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A sectional-Anosov connecting lemma

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Dedicated to the memory of Edward Norton Lorenz 1917–2008

Abstract. The Anosov flows on compact manifolds M satisfy the following property: if p, q are points such that for all positive ϵ there is a trajectory from a point ϵ-close to p to a point ϵ-close to q, then there is a point whose α-limit set is that of p and whose ω-limit set is that of q. Here we give a version of this property for sectional-Anosov flows, namely, vector fields inwardly transverse to the boundary whose maximal invariant set is sectional-hyperbolic. Indeed, if in addition M is three-dimensional and p has non-singular α-limit set, then there is a point whose α-limit set is that of p and whose ω-limit set is either a singularity or that of q.

1. *Introduction*

A fundamental theory in dynamical systems is concerned with the study of *Anosov flows*, namely, flows where there is a hyperbolic structure on the whole ambient manifold. Some generalizations of Anosov flows have been appeared in recent years as the *pseudo-Anosov flows* [**9**], where the hyperbolic structure is required not in the whole manifold but outside finitely many p-prong-type periodic orbits, or the *projectively Anosov flows* [**28**], where a dominated but not necessarily hyperbolic splitting is required instead. One could also mention the so-called *generalized hyperbolic attractors* [**26, 27**] but these are attractors for maps, not for flows.

In this paper we are concerned with the *sectional-Anosov flow* introduced in [**17**], one more generalization of Anosov flows where it is required that the flow be transverse to the boundary (if non-empty) and that the maximal invariant set of the flow be sectional-hyperbolic in the sense of [**16**]. They represent a broader class including not only Anosov flows or hyperbolic attractors but also the *geometric* and *multidimensional Lorenz attractors* [**1, 6, 10, 15**]. Remarkably the classes of sectional-Anosov and Anosov flows

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coincide in the boundaryless case, and so, one can expect that all properties of the latter flows can be extended to the former. This idea has been explored elsewhere as in [**17**], where the topology of surface transverse to sectional-Anosov flow on compact 3-manifolds were proved to satisfy some properties resembling Anosov flows; or in [**18**] where every recurrent point of a sectional-Anosov flow on a compact 3-manifold was approximated by periodic points or by points whose ω-limit set is a singularity. Here we look for a sectional-Anosov version of the following fact.

LEMMA 1.1. (Anosov connecting lemma) *If X is an Anosov flow on a compact manifold M and $p, q \in M$ satisfy that for all $\epsilon > 0$ there is a trajectory from a point ϵ-close to p to a point ϵ-close to q, then there is a point whose α-limit set is that of p and whose ω-limit set is that of q.*

It is natural to ask whether the sectional-Anosov flows satisfy the conclusion of this lemma as well. Nevertheless, consider the sectional-Anosov flow in the solid bi-torus depicted in Figure 1. It is a variation of an example in [**21**, Figure 2, p. 225] (details of the construction are given in [**18**, §4]). It follows from [**3**] that it has a periodic orbit O. Then, any point $p \in O$ and the equilibrium $q = \sigma_2$ in the figure satisfies the hypotheses but not the conclusion of this lemma.

However, it is possible to observe in this example that the following two properties hold:

- *p has non-singular α-limit set*; and
- *there is a point whose α-limit set is that of p and whose ω-limit set is the singularity σ_1 in the figure.*

In this paper we generalize this observation to all sectional-Anosov flows on compact 3-manifolds. Indeed, if p has non-singular α-limit set and $q \in M$ satisfies that for all $\epsilon > 0$ there is a trajectory from a point ϵ-close to p to a point ϵ-close to q, then there is a point whose α-limit set is that of p and whose ω-limit set is either a singularity or that of q. Interestingly, this alternative result is false without the non-singular hypothesis on the α-limit set of p. Let us state our result in a precise way.

Consider a smooth vector field X in M inwardly transverse to ∂M (if $\partial M \neq \emptyset$). Denote by X_t the flow generated by X in M. A subset $\Lambda \subset M$ is *singular* if it has a singularity and *invariant* if $\Lambda \subset M(X)$ and $X_t(\Lambda) = \Lambda$ for all $t \in \mathbb{R}$. The *maximal invariant set* of X is defined by

$$M(X) = \bigcap_{t>0} X_t(M).$$

Clearly $M(X) = M$ if $\partial M = \emptyset$.

If $x \in M$ we define its *ω-limit set* by $\omega(x) = \{y \in M \mid y = \lim_{n\to\infty} X_{t_n}(x)$ for some sequence $t_n \to \infty\}$ and if $x \in M(X)$ we define its *α-limit set* $\alpha(x) = \{y \in M \mid y = \lim_{n\to\infty} X_{t_n}(x)$ for some sequence $t_n \to -\infty\}$. We say that *there is a trajectory from a point x to a point y* if there is $t > 0$ such that $y = X_t(x)$.

Denote by $m(L)$ and $\mathrm{Det}(L)$ the minimum norm and the Jacobian of a linear operator L, respectively. A continuous invariant tangent bundle splitting $T_\Lambda M = E^s_\Lambda \oplus E^c_\Lambda$ over a compact invariant set Λ is *dominated* if $E^s_x \neq 0$ and $E^c_x \neq 0$ for every $x \in \Lambda$ and there are

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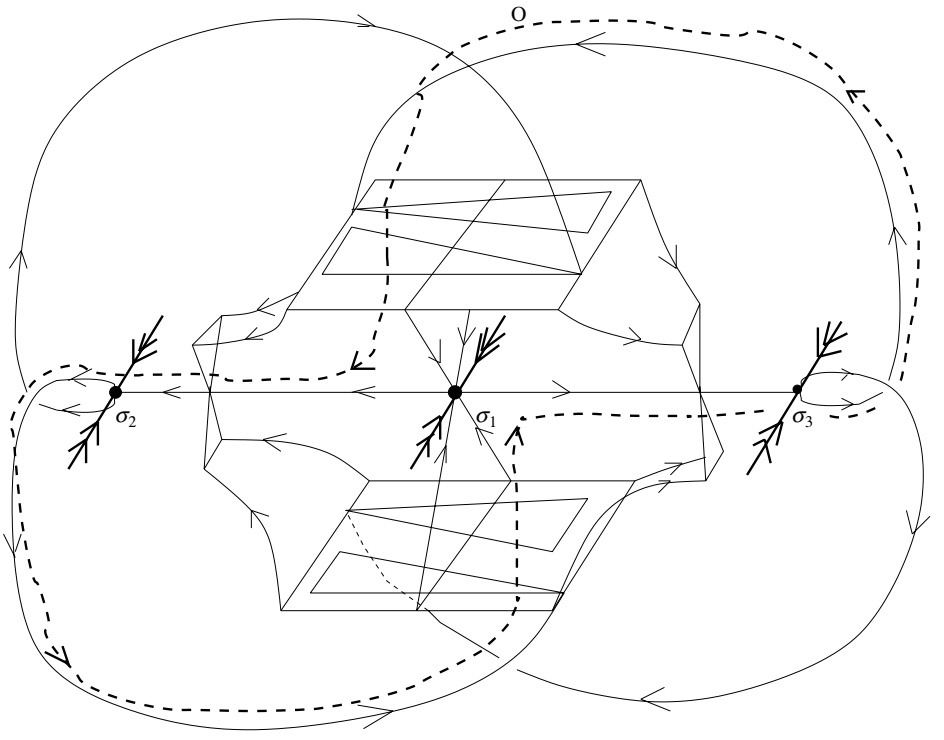


FIGURE 1. Sectional-Anosov flow in the solid bi-torus.

positive constants K, λ such that

$$\frac{\|DX_t(x)/E^s_x\|}{m(DX_t(x)/E^c_x)} \leq Ke^{-\lambda t} \quad \text{for all } t > 0, \text{ for all } x \in \Lambda. \tag{1}$$

We say that Λ is *partially hyperbolic* if it has a dominated splitting $T_\Lambda M = E^s_\Lambda \oplus E^c_\Lambda$ such that E^s_Λ is *uniformly contracting*, i.e.

$$\|DX_t(x)/E^s_x\| \leq Ke^{-\lambda t} \quad \text{for all } t > 0, \text{ for all } x \in \Lambda.$$

We say that the central direction E^c_Λ of a dominated splitting $T_\Lambda M = E^s_\Lambda \oplus E^c_\Lambda$ over Λ is *sectionally expanding* if

$$|\mathrm{Det}(DX_t(x)/L_x)| \geq K^{-1}e^{\lambda t},$$

for all $t > 0$, for all $x \in \Lambda$ and all two-dimensional subspaces $L_x \subset E^c_x$.

Definition 1.2. A compact invariant set is *sectional-hyperbolic* if it is partially hyperbolic with sectional expanding central direction and its singularities are hyperbolic. A vector field is a *sectional-Anosov flow* if its maximal invariant set is sectional-hyperbolic.

The sectional-hyperbolic sets on 3-manifolds are precisely the *singular-hyperbolic sets* defined elsewhere (see, e.g., [**24**]). Now we can state our main theorem.

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THEOREM A. (Sectional-Anosov connecting lemma) *If X is a sectional-Anosov flow on a compact 3-manifold M, $p \in M(X)$ has non-singular α-limit set and $q \in M$ satisfies that for all $\epsilon > 0$ there is a trajectory from a point ϵ-close to p to a point ϵ-close to q, then there is a point $x \in M$ such that $\alpha(x) = \alpha(p)$ and $\omega(x)$ is either a singularity or $\omega(q)$.*

This theorem is false without the hypothesis that $\alpha(p)$ be non-singular (see [**5**] for a counterexample). A direct corollary is the following. Given an invariant set H we define its *stable* and *unstable* manifolds by

$$W^s(H) = \{x \in M \mid \omega(x) \subset H\} \quad \text{and} \quad W^u(H) = \{x \in M \mid \alpha(x) \subset H\}$$

respectively. Denote by $Cl(\cdot)$ the closure operation and by $\mathrm{Sing}(X)$ the set of singularities of a vector field X.

COROLLARY 1.3. *If H is a compact invariant set of a sectional-Anosov flow on a compact 3-manifold, then $Cl(W^u(H)) \cap \mathrm{Sing}(X) \neq \emptyset$ if and only if $W^u(H) \cap W^s(\sigma) \neq \emptyset$ for some $\sigma \in \mathrm{Sing}(X)$.*

Proof. We only have to prove the direct implication since the converse is trivial. It follows from the hypothesis that there are a singularity q and a sequence $x_n \to q$ such that $\alpha(x_n) \subset H$ for all n. Take a closure point p of $\cup_n \alpha(x_n)$. Then, p and q satisfy the hypothesis of Theorem A and q is a singularity. So, there is $x \in M$ such that $\alpha(x) = \alpha(p)$ and $\omega(x)$ is a singularity σ. In particular, $x \in W^s(\sigma)$ and, since H is compact, we have $p \in H$ therefore $\alpha(x) = \alpha(p) \subset H$ so $x \in W^u(H)$. Hence, $x \in W^u(H) \cap W^s(\sigma)$ and the result follows. □

Applying this corollary to a periodic orbit O we obtain the following statement.

COROLLARY 1.4. *If O is a periodic orbit of a sectional-Anosov flow X on a compact manifold satisfying $Cl(W^u(O)) \cap \mathrm{Sing}(X) \neq \emptyset$, then there is $\sigma \in \mathrm{Sing}(X)$ such that $W^u(O) \cap W^s(\sigma) \neq \emptyset$.*

Such a corollary is closely related to [**20**, Theorem 4.1] which claims that for every sectional-Anosov flow on a compact 3-manifold M *with a dense orbit in* $M(X)$ one has $W^u(O) \cap W^s(\sigma) \neq \emptyset$ for every periodic orbit O and every $\sigma \in \mathrm{Sing}(X)$. Indeed, such a theorem is false for a counterexample can be obtained by taking the periodic orbit O and the equilibrium $\sigma = \sigma_2$ in Figure 1. Despite this, the main results in [**20**] are correct since they are based on [**20**, Theorem C], proof of which uses the conclusion of Corollary 1.4 only (see [**20**, p. 364]). Further applications of the above statement can be found in [**7, 19, 21**] or [**22**].

The paper is organized as follows. We give some preliminary results in §2. In §3 we state Theorem B which claims the existence of ∂-adapted singular cross sections close to the maximal invariant set of a sectional-Anosov flow. Its proof will be given in Appendix A for the sake of completeness. We prove Theorem A in §4.

2. *Preliminary facts*

Hereafter M denotes a compact 3-manifold with possibly non-empty boundary ∂M. The following is [**3**, Corollary 1, p. 5]. It will be used to construct stable foliations and to describe the singularities in a sectional-hyperbolic attracting set.

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THEOREM 2.1. *Let X be a sectional-Anosov flow on a compact 3-manifold M. If $T_{M(X)}MM = E^s_{M(X)} \oplus E^c_{M(X)}$ is the sectional-hyperbolic splitting of X, then $X(x) \subset E^c_x$ and $\dim(E^s_x) = 1$ for all $x \in M(X)$.*

If L is either a linear space or a submanifold of M we denote by $\dim(L)$ the dimension of L.

Recall that a compact invariant set H of a vector field X in M is *hyperbolic* if there is a continuous invariant splitting $T_H M = E^s_H \oplus E^X_H \oplus E^u_X$ such that E^s_H is contracting, E^u_H is expanding (i.e. contracting for the time-reversed vector field $-X$) and E^X_H is tangent to X. The invariant manifold theory [**12**] asserts that if H is a hyperbolic set and $p \in H$, then the sets

$$W^{ss}(p) = \left\{ x \in M \;\middle|\; \lim_{t\to\infty} d(X_t(x), X_t(p)) = 0 \right\}$$

and

$$W^{uu}(p) = \left\{ x \in M \;\middle|\; \lim_{t\to-\infty} d(X_t(x), X_t(p)) = 0 \right\}$$

are immersed submanifolds of M of class C^1. These manifolds are called *strong stable* and *strong unstable* manifolds of p. It turns out that $W^{ss}(p)$ and $W^{uu}(p)$ are tangent to the linear spaces E^s_p and E^u_p at p, respectively. In such a case we have

$$W^s(O(p)) = \bigcup_{t\in\mathbb{R}} W^{ss}(X_t(p)); \quad W^u(O(p)) = \bigcup_{t\in\mathbb{R}} W^{uu}(X_t(p)).$$

With this notation we have the following lemma.

LEMMA 2.2. *If X is a sectional-Anosov flow on M, then $M(X) \cap W^{ss}(\sigma) = \{\sigma\}$ for every $\sigma \in \mathrm{Sing}(X)$.*

Proof. By contradiction suppose that there is $\sigma \in \mathrm{Sing}(X)$ such that $M(X) \cap W^{ss}(\sigma) \neq \{\sigma\}$. Then, we can choose $x_0 \in M(X) \cap (W^{ss}(\sigma) \setminus \{\sigma\})$. Clearly $X(x_0) \in E^s_{x_0}$ and so $X(x_0) \in E^s_{x_0} \cap E^c_{x_0}$ by Theorem 2.1. However, the sum $T_{x_0}M = E^s_{x_0} \oplus E^c_{x_0}$ is direct so $X(x_0) = 0$ contradicting $x_0 \in W^{ss}(\sigma) \setminus \{\sigma\}$. This proves the result. □

Definition 2.3. A singularity is *Lorenz-like* if its eigenvalues λ_1, λ_2, λ_3 are real and satisfy $\lambda_2 < \lambda_3 < 0 < -\lambda_3 < \lambda_1$ up to some order.

All Lorenz-like singularities σ are hyperbolic. Then, the invariant manifolds $W^s(\sigma)$, $W^u(\sigma)$ are both well defined and tangent to the eigenspaces associated with the set of eigenvalues $\{\lambda_2, \lambda_3\}$ and $\{\lambda_1\}$, respectively. In particular, $W^s(\sigma)$ is two-dimensional and $W^u(\sigma)$ is one-dimensional. A further invariant manifold $W^{ss}(\sigma)$ tangent to the eigenvalue direction associated with $\{\lambda_2\}$ is well defined. If γ is a singularity with two positive eigenvalues of X we use the notation $W^{ss}(\gamma) = W^s(\gamma)$.

As observed in [**23**] every singularity of a sectional-Anosov flow X on a compact 3-manifold M with a dense orbit in $M(X)$ is Lorenz-like but not in general (see [**3**]). The following result gives sufficient conditions for a singularity of a sectional-Anosov flow to be Lorenz-like. Its proof is similar to that of [**3**, Theorem 4, p. 6].

THEOREM 2.4. *Every singularity in the closure of the unstable manifold of a periodic orbit of a sectional-Anosov flow on a compact 3-manifold is Lorenz-like.*

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Proof. Let X be a sectional-Anosov flow on a compact 3-manifold M and let O be a periodic orbit of X. Fix $\sigma \in Cl(W^u(O))$. If σ were not Lorenz-like, then $W^s(\sigma) = W^{ss}(\sigma)$ would be one-dimensional. As $\sigma \in Cl(W^u(O))$, and obviously $\sigma \notin O$, we would have that $Cl(W^u(O))$ also contains one of the two connected components of $W^s(\sigma) \setminus \{\sigma\}$. However, such connected components exit M by Lemma 2.2, which gives us a contradiction. Hence, σ is Lorenz-like and the proof follows. □

3. *∂-adapted singular cross section*

In this section we construct a special class of cross sections associated with Lorenz-like singularities called *singular cross sections*. Afterwards we define *∂-adapted singular cross sections* as those singular cross sections for which the positive orbits starting at their boundaries do not intersect their corresponding interior. The main result of this section, Theorem B, proves the existence of adapted singular cross sections close to a Lorenz-like singularity of a sectional-hyperbolic attracting set.

Let M be a compact connected 3-manifold. Given a submanifold W of M we denote by ∂W and $\mathrm{Int}(W)$ its boundary and interior, respectively. Let X be a C^1 vector field in M. A *cross section* of X is a compact codimension-one submanifold Σ for which there is a submanifold W with $\Sigma \subset \mathrm{Int}(W)$ such that X is transverse to $\mathrm{Int}(W)$.

Now we introduce the *return map* Π_Σ associated with a cross section Σ. Its domain is defined by

$$\mathrm{Dom}(\Pi_\Sigma) = \{x \in \Sigma \mid X_t(x) \in \Sigma \text{ for some } t > 0\}.$$

We define the return time

$$t_\Sigma : \mathrm{Dom}(\Pi_\Sigma) \to (0, \infty),$$
$$t_\Sigma(x) = \inf\{t > 0 \mid X_t(x) \in \Sigma\}$$

and the return map

$$\Pi_\Sigma : \mathrm{Dom}(\Pi_\Sigma) \subset \Sigma \to \Sigma,$$
$$\Pi_\Sigma(x) = X_{t_\Sigma(x)}(x).$$

Analogously we define the *holonomy map* $\Pi_{D,R}$ from a subset $D \subset M$ (disjoint from Σ) into a subset $R \subset \Sigma$. Its domain is

$$\mathrm{Dom}(\Pi_{D,R}) = \{x \in D \mid X_t(x) \in R \text{ for some } t > 0\},$$

the flight time is

$$t_{D,R} : \mathrm{Dom}(\Pi_R) \to (0, \infty),$$
$$t_{D,R}(x) = \inf\{t > 0 \mid X_t(x) \in R\}$$

and the holonomy map is

$$\Pi_{D,R} : \mathrm{Dom}(\Pi_{D,R}) \subset D \to R,$$
$$\Pi_{D,R}(x) = X_{t_{D,R}(x)}(x).$$

Observe that the holonomy map $\Pi_{D,\Sigma}$ may be discontinuous for the positive trajectory in between $x \in \mathrm{Dom}(\Pi_{D,\Sigma})$ and $\Pi_{D,\Sigma}(x)$ may hit $\partial\Sigma$ in an intermediary point z (see Figure 2). To avoid such intermediary points we introduce the following definition.

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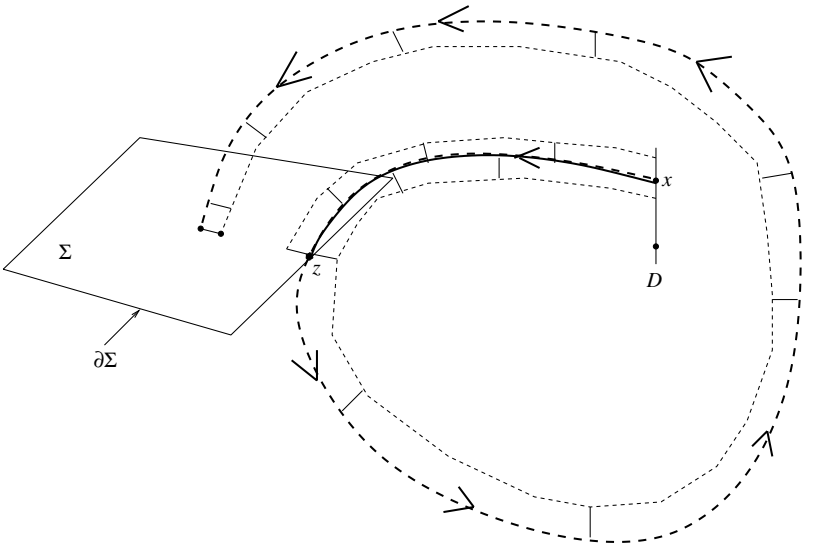


FIGURE 2. The intermediary point z (see the text).

Definition 3.1. Let Σ be a cross section of X. Given $R \subset \Sigma$ we say that Σ is *R-adapted* if the positive orbit of every point $z \in R$ does not intersect $\Sigma \setminus R$. In other words, $X_t(z) \notin \Sigma \setminus R$ for every $(z, t) \in R \times (0, \infty)$.

Note that Σ in Figure 2 is *not* $\partial\Sigma$-related for the positive orbit of the point $z \in \partial\Sigma$ in the figure intersects $\mathrm{Int}(\Sigma) = \Sigma \setminus \partial\Sigma$. The following is a simple but important sufficient condition for the continuity of holonomy maps using R-adapted cross sections.

LEMMA 3.2. *Let R be a closed subset of a cross section Σ such that Σ is R-adapted. If $D \subset M$ is disjoint from Σ, then $\mathrm{Dom}(\Pi_{D,\Sigma\setminus R})$ is open in D and $\Pi_{D,\Sigma\setminus R}$ is continuous.*

Proof. Denote by $B_\delta(x)$ the δ-ball in M centered at $x \in M$. We write $\Pi = \Pi_{D,\Sigma\setminus R}$ and $t(x) = t_{D,\Sigma\setminus R}(x)$ for simplicity. For all $A \subset M$ and $B \subset \mathbb{R}$ we define

$$X_B(A) = \{X_t(x) \mid (x, t) \in A \times B\}.$$

We write $X_B(x)$ instead of $X_B(\{x\})$ when $A = \{x\}$ consists of a single point x.

For all $x \in \mathrm{Dom}(\Pi)$ we have

$$X_{[0,t(x)]}(x) \cap R = \emptyset$$

since Σ is R-adapted and $X_{t(x)}(x) \in \Sigma \setminus R$. However, $X_{[0,t(x)]}$ and R are compact, so there is $\delta > 0$ so that

$$X_{[0,t(x)]}(B_\delta(x)) \cap R = \emptyset.$$

Now $\Sigma \setminus R$ is open in Σ (which is a cross section) and

$$X_{t(x)}(x) \in X_{[0,t(x)]}(B_\delta(x)) \cap (\Sigma \setminus R)$$

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thus we can select δ so that

$$D \cap B_\delta(x) \subset \mathrm{Dom}(\Pi) \quad \text{and} \quad X_{[0,t(y)]}(y) \cap R = \emptyset, \quad \text{for all } y \in D \cap B_\delta(x).$$

Then, x is an interior point of $\mathrm{Dom}(\Pi)$ and Π is continuous at x by the tubular flow-box theorem [**8**]. As $x \in \mathrm{Dom}(\Pi)$ is arbitrary we obtain the result. □

Note that $\Pi_{D,\Sigma\setminus R}$ above is not only continuous but also C^1 in the case when D is a submanifold satisfying $X(x) \notin T_x D$, for all $x \in D$.

We separate a particular case of Definition 3.1 to be used in the proof of Theorem A.

Definition 3.3. A *∂-adapted cross section* of X is a cross section Σ which is $\partial\Sigma$-adapted. Equivalently, the positive orbits of X through $\partial\Sigma$ do not meet $\mathrm{Int}(\Sigma)$.

In [**18**] an *adapted cross section* was defined as a cross section Σ satisfying $X_t(x) \notin \partial\Sigma$ for all $(x, t) \in \mathrm{Dom}(\Pi_\Sigma) \times (0, t_\Sigma(x)]$. Evidently a ∂-adapted cross section is an adapted cross section but the converse does not hold.

Now we restate Lemma 3.2 in terms of ∂-adapted cross sections.

COROLLARY 3.4. *If Σ is a ∂-adapted cross section and $D \subset M$ is disjoint from Σ, then $\mathrm{Dom}(\Pi_{D,\mathrm{Int}(\Sigma)})$ is open in D and $\Pi_{D,\mathrm{Int}(\Sigma)}$ is continuous.*

Proof. Apply Lemma 3.2 to $R = \partial\Sigma$. □

Next we recall the concept of a singular cross section associated with a Lorenz-like singularity σ of X. First note that it is possible to choose a coordinate system (x_1, x_2, x_3) around σ to which X is restricted, as shown in Figure 3. In this figure the boxes S^+, S^-, S^l, S^r correspond to the cross sections $\{x_3 = 1\}$, $\{x_3 = -1\}$, $\{x_1 = -1\}$, $\{x_1 = 1\}$ of X, respectively. We note that the set $\{w \in S^+ \cup S^- \mid O^+(w) \cap (S^l \cup S^r) = \emptyset\}$ is formed by two vertical curves $l^+ \subset S^+$, $l^- \subset S^-$ both contained in $W^s(\sigma)$.

Definition 3.5. A *singular cross section associated with σ* is a cross section S is equal to either S^+ or S^-. The curve l_S is equal to either l^+ or $= l^-$ will be called the *singular curve* of S. The vertical and horizontal boundaries of a singular cross section S will be denoted by $\partial^v S$ and $\partial^h S$, respectively.

As every singular cross section S is a cross section we can state the following definition.

Definition 3.6. A *∂-adapted singular cross section* is a singular cross section which is also a ∂-adapted cross section.

The following is the main technical result behind the proof of Theorem A.

THEOREM B. *Let X be a sectional-Anosov flow on a compact 3-manifold. Then, for every singular cross section associated with (and close to) a Lorenz-like singularity of X satisfying $M(X) \cap \partial^h S = \emptyset$ there is a ∂-adapted singular cross section $S^* \subset S$ such that:*

(1) $l_{S^*} = l_S$; *and*

(2) $\partial^h S^* \subset \partial^h S$.

We prove this later in Appendix A for the sake of completeness.

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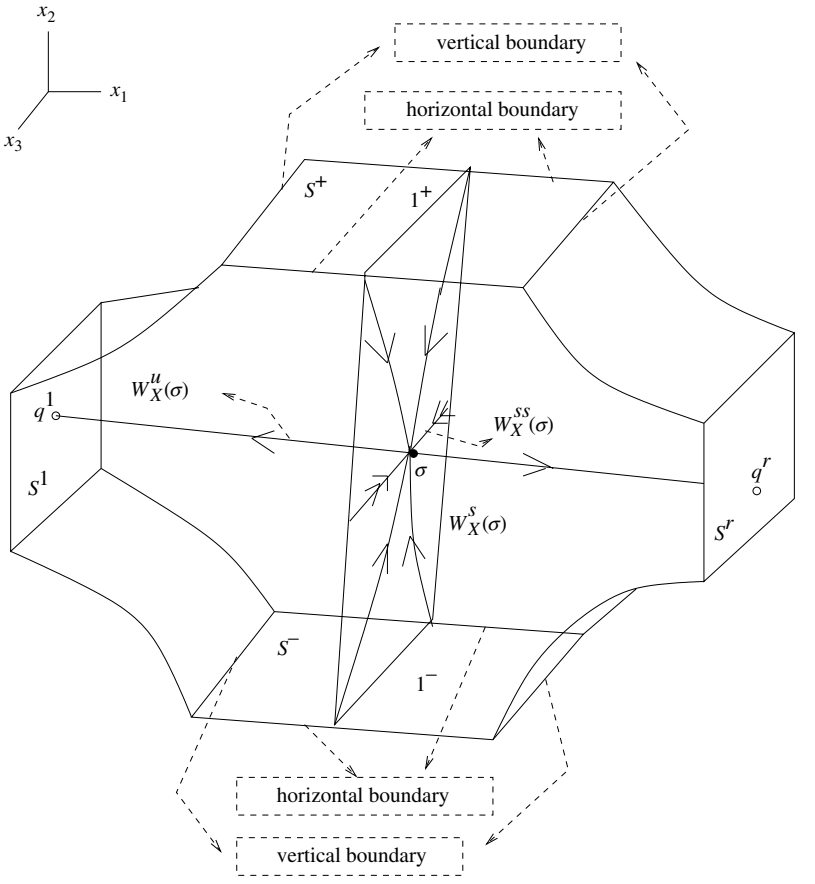


FIGURE 3. A coordinate system (x_1, x_2, x_3) around σ to which X is restricted.

4. *Proof of Theorem A*

Let X be a C^1 vector field in M inwardly transverse to ∂M. Denote by $O^+(x) = \{X_t(x) \mid t \geq 0\}$ the positive orbit of $x \in M$. The definition and theorem below can be found in [**4**].

Definition 4.1. Given $\Sigma \subset M$ we say that $q \in M$ satisfies Property $(\mathrm{P})_\Sigma$ if:

(1) $Cl(O^+(q)) \cap \Sigma = \emptyset$;

(2) there is an open arc $I \subset M$ with $q \in \partial I$ such that $O^+(x) \cap \Sigma \neq \emptyset$ for all $x \in I$.

THEOREM 4.2. *If $q \in M$ and $\omega(q)$ is sectional-hyperbolic, then $q \in M$ satisfies Property $(P)_\Sigma$ for some closed subset Σ if and only if $\omega(q)$ is a periodic orbit or a singularity.*

The next lemma is a direct consequence of the invariant manifold theory [**12**].

LEMMA 4.3. *Let H be a hyperbolic set of X. If $p, q \in H$ and there are sequences $z_n \in H \to p$, $t_n > 0$ such that $X_{t_n}(z_n) \to q$, then there is $x \in M$ such that $\alpha(x) = \alpha(p)$ and $\omega(x) = \omega(q)$.*

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Now we state a particular case of Theorem A. We say that $p \in M$ is *periodic* if its full orbit $O(p)$ is a periodic orbit of X.

From now on we consider the property (C) below for a pair $(p, q) \in M \times M$.

(C) *For every $\epsilon > 0$ there is a trajectory from a point ϵ-close to p to a point ϵ-close to q.*

PROPOSITION 4.4. *Let X be a sectional-Anosov flow on a compact 3-manifold M. If a periodic point p and a singularity $q = \sigma$ of X satisfy property (C), then there is $x \in M$ such that $\alpha(x) = \alpha(p)$ and $\omega(x)$ is a singularity.*

Proof. The proof uses Theorem B and the arguments of the proof of Theorem 4.1 of [**20**, p. 365]. Denote the periodic orbit containing p by $O = O(p)$.

Let U and V be fixed (but arbitrary) neighborhoods of p and σ, respectively. By property (C) there are $t > 0$ and $z \in X_t(U) \cap V$. Choosing U close to p we ensure that the strong stable manifold through $X_{-t}(z)$ intersects $W^u(O)$ at some point z'. On the other hand, $t > 0$ can be made large since $\sigma \notin O$. Then, as z and z' belong to the same strong stable manifold, we have that $X_t(z') \in V$. However, $z' \in W^u(O)$ which is invariant, hence $z' \in W^u(O) \cap V$. Thus, $W^u(O) \cap V \neq \emptyset$ and so $\sigma \in Cl(W^u(O))$ since V is arbitrary. Then, σ is Lorenz-like by Theorem 2.4.

By Lemma 2.2 we have $M(X) \cap W^{ss}(\sigma) = \{\sigma\}$. From this we can select as in [**22**, Lemma 2.2] a singular cross section S associated with and close to σ (hence disjoint from O) such that $M(X) \cap \partial^h S = \emptyset$ and $W^u(O) \cap \mathrm{Int}(S)$ has an accumulation point in the singular leaf l_S. Applying Theorem B to this section we can select two ∂-adapted singular cross sections Σ_0, $\Sigma_1 \subset S$ associated with σ with $\Sigma_0 \subset \mathrm{Int}(\Sigma_1)$.

In particular, $O \cap \Sigma_1 = \emptyset$ and so we can choose a fundamental domain D^u of $W^{uu}(p)$ disjoint from Σ_1 (see [**8**] for the definition of fundamental domain). However, $W^{uu}(p)$ is one-dimensional; therefore D^u can be identified with a compact interval $[a, b]$.

For simplicity we write $\Pi = \prod_{D^u, \mathrm{Int}(\Sigma_1)}$ for the holonomy map from D^u to $\mathrm{Int}(\Sigma_1)$.

We have that $\mathrm{Dom}(\Pi)$ is open and Π is continuous by Lemma 3.4 applied to $\Sigma = \Sigma_1$ and $D = D^u$.

Now, $W^u(O)$ has an accumulation point in l_S which is the singular leaf of Σ_1. Thus, we can assume without loss of generality that $a, b \in \mathrm{Dom}(\Pi)$. Moreover,

$$\Pi(a) = \Pi(b) \in \mathrm{Int}(\Sigma_0)$$

since b is in the positive orbit of a.

Define

$$q^* = \sup\{s \in [a, b] \mid [a, s] \subset \mathrm{Dom}(\Pi),\ \Pi([a, s]) \subset \Sigma_0 \text{ and } \Pi([a, s]) \text{ is connected}\}.$$

Clearly q^* is well defined and $a < q^*$ since $\Pi(a) \in \mathrm{Int}(\Sigma_0)$.

If $q^* \notin \mathrm{Dom}(\Pi)$, then q^* satisfies Property $(\mathrm{P})_{\Sigma_0}$ with $I = (a, q^*)$. In addition, $\omega(q^*)$ is sectional-hyperbolic since q^* lies in $M(X)$ which is sectional-hyperbolic.

Then, $\omega(q^*)$ is a periodic orbit or a singularity by Theorem 4.2.

However, $\omega(q^*)$ cannot be a periodic orbit (this follows from the argument in the last part of the proof of Lemma 5.4 in [**20**]). So, $q^* \in W^s(\sigma^*)$ for some singularity σ^*.

As $q^* \in D^u \subset W^u(O)$ we have that $x = q^*$ works. In this case we are done.

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Now assume that $q^* \in \mathrm{Dom}(\Pi)$. Since Π is continuous we have that $\Pi([a, q^*]) \subset \Sigma_0$ is a connected arc joining $\Pi(a)$ with $\Pi(q^*) \in \partial\Sigma_0$ transversal to the stable subbundle E^s of $M(X)$.

By taking the analogous point q^{**}, obtained by replacing a in the definition of q^* by b, we have that $\Pi([q^{**}, b])$ is a connected arc joining $\Pi(b)$ with $\Pi(q^{**}) \in \partial\Sigma_0$ transversal to E^s.

However, $\Pi(a) = \Pi(b)$, so the set

$$l = \Pi([a, q^*]) \cup \Pi([q^{**}, b])$$

is a connected arc transverse to E^s which joins the points $\Pi(q^*)$, $\Pi(q^{**}) \in \partial\Sigma_0$.

As Σ_0 is adapted we have $\Pi(q^*)$, $\Pi(q^{**}) \in \partial^v\Sigma_0$, so, l is indeed a connected arc jointing two points in $\partial^v\Sigma_0$.

As l is also transversal to E^s we conclude that l intersects the singular curve of Σ_0 at some point x.

As the singular curve belongs to $W^s(\sigma)$ and $l \subset W^u(O)$ we have $x \in W^s(\sigma) \cap W^u(O)$, thus x works. This proves the result. □

The next proposition extends the previous result to the case when $\alpha(p)$ is non-singular.

PROPOSITION 4.5. *Let X be a sectional-Anosov flow on a compact 3-manifold M. If $p \in M(X)$ and $q = \sigma \in \mathrm{Sing}(X)$ satisfy property (C) with $\alpha(p)$ non-singular, then there is $x \in M$ such that $\alpha(x) = \alpha(p)$ and $\omega(x)$ is a singularity.*

Proof. We have that $\alpha(p)$ is a hyperbolic set since it is non-singular [**23**]. Fix $y \in \alpha(p)$, i.e. there is a real number sequence $t_n \to \infty$ such that $X_{-t_n}(p) \to y$.

As the negative orbit of p remains close $\alpha(p)$ which is hyperbolic, we can apply graph transformed techniques [**11, 12**] to find $\epsilon > 0$ and a sequence of open intervals $I_n = (X_{t_n}(p) - \epsilon, X_{t_n}(p) + \epsilon) \subset W^{uu}(X_{-t_n}(p))$ converging to the open interval $I = (y - \epsilon, y + \epsilon) \subset W^{uu}(y)$.

On the other hand, as in the proof of Lemma 3.1 of [**7**, p. 652] we can choose a sequence of periodic points $p_n \to y$ whose strong unstable manifolds $W^{uu}(p_n)$ have uniformly large size. Moreover, $W^{uu}(p_n)$ approaches I as $n \to \infty$. Then, both $W^{uu}(p_n)$ and I_n approach the common interval I. This allows us to fix positive integers n_0, n_1 with the following property (to be applied twice below).

(Q) The stable manifold of every point close to $X_{-t_{n_0}}(p)$ intersects $W^{uu}(p_{n_1})$ and, conversely, the stable manifold of every point close to p_{n_1} intersects I_{n_0} (see Figure 4).

As the pair (p, σ) satisfies property (C) we have that the pair $(X_{t_{n_0}}(p), \sigma)$ also does. So, there are sequences $z_m \to X_{t_{n_0}}(p)$ and $t_m > 0$ such that $X_{t_m}(z_m) \to \sigma$. Then, property (Q) implies that there is a corresponding sequence $z'_m \in W^{uu}(p_{n_1})$ in the stable manifold of z_m. Then, $X_{t_m}(z'_m) \to \sigma$ and so the pair (p_{n_1}, σ) satisfies property (Q). Hence, Proposition 4.4 implies that there is x^* as in Figure 4 such that $\alpha(x^*) = \alpha(p_{n_1})$ and $\omega(x^*)$ is a singularity σ^*. By taking the backward orbit of x^* if necessary we can assume x^* to be close to p_{n_1}. Then, property (Q) once more implies that the stable manifold of x^*

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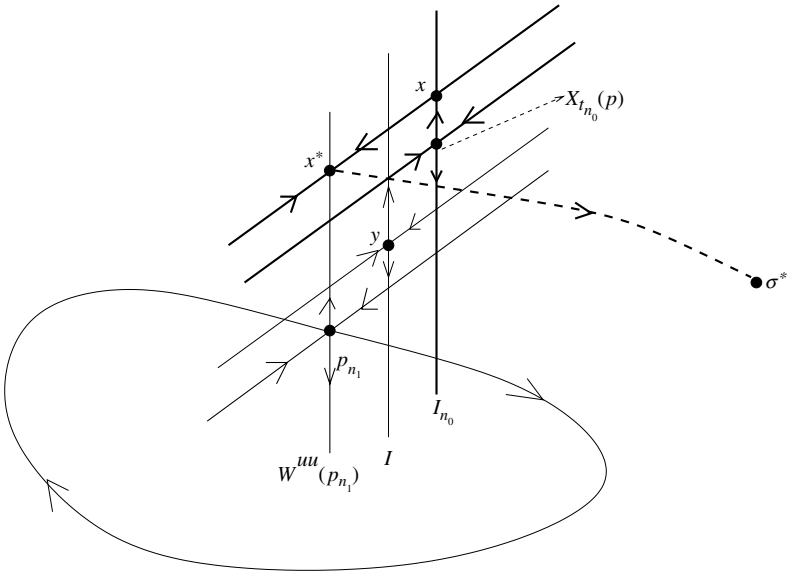


FIGURE 4. Illustration of property (Q) (see the text).

intersects I_{n_0} at some point x (see Figure 4). Hence, $\alpha(x) = \alpha(p)$ (for $I_{n_0} \subset W^{uu}(p)$) and $\omega(x) = \omega(x^*) = \sigma^*$ is a singularity. This proves the result. □

Proof of Theorem A. Let X a sectional-Anosov flow on a compact 3-manifold M and $p, q \in M(X)$ be such that $\alpha(p)$ is non-singular. Suppose that for every neighborhood U and V of p and q, respectively, there is $t > 0$ such that $X_t(U) \cap V \neq \emptyset$. We prove that there is $x \in M$ such that $\alpha(x) = \alpha(p)$ and $\omega(x)$ is either a singularity or $\omega(q)$.

For this we observe that the pair (p, q) satisfies property (C). If q belongs to the positive orbit of p, then $x = p$ satisfies the conclusion of the theorem. So, we can assume that q does not belong to the positive orbit of p.

If either $\omega(p)$ or $\omega(q)$ contains a singularity σ, then the pair (p, σ) satisfies property (C) so the existence of x follows from Proposition 4.5. If $\alpha(q)$ contains a singularity σ, then the continuity of the flow X_t and the fact that q is not in the positive orbit of p imply that the pair (p, σ) satisfies property (C). In such a case we are done by Proposition 4.5.

Hence, we can assume that the union $\omega(p) \cup \alpha(q) \cup \omega(q)$ is non-singular. In this case we prove that there is x such that $\alpha(x) = \alpha(p)$ and $\omega(x) = \omega(q)$. This is basically a consequence of Lemma 4.3.

To start with take an isolating block U of Λ. We can assume that U is *positively invariant*, namely, $X_t(U) \subset U$ for all $t \geq 0$. For all $\delta > 0$ we denote by $B_\delta(A)$ the open δ-ball centered at A.

As both $\alpha(p)$ and $\omega(p) \cup \alpha(q) \cup \omega(q)$ are non-singular we have that there is $\delta_1 > 0$ such that p and q belongs to H_1 defined by

$$H_1 = \bigcap_{t \in \mathbb{R}} X_t(U \setminus B_{\delta_1}(\mathrm{Sing}(X)).$$

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Clearly H_1 is a hyperbolic set since it is non-singular [**23**], so $W^{uu}(p)$ is a well-defined one-dimensional manifold containing p.

Now the pair (p, q) satisfies property (C) then there are sequences $z_n \to p$ and $t_n > 0$ such that $X_{t_n}(z_n) \to q$. We can take $t_n \to \infty$ since q is not in the positive orbit of p. However, the size of the strong stable manifold $W^{ss}(z_n)$ is bounded away from zero and $W^{uu}(p)$ is a one-dimensional manifold containing p. Then, we can assume that there is a sequence $z'_n \in W^{ss}(z_n) \cap W^{uu}(p)$ so that $z'_n \to p$. As $t_n \to \infty$ we obtain

$$X_{t_n}(z'_n) \to q. \tag{2}$$

Suppose for a while that for every $k \in \mathbb{N}$ there is $\sigma_k \in \mathrm{Sing}(X)$ such that

$$\sigma_k \in Cl\left(\bigcup_{n=k}^{\infty} O^+(z'_n)\right).$$

As the number of equilibria in Λ is finite we can assume that $\sigma = \sigma_k$ does not depend on k. As $z'_n \to p$ we conclude that the pair (p, σ) satisfies property (C) so Proposition 4.5 applies.

Then, we can assume that there are $k_0 \in \mathbb{N}$ and $0 < \delta_2 < \delta_1$ such that

$$\left(\bigcup_{n=k}^{\infty} O^+(z'_n)\right) \cap B_{\delta_2}(\mathrm{Sing}(X)) = \emptyset.$$

Observe that $O^+(z'_n) \subset U$ since U is positively invariant. This together with the above equality imply

$$O^+(z'_n) \subset U \setminus B_{\delta_2}(\mathrm{Sing}(X)) \quad \text{for all } n \geq k_0.$$

On the other hand, $z'_n \in W^{uu}(p)$ so $\alpha(z'_n) = \alpha(p)$ is non-singular. Then, as $z'_n \to p$, we conclude that there is $0 < \delta_3 < \delta_2$ such that

$$O^-_X(z'_n) \subset U \setminus B_{\delta_3}(\mathrm{Sing}(X)) \quad \text{for all } n \geq k_0,$$

where $O^-(z)$ is the negative orbit of z,

$$O^-(z) = \{X_t(z) \mid t \leq 0\}.$$

Consequently each z'_n (with $n \geq k_0$) as well as p and q belong to

$$H = \bigcap_{t \in \mathbb{R}} X_t(U \setminus B_{\delta_3}(\mathrm{Sing}(X))$$

which is hyperbolic for it is non-singular [**23**]. Then, (2) and Lemma 4.3 yield x such that $\alpha(x) = \alpha(p)$ and $\omega(x) = \omega(q)$ proving the result. □

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A. *Appendix*

In this appendix we give a proof of Theorem B. This will be done in Appendix A.5. Previously we present some preliminary results. Hereafter we fix sectional-Anosov flow X on a compact 3-manifold M. To simplify the notation we use $\Lambda = M(X)$.

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A.1. *Strong stable foliation.* Let $T_\Lambda M = E^s_\Lambda \oplus E^c_\Lambda$ be the sectional-hyperbolic splitting over Λ. By Theorem 2.1 one has:

(1) $\dim(E^s_x) = 1$ for all $x \in \Lambda$;

(2) $X(x) \in E^c_x$ for all $x \in \Lambda$.

It follows from the stable manifold theory [**12**] that E^s_Λ extends to a subbundle E^s_U defined in a neighborhood U of Λ. Furthermore, E^s_U is tangent to a C^0 one-dimensional *contracting* foliation $\mathcal{F}^{ss}$ defined in U. The last means that if $\mathcal{F}^{ss}_x$ denotes the leaf of $\mathcal{F}^{ss}$ containing $x \in U$, then $T_x\mathcal{F}^{ss}_x = E^s_x$, for all $x \in \Lambda$. We consider also the two-dimensional foliation $\mathcal{F}^s$ defined in $U \setminus \mathrm{Sing}(X)$ whose leaves are

$$\mathcal{F}^s_x = \bigcup_{t\in\mathbb{R}} \mathcal{F}^{ss}_{X_t(x)}.$$

Hereafter we fix such a neighborhood U. If $D \subset U$ is a cross section of X we can project $\mathcal{F}^s$ onto D to obtain a one-dimensional foliation $\mathcal{F}^s_D = \mathcal{F}^s \cap D$ in D whose leaves are the connected component of the leaves of $\mathcal{F}^s$ intersected with D.

A.2. *Stable foliation on singular cross sections.* Let $\sigma \in \Lambda$ be a Lorenz-like singularity of X. Let $S \subset U$ be a singular cross section associated with σ. Let $\mathcal{F}^s_S$ be the projection of $\mathcal{F}^{ss}$ in S defined in §A.1. Clearly l_S is a leaf of $\mathcal{F}^s_S$. The vertical boundary curves of S can be assumed to be leaves of $\mathcal{F}^s_S$. It turns out that the leaves' space of $\mathcal{F}^s_S$ is the compact interval I (equal to $[-1, 1]$ say). In particular, we can define a natural order '$<$' in the leaves' space I. If $L' < L''$ are leaves of $\mathcal{F}^s_S$, then there is a well-defined rectangle $S_{L',L''}$ in S satisfying the following properties:

- $\partial^v S_{L',L''} = L' \cup L''$;
- $\partial^h S_{L',L''} \subset \partial^h S$.

Furthermore, if $L' < l_S < L''$, then $S_{L',L''}$ is a singular cross section with singular curve $l_{S_{L',L''}} = l_S$. We denote by

$$\Pi_{L',L''} : \mathrm{Dom}(\Pi_{L',L''}) \subset S_{L',L''} \to S_{L',L''}$$

the return map $\Pi_{L',L''} = \Pi_{S_{L',L''}}$ induced by X in $S_{L',L''}$, where $\mathrm{Dom}(\cdot)$ stands for the domain operator.

A.3. *Cone fields and refinement.* In this appendix we define a cone field in S which is useful in proving Theorem B. For this we extend the central direction E^c of Λ to a (not necessarily invariant) sectional expanding direction, still denoted by E^c, on a neighborhood of Λ. Let $S \subset U$ be a singular cross section associated with a Lorenz-like singularity $\sigma \in \Lambda$ of X. It follows that E^c is also defined in S. This allows us to consider the line field F_S in S given by

$$F_x = T_xS \cap E^c_x \quad \text{for all } x \in S.$$

It is clear that F_S is transverse to $\mathcal{F}^s_S$. This line field induces a *cone field* in S. Indeed, fix $\alpha > 0$ small and for every $x \in S$ define the cone

$$C_\alpha(x) = \{v_x \in T_xS \mid \angle(v_x, F_x) \le \alpha\},$$

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where $\angle$ denotes the angle. The map

$$C_\alpha : x \in S \mapsto C_\alpha(x)$$

is a cone field with slope α in S centered at F_S. If the slope α is chosen to be small, then the cone field is transverse to the foliation $\mathcal{F}^s_S$. By a curve in S we mean a differentiable map $c : \mathrm{Dom}(c) \subset \mathbb{R} \to S$ whose domain $\mathrm{Dom}(c)$ is a compact interval. We denote by c the image set of c. We say that a curve *c is tangent to the cone field C_α* if

$$c'(t) \in C_\alpha(c(t)) \quad \text{for all } t \in \mathrm{Dom}(c).$$

Using a coordinate system (x_1, x_2, x_3) around σ we can define a coordinate system (x_1, y_2) in S such that

$$S = [-1, 1] \times [-1, 1].$$

With respect to this coordinate system we have that the leaves L of $\mathcal{F}^s_S$ are almost vertical curves, i.e.

$$L = \mathrm{Graph}(\phi) = \{(\phi(x_2), x_2) \mid x_2 \in [-1, 1]\},$$

where $\phi = \phi_L : [-1, 1] \to [-1, 1]$ is a C^1 map with uniformly small derivative (going to zero as $L \to l_S$). By using the dominance condition (1) of the sectional-hyperbolic splitting $E^s \oplus E^c$ we can also assume that the cone field C_α in S is almost horizontal, i.e. if c is a curve tangent to C_α, then

$$c = \mathrm{Graph}(\psi) = \{(x_1, \psi(x_1)) \mid x_1 \in \mathrm{Dom}(\psi)\}$$

where $\psi = \psi_c$ is a C^1 map with uniformly small derivative. We can use that coordinate system (x_1, x_2) in order to refine S in the following way. For every $\delta \in (0, 1]$ we define $x^-_\delta = (-\delta, 0)$ and $x^+_\delta = (\delta, 0)$. Denote by L^-_δ and L^+_δ the leaves of $\mathcal{F}^s_S$ containing x^-_δ, x^+_δ, respectively. We define

$$S_\delta = S_{L^-_\delta, L^+_\delta}.$$

Clearly $\{S_\delta \mid \delta \in (0, 1]\}$ is a family of singular cross sections with the same singular curve l_S. We denote by

$$\Pi_\delta = \Pi_{S_\delta}$$

the return map associated with the refinement S_δ.

If C_α is the cone field in S above, then C_α is defined in S_δ (for all δ) by restriction. Given $\epsilon > 0$ we can arrange S such that if $c = \mathrm{Graph}(\psi)$ is a curve tangent to C_α in S_δ and connecting the two connected components of $\partial^v S_\delta$, then $|\psi'| \le \epsilon$. It follows that $|c|$, the length of c, satisfies

$$(2 - \epsilon)\delta \le |c| \le (2 + \epsilon)\delta. \tag{3}$$

We can assume that the horizontal curve $h : [-1, 1] \to [-1, 1]$ given by

$$h(x_1) = (x_1, 0)$$

is tangent to C_α and, moreover,

$$|h/_J| = |J|,$$

for every interval $J \subset [-1, 1]$. In particular, the curves

$$h^+_\delta = h/_{(0,\delta]} \quad h^-_\delta = h/_{[-\delta,0)}$$

satisfy

$$|h^+_\delta| = |h^-_\delta| = \delta.$$

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A.4. *Return maps for singular cross sections.* We say that $B \subset S$ is $\mathcal{F}_S^s$*-invariant* if every leaf of $\mathcal{F}_S^s$ intersecting B is contained in B.

The following technical lemma summarizes the properties of the return maps Π_δ associated with the refinement S_δ that we need.

LEMMA A.1. *Let $S \subset U$ be a singular cross section associated with a Lorenz-like singularity σ of X satisfying*

$$\Lambda \cap \partial^h S = \emptyset \tag{4}$$

and $\{S_\delta\}$ be the refinement of S in §A.3. Then, there is $\delta_0 > 0$ such that if $0 < \delta \leq \delta_0$, then the return map Π_δ satisfies the following properties:

(1) $\mathrm{Dom}(\Pi_\delta)$ *is $\mathcal{F}_S^s$-invariant;*

(2) *for every leaf L of $\mathcal{F}_S^s$ contained in $\mathrm{Dom}(\Pi_\delta)$, there is a unique leaf $f(L)$ of $\mathcal{F}_S^s$ such that $\Pi_\delta(L) \subset f(L)$ and $\Pi_\delta/_L : L \to f(L)$ is continuous;*

(3) *there is a positive number λ (larger than four say) such that if $x \in \mathrm{Dom}(\Pi_\delta)$ and Π_δ is differentiable in x, then*

$$D\Pi_\delta(x)(C_\alpha(x)) \subset \mathrm{Int}(C_{\alpha/2}(\Pi_\delta(x)))$$

and

$$\|D\Pi_\delta(x) \cdot v_x\| \geq \lambda \cdot \|v_x\|,$$

for all $v_x \in C_\alpha(x)$;

(4) *if $L_\delta^- \leq L_0 < l_S$ is a leaf of $\mathcal{F}_S^s$ and $S_{l_S, L_\delta^+} \setminus l_S \subset \mathrm{Dom}(\Pi_{L_0, L_\delta^+})$, then*

$$\Pi_{L_0, L_\delta^+}(S_{l_S, L_\delta^+} \setminus l_S) \cap \partial^v S_{L_0, L_\delta^+} \neq \emptyset;$$

(5) *if $l_S < L_0 \leq L_\delta^+$ is a leaf of $\mathcal{F}_S^s$ and $S_{L_\delta^-, l_S} \setminus l_S \subset \mathrm{Dom}(\Pi_{L_\delta^-, L_0})$, then*

$$\Pi_{L_\delta^-, L_0}(S_{L_\delta^-, l_S} \setminus l_S) \cap \partial^v S_{L_\delta^-, L_0} \neq \emptyset.$$

Proof. The properties (1) to (3) were proved in [**3**] using the identity (4) in the statement. To prove (4) suppose by contradiction that it is not true. Then, there is a leaf $L_\delta^- \leq L_0 < l_S$ such that

$$\Pi_{L_0, L_\delta^+}(S_{l_S, L_\delta^+} \setminus l_S) \cap \partial^v S_{L_0, L_\delta^+} = \emptyset.$$

Consequently Π_{L_0, L_δ^+} is a C^1 embedding in $S_{l_S, L_\delta^+} \setminus l_S$ by the tubular flow-box theorem [**8**]. Now consider the curve $h_\delta^+ = h/_{(0,\delta]}$ defined at the end of §A.3. Since h_δ is tangent to C_α we have that $c = \Pi_{L_0, L_\delta^+}(h_\delta^+)$ is also tangent to C_α by property (3). Again by property (3), and the definition of h_δ, one has

$$|c| \geq \lambda |h_\delta^+| = \lambda \cdot \delta > 4\delta.$$

Then, the inequality (3) applied to c yields

$$4\delta < |c| \leq (2 + \epsilon)\delta$$

which is absurd for $\epsilon = 1/2$ (say). This proves the result. To prove property (5) we use a similar argument but with h_δ^- instead of h_δ^+. □

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A.5. *Proof of Theorem B.* Recall that X denotes a sectional-Anosov flow on a compact 3-manifold M and that $\Lambda = M(X)$. Let S be a singular cross section associated (and close) to a Lorenz-like singularity $\sigma \in \Lambda$ such that

$$\Lambda \cap \partial^h S = \emptyset.$$

If the singular curve l_S of S is not accumulated by periodic orbits, then [**18**, Theorem 3.9] implies that S satisfies Theorem B. (Indeed, [**18**, Lemma 3.8] implies that the closure $\overline{Q}$ of the adapted vertical band Q found in such a theorem is in fact a ∂-adapted cross section.)

Therefore, we can assume that the following hypothesis holds:

(H) *l_S is accumulated by periodic orbits.*

Since S is close to $\sigma \in \Lambda$ we can assume that $S \subset U$ where U is the domain of the strong stable foliation (§A.1). As $\Lambda \cap \partial^h S = \emptyset$ we have that S satisfies the identity (4) in Lemma A.1.

The following lemma reduces the proof of Theorem B to find leaves $L_0^- < l_S < L_0^+$ in $\mathcal{F}^s_S$ satisfying identity (5) below.

LEMMA A.2. *Assume that S is sufficiently close to σ. If there are leaves $L_0^- < l_S < L_0^+$ of $\mathcal{F}^s_S$ satisfying*

$$(L_0^- \cup L_0^+) \cap \left(\bigcup_{n \geq 1} \Pi^{-n}_{L_0^-, L_0^+}(\mathrm{Int}(S_{L_0^-, L_0^+})) \right) = \emptyset, \tag{5}$$

then the conclusion of Theorem B holds.

Proof. Define $S^* = S_{L_0^-, L_0^+}$. Then, $l_{S^*} = l_S$, $\partial^v S^* = L_0^- \cup L_0^+$ and $\partial^h S^* \subset \partial^h S$ by the definition of $S_{L_0^-, L_0^+}$.

Now we prove that S^* is ∂-adapted. Since S is close to σ we have that the return time defining $\Pi_{L_0^-, L_0^+}$ is uniformly large. Then, an intersection point between a positive orbit through S^* with ∂S^* is close to Λ hence far from $\partial^h S$ (recall $\Lambda \cap \partial^v S = \emptyset$). Consequently, all such intersection points actually belong to $\partial^v S^* = L_0^- \cup L_0^+$. Therefore, S^* is ∂-adapted since the positive orbits through $\partial^v S^*$ do not meet $\mathrm{Int}(S^*)$ because of identity (5).

Summarizing, S^* is ∂-adapted, $\partial^h S^* \subset \partial^h S$ and $l_{S^*} = l_S$. Then S^* satisfies Theorem B. □

To find L_0^-, L_0^+ satisfying the identity (5) we use the refinement S_δ of S in §A.3. The arguments below follow closely those used in the proof of Theorem 3.9 of [**18**].

Let δ_0 be as in Lemma A.1. *Hereafter we fix $0 < \delta < \delta_0$ and use the notation in Lemma A.1.* For simplicity we denote

$$S = S_\delta, \quad \Pi_S = \Pi_\delta, \quad L^- = L^-_\delta, \quad L^+ = L^+_\delta.$$

Then, the return map $\Pi_S : \mathrm{Dom}(\Pi_S) \subset S \to S$ satisfies the properties (1), (2) and (4) in Lemma A.1. In particular, by Lemma A.1(2), there is a one-dimensional map

$$f : \mathrm{Dom}(f) \subset I \to I$$

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such that $\Pi_S(L) \subset f(L)$ where $f(L)$ is a leaf of $\mathcal{F}^s_S$ and $I = [0, 1]$ is the leaves' space of $\mathcal{F}^s_S$.

Now we need the following auxiliary lemma.

LEMMA A.3. One *of the following* two *alternatives occurs:*

(1) *there is a leaf* $L^* < l_S$ *of* $\mathcal{F}^s_S$ *such that*

$$L^* \cap \left(\bigcup_{n\geq 1} \Pi^{-n}_{L^*,L^+}(\mathrm{Int}(S_{L^*,L^+})) \right) = \emptyset; \tag{6}$$

(2) *there is a leaf* $L^* > l_S$ *of* $\mathcal{F}^s_S$ *such that*

$$L^* \cap \left(\bigcup_{n\geq 1} \Pi^{-n}_{L^-,L^*}(\mathrm{Int}(S_{L^-,L^*})) \right) = \emptyset. \tag{7}$$

Proof. By property (H) we can assume that Π_S has a periodic leaf L'. We have the following subcases for the orbit $\{\Pi^n_S(L') \mid n \geq 1\}$ of L':

- $(\bigcup_{n\geq 1} \Pi^n_S(L')) \cap (l_S, L^+] = \emptyset$;
- $(\bigcup_{n\geq 1} \Pi^n_S(L')) \cap [L^-, l_S) = \emptyset$;
- $(\bigcup_{n\geq 1} \Pi^n_S(L')) \cap [L^-, l_S) \neq \emptyset$ and $(\bigcup_{n\geq 1} \Pi^n_S(L')) \cap (l_S, L^+] \neq \emptyset$.

In the first two subcases we simply choose L^* as the leaf realizing the minimal distance between l_S and the leaves in $\{\Pi^n_S(L') \mid n \geq 1\}$. With this choice we obtain (6) or (7) depending on whether the first or the second subcase holds. Now we consider the third subcase. Let L^*_1 be the leaf realizing the minimal distance between l_S and the leaves in $\{\Pi^n_S(L') \mid n \geq 1\}$ *contained in* $[L^-, l_S)$. Let L^*_2 be the leaf realizing the minimal distance between l_S and the leaves in $\{\Pi^n_S(L') \mid n \geq 1\}$ *contained in* $(l_S, L^+]$. With this choice we obtain (5) with $L^-_0 = L^*_1$ and $L^+_0 = L^*_2$. The lemma is proved. □

Hereafter we assume that the first alternative in Lemma A.3 holds since the proof for the second is analogous.

Under such an assumption we define

$$L^-_0 = L^*. \tag{8}$$

To define L^+_0 we consider the following cases:

$$S_{l_S,L^+} \setminus l_S \subset \mathrm{Dom}(\Pi_{L^-_0,L^+}) \quad \text{or} \quad S_{l_S,L^+} \setminus l_S \not\subset \mathrm{Dom}(\Pi_{L^-_0,L^+}).$$

If $S_{l_S,L^+} \setminus l_S \not\subset \mathrm{Dom}(\Pi_{L^-_0,L^+})$, then there is a leaf $l_S < L^{**} \leq L^+$ such that $L^{**} \cap \mathrm{Dom}(\Pi_{L^-_0,L^+}) = \emptyset$. So, we define $L^+_0 = L^{**}$. Hence (6) and (8) imply (5). Then, Theorem B follows from Lemma A.2.

Consequently we can assume

$$S_{l_S,L^+} \setminus l_S \subset \mathrm{Dom}(\Pi_{L^-_0,L^+}). \tag{9}$$

In particular, $\Pi_{L^-_0,L^+}(L^+)$ exists. We have two subcases for $\Pi_{L^-_0,L^+}(L^+)$:

$$\Pi_{L^-_0,L^+}(L^+) \subset \partial^v S_{L^-_0,L^+} \quad \text{or} \quad \Pi_{L^-_0,L^+}(L^+) \not\subset \partial^v S_{L^-_0,L^+}.$$

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If $\Pi_{L_0^-,L^+}(L^+) \subset \partial^v S_{L_0^-,L^+}$, we define $L_0^+ = L^+$. Hence (6) and (8) imply (5). Then, Theorem B follows from Lemma A.2.

Consequently we can assume $\Pi_{L_0^-,L^+}(L^+) \not\subset \partial^v S_{L_0^-,L^+}$ and so

$$\Pi_{L_0^-,L^+}(L^+) \subset \mathrm{Int}(S_{L_0^-,L^+}). \tag{10}$$

On the other hand, (9) and Lemma A.1-(4) imply

$$\Pi_{L_0^-,L^+}(S_{l_S,L^+} \setminus l_S) \cap \partial^v S_{L_0^-,L^+} \neq \emptyset.$$

Then, there is a leaf $l_S < L^{**} \leq L^+$ of $\mathcal{F}_S^s$ such that if we define

$$\partial^v S_{L_0^-,L^+} = L_0^- \cup L^+,$$

then

$$\Pi_{L_0^-,L^+}(L^{**}) \subset \partial^v S_{L_0^-,L^+}.$$

Observe that $L^{**} \neq L^+$ because of (10). Then $l_S < L^{**} < L^+$ and so the rectangle S_{L^{**},L^+} is well defined (see §A.2). By choosing the first leaf L^{**} with this property we can further assume that the map $\Pi_{L_0^-,L^+}/_{(S_{L^{**},L^+} \setminus L^{**})}$ is continuous (in fact C^1).

We have the following possibilities for $\Pi_{L_0^-,L^+}(L^{**})$:

$$\Pi_{L_0^-,L^+}(L^{**}) \subset L_0^- \quad \text{or} \quad \Pi_{L_0^-,L^+}(L^{**}) \subset L^+.$$

If $\Pi_{L_0^-,L^+}(L^{**}) \subset L_0^-$, then (6) implies (5) choosing L_0^- as in (8) and $L_0^+ = L^{**}$. Then, Theorem B follows from Lemma A.2. So, we can assume

$$\Pi_{L_0^-,L^+}(L^{**}) \subset L^+. \tag{11}$$

However, according to Lemma A.1(2), for every leaf L of $\mathcal{F}_S^s$ there is a unique leaf $f(L)$ such that

$$\Pi_{L_0^-,L^+}(L) \subset f(L).$$

This defines a one-dimensional map

$$f : \mathrm{Dom}(f) \subset I \to I,$$

where $I = [0, 1]$ is the leaves' space of $\mathcal{F}_S^s$.

Now (9) implies $(l_S, L^+] \subset \mathrm{Dom}(f)$ and (10) implies that $f(L^+) \subset \mathrm{Int}(I)$ and f is continuous in L^+. We also have that $f/_{(L^{**},L^+]}$ is *continuous* (because $\Pi_{L_0^-,L^+}/(S_{L^{**},L^+} \setminus L^{**})$ is continuous) and *monotone* (because $\mathrm{Dom}(\Pi_S)$ is $\mathcal{F}_S^s$-invariant by Lemma A.1(1)).

Therefore, the lateral limits below

$$f(L^+-) = \lim_{L \to (L^+)^-} f(L); \quad f(L^{**}+) = \lim_{L \to (L^{**})^+} f(L)$$

exist. Since f is continuous at L^+ we obtain

$$f(L^+-) = f(L^+) \in \mathrm{Int}(I). \tag{12}$$

Now we state the following result.

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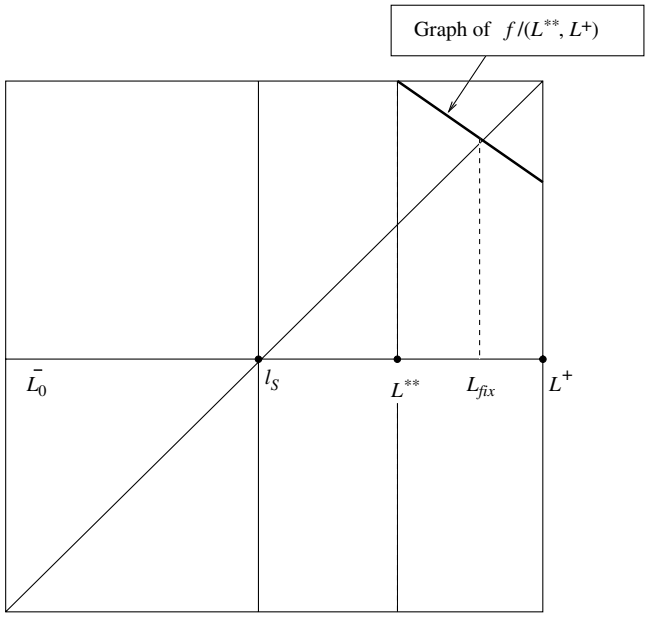


FIGURE A.1.

LEMMA A.4. *We know that* $f/_{(L^{**},L^{+}]}$ *has a fixed point.*

Proof. First we prove

$$f(L^{**}+) = L^{+}. \tag{13}$$

For this observe that $f(L^{**}) = L^{+}$ by (11). Then, (10) and the tubular flow-box theorem as in the proof of Proposition 2 in [**3**, p. 287] imply either

$$f(L^{**}+) = L^{+} \quad \text{or} \quad f(L^{**}+) = f(L^{+}).$$

However, $f(L^{**}+) = f(L^{+})$ together with (12) would imply

$$f(L^{**}+) = f(L^{+}-) = f(L^{+}),$$

which contradicts the fact that $f/(L^{**}, L^{+}]$ is monotone. Consequently (13) holds, and so (12) implies that $f/(L^{**}, L^{+}]$ is orientation-reversing. Hence, $f/(L^{**}, L^{+}]$ has a fixed point L_{fix} by (13) (see Figure A.1). □

With this lemma we conclude the proof of Theorem B as follows. Let L_{fix} be the fixed point in Lemma A.4. Choosing L_0^{-} as in (8) and $L_0^{+} = L_{\text{fix}}$ we obtain (5) from (6). Then, Lemma A.2 applies and the proof follows. □

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