cunningham, W. R. Pulleyblank and A. Schrijver, *Combinatorial Optimization*, Wiley, 1998.
12. Z. Dvořák, K. Kawarabayashi and D. Král, Packing six T-joins in plane graphs, manuscript.
13. J. Edmonds, Maximum matching and a polyhedron with 0,1-vertices, *J. Research Nat. Bureau of Standards (B)* **69** (1965), 125–130.
14. K. Edwards, *Optimization and Packings of T-joins and T-cuts*, M.Sc. thesis, McGill University, 2011.
15. P. Erdős and R. J. Wilson, On the chromatic index of almost all graphs, *J. Combin. Theory (B)* **23** (1977), 255–257.
16. L. M. Favrholdt, M. Stiebitz and B. Toft, *Graph Edge Colouring: Vizing's Theorem and Goldberg's Conjecture*, Preprints 2006, No. 20, IMADA, University of Southern Denmark, 91 pages.
17. S. Fiorini and R. J. Wilson, *Edge-colourings of Graphs*, Research Notes in Mathematics **16**, Pitman, 1977.
18. M. K. Goldberg, On multigraphs with almost-maximal chromatic class [in Russian], *Diskret. Analiz* **23** (1973), 3–7.
19. M. K. Goldberg, A remark on the chromatic class of a multigraph [in Russian], *Vyčisl. Mat. i. Vyčisl. Tehn. (Kharkow)* **5** (1974), 128–130.
20. M. K. Goldberg, Structure of multigraphs with restrictions on the chromatic class [in Russian], *Metody Diskret. Analiz* **30** (1977), 3–12.
21. M. K. Goldberg, Edge-coloring of multigraphs: recolouring technique, *J. Graph Theory* **8** (1984), 123–137.
22. M. Grötschel, L. Lovász and A. Schrijver, *Geometric Algorithms and Combinatorial Optimization*, Springer-Verlag, 1988.
23. S. Grünewald, *Chromatic Index Critical Graphs and Multigraphs*, Ph.D. thesis, Universität Bielefeld, 2000.
24. S. Grünewald and E. Steffen, Chromatic-index critical graphs of even order, *J. Graph Theory* **30** (1999), 27–36.
25. S. Grünewald and E. Steffen, Independent sets and sets and 2-factors in edge-chromatic-critical graphs, *J. Graph Theory* **45** (2004), 113–118.
26. B. Guenin, Packing T-joins and edge-colouring in planar graphs, manuscript.
27. R. P. Gupta, *Studies in the Theories of Graphs*, Ph.D. thesis, Tata Institute, Bombay, 1967.
28. P. Haxell and J. McDonald, On characterizing Vizing's edge-colouring bound, *J. Graph Theory* **69** (2012), 160–168.
29. A. J. W. Hilton and P. D. Johnson, Graphs which are vertex-critical with respect to the edge-chromatic number, *Math. Proc. Cambridge Philos. Soc.* **102** (1987), 211–221.
30. A. J. W. Hilton and R. J. Wilson, Edge-colorings of graphs: a progress report, *Graph Theory and its Applications: East and West (Jinan, 1986)* (eds. M. Capobianco *et al.*), Ann. New York Acad. Sci. **576** (1989), 241–249.
31. A. J. W. Hilton and Y. Zhao, One the edge-colouring of graphs whose core has maximum degree two, *J. Combin. Math. Combin. Comput.* **21** (1996), 97–108.
32. D. Hoffman, Cores of class II graphs, *J. Graph Theory* **20** (1995), 397–402.
33. D. Hoffman and C. Rodger, Class 1 graphs, *J. Combin. Theory (B)* **44** (1988), 372–376.
34. D. Hoffman and C. Rodger, The chromatic index of complete multipartite graphs, *J. Graph Theory* **16** (1992), 159–163.
35. I. Holyer, The NP-completeness of edge-colouring, *SIAM J. Comput.* **10** (1981), 718–720.

https://doi.org/10.1017/CBO9781139519793.008 Published online by Cambridge University Press

36. I. T. Jakobsen, Some remarks on the chromatic index of a graph, *Arch. Math. (Basel)* **24** (1973), 440–448.
37. J. Kahn, Asymptotics of the chromatic index for multigraphs, *J. Combin. Theory (B)* **68** (1996), 223–254.
38. H. A. Kierstead, On the chromatic index of multigraphs without large triangles, *J. Combin. Theory (B)* **36** (1984), 156–160.
39. D. König, Über Graphen und ihre Anwendung auf Determinantentheorie und Mengenlehre, *Math. Ann.* **77** (1916), 453–465.
40. A. Kostochka and M. Stiebitz, Edge-colouring: a new fan equation, manuscript, 2006.
41. O. Kurt, *On the Edge Coloring of Graphs*, Ph.D. thesis, Ohio State University, 2009.
42. R. Luo and Y. Zhao, Finding the exact bound of the maximum degrees of class two graphs embeddable in a surface of characteristic $\varepsilon \in \{-1, -2, -3\}$, *J. Combin. Theory (B)* **306** (2008), 1788–1790.
43. O. Marcotte, On the chromatic index of multigraphs and a conjecture of Seymour, II, *Proc. DIMACS Workshop, Morristown, N. J.* (eds. W. Cook and P. D. Seymour), American Math. Soc. (1990), 245–279.
44. O. Marcotte, Optimal edge-colourings for a class of planar multigraphs, *Combinatorics* **21** (2001), 361–394.
45. J. McDonald, *Multigraphs With High Chromatic Index*, Ph.D. thesis, University of Waterloo, 2009.
46. J. McDonald, Achieving maximum chromatic index in multigraphs, *Discrete Math.* **309**(8) (2009) 2077–2084.
47. J. McDonald, On multiples of simple graphs and Vizing's theorem, *Discrete Math.* **310** (2010), 2212–2214.
48. J. McDonald, On a theorem of Goldberg, *J. Graph Theory* **68**(1) (2011), 8–21.
49. J. McDonald, B. Mohar and D. Scheide, Kempe equivalence of edge-colourings in subcubic and subquartic graphs, *J. Graph Theory* **70**(2) (2012), 226–239.
50. B. Mohar, Kempe equivalence of colorings, *Graph Theory in Paris: Conf. in Honour of Claude Berge* (eds. J. A. Bondy *et al.*), Birkhäuser (2006), 287–297.
51. T. Nishizeki and K. Kashiwagi, On the 1.1 edge-coloring of multigraphs, *SIAM J. Discrete Math.* **3(3)** (1990), 391–410.
52. T. Nissen and L. Volkmann, Class 1 conditions depending on the minimum degree and the number of vertices of maximum degree, *J. Graph Theory* **14** (1990), 225–246.
53. O. Ore, *The Four-color Problem*, Academic Press, 1967.
54. M. W. Padberg and M. R. Rao, Odd minimum cut-sets and *b*-matchings, *Math. Oper. Res.* **7**(1) (1982), 67–80.
55. L. Perković and B. Reed, Edge coloring regular graphs of high degree, *Discrete Math.* **165/166** (1997), 567–578.
56. M. Plantholt, A sublinear bound on the chromatic index of multigraphs, *Discrete Math.* **202** (1999), 201–213.
57. M. Plantholt, Overfull conjecture for graphs with high minimum degree, *J. Graph Theory* **47** (2004), 73–80.
58. M. Plantholt, A combined logarithmic bound on the chromatic index of multigraphs, *J. Graph Theory* **73** (2013), 239–259.
59. M. Plantholt and S. Tipnis, The chromatic index of multigraphs of order at most 10, *Discrete Math.* **177** (1997), 185–193.
60. M. Plantholt and S. Tipnis, All regular multigraphs of even order and high degree are 1-factorable, *Electron. J. Combin.* **8**(1) (2001), R41.
61. N. Robertson, P. Seymour and R. Thomas, Tutte's edge-coloring conjecture, *J. Combin. Theory (B)* **70** (1997), 166–183.

https://doi.org/10.1017/CBO9781139519793.008 Published online by Cambridge University Press

62. D. Sanders, P. Seymour and R. Thomas, Edge three-coloring cubic doublecross graphs, in preparation.
63. D. Sanders and R. Thomas, Edge three-coloring cubic apex graphs, in preparation.
64. D. Sanders and Y. Zhao, On the size of edge chromatic critical graphs, *J. Combin. Theory (B)* **86** (2002), 408–412.
65. D. Sanders and Y. Zhao, Coloring edges of graphs embedded in a surface of characteristic zero, *J. Combin. Theory (B)* **87** (2003), 254–263.
66. D. Scheide, On the 15/14 edge-colouring of multigraphs, Preprints 2007 No. 11, IMADA, University of Southern Denmark, 49 pages.
67. D. Scheide, *Edge-colourings of Multigraphs*, Ph.D. thesis, TU Ilmenau, 2009.
68. D. Scheide, A polynomial-time $\Delta + \sqrt{(\Delta - 1)/2}$-edge-colouring algorithm, Preprints 2009 No. 4, IMADA, University of Southern Denmark, 15 pages.
69. D. Scheide, Graph edge-colouring: Tashkinov trees and Goldberg's conjecture, *J. Combin. Theory (B)* **100** (2010), 68–96.
70. D. Scheide and M. Stiebitz, On Vizing's bound for the chromatic index of a multigraph, *Discrete Math.* **309** (2009), 4920–4925.
71. A. Schrijver, *Theory of Linear and Integer Programming*, Wiley, 1986.
72. P. Seymour, *Matroids, Hypergraphs and the Max.-Flow Min.-Cut Theorem*, D.Phil. Thesis, Oxford University, 1975.
73. P. Seymour, On multi-colourings of cubic graphs, and conjectures of Fulkerson and Tutte, *Proc. London Math. Soc.* **3** (1979), 423–460.
74. P. Seymour, Colouring series-parallel graphs, *Combinatorica* **10** (1990), 379–392.
75. C. E. Shannon, A theorem on colouring the lines of a network, *J. Math. Phys.* **28** (1949), 148–151.
76. E. Steffen, A refinement of Vizing's theorem, *Discrete Math.* **218** (2000), 289–291.
77. M. Stiebitz, D. Scheide, B. Toft and L. Favrholt, *Graph Edge-colouring: Vizing's Theorem and Goldberg's Conjecture*, Wiley, 2012.
78. V. A. Tashkinov, On an algorithm to color the edges of a multigraph (in Russian), *Diskret. Analiz* **7** (2000), 72–85.
79. C. Thomassen, Hajós' conjecture for line graphs, *J. Combin. Theory (B)* **97** (2007), 156–157.
80. E. Vaughan, An asymptotic version of the multigraph 1-factorization conjecture, *J. Graph Theory* **72** (2013), 19–29.
81. V. G. Vizing, On an estimate of the chromatic class of a *p*-graph, *Diskret. Analiz* **3** (1964), 25–30.
82. V. G. Vizing, Critical graphs with a given chromatic class, *Diskret. Analiz* **5** (1965), 9–17.
83. V. G. Vizing, The chromatic class of a multigraph, *Kibernetika* (Kiev) **1** (1965) 29–39 [in Russian]; English transl.: *Cybernetics* **1** (1965), 32–41.
84. V. G. Vizing, Some unsolved problems in graph theory (in Russian), *Uspekhi Mat. Nauk.* **23** (1968), 117–134; English transl.: *Russian Math. Surveys* **23** (1968), 125–141.
85. D. R. Woodall, The average degree of an edge-chromatic critical graph II, *J. Graph Theory* **56** (2007), 194–218.
86. D. R. Woodall, The independence number of an edge-chromatic critical graph, *J. Graph Theory* **66** (2011), 98–103.
87. H. P. Yap, *Some Topics in Graph Theory*, London Math. Soc. Lecture Notes **108**, Cambridge Univ. Press, 1986.
88. L. Zhang, Every planar graph with maximum degree 7 is class 1, *Graphs Combin.* **16** (2000), 467–495.

https://doi.org/10.1017/CBO9781139519793.008 Published online by Cambridge University Press