Hanoi revisited

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1. *Introduction*

We consider a simple extension of the familiar *Tower of Hanoi* puzzle. There are three vertical pegs lined up in a row and n discs. The discs have holes in their centres so that they can be threaded on to the pegs. Initially, all n discs are placed on the left-hand peg in non-increasing order of radius to form a tower, as in Figure 1. The object of the game is to transfer the entire tower to the right-hand peg by moving discs from peg to peg, one at a time according to the rules:

(1) only a disc at the top of a tower may be moved;

(2) a disc must never be placed on top of a smaller disc.

Three questions naturally arise.

(i) What is the smallest number of moves required to transfer a tower of n discs?

(ii) For a given number of discs, how many distinct types of starting position are there?

(iii) Depending on the distribution of the radii, the minimum number of moves for a tower of n discs varies from n (when all discs have the same radius) to $2^n - 1$ (n distinct radii). How many numbers in this interval are represented?

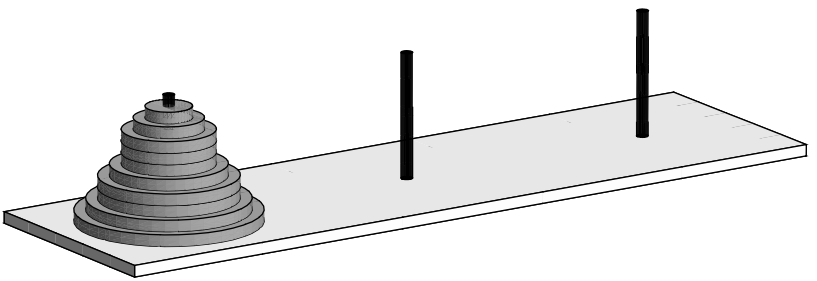


FIGURE 1: The extended Tower of Hanoi puzzle

In the original version of the puzzle, as described in [1, p. 55], for example, the discs have *distinct* radii and the solution is well known. Denote by N_t the number of moves necessary to transfer a tower of t discs. Assuming $t > 1$, first transfer the top $t - 1$ discs from the left-hand peg to the middle peg (N_{t-1} moves), and then transfer the largest disc to the right-hand peg (one move) followed by rest of the discs (another N_{t-1} moves). Therefore $N_t = 2N_{t-1} + 1$, which when combined with $N_1 = 1$ gives $N_t = 2^t - 1$. Moreover, analysis of the way this procedure works reveals that in the optimum solution the kth disc, $k = 1, 2, \ldots, t$, numbering from

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the top of the tower, gets moved 2^{t-k} times; see [2], for example.

The main purpose of this article is our analysis of question (iii), where we find a surprising and interesting connection with the Fibonacci numbers. But for completeness we first deal with questions (i) and (ii), which are both answered quite easily.

2. *Questions (i) and (ii)*

In what follows we assume without loss of generality that there are $t \geqslant 1$ distinct radii, $1, 2, \ldots, t$, and we represent the configuration of the initial tower by $(r_1, r_2, \ldots, r_t)$ if it has $r_k \geqslant 1$ discs of radius k, $k = 1, 2, \ldots, t$. The number of discs is $n = r_1 + r_2 + \ldots + r_t$.

Denote by $N((r_1, r_2, \ldots, r_t))$ the number of moves required to transfer the tower represented by $(r_1, r_2, \ldots, r_t)$. Unless $t = 1$, in which case $N((r_1)) = r_1$, it is necessary to expose the r_t discs of radius t on the left-hand peg and leave the right-hand peg empty. Hence we must transfer the smaller discs to the middle peg, and this will require $N((r_1, r_2, \ldots, r_{t-1}))$ moves. Clearly, when the largest discs are exposed the best strategy is to transfer all of them immediately to the right-hand peg: r_t moves. And to complete the solution we transfer the remaining discs to the right-hand peg with another $N((r_1, r_2, \ldots, r_{t-1}))$ moves. Hence

$$N((r_1, r_2, \ldots, r_t)) = 2N((r_1, r_2, \ldots, r_{t-1})) + r_t.$$

Therefore

$$N((r_1, r_2)) = 2N((r_1)) + r_2 = 2r_1 + r_2,$$

$$N((r_1, r_2, r_3)) = 2N((r_1, r_2)) + r_3 = 4r_1 + 2r_2 + r_3, \ldots,$$

leading to the formula

$$N((r_1, r_2, \ldots, r_t)) = \sum_{k=1}^{t} r_k 2^{t-k}, \tag{1}$$

which of course simplifies to $2^t - 1$ for the original Tower of Hanoi puzzle when $r_1 = r_2 = \ldots = r_t = 1$.

For example, starting with discs of radii 1, 1, 1, 2, 2, 2, 2, 3, 3, 4, 4, 4, 4 and 5, we have $n = 14, t = 5$, and

$$N((3, 4, 2, 4, 1)) = 3 \times 2^4 + 4 \times 2^3 + 2 \times 2^2 + 4 \times 2 + 1 = 97.$$

Since it is not needed for our purpose we omit a description of the actual sequence of 97 moves. We leave it as an entertaining puzzle for the interested reader to solve for himself or herself.

The number of starting positions for n discs is 2^{n-1}. To see this, try building the initial tower from the top down. The top disc has radius 1. Assume some discs have been assembled and suppose the bottom disc has radius s. Then there are precisely two ways to extend the pile one disc further downwards: we can add a disc of radius either s or $s + 1$. Thus we double the number of possibilities $n - 1$ times from 1 to 2^{n-1}.

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Moreover, we can represent the arrangement of the discs in the initial tower by an *ordered partition* of n; that is, a finite sequence of positive integers, $(r_1, r_2, \ldots, r_t)$ such that $r_1 + r_2 + \ldots + r_t = n$. (Order is relevant and the elements $r_1, r_2, \ldots, r_t$ are not necessarily distinct.) By the previous argument we see that the number of ordered partitions of n is 2^{n-1}. For illustration, in Table 1 we determine the eight starting configurations corresponding to $n = 4$.

Ordered partition	(4)	(1,3)	(2,2)	(3,1)	(1,1,2)	(1,2,1)	(2,1,1)	(1,1,1,1)
Disc radii	1111	1222	1122	1112	1233	1223	1123	1234
Number of moves	4	5	6	7	8	9	11	15

TABLE 1: Analysis of 4-disc puzzles

3. *Question (iii)*

If S is a set, we use the notation $|S|$ for the cardinality of S, i.e. the number of elements in S. Also we write $\lfloor x \rfloor$ for the largest integer less than or equal to x.

We denote by $m(n)$ the number of distinct values of $N(R)$ as R ranges over all of the 2^{n-1} ordered partitions of n. Consider the construction of the initial tower as described in section 2. Suppose we have a starting configuration with largest disc of radius s, and let x be the smallest number of moves needed to transfer the tower. Adding another disc of radius s increases the optimum solution to $x + 1$ moves. However, if we add a disc of radius $s + 1$, the number of moves increases to $2x + 1$. So the set of minimum numbers of moves for initial towers of n discs is M_n, defined by

$$M_1 = \{1\},$$

$$M_n = \{x + 1 : x \in M_{n-1}\} \cup \{2x + 1 : x \in M_{n-1}\}, n > 1,$$

and $m(n) = |M_n|$. Thus

$$M_2 = \{2, 3\},$$

$$M_3 = \{3, 4, 5, 7\},$$

$$M_4 = \{4, 5, 6, 7, 8, 9, 11, 15\},$$

$$M_5 = \{5, 6, 7, 8, 9, 10, 11, 12, 13, 15, 16, 17, 19, 23, 31\}$$

and so on. Observe that 9 arises in M_5 from M_4 either as $8 + 1$ or as $2 \times 4 + 1$; so it is not necessarily true that $m(n + 1) = 2m(n)$. The smallest element of M_n is n and the largest is $2^n - 1$. For $n = 1, 2, \ldots, 15$, we determine $m(n)$ by direct computation:

$$m(n) : 1, 2, 4, 8, 15, 27, 47, 80, 134, 222, 365, 597, 973, 1582, 2568.$$

These numbers constitute the first fifteen elements of entry A000126 in *The On-line Encyclopedia of Integer Sequences*, [3], namely $L(n, 4)$, where

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$$L(n, r) = \sum_{i=0}^{r-1} \binom{n-1}{i} + \sum_{j=1}^{\lfloor \frac{1}{2}(n-r) \rfloor} \binom{n-j-1}{j+r-1}$$

with the usual conventions $\binom{0}{0} = 1$ and $\binom{u}{v} = 0$ whenever $u < v$. The function $L(n, r)$ is described in [4] as a generalisation of the Fibonacci sequence $F_n = L(n, 1) : 1, 1, 2, 3, 5, 8, 13, 21, \ldots$, with $F_1 = F_2 = 1$. Furthermore, it is proved in [4] that $L(n, 4) = F_{n+3} - n - 1$. Our aim is to prove Theorem 1, below, which confirms the identification of $m(n)$ with $L(n, 4)$, thus resolving a conjecture stated in the comments associated with [3, A000126].

Theorem 1: For $n = 1, 2, \ldots$,

$$m(n) = F_{n+3} - n - 1. \tag{2}$$

Making use of the Fibonacci property, $F_{j+2} = F_{j+1} + F_j$, we have a similar recursive formula for $F_{n+3} - n - 1$:

$$F_{(n+2)+3} - (n+2) - 1$$

$$= F_{(n+1)+3} - (n+1) - 1 + F_{n+3} - n - 1 + n. \tag{3}$$

Therefore, since $m(1) = L(1, 4) = 1$ and $m(2) = L(2, 4) = 2$, to prove Theorem 1 it suffices to show that $m(n)$ satisfies the same equality (3) as $F_{n+3} - n - 1$:

$$m(n+2) = m(n+1) + m(n) + n, \qquad n = 1, 2, \ldots . \tag{4}$$

For the remainder of the article, we concentrate on proving (4). We consider even and odd elements of M_n separately. Let

$$E_n = \{x : x \in M_n, x \text{ even}\}, \qquad O_n = \{x : x \in M_n, x \text{ odd}\}.$$

The only way to get an even number in M_n is by adding 1 to an odd number in M_{n-1}. Hence

$$|E_n| = |O_{n-1}|, \qquad n = 2, 3, \ldots . \tag{5}$$

Our next goal is to obtain a similar kind of relation for $|O_n|$. However, the details are more complicated. The odd numbers O_n for $n > 1$ arise as the union of three sets, $O_n = U_n \cup V_n \cup W_n$, where

$$U_n = \{2x + 1 : x \in M_{n-1}\},$$

$$V_n = \{x + 1 : x \in E_{n-1}, x \leqslant 2n - 4\},$$

and $$W_n = \{x + 1 : x \in E_{n-1}, x \geqslant 2n - 2\}.$$

To see what the various subsets of M_n look like for small n, we have computed the values presented in Table 2. We have split odd numbers arising from the '$x + 1$' process into two sets, V_n and W_n, where the upper

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limit $x \leqslant 2n - 4$ in the definition of V_n ensures that its members do not clash with elements of U_n. Also we find that there is a simple formula for $|V_n|$.

$$
\begin{array}{llll}
E_1 = \{\}, & O_1 = \{1\}, & & \\
E_2 = \{2\}, & U_2 = \{3\}, & V_2 = W_2 = \{\}, & \\
E_3 = \{4\}, & U_3 = \{5,7\}, & V_3 = \{3\}, & W_3 = \{\}, \\
E_4 = \{4,6,8\}, & U_4 = \{7,9,11,15\}, & V_4 = \{5\}, & W_4 = \{\}, \\
E_5 = \{6,8,10,12,16\}, & & & \\
U_5 = \{9,11,13,15,17,19,23,31\}, & & & \\
V_5 = \{5,7\}, & W_5 = \{9\} \subseteq U_5, & & \\
E_6 = \{6,8,10,12,14,16,18,20,24,32\}, & & & \\
U_6 = \{11,13,15,17,19,21,23,25,27,31,33,35,39,47,63\}, & & & \\
V_6 = \{7,9\}, & W_6 = \{11,13,17\} \subseteq U_6, & & \\
E_7 = \{8,10,12,14,16,18,20,22,24,26,28,32,34,36,40,48,64\}, & & & \\
U_7 = \{13,15, \ldots ,43,47,49,51,55,63,65,67,71,79,95,127\}, & & & \\
V_7 = \{7,9,11\}, W_7 = \{13,15,17,19,21,25,33\} \subseteq U_7. & & &
\end{array}
$$

TABLE 2: Subsets of M_n for small n

Lemma 1: For $n = 2, 3, \ldots, |V_n| = \left\lfloor \frac{n-1}{2} \right\rfloor$ and $V_n \cap U_n = \{\}$.

Proof: Since V_2 is empty, $V_3 = \{3\}$ and $U_3 = \{5,7\}$, the lemma holds for $n = 2$ and $n = 3$.

Assume $n \geqslant 4$ and consider the initial towers represented by the ordered partitions $(n-1), (1, n-2), (2, n-3), \ldots, (n-3,2)$. Using (1), we compute the corresponding numbers of moves to be $n-1, n, n+1, \ldots, 2n-4$ respectively. So all even numbers in this range and no numbers less than $n-1$ are in E_{n-1}. Hence V_n is the set of the $\lfloor \frac{1}{2}(n-1) \rfloor$ odd numbers in the range $n, n+1, \ldots, 2n-3$. Also the smallest element of U_n is $2n - 1 > 2n - 3$, and hence V_n and U_n are disjoint.

A consequence of Lemma 1 is that $|O_n| \geqslant |U_n| + |V_n|$. However, the evidence from the finite number of cases at our disposal suggests that equality holds. Indeed, we shall prove in Lemma 4, below, that $W_n \subseteq U_n$. This is achieved in stages with two more lemmas, each of which might have some independent interest. However, their proofs are rather technical. The first (Lemma 2) says that no numbers apart from n are lost in going from M_n to M_{n+1}.

Lemma 2: For $n = 1, 2, \ldots$, all elements of M_n other than n belong to M_{n+1}; that is, in symbols,

$$M_n \backslash \{n\} \subseteq M_{n+1}. \tag{6}$$

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Proof: We use induction. By direct computation we see that (6) holds for $n = 1$ and $n = 2$; so we assume (6) holds up to some $n \geqslant 2$.

Suppose $x \in M_{n+1}$ and $x \neq n+1$. If x comes from $x - 1 \in M_n$, then $x - 1 \neq n$ and so by the induction hypothesis $x - 1 \in M_{n+1}$; hence $x \in M_{n+2}$, as required. If x comes from $\frac{1}{2}(x-1) \in M_n$ and $x \neq 2n+1$, then $\frac{1}{2}(x-1) \neq n$; hence $\frac{1}{2}(x-1) \in M_{n+1}$ by the induction hypothesis and therefore $x \in M_{n+2}$. The only other possibility is $x = 2n+1$. But $2n+1 \in M_{n+2}$ since the initial tower based on the ordered partition $(n-1,3)$ has $n+2$ discs and requires $2n+1$ moves. Therefore $M_n \backslash \{n+1\} \subseteq M_{n+2}$, and the lemma follows by induction.

For the next lemma, we require a further definition:

$$H_n = \left\{\tfrac{1}{2}x : x \in E_n, x \geqslant 2n\right\}, \qquad n = 1, 2, \ldots .$$

Thus

$$H_1 = H_2 = H_3 = \{\}, \qquad H_4 = \{4\}, \qquad H_5 = \{5,6,8\}$$
$$H_6 = \{6,7,8,9,10,12,16\}, \qquad \ldots .$$

Lemma 3: For $n = 1, 2, \ldots, H_n \subseteq M_n$.

Proof: We use induction. By direct computation, Lemma 3 holds for $n = 1, 2, 3$ and 4. Assume Lemma 3 holds up to some $n \geqslant 4$.

Suppose $x \in H_{n+1}$. Then $2x \in E_{n+1}$ and $2x \geqslant 2n+2$; hence $2x - 1 \in M_n$. If $2x - 1$ arises from adding 1 to $2x - 2 \in E_{n-1}$, then $x - 1 \in H_{n-1}$ since $2x - 2 \geqslant 2(n-1)$, and hence $x - 1 \in M_{n-1}$ by the induction hypothesis. Otherwise $2x - 1$ comes from $x - 1 \in M_{n-1}$. In either case $x - 1 \in M_{n-1}$ and hence $x \in M_n$. But $x > n$; so by Lemma 2, $x \in M_{n+1}$. Hence $H_{n+1} \subseteq M_{n+1}$.

Lemma 4: For $n = 2, 3, \ldots, W_n \subseteq U_n$.

Proof: We may assume $n \geqslant 5$ (otherwise W_n is empty). Suppose $x \in W_n$. Then $x - 1 \in E_{n-1}$, $x - 1 \geqslant 2n - 2$ and hence $\frac{1}{2}(x-1) \in H_{n-1}$. So $\frac{1}{2}(x-1) \in M_{n-1}$ by Lemma 3. Hence $x \in U_n$.

This is the result we want. From Lemma 4 we see that members of W_n do not contribute to the size of O_n. Thus $|O_n| = |U_n| + |V_n|$ and then from Lemma 1 and the observation that $|U_n| = |M_{n-1}|$ we obtain the sought-after recursive formula involving the odd integers in M_n:

$$|O_n| = |M_{n-1}| + \left\lfloor \frac{n-1}{2} \right\rfloor. \tag{7}$$

From (5) and (7) we obtain

$$m(n+2) = |E_{n+2}| + |O_{n+2}| = |O_{n+1}| + |M_{n+1}| + \left\lfloor \frac{n+1}{2} \right\rfloor$$

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$$= |M_n| + \left\lfloor \frac{n}{2} \right\rfloor + |M_{n+1}| + \left\lfloor \frac{n+1}{2} \right\rfloor$$
$$= m(n+1) + m(n) + \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n+1}{2} \right\rfloor$$
$$= m(n+1) + m(n) + n.$$

Therefore (4) holds. Hence (2) follows and Theorem 1 is proved.

References

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The answers to the *Nemo* page from July on spheres were:

1.	Charles Dickens	Martin Chuzzlewit	Chapter 17
2.	William Shakespeare	King John	Act V Scene VII
3.	George Eliot	Adam Bede	
4.	William Habington	Nox Nocti Indicat Scientam	
5.	Robert Burns	The Cotter's Saturday Night	
6.	Emily Dickinson	Nature: LX	

Congratulations to Ian Anderson for finding all of these quotations.

This month we take a look at measuring, an activity often viewed with some suspicion by poets and writers. The quotations are to be identified by reference to author and work. Solutions are invited to the Editor by 31st January 2017.

1. For Mercy, Courage, Kindness, Mirth,
   There is no measure upon earth.
   Nay, they wither, root and stem,
   If an end be set to them.

2. The Countess was not very exact at measurements, but it seemed to her that if Isabel should draw herself up she would be the taller spirit of the two.

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