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A nonabelian Fourier transform for tempered unipotent representations

Anne-Marie Aubert, Dan Ciubotaru and Beth Romano

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Abstract

We define an involution on the elliptic space of tempered unipotent representations of inner twists of a split simple p-adic group G and investigate its behaviour with respect to restrictions to reductive quotients of maximal compact open subgroups. In particular, we formulate a precise conjecture about the relation with a version of Lusztig's nonabelian Fourier transform on the space of unipotent representations of the (possibly disconnected) reductive quotients of maximal compact subgroups. We give evidence for the conjecture, including proofs for SL_n and PGL_n.

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1. Introduction

The local Langlands correspondence (LLC) predicts that the depth-zero irreducible smooth representations of a reductive p-adic group G are controlled by the geometry of the Langlands dual group $G^\vee$. This idea is most well developed for the class of unipotent representations (or representations with unipotent reduction) of G defined by Lusztig in [Lus95], which contains, in particular, all of the irreducible representations of G with vectors fixed under an Iwahori subgroup [IM65, KL87]. A correspondence for unipotent representations has now been defined that satisfies many of the desired properties of the LLC. These results have come from many years of developments, starting with the seminal papers of Kazhdan and Lusztig [KL87] for Iwahori-spherical representations of split adjoint groups, Lusztig [Lus95, Lus02] for unipotent representations of adjoint groups, Reeder [Ree02] for Iwahori-spherical representations of split groups of arbitrary isogeny and, finally, the recent papers of Solleveld [Sol23a, Sol23b] (building on [AMS18, AMS17, FOS20]) for all reductive p-adic groups. Yet not all desired properties of the LLC have been verified in full generality, and one of the main outstanding questions is to understand stability in L-packets.

To be more precise, assume for simplicity that G is the group of F-points of an absolutely simple, split connected reductive group over a non-Archimedean local field F with finite residue field. In the correspondence mentioned above, the irreducible representations of G (and of its inner forms) are partitioned into L-packets indexed by the conjugacy classes $G^\vee \cdot x$ for $x \in G^\vee$. From the perspective of abstract harmonic analysis, to understand these L-packets, the most basic case to consider is that of *tempered* L-packets, which correspond to the conjugacy classes $G^\vee \cdot x$ where the semisimple part of x is compact. Many representation-theoretic questions can be reduced further to the case of *elliptic tempered* L-packets, as defined by Arthur: these consist of tempered representations that are not irreducibly parabolically induced from proper parabolic subgroups. As mentioned above, while most of the predicted properties of unipotent L-packets have now been verified, it is not yet known which linear combinations of Harish-Chandra

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distribution characters of the representations in a given (elliptic) tempered L-packet are stable, in the sense of being constant on geometric (compact) semisimple conjugacy classes.

To approach this question, a natural first step is to consider the restriction of unipotent representations to maximal compact open subgroups, as in Mœglin–Waldspurger's tour de force [MW03], which tackles the question of stability for elliptic tempered L-packets for the group SO_{2n+1}. These maximal compact subgroups allow us to pass from representations of p-adic groups to unipotent representations of certain finite reductive groups, which have a rich structure (see, e.g., [Luz84a]). In particular, while the characters of irreducible representations of a finite connected reductive group do not have good intrinsic stability properties, Lusztig's *almost characters*, certain class functions defined in terms of traces of character sheaves, do. The transition matrix between characters and almost characters is Lusztig's famous *nonabelian Fourier transform* [Luz84a, Lus18]. If we can lift this Fourier transform to the setting of p-adic groups, we might be able to lift stability properties of combinations of almost characters of finite reductive groups, as in [MW03, Theorem 4.3].

With this idea in mind, in this paper, we formulate a conjecture that relates a nonabelian Fourier transform for pure inner twists of a (possibly disconnected) finite reductive group and an elliptic Fourier transform $\mathrm{FT}^\vee_{\mathrm{ell}}$ (cf. [CO17, Ciu20]) for pure inner twists of G. In addition to Mœglin and Waldspurger's work on the elliptic representations of the special orthogonal groups [MW03, Wal18], our approach is also inspired by Lusztig's articles proposing a theory of almost characters for p-adic groups [Lus15, Lus14]. We are also influenced by Reeder's [Ree01] and Waldspurger's [Wal07] ideas relating the classification of elliptic tempered unipotent representations and the geometry of $G^\vee$.

We need two main innovations to formulate a precise conjecture. To understand the first, note that if we take a maximal compact open subgroup K of G with reductive quotient $\overline{K}$, then $\mathrm{FT}^\vee_{\mathrm{ell}}$ does not necessarily induce a well-defined linear map on the unipotent representation space for $\overline{K}$. Instead we must look at maximal compact open subgroups in all pure inner twists of G at the same time, and so we form the space $\mathcal{C}(G)_{\mathrm{cpt,un}}$ defined below. The reductive quotients $\overline{K}$ are not necessarily connected, and so the second innovation is to extend the definition of Lusztig's nonabelian Fourier transform to disconnected finite reductive groups. To do this, we must look at all pure inner twists of a disconnected finite reductive group, and the Fourier transform will mix the corresponding representation spaces. These two ideas are related: for every pure inner twist H of $\overline{K}$, there is a pure inner twist G' of G and a maximal compact open subgroup K' of G' such that the reductive quotient $\overline{K}'$ is H.

1.1 Main results

We now describe our work in more detail. As above, let us assume that $\mathbf{G}$ is a simple, split group over F and $G = \mathbf{G}(F)$. Let $\mathrm{InnT}^p(G)$ denote the set of equivalence classes of pure inner twists of G. Then the LLC (see § 4) states that the L-packets of irreducible tempered unipotent representations of the groups $G' \in \mathrm{InnT}^p(G)$ are in one-to-one correspondence with $G^\vee$-conjugacy classes of elements $x = su \in G^\vee$ (Jordan decomposition) such that s is compact. The elements in the L-packet are parametrized by irreducible representations ϕ of the group of components $A_{G^\vee}(x)$ of the centralizer of x in $G^\vee$. Hence, an L-packet is a collection $\{\pi(su,\phi) \mid \phi \in \widehat{A_{G^\vee}(su)}\}$. Let Γ_u denote the reductive part of the centralizer of u in $G^\vee$. In [Wal18, Ciu20], one considered the set $\mathcal{Y}(\Gamma_u)$ of pairs $(s,h) \in \Gamma_u^2$ of commuting semisimple elements and the subset $\mathcal{Y}(\Gamma_u)_{\mathrm{ell}}$ of elliptic pairs (see § 8.3). These will play a role below in the Langlands parametrization.

Each group $G' \in \mathrm{InnT}^p(G)$ has a finite collection of conjugacy classes of maximal compact open subgroups $\max(G')$. These are classified in terms of the theory of [BT72, IM65] (see § 7).

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A compact group $K' \in \max(G')$ has a finite quotient $\overline{K}'$ that is the group of k-points of a (possibly disconnected) reductive group over a finite field k. Write $R_{\mathrm{un}}(\overline{K}')$ for the $\mathbb{C}$-vector space spanned by the irreducible unipotent representations of $\overline{K}'$. As mentioned above, for connected finite reductive groups, Lusztig [Luz84a] defined the nonabelian Fourier transform, which is the change-of-basis matrix between the basis of irreducible unipotent characters and the basis of unipotent almost characters. This is recalled in § 5. We need to define an extension of this map to disconnected finite groups in the spirit of [Lus86] and [DM90, § 5]. To fit with our picture, we define a nonabelian Fourier transform for the representations of the pure inner twists of the finite (possibly disconnected) reductive group $\overline{K}$, where $K \in \max(G)$. See § 6. The point is that this transform gives an involution

$$\mathrm{FT}_{\mathrm{cpt,un}} \colon \mathcal{C}(G)_{\mathrm{cpt,un}} \to \mathcal{C}(G)_{\mathrm{cpt,un}}, \tag{1.1}$$

on the space

$$\mathcal{C}(G)_{\mathrm{cpt,un}} = \bigoplus_{G' \in \mathrm{Inn T}^p(G)} \bigoplus_{K' \in \max(G')} R_{\mathrm{un}}(\overline{K}'),$$

which we can think of as the sum over $K \in \max(G)$ of the unipotent representation spaces of the pure inner twists of $\overline{K}$. See (7.2) and Definition 7.1. It is important to note that, in general, $\mathrm{FT}_{\mathrm{cpt,un}}$ mixes the pure inner twists of a given $\overline{K}$.

Since parabolic induction of characters is generally well understood, of particular interest is the space of elliptic (unipotent) tempered representations for all pure inner twists

$$\mathcal{R}^p_{\mathrm{un,ell}}(G) = \bigoplus_{G' \in \mathrm{Inn T}^p(G)} \overline{R}_{\mathrm{un}}(G')$$

(see § 9.1). Generalizing [Ree01], we prove the following theorem.

Theorem 1.1 (Theorem 11.1). *Suppose that G is split and adjoint. The local Langlands correspondence induces an isometric isomorphism*

$$\overline{\mathsf{LLC}^p}_{\mathrm{un}} : \bigoplus_u \mathbb{C}[\mathcal{Y}(\Gamma_u)_{\mathrm{ell}}]^{\Gamma_u} \longrightarrow \mathcal{R}^p_{\mathrm{un,ell}}(G), \ (s,h) \mapsto \Pi(u,s,h), \tag{1.2}$$

where the left-hand side has a natural elliptic inner product while the right-hand side is endowed with the Euler–Poincaré product. The element u ranges over representatives of unipotent conjugacy classes in $G^\vee$ and $\Pi(u,s,h)$ is defined in (9.9).

We remark that $\Pi(u,s,1)$ is expected to be the stable combination of characters in the L-packet, while in general $\Pi(u,s,h)$ are expected to satisfy the endoscopic identities.

The proof of Theorem 11.1 in § 11 applies in more generality, for example for Iwahori-spherical representations of groups of arbitrary isogeny (see § 11.4). Since the left-hand side has an obvious involution given by the flip $(s,h) \to (h,s)$, this defines an involution, the dual elliptic nonabelian transform

$$\mathrm{FT}^\vee_{\mathrm{ell}} \colon \mathcal{R}^p_{\mathrm{un,ell}}(G) \to \mathcal{R}^p_{\mathrm{un,ell}}(G). \tag{1.3}$$

We note that $\mathrm{FT}^\vee_{\mathrm{ell}}$ mixes representations of the pure inner twists of G. We expect that there is a commutative diagram as follows.

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Conjecture 1.2 (Conjecture 9.7). Up to certain roots of unity (see Remark 9.8), the following diagram commutes.

$$\begin{array}{ccc} \mathcal{R}^p_{\mathrm{un,ell}}(G) & \xrightarrow{\mathrm{FT}^\vee_{\mathrm{ell}}} & \mathcal{R}^p_{\mathrm{un,ell}}(G) \\ \Big\downarrow{\scriptstyle \mathrm{res}_{\mathrm{cpt,un}}} & & \Big\downarrow{\scriptstyle \mathrm{res}_{\mathrm{cpt,un}}} \\ \mathcal{C}(G)_{\mathrm{cpt,un}} & \xrightarrow[\mathrm{FT}_{\mathrm{cpt,un}}]{} & \mathcal{C}(G)_{\mathrm{cpt,un}} \end{array}$$

Here the vertical arrows are defined by taking invariants by the pro-unipotent radicals of maximal compact subgroups.

It is also natural to expect that the images of irreducible elliptic tempered characters under $\mathrm{FT}^\vee_{\mathrm{ell}}$ are the 'almost characters' (on elliptic elements) defined in [Lus15]: see, for comparison, [Lus14, Conjecture 2.2(c)].

Conjecture 1.2 is a generalization of [Ciu20, Conjecture 1.3] with an important difference: we remark that the role of maximal compact subgroups (rather than maximal parahoric subgroups) and, hence, of a Fourier transform for pure inner twists of disconnected finite reductive groups in the conjecture is essential for treating all pure inner twists of G. We verify the conjecture in some examples. In particular, we have the following theorem.

Theorem 1.3. *If* $\mathbf{G} = \mathsf{SL}_n$ *or* PGL_n, *Conjecture 1.2 holds.*

See §§ 13 and 14. We also verify the conjecture for $\mathbf{G} = \mathsf{Sp}_4$ (§ 12). The results of Waldspurger [Wal18] show that this conjecture holds when $\mathbf{G} = \mathsf{SO}_{2n+1}$.

In future work, we will consider a generalization of Conjecture 1.2 to the space of compact/rigid tempered representations defined in [CH17, CH21].

1.2 Structure of the paper

In §§ 2, 3, and 4, we review relevant background about inner twists of p-adic groups, the generalized Springer correspondence, and the LLC. In § 5, we recall Lusztig's parametrization of unipotent representations of a connected reductive group over a finite field and the definition of the nonabelian Fourier transform on the space spanned by these representations. We then extend Lusztig's parametrization: for the (possibly disconnected) groups $\overline{K}$ that arise as reductive quotients of subgroups $K \in \max(G)$ as defined above, we parametrize the union over all pure inner twists $\overline{K}'$ of $\overline{K}$ of the set of unipotent representations of $\overline{K}'$, and we then define a nonabelian Fourier transform on the space spanned by these representations (see § 6).

In § 7, we return to the setting of p-adic groups. We review the parametrization of maximal compact open subgroups of $G' \in \mathrm{Inn}\mathrm{T}^p(G)$, under the assumption G is F-split. We define the space $\mathcal{C}(G)_{\mathrm{cpt,un}}$ in terms of these subgroups, and we use the Fourier transform of § 6 to define an involution $\mathrm{FT}_{\mathrm{cpt,un}}$ on $\mathcal{C}(G)_{\mathrm{cpt,un}}$. In § 8, we review the definition of $\mathcal{Y}(\Gamma)_{\mathrm{ell}}$ for a complex reductive group Γ. We also review the definition of the elliptic pairing on the Grothendieck group of a finite group.

Section 9 contains the conjectures outlined above. We first review the Euler–Poincaré pairing and state Conjecture 9.1, which predicts that the LLC induces an isometric isomorphism at the level of elliptic spaces. We then define a restriction map $\mathrm{res}_{\mathrm{cpt,un}} \colon \mathcal{R}^p_{\mathrm{un,ell}}(G) \to \mathcal{C}(G)_{\mathrm{cpt,un}}$ and state Conjecture 9.7, which predicts that the elliptic nonabelian Fourier transform $\mathrm{FT}^\vee_{\mathrm{ell}}$ is

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compatible with $\mathrm{FT}_{\mathrm{cpt,un}}$ under $\mathrm{res}_{\mathrm{cpt,un}}$. We give evidence for this conjecture in Proposition 9.10, which considers linear combinations of twists of Steinberg representations.

In § 10, we present an alternative definition of the elliptic nonabelian Fourier transform motivated by Lusztig's pairing [Lus14, § 1.3].

In § 11, we prove Conjecture 9.1 in the case when G is simple, split, and adjoint. In § 11.4, we indicate how the proof can be extended to the non-adjoint case. In the final three sections, we verify the conjectures for explicit examples: in § 12, we consider the group $\mathsf{Sp}_4(F)$; in § 13, we consider $\mathsf{SL}_n(F)$; and in § 14, we consider $\mathsf{PGL}_n(F)$.

1.3 Notation and conventions

Given a complex Lie group $\mathcal{G}$, we write $\mathrm{Z}_{\mathcal{G}}$ for the center of $\mathcal{G}$. Given $x \in \mathcal{G}$, we write $\mathrm{Z}_{\mathcal{G}}(x)$ for the centralizer of x in $\mathcal{G}$. Similarly, if H is a subgroup of $\mathcal{G}$, we write $\mathrm{Z}_{\mathcal{G}}(H)$ for the centralizer of H in $\mathcal{G}$, and if φ is a homomorphism with image in $\mathcal{G}$, we write $\mathrm{Z}_{\mathcal{G}}(\varphi)$ for $\mathrm{Z}_{\mathcal{G}}(\mathrm{im}\ \varphi)$. We write $\mathcal{G}^\circ$ for the identity component of $\mathcal{G}$. If $x \in \mathcal{G}$, we write $A_{\mathcal{G}}(x) = \mathrm{Z}_{\mathcal{G}}(x)/\mathrm{Z}_{\mathcal{G}}(x)^\circ$ for the component group of $\mathrm{Z}_{\mathcal{G}}(x)$. If $u \in \mathcal{G}$ is unipotent, we write Γ_u for the reductive part of $\mathrm{Z}_{\mathcal{G}}(u)$. Given a torus $\mathcal{T}$, we write $X^*(\mathcal{T})$ for the character group of $\mathcal{T}$.

Given a finite group A, we write $\widehat{A}$ for the set of irreducible characters of A, and we write $R(A)$ for the $\mathbb{C}$-vector space with basis given by (isomorphism classes of) irreducible representations of A. Given a finite set S, we write $\mathbb{C}[S]$ for the $\mathbb{C}$-vector space of functions $S \to \mathbb{C}$.

2. Recollection on inner twists

2.1 Inner twists

Let F be a non-Archimedean local field with finite residue field $k_F = \mathbb{F}_q$. We denote by $\mathfrak{o}_F$ the ring of integers of F. Let F_{s} be a fixed separable closure of F, and let Γ_F denote the Galois group of F_{s}/F. Let $F_{\mathrm{un}} \subset F_{\mathrm{s}}$ be the maximal unramified extension of F. Let Frob be the geometric Frobenius element of $\mathrm{Gal}(F_{\mathrm{un}}/F) \simeq \widehat{\mathbb{Z}}$, i.e. the topological generator that induces the inverse of the automorphism $x \mapsto x^q$ of k_F. We denote by $\mathrm{Fr}_{\mathbf{G}}$ the action of Frob on a connected reductive F-group $\mathbf{G}$. We now review definitions related to inner twists and pure inner twists of a p-adic group. For details see, e.g., [Vog93, § 2], [Kal16, § 2] and [ABPS17b, § 1.3]. (Note that [Vog93] uses the term 'pure rational form' for what we call a pure inner twist.)

Let $G = \mathbf{G}(F)$. Write $\mathrm{Inn}(\mathbf{G})$ for the group of inner automorphisms of $\mathbf{G}$. Recall that given an algebraic group $\mathbf{H}$ over F, an isomorphism $\alpha\colon \mathbf{H} \to \mathbf{G}$ defined over F_s determines a 1-cocycle

$$\gamma_\alpha\colon \begin{array}{ccc} \Gamma_F & \to & \mathrm{Aut}(\mathbf{G}) \\ \sigma & \mapsto & \alpha\sigma\alpha^{-1}\sigma^{-1}. \end{array} \tag{2.1}$$

An *inner twist* of G consists of a pair (H, α), where $H = \mathbf{H}(F)$ for some connected reductive F-group $\mathbf{H}$, and $\alpha\colon \mathbf{H} \xrightarrow{\sim} \mathbf{G}$ is an isomorphism of algebraic groups defined over F_s such that $\mathrm{im}\ (\gamma_\alpha) \subset \mathrm{Inn}(\mathbf{G})$. Two inner twists $(H, \alpha), (H', \alpha')$ of G are *equivalent* if there exists $f \in \mathrm{Inn}(\mathbf{G})$ such that

$$\gamma_\alpha(\sigma) = f^{-1}\gamma_{\alpha'}(\sigma)\ \sigma f \sigma^{-1} \quad \text{for all } \sigma \in \Gamma_F. \tag{2.2}$$

Denote the set of equivalence classes of inner twists of G by $\mathrm{InnT}(G)$.

An inner twist of G is the same thing as an inner twist of the unique quasi-split inner form $G^* = \mathbf{G}^*(F)$ of G. Thus the equivalence classes of inner twists of G are parametrized by the Galois cohomology group $H^1(F, \mathrm{Inn}(\mathbf{G}^*))$:

$$\mathrm{InnT}(G) \longleftrightarrow H^1(F, \mathrm{Inn}(\mathbf{G}^*)).$$

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Example 2.1. For $G = \mathsf{SL}_n(F)$, there is a one-to-one correspondence

$$\mathrm{InnT}(\mathsf{SL}_n(F)) \longleftrightarrow \mathbb{Z}/n\mathbb{Z}. \tag{2.3}$$

This is given as follows. Let r be an integer mod n and let $m = \gcd(r, n)$. Then $n = dm$ and r/m is coprime to d. Therefore, there exists a division algebra $D_{d,r/m}$, central over F and of dimension $\dim_F D_{d,r/m} = d^2$. The corresponding inner twist is $\mathsf{SL}_m(D_{d,r/m})$.

A *pure inner twist* of G is a triple (H, α, z), where (H, α) is an inner twist and $z \in Z^1(F, G)$ such that $\alpha^{-1} \circ \gamma(\alpha) = \mathrm{Ad}(z(\gamma))$ for any $\gamma \in \Gamma_F$ (see [Kal16, § 2.3]). When G splits over an unramified extension of F such a cocycle is determined by the image $u := z(\mathrm{Frob}) \in G$. The corresponding inner twist of $\mathbf{G}$ is then defined by the functorial image $z_{\mathrm{ad}} \in Z^1(F, \mathrm{Inn}(\mathbf{G}^*))$ of z. This pure inner twist is defined by the twisted Frobenius action Fr_u on $\mathbf{G}$ given by $\mathrm{Fr}_u = \mathrm{Ad}(u) \circ \mathrm{Fr}_{\mathbf{G}}$.

In cohomological terms, the short exact sequence

$$1 \longrightarrow \mathrm{Z}_{\mathbf{G}^*} \longrightarrow \mathbf{G}^* \longrightarrow \mathrm{Inn}(\mathbf{G}^*) \longrightarrow 1$$

induces a map in cohomology $\mathrm{H}^1(F, \mathrm{Inn}(\mathbf{G}^*)) \to \mathrm{H}^2(F, \mathrm{Z}_{\mathbf{G}^*})$. An inner twist of $\mathbf{G}^*$ has a corresponding pure inner twist if and only if the corresponding element of $\mathrm{H}^2(F, \mathrm{Z}_{\mathbf{G}^*})$ is trivial [Vog93, Lemma 2.10]. Denote by $\mathrm{InnT}^p(G^*)$ the set of equivalence classes of pure inner twists of $\mathbf{G}^*$. We have [Vog93, Proposition 2.7]

$$\mathrm{InnT}^p(G^*) \longleftrightarrow \mathrm{H}^1(F, \mathbf{G}^*). \tag{2.4}$$

Example 2.2. If $\mathbf{G}^*$ is semisimple adjoint, every inner twist corresponds to a unique pure inner twist: $\mathrm{InnT}^p(G^*) = \mathrm{InnT}(G^*)$. If $\mathbf{G}^*$ is semisimple and simply connected, $\mathrm{H}^1(F, \mathrm{Inn}(\mathbf{G}^*)) \cong \mathrm{H}^2(F, \mathrm{Z}_{\mathbf{G}^*})$ and therefore there is only one class of pure inner twists, the quasi-split form, $\mathrm{InnT}^p(G^*) = \{G^*\}$. When $G = \mathsf{SL}_n(F)$, the only pure inner twist is G itself (see [Vog93, Example 2.12]).

2.2 The *L*-group

Let $G^\vee$ denote the $\mathbb{C}$-points of the dual group of $\mathbf{G}$. It is endowed with an action of Γ_F. Let W_F be the Weil group of F (relative to F_s/F) and let ${}^L G := G^\vee \rtimes W_F$ denote the L-group of G.

Kottwitz proved in [Kot84, Proposition 6.4] that there exists a natural isomorphism

$$\kappa_G \colon \mathrm{H}^1(F, \mathbf{G}) \xrightarrow{\sim} \mathrm{Irr}\Big(\pi_0\big(\mathrm{Z}_{G^\vee}^{W_F}\big)\Big). \tag{2.5}$$

Let $G_{\mathrm{sc}}^\vee$ denote the simply connected cover of the derived group $G_{\mathrm{der}}^\vee$ of $G^\vee$. We have $G_{\mathrm{sc}}^\vee = (G_{\mathrm{ad}})^\vee$, and

$$\kappa_{G_{\mathrm{ad}}^*} \colon \mathrm{H}^1(F, \mathrm{Inn}(\mathbf{G}^*)) \xrightarrow{\sim} \mathrm{Irr}\big(\mathrm{Z}_{G_{\mathrm{sc}}^\vee}^{W_F}\big). \tag{2.6}$$

All the inner twists of a given group G share the same L-group, because the action of W_F on $G^\vee$ is only uniquely defined up to inner automorphisms. This also works the other way around: from the Langlands dual group ${}^L G$ one can recover the inner-form class of G.

Example 2.3. If $G = \mathsf{Sp}_{2n}(F)$, then we have $G^\vee = \mathsf{SO}_{2n+1}(\mathbb{C})$ and $G_{\mathrm{sc}}^\vee = \mathsf{Spin}_{2n+1}(\mathbb{C})$, so $\mathrm{Z}_{G_{\mathrm{sc}}^\vee} \simeq \mathbb{Z}/2\mathbb{Z}$. An inner twist of G is determined by its Tits index [Tit65]. The group $G^* = G$ is split and its nontrivial inner twist is the group $\mathsf{SU}(n, h_r)$, where h_r is a nondegenerate Hermitian form of index $r = \lfloor n/2 \rfloor$ over the quaternion algebra Q over F (see for instance [Art13, § 9]).

We will consider G as an inner twist of G^*, so endowed with an isomorphism $\mathbf{G} \to \mathbf{G}^*$ over F_s. Via (2.6), G is labelled by a character ζ_G of $\mathrm{Z}_{G_{\mathrm{sc}}^\vee}^{W_F}$. We choose an extension ζ of ζ_G to $\mathrm{Z}_{G_{\mathrm{sc}}^\vee}$.

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3. Generalized Springer correspondence for disconnected groups

Let $\mathcal{G}$ be a possibly disconnected complex Lie group. We denote by $\mathcal{G}^\circ$ the identity component of $\mathcal{G}$. Let u be a unipotent element in $\mathcal{G}^\circ$, and let $A_{\mathcal{G}^\circ}(u)$ denote the group of components of $\mathrm{Z}_{\mathcal{G}^\circ}(u)$.

Let ϕ° be an irreducible representation of $A_{\mathcal{G}^\circ}(u)$. The pair (u,ϕ°) is called *cuspidal* if it determines a $\mathcal{G}^\circ$-equivariant cuspidal local system on the $\mathcal{G}^\circ$-conjugacy class of u as defined in [Lus84b]. In particular, if (u,ϕ°) is cuspidal, then u is a distinguished unipotent element in $\mathcal{G}^\circ$ (that is, u does not meet the unipotent variety of any proper Levi subgroup of $\mathcal{G}^\circ$), [Lus84b, Proposition 2.8]. However, in general not every distinguished unipotent element supports a cuspidal representation.

Example 3.1. For $\mathcal{G} := \mathsf{SL}_n(\mathbb{C})$, the unipotent classes in $\mathcal{G}$ are in bijection with the partitions $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ of n: the corresponding $\mathcal{G}$-conjugacy class $\mathcal{O}_\lambda$ consists of unipotent matrices with Jordan blocks of sizes $\lambda_1, \lambda_2, \dots, \lambda_r$. We identify the center $\mathrm{Z}_\mathcal{G}$ with the group μ_n of complex nth roots of unity. For $u \in \mathcal{O}_\lambda$, the natural homomorphism $\mathrm{Z}_\mathcal{G} \to A_\mathcal{G}(u)$ is surjective with kernel $\mu_{n/\gcd(\lambda)}$, where $\gcd(\lambda) := \gcd(\lambda_1, \lambda_2, \dots, \lambda_r)$. Hence the irreducible $\mathcal{G}$-equivariant local systems on $\mathcal{O}_\lambda$ all have rank one, and they are distinguished by their central characters, which range over those $\chi \in \widehat{\mu_n}$ such that $\gcd(\lambda)$ is a multiple of the order of χ. We denote these local systems by $\mathcal{E}_{\lambda,\chi}$. The unique distinguished unipotent class in $\mathcal{G}$ is the regular unipotent class $\mathcal{O}_{(n)}$, consisting of unipotent matrices with a single Jordan block. The cuspidal irreducible $\mathcal{G}$-equivariant local systems are supported on $\mathcal{O}_{(n)}$ and are of the form $\mathcal{E}_{(n),\chi}$, with $\chi \in \widehat{\mu_n}$ of order n (see [Lus84b, (10.3.2)]).

The group $A_{\mathcal{G}^\circ}(u)$ may be viewed as a subgroup of the group $A_u := A_\mathcal{G}(u)$ of components of $\mathrm{Z}_\mathcal{G}(u)$. Let ϕ be an irreducible representation of $A_\mathcal{G}(u)$. We say that (u,ϕ) is a *cuspidal pair* if the restriction of ϕ to $A_{\mathcal{G}^\circ}(u)$ is a direct sum of irreducible representations ϕ° such that one (or, equivalently, any) of the pairs (u,ϕ°) is cuspidal. Let

$$\mathbf{I}^\mathcal{G} := \{(U,\mathcal{E}) \mid U \text{ unipotent conjugacy class in } \mathcal{G},\ \mathcal{E} \text{ irreducible } \mathcal{G}\text{-equivariant local system on } U\}.$$

This set can be identified with the set of $\mathcal{G}$-orbits of pairs (u,ϕ), where $u \in \mathcal{G}$ is unipotent and $\phi \in \widehat{A}_u$. If (ϕ, V_ϕ) is an irreducible A_u-representation, we can first regard it as an irreducible $\mathrm{Z}_\mathcal{G}(u)$-representation, and then the corresponding local system is $\mathcal{E} = (\mathcal{G} \times_{\mathrm{Z}_\mathcal{G}(u)} V_\phi \to \mathcal{G}/\mathrm{Z}_\mathcal{G}(u) \cong U)$. We denote by $\mathbf{I}_\mathrm{c}^\mathcal{G}$ the subset of $\mathbf{I}^\mathcal{G}$ of cuspidal pairs. We write $\mathbf{I} := \mathbf{I}^{\mathcal{G}^\circ}$ and $\mathbf{I}_\mathrm{c} := \mathbf{I}_\mathrm{c}^{\mathcal{G}^\circ}$.

Let $\mathbf{J}^\mathcal{G}$ denote the set of $\mathcal{G}$-orbits of triples $j = (\mathcal{M}, U_\mathrm{c}, \mathcal{E}_\mathrm{c})$ such that $\mathcal{M}^\circ$ is a Levi subgroup of $\mathcal{G}^\circ$,

$$\mathcal{M} := \mathrm{Z}_\mathcal{G}(\mathrm{Z}^\circ_{\mathcal{M}^\circ}), \tag{3.1}$$

and $(U_\mathrm{c}, \mathcal{E}_\mathrm{c}) \in \mathbf{I}_\mathrm{c}^{\mathcal{M}^\circ}$. We observe that $\mathcal{M}$ has identity component $\mathcal{M}^\circ$ and that $\mathrm{Z}^\circ_\mathcal{M} = \mathrm{Z}^\circ_{\mathcal{M}^\circ}$. We set $\mathbf{J} := \mathbf{J}^{\mathcal{G}^\circ}$. We note that $\mathcal{M} = \mathcal{M}^\circ$ whenever $\mathcal{G} = \mathcal{G}^\circ$.

Let $\mathrm{Z}^\circ_{\mathcal{M}^\circ,\mathrm{reg}} = \{z \in Z^\circ_{\mathcal{M}^\circ} \mid \mathrm{Z}_\mathcal{G}(z) = \mathcal{M}^\circ\}$ and $Y_j(\mathcal{G}) = \bigcup_{x\in\mathcal{G}} x(\mathrm{Z}^\circ_{\mathcal{M}^\circ,\mathrm{reg}} U_\mathrm{c})x^{-1}$. Let $\overline{Y}_j(\mathcal{G})$ be the closure of $Y_j(\mathcal{G})$ in $\mathcal{G}$. We set $Y_j = Y_j(\mathcal{G}^\circ)$ and $\overline{Y}_j = \overline{Y}_j(\mathcal{G}^\circ)$. For example, if $j_0 = (T, 1, \mathrm{triv})$ is the trivial cuspidal pair on the maximal torus T in $\mathcal{G}^\circ$, then Y_{j_0} is the variety of regular semisimple elements in $\mathcal{G}^\circ$, hence $\overline{Y}_{j_0} = \mathcal{G}^\circ$.

Set $W_j^\circ := \mathrm{N}_{\mathcal{G}^\circ}(\mathcal{M}^\circ)/\mathcal{M}^\circ$. This is a Coxeter group due to the particular nature of the Levi subgroups in $\mathcal{G}^\circ$ that support cuspidal local systems (see [Lus84b, Theorem 9.2]).

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One constructs a $\mathcal{G}^\circ$-equivariant semisimple perverse sheaf K_j supported on $\overline{Y}_j$ that has a W_j°-action and a decomposition [Lus84b, Theorem 6.5] and [AMS18, § 5]

$$K_j = \bigoplus_{\rho^\circ \in \widehat{W_j^\circ}} V_{\rho^\circ} \otimes A_{j,\rho^\circ},$$

where $(\rho^\circ, V_{\rho^\circ})$ ranges over the (equivalence classes of) irreducible W_j°-representations and A_{j,ρ° is an irreducible $\mathcal{G}^\circ$-equivariant perverse sheaf. The perverse sheaf A_{j,ρ° has the property that there exists a (unique) pair $(U, \mathcal{E}^\circ) \in \mathbf{I}$ such that its restriction to the variety $\mathcal{G}^\circ_{\mathrm{un}}$ of unipotent elements in $\mathcal{G}^\circ$ is

$$(A_{j,\rho^\circ})|_{\mathcal{G}^\circ_{\mathrm{un}}}[-\dim(\mathrm{Z}^\circ_{\mathcal{M}^\circ})] \cong \mathrm{IC}(\overline{U}, \mathcal{E}^\circ)[\dim(U)]. \tag{3.2}$$

In particular, the hypercohomology of A_{j°,ρ° on U is concentrated in one degree, namely

$$\mathcal{H}^{a_U}(A_{j,\rho^\circ})|_U \cong \mathcal{E}^\circ, \quad \text{where } a_U = -\dim(U) - \dim(\mathrm{Z}^\circ_{\mathcal{M}^\circ}).$$

If we set $\widetilde{\mathbf{J}} = \widetilde{\mathbf{J}}^{\mathcal{G}^\circ} := \{(j, \rho^\circ) : j \in \mathbf{J}^{\mathcal{G}^\circ},\ \rho^\circ \in \widehat{W_j^\circ}\}$, the generalized Springer correspondence for $\mathcal{G}^\circ$ is the bijection

$$\nu^\circ \colon \mathbf{I}^{\mathcal{G}^\circ} \to \widetilde{\mathbf{J}}^{\mathcal{G}^\circ}, \quad (U, \mathcal{E}) \mapsto (j, \rho^\circ), \tag{3.3}$$

where the relation between (j, ρ°) and $(U, \mathcal{E})$ is given by (3.2). Let $\nu^\circ_{\mathrm{c}} \colon \mathbf{I} \to \mathbf{J}$ denote the composition of ν° with the projection from $\widetilde{\mathbf{J}}$ to $\mathbf{J}$.

We will now explain how, following [AMS18, § 4], one can extend the maps ν° and ν°_{c} to the case of disconnected groups. Let $j = (\mathcal{M}, U_{\mathrm{c}}, \mathcal{E}_{\mathrm{c}}) \in \mathbf{J}^{\mathcal{G}}$. We set $W_j := \mathrm{N}_{\mathcal{G}}(j)/\mathcal{M}^\circ$. There exists a subgroup $\mathfrak{R}_j$ of W_j such that $W_j = W_j^\circ \rtimes \mathfrak{R}_j$ (see [AMS18, Lemma 4.2]). Suppose that $\natural_j$ is a 2-cocycle

$$\natural_j \colon \mathfrak{R}_j \times \mathfrak{R}_j \to \overline{\mathbb{Q}}_\ell^\times.$$

We view $\natural_j$ as a 2-cocycle on W_j that is trivial on W_j°. Then the $\natural_j$-twisted group algebra of W_j, denoted by $\overline{\mathbb{Q}}_\ell[W_j, \natural_j]$, is defined to be the $\overline{\mathbb{Q}}_\ell$-vector space $\overline{\mathbb{Q}}_\ell[W_j, \natural_j]$ with basis $\{f_w : w \in W_j\}$ and multiplication rules

$$f_w f_{w'} = \natural_j(w, w') f_{ww'}, \quad w, w' \in W_j.$$

One constructs a $\mathcal{G}$-equivariant semisimple perverse sheaf K_j supported on $\overline{Y}_j$ that has a W_j-action and a decomposition [Lus84b, Theorem 6.5]

$$K_j = \bigoplus_{\rho \in \mathrm{Irr}(\overline{\mathbb{Q}}_\ell[W_j, \natural_j])} V_\rho \otimes A_{j,\rho},$$

where (ρ, V_ρ) ranges over the (equivalence classes of) simple modules of $\overline{\mathbb{Q}}_\ell[W_j, \natural_j]$, and $A_{j,\rho}$ is an irreducible $\mathcal{G}$-equivariant perverse sheaf.

We set

$$\widetilde{\mathbf{J}}^{\mathcal{G}} := \{(j, \rho) : j \in \mathbf{J}^{\mathcal{G}},\ \rho \in \mathrm{Irr}(\overline{\mathbb{Q}}_\ell[W_j, \natural_j])\}. \tag{3.4}$$

The generalized Springer correspondence for $\mathcal{G}$ is the bijection

$$\nu \colon \mathbf{I}^{\mathcal{G}} \to \widetilde{\mathbf{J}}^{\mathcal{G}} \tag{3.5}$$

defined in [AMS18, Theorem 5.5].

Definition 3.2. Let $\nu_{\mathrm{c}} = \nu^{\mathcal{G}}_{\mathrm{c}} \colon \mathbf{I}^{\mathcal{G}} \to \mathbf{J}^{\mathcal{G}}$ denote the composition of ν with the projection from $\widetilde{\mathbf{J}}^{\mathcal{G}}$ to $\mathbf{J}^{\mathcal{G}}$.

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Suppose $(U,\mathcal{E})\in\mathbf{I}^{\mathcal{G}}$, and suppose the $\mathcal{G}$-class U splits into $\mathcal{G}^\circ$-classes $U_1^\circ,\dots,U_\ell^\circ$, for some $\ell\geqslant 1$. If we regard $\mathcal{E}$ as a $\mathcal{G}^\circ$-equivariant local system, then it restricts as $\mathcal{E}|_{U_i^\circ}=\bigoplus_{t=1}^{k_i}\mathcal{E}_{i,t}^\circ$, $1\leqslant i\leqslant\ell$, where $\nu^\circ(U_i^\circ,\mathcal{E}_{i,t}^\circ)=(j^\circ,\rho_{i,t}^\circ)$, with $j^\circ=\nu_{\mathrm{c}}^\circ(U_1^\circ,\mathcal{E}_{1,1}^\circ)$ and $\rho|_{W_j^\circ}=\bigoplus_{i,t}\rho_{i,t}^\circ$.

Example 3.3. Let $\mathcal{G}=\mathsf{O}_{2n}(\mathbb{C})$, so $\mathcal{G}^\circ=\mathsf{SO}_{2n}(\mathbb{C})$ and $\mathcal{G}/\mathcal{G}^\circ\cong\mathbb{Z}/2\mathbb{Z}$. The unipotent classes in $\mathcal{G}$ are parametrized by partitions $\lambda=(\lambda_1,\dots,\lambda_m)$ of $2n$ such that each even part appears with even multiplicity. If U_λ is the corresponding unipotent class, then U_λ is a single $\mathcal{G}^\circ$-class unless the partition λ is 'very even' [SS70, CM93], i.e. all parts λ_i are even, in which case U_λ splits into two $\mathcal{G}^\circ$-classes, U_λ^+ and U_λ^-.

Let $j=j_0$ correspond to the trivial cuspidal local system on the torus of $\mathcal{G}^\circ$. Then $W_{j_0}^\circ=W^\circ\cong W(D_n)$ and $W_j=W\cong W(B_n)$, hence $W/W^\circ\cong\mathcal{G}/\mathcal{G}^\circ=\mathbb{Z}/2\mathbb{Z}$. (Here $W(D_n)$ denotes a Weyl group of type D_n and similarly for $W(B_n)$.) If λ is not a very even partition and $u\in U_\lambda$, then $A_u/A_{\mathcal{G}^\circ}(u)=\mathbb{Z}/2\mathbb{Z}$; if $(j_0,\rho^\circ)=\nu^\circ(U_\lambda^\circ,\phi^\circ)$, then there are two nonisomorphic ways ϕ,ϕ' in which one can extend ϕ° to A_u, and two nonisomorphic ways ρ,ρ' to extend ρ° to W, which can be chosen such that ρ corresponds to ϕ and ρ' corresponds to ϕ' under the disconnected Springer correspondence.

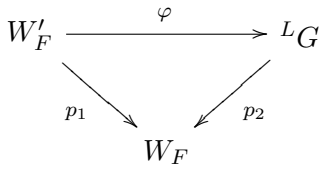
If, on the other hand, λ is a very even partition, $u=u^+$ is a representative of U_λ^+, and u^- a representative of U_λ^-, then $A_u=A_{\mathcal{G}^\circ}(u^+)=A_{\mathcal{G}^\circ}(u^-)=\{1\}$. In this case, $\nu^\circ(U^\pm,\mathbf{1})=\rho^\pm$ ($W(D_n)$-representations), where ρ is parametrized by a bipartition of n of the form $\lambda'\times\lambda'$ (necessarily n is even). Then $\nu(U,\mathbf{1})=\rho$ ($W(B_n)$-representation), where $\rho|_{W(D_n)}=\rho^+\oplus\rho^-$.

4. The Langlands parametrization

We use the notation of § 2. In addition, we write I_F for the inertia subgroup of W_F, and we set $W_F':=W_F\times\mathsf{SL}_2(\mathbb{C})$. We have natural projections from $p_1\colon W_F'\twoheadrightarrow W_F$ and $p_2\colon {}^LG\twoheadrightarrow W_F$.

4.1 Langlands parameters

A *Langlands parameter* (or *L-parameter*) for G is a continuous morphism $\varphi\colon W_F'\to{}^LG$ such that $\varphi(w)$ is semisimple for each $w\in W_F$ (that is, $r(\varphi(w))$ is semisimple for every finite-dimensional representation r of ${}^LG$), the restriction of φ to $\mathsf{SL}_2(\mathbb{C})$ is a morphism of complex algebraic groups, and the following diagram commutes.



Write $\Phi(G)$ for the set of $G^\vee$-conjugacy classes of Langlands parameters for G.

Let $\mathrm{Z}_{G^\vee}(\varphi)$ denote the centralizer in $G^\vee$ of $\varphi(W_F')$. We have

$$\mathrm{Z}_{G^\vee}(\varphi)\cap\mathrm{Z}_{G^\vee}=\mathrm{Z}_{G^\vee}^{W_F},\tag{4.1}$$

and, hence,

$$\mathrm{Z}_{G^\vee}(\varphi)/\mathrm{Z}_{G^\vee}^{W_F}\simeq\mathrm{Z}_{G^\vee}(\varphi)\mathrm{Z}_{G^\vee}/\mathrm{Z}_{G^\vee}.$$

The group $\mathrm{Z}_{G^\vee}(\varphi)\mathrm{Z}_{G^\vee}/\mathrm{Z}_{G^\vee}$ can be considered as a subgroup of $G_{\mathrm{ad}}^\vee$ and we define $\mathrm{Z}^1_{G_{\mathrm{sc}}^\vee}(\varphi)$ to be its inverse image under the canonical projection $p\colon G_{\mathrm{sc}}^\vee\to G_{\mathrm{ad}}^\vee$. The group $\mathrm{Z}^1_{G_{\mathrm{sc}}^\vee}(\varphi)$ coincides with the group introduced by Arthur in [Art06, (3.2)] (denoted there by $\widetilde{\mathcal{S}}_\varphi$). As observed in [Art06], it is an extension of $\mathrm{Z}_{G^\vee}(\varphi)/\mathrm{Z}_{G^\vee}^{W_F}$ by $\mathrm{Z}_{G_{\mathrm{sc}}^\vee}$. Let A_φ^1 denote the component group of $\mathrm{Z}^1_{G_{\mathrm{sc}}^\vee}(\varphi)$.

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Remark 4.1. When $Z^\circ_{\mathbf{G}}$ is F-split, the group A^1_φ also coincides with the group considered by Kaletha in [Kal18, § 4.6] in the parametrization of the L-packet of φ.

An *enhancement* of φ is an irreducible representation ϕ of A^1_φ. We denote by $\widehat{A}^1_\varphi$ the set of irreducible characters of A^1_φ. The pairs (φ, ϕ) are called *enhanced L-parameters*. Let $\phi \in \widehat{A}^1_\varphi$. Then ϕ determines a character ζ_ϕ of $\mathrm{Z}_{G^\vee_{\mathrm{sc}}}$. An enhanced L-parameter (ϕ, φ) is said to be *G-relevant* if $\zeta_\phi = \zeta$, where ζ is as defined in § 2.2. The set $\Phi_{\mathrm{e}}(G)$ of $G^\vee$-conjugacy classes of G-relevant enhanced L-parameters is expected to parametrize the admissible dual of G.

The group $H^1(W_F, \mathrm{Z}_{G^\vee})$ acts on $\Phi(G)$ by

$$(z\varphi)(w, x) := z'(w)\varphi(w, x), \quad \varphi \in \Phi(G),\ w \in W_F,\ x \in \mathsf{SL}_2(\mathbb{C}), \tag{4.2}$$

where $z' \colon W_F \to \mathrm{Z}_{G^\vee}$ represents $z \in H^1(W_F, \mathrm{Z}_{G^\vee})$. This extends to an action of $H^1(W_F, \mathrm{Z}_{G^\vee})$ on $\Phi_{\mathrm{e}}(G)$ that does nothing to the enhancements.

A character of G is called *weakly unramified* if it is trivial on the kernel of the Kottwitz homomorphism. Let $X_{\mathrm{wur}}(G)$ denote the group of weakly unramified characters of G. There is a natural isomorphism

$$X_{\mathrm{wur}}(G) \simeq (\mathrm{Z}_{G^\vee}^{I_F})_{\mathrm{Frob}} \subset H^1(W_F, \mathrm{Z}_{G^\vee}) \tag{4.3}$$

(see [Hai14, § 3.3.1]). Its identity component is the group $X_{\mathrm{un}}(G)$ of unramified characters of G. Via (4.2) and (4.3), the group $X_{\mathrm{wur}}(G)$ acts naturally on $\Phi_{\mathrm{e}}(G)$.

Let $\varphi \colon W_F \times \mathsf{SL}_2(\mathbb{C}) \to {}^L G$ be an L-parameter. We consider the (possibly disconnected) complex reductive group

$$\mathcal{G}_\varphi := \mathrm{Z}^1_{G^\vee_{\mathrm{sc}}}\big(\varphi|_{W_F}\big), \tag{4.4}$$

defined analogously to $\mathrm{Z}^1_{G^\vee_{\mathrm{sc}}}(\varphi)$. Denote by $\mathcal{G}^\circ_\varphi$ its identity component.

We define elements $u_\varphi, s_\varphi \in G^\vee$ by

$$(u_\varphi, 1) = \varphi\big(1, \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\big) \quad \text{and} \quad (s_\varphi, \mathrm{Frob}) = \varphi(\mathrm{Frob}, \mathrm{Id}_{\mathsf{SL}_2(\mathbb{C})}). \tag{4.5}$$

Then $u_\varphi \in \mathcal{G}^\circ_\varphi$.

We recall that by the Jacobson–Morozov Theorem any unipotent element u of $\mathcal{G}^\circ_\varphi$ determines (up to conjugation by $\mathrm{Z}_{\mathcal{G}}(u)$) a homomorphism of algebraic groups $\mathsf{SL}_2(\mathbb{C}) \to \mathcal{G}^\circ_\varphi$ taking the value u at $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Hence, any enhanced L-parameter (φ, ϕ) is completely determined, up to $G^\vee$-conjugacy, by $\varphi|_{W_F}$, u_φ and ϕ. More precisely, the map

$$(\varphi, \phi) \mapsto (\varphi|_{W_F}, u_\varphi, \phi) \tag{4.6}$$

provides a bijection between $\Phi_{\mathrm{e}}(G)$ and the set of $G^\vee$-conjugacy classes of triples $(\varphi|_{W_F}, u_\varphi, \phi)$.

We define an action of $G^\vee_{\mathrm{sc}}$ on $G^\vee$ by setting

$$h \cdot g := h' g h'^{-1} \quad \text{for } h \in G^\vee_{\mathrm{sc}} \text{ and } g \in G^\vee, \text{ where } p(h) = h'\mathrm{Z}_{G^\vee}.$$

It induces an action of $G^\vee_{\mathrm{sc}}$ on ${}^L G$ and we denote by $\mathrm{Z}_{G^\vee_{\mathrm{sc}}}(\varphi)$ the stabilizer in $G^\vee_{\mathrm{sc}}$ of $\varphi(W'_F)$ for this action.

On the other hand, the inclusion $\mathrm{Z}^1_{G^\vee_{\mathrm{sc}}}(\varphi) \hookrightarrow \mathrm{Z}^1_{G^\vee_{\mathrm{sc}}}(\varphi|_{W_F}) \cap \mathrm{Z}_{G^\vee_{\mathrm{sc}}}(u_\varphi)$ induces a group isomorphism

$$A^1_\varphi \xrightarrow{\sim} \pi_0(\mathrm{Z}^1_{G^\vee_{\mathrm{sc}}}(\varphi|_{W_F}) \cap \mathrm{Z}_{G^\vee_{\mathrm{sc}}}(u_\varphi)). \tag{4.7}$$

As observed in [AMS18, (92)], another way to formulate (4.7) is

$$A^1_\varphi \simeq A_{\mathcal{G}_\varphi}(u_\varphi) := \mathrm{Z}_{\mathcal{G}_\varphi}(u_\varphi)/\mathrm{Z}_{\mathcal{G}_\varphi}(u_\varphi)^\circ. \tag{4.8}$$

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The L-parameter φ is called:

– *discrete* if there is no proper W_F-stable Levi subgroup $L^\vee \subset G^\vee$ such that $\varphi(W_F') \subset L^\vee \rtimes W_F$;
– *bounded* if s_φ belongs to a bounded subgroup of $G^\vee$.

We say that $(\varphi, \phi) \in \Phi_{\mathrm{e}}(G)$ is *cuspidal* if φ is discrete and (u_φ, ϕ) is a cuspidal pair for $\mathcal{G}_\varphi$ (as defined in § 3). The set of G-relevant cuspidal (respectively, discrete, bounded) enhanced L-parameters is expected to correspond to the set of *supercuspidal* (respectively, *essentially square-integrable, tempered*) irreducible smooth G-representations [AMS18, § 6].

4.2 Inertial classes

For L a Levi subgroup of G and $g \in G^\vee$, the group $gL^\vee g^{-1}$ is not necessarily W_F-stable, so the group $G^\vee$ need not act on pairs of the form $({}^L L, (\varphi_{\mathrm{c}}, \phi_{\mathrm{c}}))$ with $(\varphi_{\mathrm{c}}, \phi_{\mathrm{c}})$ a cuspidal enhanced L-parameter for L. In order to deal with this, as in [AMS18, Definition 7.1], we will have to consider all the pairs $(\mathrm{Z}_{{}^L G}(\mathcal{T}), (\varphi_{\mathrm{c}}, \phi_{\mathrm{c}}))$ of the following form.

– $\mathcal{T}$ is a torus of $G^\vee$ such that the projection $\mathrm{Z}_{{}^L G}(\mathcal{T}) \to W_F$ is surjective;
– $\varphi_{\mathrm{c}} \colon W_F' \to \mathrm{Z}_{{}^L G}(\mathcal{T})$ satisfies the requirements in the definition of an L-parameter;
– let $\mathcal{L} = G^\vee \cap \mathrm{Z}_{{}^L G}(\mathcal{T})$, and let $\mathcal{L}_{\mathrm{sc}}$ be the simply connected cover of the derived group of $\mathcal{L}$. Then ϕ_{c} is an irreducible representation of $\pi_0(\mathrm{Z}^1_{\mathcal{L}_{\mathrm{sc}}}(\varphi))$ such that $(u_{\varphi_{\mathrm{c}}}, \phi)$ is a cuspidal pair for $\mathrm{Z}^1_{\mathcal{L}_{\mathrm{sc}}}(\varphi_{\mathrm{c}}|_{W_F}))$ and ϕ_{c} is G-relevant as defined in [AMS18, Definition 7.2] (that is $\zeta_\phi = \zeta$ on $\mathcal{L}_{\mathrm{sc}} \cap \mathrm{Z}^{W_F}_{G^\vee_{\mathrm{sc}}}$ and $\phi = 1$ on $\mathcal{L}_{\mathrm{sc}} \cap \mathrm{Z}^\circ_{\mathcal{L}_c}$, where $\mathcal{L}_c$ denotes the preimage of $\mathcal{L}$ under $G^\vee_{\mathrm{sc}} \to G^\vee$).

Fix such a pair $(\mathrm{Z}_{{}^L G}(\mathcal{T}), (\varphi_{\mathrm{c}}, \phi_{\mathrm{c}}))$. The group

$$X_{\mathrm{un}}(\mathrm{Z}_{{}^L G}(\mathcal{T})) := \left(\mathrm{Z}_{(G^\vee \rtimes I_F) \cap \mathrm{Z}_{{}^L G}(\mathcal{T})}\right)^\circ_{\mathrm{Frob}} \tag{4.9}$$

plays the role of unramified characters for $\mathrm{Z}_{{}^L G}(\mathcal{T})$. It acts on the enhanced L-parameters $(\varphi_{\mathrm{c}}, \phi_{\mathrm{c}})$ (see [AMS18, (110) and (111)]) and we denote by $X_{\mathrm{un}}(\mathrm{Z}_{{}^L G}(\mathcal{T})) \cdot (\varphi_{\mathrm{c}}, \phi_{\mathrm{c}})$ the orbit of $(\varphi_{\mathrm{c}}, \phi_{\mathrm{c}})$.

We denote by $\mathfrak{s}^\vee$ the $G^\vee$-conjugacy class of $(\mathrm{Z}_{{}^L G}(\mathcal{T}), X_{\mathrm{un}}(\mathrm{Z}_{{}^L G}(\mathcal{T})) \cdot (\varphi_{\mathrm{c}}, \phi_{\mathrm{c}}))$. We write

$$\mathfrak{s}^\vee = \mathfrak{s}^\vee_G = [\mathrm{Z}_{{}^L G}(\mathcal{T}), (\varphi_{\mathrm{c}}, \phi_{\mathrm{c}})]_{G^\vee},$$

and call $\mathfrak{s}^\vee$ an *inertial class* for $\Phi_{\mathrm{e}}(G)$. We denote by $\mathfrak{B}^\vee(G)$ the set of all such $\mathfrak{s}^\vee$.

Note that there exists a W_F-stable Levi subgroup $L^\vee$ of $G^\vee$ such that $\mathrm{Z}_{{}^L G}(\mathcal{T})$ is $G^\vee$-conjugate to $L^\vee \rtimes W_F$ and $\mathcal{L} = G^\vee \cap \mathrm{Z}_{{}^L G}(\mathcal{T})$ is $G^\vee$-conjugate to $L^\vee$. Conversely, every $G^\vee$-conjugate of this $L^\vee \rtimes W_F$ is of the form $\mathrm{Z}_{{}^L G}(\mathcal{T})$ for a torus $\mathcal{T}$ as above (see [AMS18, Lemma 6.2]).

We write

$$\mathfrak{s}^\vee_L = (\mathrm{Z}_{{}^L G}(\mathcal{T}), X_{\mathrm{un}}(\mathrm{Z}_{{}^L G}(\mathcal{T})) \cdot (\varphi_{\mathrm{c}}, \phi_{\mathrm{c}})). \tag{4.10}$$

We will consider the groups

$$W_{\mathfrak{s}^\vee} = \mathrm{N}_{G^\vee}(\mathfrak{s}^\vee_L)/L^\vee \quad \text{and} \quad J_{\varphi_{\mathrm{c}}} := \mathrm{Z}_{G^\vee}(\varphi_{\mathrm{c}}(I_F)). \tag{4.11}$$

The group $J_{\varphi_{\mathrm{c}}}$ is a complex (possibly disconnected) reductive group. Define $R(J^\circ_{\varphi_{\mathrm{c}}}, \mathcal{T})$ as the set of $\alpha \in X^*(\mathcal{T}) \setminus \{0\}$ that appear in the adjoint action of $\mathcal{T}$ on the Lie algebra of $J^\circ_{\varphi_{\mathrm{c}}}$. It is a root system (see [AMS17, Proposition 3.9]).

We set $W^\circ_{\mathfrak{s}^\vee} := \mathrm{N}_{J^\circ_{\varphi_{\mathrm{c}}}}(\mathcal{T})/\mathrm{Z}_{J^\circ_{\varphi_{\mathrm{c}}}}(\mathcal{T})$, where $W^\circ_{\mathfrak{s}^\vee}$ is the Weyl group of $R(J^\circ_{\varphi_{\mathrm{c}}}, \mathcal{T})$. Let $R^+(J^\circ_{\varphi_{\mathrm{c}}}, \mathcal{T})$ be the positive system defined by a parabolic subgroup $P^\circ_{\varphi_{\mathrm{c}}} \subset J^\circ_{\varphi_{\mathrm{c}}}$ with Levi factor $(L^\vee_{\varphi_{\mathrm{c}}})^\circ$. Two such parabolic subgroups $P^\circ_{\varphi_{\mathrm{c}}}$ are $J^\circ_{\varphi_{\mathrm{c}}}$-conjugate, so the choice is inessential.

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Since $W_{\mathfrak{s}^\vee}^\circ$ acts simply transitively on the collection of positive systems for $R(J_{\varphi_c}^\circ, \mathcal{T})$, we obtain a semi-direct factorization

$$W_{\mathfrak{s}^\vee} = W_{\mathfrak{s}^\vee}^\circ \rtimes \mathfrak{R}_{\mathfrak{s}^\vee},$$

where $\mathfrak{R}_{\mathfrak{s}^\vee} = \{w \in W_{\mathfrak{s}^\vee} \mid w \cdot R^+(J_{\varphi_c}^\circ, \mathcal{T}) = R^+(J_{\varphi_c}^\circ, \mathcal{T})\}$.

Definition 4.2. Let $\boldsymbol{\nu}_c \colon \Phi_e(G) \to \mathfrak{B}^\vee(G)$ be the map defined by

$$\boldsymbol{\nu}_c(\varphi, \phi) = [\mathrm{Z}_{^LG}(\mathrm{Z}_{\mathcal{M}_\varphi}^\circ), \varphi|_{W_F}, u_c, \phi_c]_{G^\vee},$$

where $(\varphi|_{W_F}, u, \phi)_{G^\vee}$ is the image of $(\varphi, \phi)_{G^\vee}$ via the bijection (4.6), (u_c, ϕ_c) corresponds to $(U_c, \mathcal{E}_c) \in \mathbf{I}_c^{\mathcal{M}_\varphi}$ and $(\mathcal{M}_\varphi, U_c, \mathcal{E}_c) := \nu_c^{\mathcal{G}_\varphi}(U, \mathcal{E})$ is the image under the map $\nu_c^{\mathcal{G}_\varphi}$ from Definition 3.2 of the pair $(U, \mathcal{E}) \in \mathbf{I}^{\mathcal{G}_\varphi}$ associated with (u, ϕ).

We have the following decomposition (see [AMS18, (115)]):

$$\Phi_e(G) = \bigsqcup_{\mathfrak{s}^\vee \in \mathfrak{B}^\vee(G)} \Phi_e(G)^{\mathfrak{s}^\vee}, \quad \text{where } \Phi_e(G)^{\mathfrak{s}^\vee} := \boldsymbol{\nu}_c^{-1}(\mathfrak{s}^\vee). \tag{4.12}$$

Let $\mathrm{Irr}(G)$ be the set of isomorphism classes of irreducible smooth G-representations. For L a Levi subgroup of G, we denote by $\mathrm{Irr}_{\mathrm{cusp}}(L)$ the set of isomorphism classes of supercuspidal irreducible smooth L-representations.

Let $\sigma \in \mathrm{Irr}_{\mathrm{cusp}}(L)$. We call (L, σ) a *supercuspidal pair*, and we consider such pairs up to *inertial equivalence*. This is the equivalence relation generated by:

- unramified twists, $(L, \sigma) \sim (L, \sigma \otimes \chi)$ for $\chi \in X_{\mathrm{un}}(L)$;
- G-conjugation, $(L, \sigma) \sim (gLg^{-1}, g \cdot \sigma)$ for $g \in G$.

We denote the set of all inertial equivalence classes for G by $\mathfrak{B}(G)$ and a typical inertial equivalence class by $\mathfrak{s} := [L, \sigma]_G$.

In [Ber84], Bernstein attached to every $\mathfrak{s} \in \mathfrak{B}(G)$ a block $\mathfrak{R}(G)^{\mathfrak{s}}$ in the category $\mathfrak{R}(G)$ of smooth G-representations as follows. Denote by I_P^G the normalized parabolic induction functor, where P is a parabolic subgroup of G with Levi subgroup L. If $\pi \in \mathrm{Irr}(G)$ is a constituent of $I_P^G(\tau)$ for some $\sigma \in \mathrm{Irr}(L)$ such that $[L, \sigma]_G = \mathfrak{s}$, then $\mathfrak{s}$ is called the *inertial supercuspidal support* of π. We set

$$\mathrm{Irr}(G)^{\mathfrak{s}} := \{\pi \in \mathrm{Irr}(G) : \pi \text{ has inertial supercuspidal support } \mathfrak{s}\},$$
$$\mathfrak{R}(G)^{\mathfrak{s}} := \{\pi \in \mathfrak{R}(G) \ : \ \text{every irreducible constituent of } \pi \text{ belongs to } \mathrm{Irr}(G)^{\mathfrak{s}}\}.$$

4.3 Unipotent representations

An irreducible smooth representation (π, V) of G is called *unipotent* if there exists a parahoric subgroup K of G such that the subspace V^{K^+} of the vectors in V that are fixed by the pro-unipotent radical K^+ of K contains an irreducible unipotent representation of the finite reductive group $\overline{K} := K/K^+$. We denote by $\mathrm{Irr}_{\mathrm{un}}(G)$ the set of isomorphism classes of irreducible unipotent G-representations.

For the rest of the section we will assume that the quasi-split inner form G^* of G is F-split.

Definition 4.3. An L-parameter $\varphi \colon W_F \times \mathsf{SL}_2(\mathbb{C}) \to {}^LG$ is called *unipotent* if $\varphi(w, 1) = (1, w)$ for any element w of the inertia subgroup I_F of W_F.

Denote by $\Phi_{\mathrm{un}}({}^LG)$ the set of $G^\vee$-conjugacy classes of unipotent L-parameters φ and by $\Phi_{e,\mathrm{un}}(G)$ the set of unipotent enhanced G-relevant parameters, i.e. the subset of the

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$(\varphi,\phi)\in\Phi_{\mathrm{e}}(G)$ such that φ is unipotent. We then set

$$\Phi_{\mathrm{e,un}}({}^LG)=G^\vee\backslash\{(\varphi,\phi)\mid \varphi \text{ unipotent}, \phi\in\widehat{A^1_\varphi}\}.$$

Given $\varphi\in\Phi_{\mathrm{un}}({}^LG)$, since G^* is F-split, every $\phi\in\widehat{A^1_\varphi}$ is G'-relevant for some $G'\in\mathrm{InnT}(G)$. We thus have

$$\Phi_{\mathrm{e,un}}({}^LG)=\bigsqcup_{G'\in\mathrm{InnT}(G)}\Phi_{\mathrm{e,un}}(G').$$

Given $\varphi\in\Phi_{\mathrm{un}}({}^LG)$, let A_φ be the component group of $Z_{G^\vee}(\varphi)$. We then set

$$\Phi^p_{\mathrm{e,un}}({}^LG)=G^\vee\backslash\{(\varphi,\phi)\mid \varphi \text{ unipotent}, \phi\in\widehat{A_\varphi}\}.$$

The set $\Phi_{\mathrm{e,un}}(G)$ is known to parametrize $\mathrm{Irr}_{\mathrm{un}}(G)$: such a parametrization was defined by Lusztig in [Lus95, Lus02] in the case when $\mathbf{G}$ is simple adjoint, and extended by Solleveld in [Sol23a, Sol23b] to the case when G is arbitrary. In the case when $G=\mathsf{GL}_n(F)$ or $\mathsf{SL}_n(F)$, it also follows from [HS12, ABPS16]. This parametrization induces a bijection:

$$\mathsf{LLC}\colon \Phi_{\mathrm{e,un}}({}^LG)\longleftrightarrow\bigsqcup_{G'\in\mathrm{InnT}(G)}\mathrm{Irr}_{\mathrm{un}}(G'). \tag{4.13}$$

This correspondence sends cuspidal (respectively, discrete, bounded) parameters to supercuspidal (respectively, essentially square-integrable, tempered) irreducible unipotent representations.

Let $x\in G^\vee$ with Jordan decomposition $x=su$. There is an unipotent L-parameter φ (unique up to $G^\vee$-conjugation) such that $u=u_\varphi$ and $s=s_\varphi$. We set

$$\mathcal{G}_s=\mathrm{Z}^1_{G^\vee_{\mathrm{sc}}}(\varphi|_{W_F}). \tag{4.14}$$

Note that $\varphi|_{W_F}$ depends only on s, which explains the notation. By (4.8),

$$A^1_\varphi\cong A_{\mathcal{G}_s}(u).$$

We set

$$\Phi_{\mathrm{e,un}}({}^LG,s)=G^\vee\backslash\{(\varphi',\phi)\in\Phi_{\mathrm{e,un}}({}^LG)\mid \varphi'(\mathrm{Frob},1)=(s',\mathrm{Frob}),\ s'\in G^\vee\cdot s\}.$$

Then

$$\begin{aligned}\Phi_{\mathrm{e,un}}({}^LG,s)&=\mathrm{Z}_{G^\vee}(\varphi|_{W_F})\text{-orbits in }\{(u',\phi)\mid u'\in\mathcal{G}_s^\circ \text{ unipotent},\ \phi\in\widehat{A_{\mathcal{G}_s}(u')}\}\\&=\mathcal{G}_s\text{-orbits in }\{(u',\phi)\mid u'\in\mathcal{G}_s^\circ \text{ unipotent},\ \phi\in\widehat{A_{\mathcal{G}_s}(u')}\}.\end{aligned} \tag{4.15}$$

The second equality follows from the fact that conjugation of unipotent elements is insensitive to isogenies. This allows us to rephrase the unipotent LLC as follows. Let $\mathcal{C}(\mathcal{G})$, $\mathcal{C}(\mathcal{G})_{\mathsf{ss}}$, $\mathcal{C}(\mathcal{G})_{\mathrm{un}}$ denote the set of conjugacy classes, respectively semisimple, unipotent conjugacy classes in a complex group $\mathcal{G}$. Let $R_{\mathrm{un}}(G')$ be the $\mathbb{C}$-span of $\mathrm{Irr}_{\mathrm{un}}(G')$. Then (4.13) can be written as the bijection

$$\mathsf{LLC}_{\mathrm{un}}:\bigsqcup_{s\in\mathcal{C}(G^\vee)_{\mathsf{ss}}}\bigsqcup_{u\in\mathcal{C}(\mathcal{G}_s)_{\mathrm{un}}}\widehat{A_{\mathcal{G}_s}(u)}\longleftrightarrow\bigsqcup_{G'\in\mathrm{InnT}(G)}\mathrm{Irr}_{\mathrm{un}}(G'),$$

which induces a linear isomorphism

$$\mathsf{LLC}_{\mathrm{un}}\colon R(\Phi_{\mathrm{e,un}}({}^LG)):=\bigoplus_{s\in\mathcal{C}(G^\vee)_{\mathsf{ss}}}\bigoplus_{u\in\mathcal{C}(\mathcal{G}_s)_{\mathrm{un}}}R(A_{\mathcal{G}_s}(u))\longrightarrow\bigoplus_{G'\in\mathrm{InnT}(G)}R_{\mathrm{un}}(G'). \tag{4.16}$$

If we instead consider pure inner twists, then we need to replace the group $\mathcal{G}_s$ by the group

$$\mathcal{G}_s^p=\mathrm{Z}_{G^\vee}(\varphi|_{W_F}), \tag{4.17}$$

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and the correspondence becomes

$$\mathsf{LLC}^p_{\mathrm{un}}\colon R(\Phi^p_{\mathrm{e,un}}({}^LG)) := \bigoplus_{s\in\mathcal{C}(G^\vee)_{\mathrm{ss}}} \bigoplus_{u\in\mathcal{C}(\mathcal{G}^p_s)_{\mathrm{un}}} R(A_{\mathcal{G}^p_s}(u)) \longrightarrow \bigoplus_{G'\in \mathrm{Inn T}^p(G)} R_{\mathrm{un}}(G'). \tag{4.18}$$

Given $\phi \in R(A_{\mathcal{G}^p_s}(u))$, write $\pi(s,u,\phi)$ for the image of ϕ under $\mathsf{LLC}^p_{\mathrm{un}}$.

Remark 4.4. Note that the existing LLC for unipotent representations is not entirely canonical (see the discussion in [Sol23a, Introduction]) but for the rest of the paper, we fix a map $\mathsf{LLC}^p_{\mathrm{un}}$ as above satisfying the usual properties (described, for example, in [Sol23a, Theorem 1]).

Example 4.5. For $G = \mathsf{SL}_n(F)$, recall that there is a one-to-one correspondence between $\mathrm{InnT}(\mathsf{SL}_n(F))$ and $\mathbb{Z}/n\mathbb{Z}$, where the inner twists are $\mathsf{SL}_m(D_{d,r/m})$, $m = \gcd(r,n)$, $r \in \mathbb{Z}/n\mathbb{Z}$.

The dual Langlands group is $G^\vee = \mathsf{PGL}_n(\mathbb{C})$. The correspondence (4.13) takes the form:

$$\bigsqcup_{r\in\mathbb{Z}/n\mathbb{Z}} \mathrm{Irr}_{\mathrm{un}}(\mathsf{SL}_m(D_{d,r/m})) \longleftrightarrow \mathsf{PGL}_n(\mathbb{C})\backslash\{(x,\phi)\,|\,x\in\mathsf{PGL}_n(\mathbb{C}), \phi\in\widehat{A^1_x}\}, \tag{4.19}$$

in particular,

$$\mathrm{Irr}_{\mathrm{un}}(\mathsf{SL}_n(F)) \longleftrightarrow \mathsf{PGL}_n(\mathbb{C})\backslash\{(x,\phi)\,|\,x\in\mathsf{PGL}_n(\mathbb{C}), \phi\in\widehat{A_x}\}.$$

In this case, $G^\vee_{\mathrm{sc}} = \mathsf{SL}_n(\mathbb{C})$ and $\mathrm{Z}_{G^\vee_{\mathrm{sc}}} = C_n$. The irreducible central characters are therefore $\widehat{\mathrm{Z}_{\mathsf{SL}_n(\mathbb{C})}} = \{\zeta_r \mid r\in\mathbb{Z}/n\mathbb{Z}\}$. A Langlands parameter (x,ϕ) parametrizes an irreducible unipotent representation of $\mathsf{SL}_m(D_{d,r/m})$ if and only if $\zeta_\phi = \zeta_r$. In particular, the unipotent representations of $\mathsf{SL}_n(F)$ correspond to central characters $\zeta_0 = 1$.

Moreover, for $x\in\mathsf{PGL}_n(\mathbb{C})$, A^1_x is the group of components of $\mathrm{Z}^1_{\mathsf{SL}_n(\mathbb{C})}(x) = \{g\in\mathsf{SL}_n(\mathbb{C}) \mid gxg^{-1} = x\}$.

Remark 4.6. Note the Langlands parameters we call unipotent here are also known as *unramified* Langlands parameters (cf. [Vog93, Sol23a]).

5. Lusztig's nonabelian Fourier transform for finite groups

We recall the definition of the nonabelian Fourier transform [Luz84a]. For the background material, we follow [Luz84a, GM20, DM90].

5.1 Fourier transforms

For a finite group Γ, define $\mathcal{M}(\Gamma)$ to be the set

$$\{(x,\sigma)\mid x\in\Gamma,\ \sigma\in\widehat{\mathrm{Z}_\Gamma(x)}\}, \tag{5.1}$$

modulo the equivalence relation given by conjugation by Γ: $g\cdot(x,\sigma) = (gxg^{-1},\sigma^g)$, where $\sigma^g(y) = \sigma(g^{-1}yg)$ for all $g\in\Gamma$, $y\in\mathrm{Z}_\Gamma(gxg^{-1})$. Define also

$$\mathcal{Y}(\Gamma) = \{(y,z)\in\Gamma\times\Gamma \mid yz = zy\}. \tag{5.2}$$

Write $\Gamma\setminus\mathcal{Y}(\Gamma)$ for the set of Γ-orbits on $\mathcal{Y}(\Gamma)$. Let $\mathsf{Sh}^\Gamma(\Gamma)$ be the category of Γ-equivariant coherent sheaves of Γ (Γ acting on itself by conjugation). The irreducible objects in $\mathsf{Sh}^\Gamma(\Gamma)$ are parametrized by $\mathcal{M}(\Gamma)$: for every pair $(x,\sigma)\in\mathcal{M}(\Gamma)$, let $\mathcal{V}((x,\sigma)) = \Gamma\times_{\mathrm{Z}_\Gamma(x)}\sigma$ be the corresponding irreducible Γ-equivariant sheaf. This means that there is a natural isomorphism $\mathbb{C}[\mathcal{M}(\Gamma)]\cong K(\mathsf{Sh}^\Gamma(\Gamma))_{\mathbb{C}}$, where $K(\ )_{\mathbb{C}}$ is the complexification of the K-group. Moreover, there is

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an isomorphism

$$\kappa\colon \mathbb{C}[\mathcal{M}(\Gamma)] \cong K(\mathsf{Sh}^{\Gamma}(\Gamma))_{\mathbb{C}} \to \mathbb{C}[\Gamma \setminus \mathcal{Y}(\Gamma)], \quad \mathcal{V} \mapsto ((y,z) \mapsto \mathrm{tr}(z, \mathcal{V}|_y)).$$

Lusztig [Luz84a] defined a pairing on $\mathcal{M}(\Gamma)$:

$$\{(x,\sigma),(y,\tau)\} = \frac{1}{|\mathrm{Z}_\Gamma(x)||\mathrm{Z}_\Gamma(y)|} \sum_{\substack{g\in\Gamma \\ xgyg^{-1}=gyg^{-1}x}} \sigma(gyg^{-1})\tau(g^{-1}x^{-1}g), \tag{5.3}$$

which extends to a Hermitian pairing on $\mathbb{C}[\mathcal{M}(\Gamma)]$. He also defined a linear map, the Fourier transform for Γ,

$$\mathrm{FT}_\Gamma\colon \mathbb{C}[\mathcal{M}(\Gamma)] \to \mathbb{C}[\mathcal{M}(\Gamma)], \quad \mathrm{FT}_\Gamma(f)(x,\sigma) = \sum_{(y,\tau)\in\mathcal{M}(\Gamma)} \{(x,\sigma),(y,\tau)\} f(y,\tau). \tag{5.4}$$

See Lemma 5.1 for the interpretation of FT_Γ in terms of $\mathcal{Y}(\Gamma)$.

Now consider the following generalization. Suppose $\widetilde{\Gamma} = \Gamma \rtimes \langle\alpha\rangle$, where α has order c. Set $\Gamma' = \Gamma\alpha \subset \widetilde{\Gamma}$. As in [Luz84a, § 4.16], define two sets $\mathcal{M} = \mathcal{M}(\Gamma \trianglelefteq \widetilde{\Gamma})$ and $\overline{\mathcal{M}} = \overline{\mathcal{M}}(\Gamma \trianglelefteq \widetilde{\Gamma})$ as follows:

$$\begin{aligned} \mathcal{M} &= \{(x,\sigma) \mid x \in \Gamma \text{ such that } \mathrm{Z}_{\widetilde{\Gamma}}(x) \cap \Gamma' \neq \emptyset,\ \sigma \in \widehat{\mathrm{Z}_{\widetilde{\Gamma}}(x)} \text{ with } \sigma|_{\mathrm{Z}_\Gamma(x)} \text{ irreducible}\}, \\ \overline{\mathcal{M}} &= \{(x,\bar\sigma) \mid x \in \Gamma',\ \bar\sigma \in \widehat{\mathrm{Z}_\Gamma(x)}\}, \end{aligned} \tag{5.5}$$

in each case modulo the equivalence relation given by conjugation by $\widetilde{\Gamma}$.

In addition, the cyclic group $\langle\alpha\rangle$ acts on $\mathcal{M}$ by twists in the second entry of the pair (x,σ). Denote by $\sim_c$ the corresponding equivalence relation.

The set $\mathcal{M}$ is a subset of $\mathcal{M}(\widetilde{\Gamma})$. Given $(x,\bar\sigma) \in \overline{\mathcal{M}}$, we have that $(x,\sigma) \in \mathcal{M}(\widetilde{\Gamma})$ for any extension σ of $\bar\sigma$ to $\mathrm{Z}_{\widetilde{\Gamma}}(x)$. Thus, the pairing $\{\ ,\ \}$ on $\mathcal{M}(\widetilde{\Gamma})$ induces a pairing

$$\{\ ,\ \} : \overline{\mathcal{M}} \times \mathcal{M} \to \mathbb{C}, \quad \{(x,\bar\sigma),(y,\tau)\} := c\{(x,\sigma),(y,\tau)\}, \tag{5.6}$$

for any fixed extension σ of $\bar\sigma$ to $\mathrm{Z}_{\widetilde{\Gamma}}(x)$.

Let $\mathcal{P} = \mathcal{P}(\Gamma \trianglelefteq \widetilde{\Gamma})$ and $\overline{\mathcal{P}} = \overline{\mathcal{P}}(\Gamma \trianglelefteq \widetilde{\Gamma})$ be the spaces of functions on $\mathcal{M}(\widetilde{\Gamma})$ with support in $\mathcal{M}$ and $\overline{\mathcal{M}}$, respectively. The operator [Luz84a, (4.16.1)] (see also [GM20, § 4.2.14])

$$\mathrm{FT}_{\Gamma\trianglelefteq\widetilde{\Gamma}} : \mathcal{P} \to \overline{\mathcal{P}},\ \mathrm{FT}_{\Gamma\trianglelefteq\widetilde{\Gamma}} f(x,\bar\sigma) = \sum_{(y,\tau)\in\mathcal{M}/\sim_c} \{(x,\bar\sigma),(y,\tau)\} f(y,\tau) \tag{5.7}$$

is an isomorphism with inverse $\mathrm{FT}^{-1}_{\Gamma\trianglelefteq\widetilde{\Gamma}} f(y,\tau) = \sum_{(x,\bar\sigma)\in\overline{\mathcal{M}}} \{(x,\bar\sigma),(y,\tau)\} f(x,\bar\sigma)$.

5.2 Families of Weyl group representations

Let W be a finite Weyl group with the set of simple generators S. The partition of $\widehat{W}$ into families is defined in [Luz84a, § 4.2] as follows. Let sgn denote the sign character of W. If $W = \{1\}$, there is only one family consisting of the trivial representation. Otherwise, assume that the families have been defined for all proper parabolic subgroups of W. Then $\mu, \mu' \in \widehat{W}$ belong to the same family of W if there exists a sequence $\mu = \mu_0, \mu_1, \ldots, \mu_m = \mu'$, $\mu_i \in \widehat{W}$, such that for each i there exists a parabolic subgroup $W_i \subsetneq W$ and $\mu_i', \mu_i'' \in \widehat{W}_i$ in the same family of W_i such that either

$$\langle \mu_i', \mu_{i-1}\rangle_{W_i} \neq 0,\ a_{\mu_i'} = a_{\mu_{i-1}}, \quad \langle \mu_i'', \mu_i\rangle_{W_i} \neq 0,\ a_{\mu_i''} = a_{\mu_i}$$

or

$$\langle \mu_i', \mu_{i-1}\otimes\mathrm{sgn}\rangle_{W_i} \neq 0,\ a_{\mu_i'} = a_{\mu_{i-1}\otimes\mathrm{sgn}}, \quad \langle \mu_i'', \mu_i\otimes\mathrm{sgn}\rangle_{W_i} \neq 0,\ a_{\mu_i''} = a_{\mu_i\otimes\mathrm{sgn}}.$$

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Here a_μ is the a-invariant of μ defined in [Luz84a, § 4.1]. It follows from the definition that if $\mathcal{F} \subset \widehat{W}$ is a family, then so is $\mathcal{F} \otimes \mathrm{sgn}$ and the families for $W_1 \times W_2$ are $\mathcal{F}_1 \boxtimes \mathcal{F}_2$, where $\mathcal{F}_i$ is a family for W_i, $i = 1, 2$.

Suppose in addition that we have a Coxeter group automorphism $\sigma\colon W \to W$, i.e. $\sigma \in \mathrm{Aut}(W)$ such that $\sigma(S) = S$. Such an automorphism is called ordinary if, on each irreducible component of W, it is not the nontrivial graph automorphism of type B_2, G_2 or F_4. The automorphism σ acts on $\widehat{W}$ and it permutes the families $\mathcal{F}$. An important observation [Luz84a, § 4.17] is that if σ is ordinary and $\mathcal{F}$ is σ-stable, then every element of $\mathcal{F}$ is σ-stable.

5.3 Families of unipotent representations

Let $\mathbb{G}$ be a connected reductive algebraic group over $\overline{\mathbb{F}}_q$ with a Frobenius map $\mathrm{Fr}\colon \mathbb{G} \to \mathbb{G}$ such that there exists a maximal torus $\mathbb{T}_0$ with the property $\mathrm{Fr}(t) = t^q$, for all $t \in \mathbb{T}_0$. Let $W = N_{\mathbb{G}}(\mathbb{T}_0)/\mathbb{T}_0$ be the Weyl group. Recall that an irreducible representation $\rho \in \mathrm{Irr}\ \mathbb{G}^{\mathrm{Fr}}$ is called *unipotent* if $\langle \rho, R_{\mathbb{T}}^{\mathbb{G}}(1)\rangle_{\mathbb{G}^{\mathrm{Fr}}} \neq 0$ for some Fr-stable maximal torus $\mathbb{T}$ of $\mathbb{G}$. Here $R_{\mathbb{T}}^{\mathbb{G}}$ is Deligne–Lusztig induction [DL76, § 7.8]. Let $\mathrm{Irr}_{\mathrm{un}}\mathbb{G}^{\mathrm{Fr}}$ denote the set of irreducible unipotent $\mathbb{G}^{\mathrm{Fr}}$-representations. By the results of Lusztig, the classification of $\mathrm{Irr}_{\mathrm{un}}\mathbb{G}^{\mathrm{Fr}}$ is reduced to the case when $\mathbb{G}$ is adjoint simple (see, for example, the exposition in [GM20, Remark 4.2.1]). More precisely, if $\pi\colon \mathbb{G} \to \mathbb{G}_{\mathrm{ad}}$ is the surjective homomorphism with central kernel ($\mathbb{G}_{\mathrm{ad}}$ is the semisimple adjoint group isogeneous to $\mathbb{G}/\mathrm{Z}_{\mathbb{G}}$), there exists a Frobenius map $\mathrm{Fr}_{\mathrm{ad}}$ such that $\mathrm{Fr}_{\mathrm{ad}} \circ \pi = \pi \circ \mathrm{Fr}$ such that the resulting group homomorphism $\pi\colon \mathbb{G}^{\mathrm{Fr}} \to \mathbb{G}_{\mathrm{ad}}^{\mathrm{Fr}_{\mathrm{ad}}}$ induces a bijection

$$\mathrm{Irr}_{\mathrm{un}}\mathbb{G}^{\mathrm{Fr}} \leftrightarrow \mathrm{Irr}_{\mathrm{un}}\mathbb{G}_{\mathrm{ad}}^{\mathrm{Fr}_{\mathrm{ad}}}.$$

Furthermore, write $\mathbb{G}_{\mathrm{ad}} = \mathbb{G}_1 \times \cdots \times \mathbb{G}_r$ for the decomposition into factors such that each $\mathbb{G}_i$ is semisimple adjoint, $\mathrm{Fr}_{\mathrm{ad}}$-stable, and a direct product of simple algebraic groups that are cyclically permuted by $\mathrm{Fr}_{\mathrm{ad}}$. Let $\mathbb{H}_i$ be one of the simple factors in $\mathbb{G}_i$: if h_i is the number of copies of $\mathbb{H}_i$, then $\mathrm{Fr}_{\mathrm{ad}}^{h_i}$ preserves $\mathbb{H}_i$. Denote by Fr_i the restriction to $\mathbb{H}_i$. Then

$$\mathrm{Irr}_{\mathrm{un}}\mathbb{G}_{\mathrm{ad}}^{\mathrm{Fr}_{\mathrm{ad}}} \cong \prod_{i=1}^{r} \mathrm{Irr}_{\mathrm{un}}\mathbb{H}_i^{\mathrm{Fr}_i}.$$

The Frobenius map Fr induces a Coxeter group automorphism σ of W. Define a graph with vertices $\mathrm{Irr}_{\mathrm{un}}\mathbb{G}^{\mathrm{Fr}}$ as follows: $\rho_1, \rho_2 \in \mathrm{Irr}_{\mathrm{un}}\mathbb{G}^{\mathrm{Fr}}$ are joined by an edge if and only if there is σ-stable $\mu \in \widehat{W}$ such that $\langle \rho_i, R_{\widetilde{\mu}}\rangle_{\mathbb{G}^{\mathrm{Fr}}} \neq 0$ for $i = 1, 2$ where $R_{\widetilde{\mu}}$ is the almost character associated to a fixed extension $\widetilde{\mu}$ of μ to $\widetilde{W} = W \rtimes \langle\sigma\rangle$ as defined in [Luz84a, (3.7.1)]. Each connected component of this graph is called a *family* in $\mathrm{Irr}_{\mathrm{un}}\mathbb{G}^{\mathrm{Fr}}$. One can define an equivalence relation on the set $\widehat{W}^\sigma$ of σ-stable irreducible W-representations: μ and μ' are equivalent if $R_{\widetilde{\mu}}$ and $R_{\widetilde{\mu}'}$ have unipotent constituents in the same family. By [Luz84a] (see also [GM20, Proposition 4.2.3]), the equivalence classes are the same as the σ-stable families in $\widehat{W}$, when σ is ordinary.

To each family $\mathcal{U} \subset \mathrm{Irr}_{\mathrm{un}}\mathbb{G}^{\mathrm{Fr}}$ corresponding to the σ-stable family $\mathcal{F} \subset \widehat{W}^\sigma$, Lusztig [Luz84a, § 4] attached finite groups $\Gamma_{\mathcal{U}} \trianglelefteq \widetilde{\Gamma}_{\mathcal{U}}$ such that $\widetilde{\Gamma}_{\mathcal{U}} = \Gamma_{\mathcal{U}}\langle\sigma\rangle$, a bijection

$$\mathcal{U} \longleftrightarrow \overline{\mathcal{M}}(\Gamma_{\mathcal{U}} \trianglelefteq \widetilde{\Gamma}_{\mathcal{U}}), \quad \rho \mapsto \bar{x}_\rho, \tag{5.8}$$

scalars $\Delta(\bar{x}_\rho) \in \{\pm 1\}$ (see [Luz84a, § 6.7]), and an injection

$$\mathcal{F} \longrightarrow \mathcal{M}(\Gamma_{\mathcal{U}} \trianglelefteq \widetilde{\Gamma}_{\mathcal{U}}), \quad \mu \mapsto x_\mu, \tag{5.9}$$

such that, when σ is ordinary, [Luz84a, Theorem 4.23] states that

$$\langle \rho, R_{\widetilde{\mu}}\rangle_{\mathbb{G}^{\mathrm{Fr}}} = \Delta(\bar{x}_\rho)\{\bar{x}_\rho, x_\mu\}. \tag{5.10}$$

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Define the *unipotent almost characters* of $\mathbb{G}^{\mathrm{Fr}}$ to be the set of orthonormal class functions

$$R_x = \sum_{\rho \in \mathcal{U}} \Delta(\bar{x}_\rho)\{\bar{x}_\rho, x\}\rho, \quad x \in \mathcal{M}(\Gamma_{\mathcal{U}} \trianglelefteq \widetilde{\Gamma}_{\mathcal{U}}). \tag{5.11}$$

Hence, the unipotent nonabelian Fourier transform of $\mathbb{G}^{\mathrm{Fr}}$

$$\mathrm{FT}_{\mathbb{G}^{\mathrm{Fr}}} := \bigoplus_{\mathcal{U} \subset \mathrm{Irr}_{\mathrm{un}}\mathbb{G}^{\mathrm{Fr}}} \mathrm{FT}_{\Gamma_{\mathcal{U}} \trianglelefteq \widetilde{\Gamma}_{\mathcal{U}}} \tag{5.12}$$

gives the change of bases matrix, up to the signs $\Delta(\bar{x}_\rho)$, between irreducible unipotent characters and almost characters.

Assume $\mathbb{G}$ is Fr-split, so that σ is trivial and a family $\mathcal{U}$ is parametrized by $\mathcal{M}(\Gamma_{\mathcal{U}})$. Let $x \in \Gamma = \Gamma_{\mathcal{U}}$, and define the virtual combinations of unipotent characters

$$\Pi_{\mathcal{U}}(x, y) = \sum_{\sigma \in \widehat{\mathrm{Z}_\Gamma(x)}} \sigma(y^{-1})\rho_{(x,\sigma)}, \quad \text{where } y \in \mathrm{Z}_\Gamma(x), \tag{5.13}$$

where $\rho_{(x,\sigma)}$ is the representation in $\mathcal{U}$ parametrized by $(x, \sigma) \in \mathcal{M}(\Gamma_{\mathcal{U}})$. If Γ is abelian (which is often the case), we may also define

$$\Pi_{\mathcal{U}}(\sigma, \tau) = \sum_{y \in \Gamma} \tau(y)\rho_{(y,\sigma)}, \quad \text{if } \sigma, \tau \in \widehat{\Gamma}. \tag{5.14}$$

Lemma 5.1 (Cf. [DM90]). *With the notation of (5.13)–(5.14) and $\Gamma = \Gamma_{\mathcal{U}}$,*

$$\mathrm{FT}_\Gamma(\Pi_{\mathcal{U}}(x, y)) = \Pi_{\mathcal{U}}(y, x), \quad \mathrm{FT}_\Gamma(\Pi_{\mathcal{U}}(\sigma, \tau)) = \Pi_{\mathcal{U}}(\tau, \sigma),$$

the latter when Γ is abelian.

Proof. We verify the first formula. The second is analogous (or it follows by change of bases). Denote by C_y the conjugacy class of y in $\mathrm{Z}_\Gamma(x)$. Then,

$$\begin{aligned}
&\mathrm{FT}_{\Gamma_{\mathcal{U}}}(\Pi_{\mathcal{U}}(x, y)) \\
&\quad = \sum_\sigma \sigma(y^{-1}) \sum_{(z,\tau)} \{(x, \sigma), (z, \tau)\}\rho_{(z,\tau)} \\
&\quad = \sum_\sigma \sigma(y^{-1}) \sum_{(z,\tau)} \frac{1}{|\mathrm{Z}_\Gamma(x)||\mathrm{Z}_\Gamma(z)|} \sum_{g \in \Gamma,\ gzg^{-1} \in \mathrm{Z}_\Gamma(x)} \sigma(gzg^{-1})\tau(g^{-1}x^{-1}g)\rho_{(z,\tau)} \\
&\quad = \sum_{(z,\tau)} \frac{1}{|\mathrm{Z}_\Gamma(z)|} \Bigg(\sum_{g \in \Gamma,\ gzg^{-1} \in \mathrm{Z}_\Gamma(x)} \frac{1}{|\mathrm{Z}_\Gamma(x)|} \sum_\sigma \sigma(y^{-1})\sigma(gzg^{-1}) \Bigg) \tau(g^{-1}x^{-1}g)\rho_{(z,\tau)} \\
&\quad = \sum_{g,z \in \Gamma,\ gzg^{-1} \in C_y} \frac{1}{|C_y|} \frac{1}{|\mathrm{Z}_\Gamma(z)|} \sum_{\tau \in \widehat{\mathrm{Z}_\Gamma(z)}} \tau(g^{-1}x^{-1}g)\rho_{(z,\tau)} \quad \text{(by character orthogonality)} \\
&\quad = \frac{1}{|C_y|} \sum_{y' \in C_y} \sum_{\tau \in \mathrm{Z}_\Gamma(\widehat{g^{-1}y'g})} \tau(g^{-1}x^{-1}g)\rho_{(g^{-1}y'g,\tau)} \quad \text{(setting } y' = gzg^{-1}\text{)} \\
&\quad = \frac{1}{|C_y|} \sum_{y' \in C_y} \Pi_{\mathcal{U}}(g^{-1}y'g, g^{-1}xg) = \Pi_{\mathcal{U}}(y, x),
\end{aligned}$$

where we used column orthogonality of characters and that (y, x) is Γ-conjugate to $(g^{-1}y'g, g^{-1}xg)$ when $y' \in C_y$. □

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6. Disconnected groups over finite fields

Suppose that $\mathbb{G}$ is a disconnected reductive group over $\overline{\mathbb{F}}_p$ with Frobenius map $\mathrm{Fr}\colon \mathbb{G} \to \mathbb{G}$ and identity component $\mathbb{G}^\circ$ such that $A = \mathbb{G}/\mathbb{G}^\circ$ is abelian. (In our applications, $\mathbb{G}/\mathbb{G}^\circ$ will almost always be a cyclic group.) By definition, the irreducible unipotent $\mathbb{G}^{\mathrm{Fr}}$-representations $\mathrm{Irr}_{\mathrm{un}}\mathbb{G}^{\mathrm{Fr}}$ are the constituents of all induced representations $\mathrm{Ind}_{\mathbb{G}^{\circ\mathrm{Fr}}}^{\mathbb{G}^{\mathrm{Fr}}}\rho$, where $\rho \in \mathrm{Irr}_{\mathrm{un}}\mathbb{G}^{\circ\mathrm{Fr}}$. See [GM20, Proposition 4.8.19] for the compatibility with the definition in terms of the appropriate version of $R_{\mathbb{T}}^{\mathbb{G}}(1)$. The parametrization of $\mathrm{Irr}_{\mathrm{un}}\mathbb{G}^{\mathrm{Fr}}$ follows from that of $\mathrm{Irr}_{\mathrm{un}}\mathbb{G}^{\circ\mathrm{Fr}}$ via Mackey induction using the explicit results for simple groups, e.g. [GM20, Theorems 4.5.11 and 4.5.12].

We are interested in studying the irreducible unipotent representations for groups $\mathbb{G}^{\mathrm{Fr}}$ that are related via the structure theory of p-adic groups.

Let $\mathbb{G}$ be a reductive algebraic group over $\overline{\mathbb{F}}_p$ with identity component $\mathbb{G}^\circ$ and such that $\mathbb{G}/\mathbb{G}^\circ = A$ is a finite abelian group. Let Fr_0 be a Frobenius map on $\mathbb{G}$ and assume that $\mathbb{G}^{\mathrm{Fr}_0}$ is split. Given $a \in A$, conjugation by a defines an outer automorphism of $\mathbb{G}^\circ$, which induces an isomorphism, call it σ_a, of the based root datum of $\mathbb{G}^\circ$. For every a, define the Frobenius automorphism $\mathrm{Fr}_a = \mathrm{Fr}_0 \circ \sigma_a$. By analogy with § 2, we write

$$\mathrm{Inn}\mathrm{T}^p\mathbb{G} = \{\mathbb{G}^{\mathrm{Fr}_a} \mid a \in A\}$$

and call this the set of pure inner twists of $\mathbb{G}^{\mathrm{Fr}_0}$. Just as in the p-adic case, this set is in one-to-one correspondence with the first Galois cohomology group

$$\mathrm{Inn}\mathrm{T}^p\mathbb{G} \leftrightarrow H^1(\mathbb{F}_q, \mathbb{G}) \cong H^1(\mathbb{F}_q, \mathbb{G}/\mathbb{G}^\circ) = H^1(\mathbb{F}_q, A) \cong A,$$

using the fact that $H^1(\mathbb{F}_q, \mathbb{G}^\circ) = 0$ by Lang's Theorem (see, for example, [Ser02, III.§ 2, Corollary 3]), and the assumption that Fr_0 acts trivially on A.

By (5.8), every unipotent family $\mathcal{U} \subset \mathrm{Irr}_{\mathrm{un}}(\mathbb{G}^{\circ\mathrm{Fr}_0})$ has an associated finite group $\Gamma_{\mathcal{U}} = \widetilde{\Gamma}_{\mathcal{U}}$ (since $\mathbb{G}^{\circ\mathrm{Fr}_0}$ is split). The group A acts on the set of families $\mathcal{U}$. For every orbit $\mathcal{O}_A = A \cdot \mathcal{U}$ with representative $\mathcal{U}$, let $\mathrm{Z}_A(\mathcal{U})$ be the corresponding isotropy group. Then $\mathrm{Z}_A(\mathcal{U})$ permutes the elements of $\mathcal{U}$, hence the corresponding parameters $\mathcal{M}(\Gamma_{\mathcal{U}})$. If $\Gamma_{\mathcal{U}}$ is abelian, which turns out to be the case in all of the examples of interest to us when $A \neq \{1\}$, this automatically defines an action of $\mathrm{Z}_A(\mathcal{U})$ on $\Gamma_{\mathcal{U}}$, hence a group

$$\widetilde{\Gamma}_{\mathcal{U}}^A = \Gamma_{\mathcal{U}} \rtimes \mathrm{Z}_A(\mathcal{U}). \tag{6.1}$$

See [DM90, § 5] and [Lus86, § 17] for more details.

Proposition 6.1 (Cf. [DM90, Proposition 5.2]). *As above, assume $\mathbb{G}$ is Fr_0-split. The parametrization (5.8) induces a bijection*

$$\bigsqcup_{a \in A} \mathrm{Irr}_{\mathrm{un}}(\mathbb{G}^{F_a}) \longleftrightarrow \bigsqcup_{\mathcal{U} \subset A\backslash \mathrm{Irr}_{\mathrm{un}}(\mathbb{G}^{\circ F_0})} \mathcal{M}(\widetilde{\Gamma}_{\mathcal{U}}^A),$$

where $\mathcal{U}$ in the right-hand side ranges over a set of representatives of the A-orbits of families in $\mathrm{Irr}_{\mathrm{un}}(\mathbb{G}^{\circ F_0})$.

Proof. This can be viewed as a particular case of [DM90, Proposition 5.2]. There the right-hand side of the bijection involves the groups $\overline{\mathcal{M}}(\widetilde{\Gamma}_{\mathcal{U}}^A \subset \widetilde{\Gamma}_{\mathcal{U}}^A \rtimes \langle \mathrm{Fr}_0 \rangle)$, but since we are assuming $\mathbb{G}$ is Fr_0-split, $\overline{\mathcal{M}}(\widetilde{\Gamma}_{\mathcal{U}}^A \subset \widetilde{\Gamma}_{\mathcal{U}}^A \rtimes \langle \mathrm{Fr}_0 \rangle) \cong \mathcal{M}(\widetilde{\Gamma}_{\mathcal{U}}^A)$. □

Remark 6.2. The bijection in Proposition 6.1 is not unique. To account for this (and in order to be able to carry out computations later on), we will work out explicit parametrizations in Examples 6.3–6.8. These cases are relevant for the branching computations for the unipotent representations of reductive p-adic groups in the inner class of the split group.

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Let $R_{\mathrm{un}}(\mathbb{G}^{F_a})$ be the $\mathbb{C}$-span of $\mathrm{Irr}_{\mathrm{un}}(\mathbb{G}^{F_a})$. The bijection of Proposition 6.1 induces a linear isomorphism

$$\bigoplus_{a\in A} R_{\mathrm{un}}(\mathbb{G}^{\mathrm{Fr}_a}) \to \bigoplus_{\mathcal{U}\subset A\backslash \mathrm{Irr}_{\mathrm{un}}\mathbb{G}^{\circ\mathrm{Fr}_0}} \mathbb{C}[\mathcal{M}(\widetilde{\Gamma}_{\mathcal{U}}^{A})]. \tag{6.2}$$

The right-hand side of (6.2) has the involution given by (5.4). Define

$$\mathrm{FT}_{\mathbb{G}}\colon \bigoplus_{a\in A} R_{\mathrm{un}}(\mathbb{G}^{\mathrm{Fr}_a}) \to \bigoplus_{a\in A} R_{\mathrm{un}}(\mathbb{G}^{\mathrm{Fr}_a}), \tag{6.3}$$

to be the corresponding involution on the left-hand side.

In the examples below, when A is clear from the context, we may write $\widetilde{\Gamma}_{\mathcal{U}}$ in place of $\widetilde{\Gamma}_{\mathcal{U}}^{A}$ for simplicity of notation.

Example 6.3. Let $\mathbb{H}$ be a connected almost simple $\mathbb{F}_q$-split group and $\mathbb{G} = (\mathbb{H}\times\mathbb{H})\rtimes \mathbb{Z}/2\mathbb{Z}$, where the nontrivial element δ of $A = \mathbb{Z}/2\mathbb{Z}$ acts by flipping the two copies of $\mathbb{H}$. There are two pure inner twists:

$$\mathrm{InnT}^p\mathbb{G} = \{\mathbb{H}(\mathbb{F}_q)^2\rtimes\mathbb{Z}/2\mathbb{Z}, \mathbb{H}(\mathbb{F}_{q^2})\rtimes\mathbb{Z}/2\mathbb{Z}\},$$

the second for the Frobenius map $\mathrm{Fr}_1(h_1,h_2) = (\mathrm{Fr}_0(h_2),\mathrm{Fr}_0(h_1))$, $h_1,h_2\in\mathbb{H}$. A family of $\mathbb{G}^\circ(\mathbb{F}_q) = \mathbb{H}(\mathbb{F}_q)^2$ is $\mathcal{U}_1\boxtimes\mathcal{U}_2$, where $\mathcal{U}_1$, $\mathcal{U}_2$ are unipotent families of $\mathbb{H}$. The A-orbits are either $\{\mathcal{U}_1\boxtimes\mathcal{U}_2, \mathcal{U}_2\boxtimes\mathcal{U}_1\}$ for $\mathcal{U}_1\neq\mathcal{U}_2$ or $\{\mathcal{U}\boxtimes\mathcal{U}\}$. Assume that all $\Gamma_{\mathcal{U}}$ are abelian. Set

$$\widetilde{\Gamma}^{\mathbb{Z}/2\mathbb{Z}}_{\mathcal{U}_1\boxtimes\mathcal{U}_2} = \Gamma_{\mathcal{U}_1}\times\Gamma_{\mathcal{U}_2},\ \mathcal{U}_1\neq\mathcal{U}_2,\quad \widetilde{\Gamma}^{\mathbb{Z}/2\mathbb{Z}}_{\mathcal{U}\boxtimes\mathcal{U}} = \Gamma^2_{\mathcal{U}}\rtimes\mathbb{Z}/2\mathbb{Z},$$

with the flip action of δ. There are $\ell(\ell+3)/2$ conjugacy classes in $\widetilde{\Gamma}_{\mathcal{U}\boxtimes\mathcal{U}}$, $\ell = |\Gamma_{\mathcal{U}}|$, and they are represented by:

- $(x,x')\sim(x',x)$ if $x\neq x'\in\Gamma_{\mathcal{U}}$, $\mathrm{Z}_{\widetilde{\Gamma}_{\mathcal{U}\boxtimes\mathcal{U}}}((x,x')) = \Gamma^2_{\mathcal{U}}$;
- (x,x), $x\in\Gamma_{\mathcal{U}}$, $\mathrm{Z}_{\widetilde{\Gamma}_{\mathcal{U}\boxtimes\mathcal{U}}}((x,x)) = \Gamma^2_{\mathcal{U}}\rtimes\mathbb{Z}/2\mathbb{Z}$;
- $(x,1)\delta$, $x\in\Gamma_{\mathcal{U}}$, $\mathrm{Z}_{\widetilde{\Gamma}_{\mathcal{U}\boxtimes\mathcal{U}}}((x,1)\delta) = \langle\Gamma^{\Delta}_{\mathcal{U}}, (x,1)\delta\rangle$, where $\Gamma^{\Delta}_{\mathcal{U}}$ is the diagonal copy of $\Gamma_{\mathcal{U}}$.

When $\mathcal{U}_1\neq\mathcal{U}_2$, if $\rho_1\in\mathcal{U}_1$, $\rho_2\in\mathcal{U}_2$, then $\rho_1\times\rho_2 := \mathrm{Ind}_{\mathbb{G}^\circ(\mathbb{F}_q)}^{\mathbb{G}(\mathbb{F}_q)}(\rho_1\boxtimes\rho_2)$ is parametrized by $(\bar{x}_{\rho_1},\bar{x}_{\rho_2})\in\mathcal{M}(\widetilde{\Gamma}_{\mathcal{U}_1\boxtimes\mathcal{U}_2})$.

In the second case, let $\rho,\rho'\in\mathcal{U}$. If $\rho\neq\rho'$, then $\rho\times\rho'\cong\rho'\times\rho$ is an irreducible representation of $\mathbb{G}(\mathbb{F}_q)$. If $\bar{x}_\rho = (x,\sigma)$, $\bar{x}_{\rho'} = (x',\sigma')$ are the corresponding parameters of ρ, ρ' in $\mathcal{M}(\Gamma_u)$, then the parameter for $\rho\times\rho'$ in $\mathcal{M}(\widetilde{\Gamma}_{\mathcal{U}\boxtimes\mathcal{U}})$ is $((x,x'),\sigma\boxtimes\sigma')$, if $x\neq x'$, or $((x,x),\sigma\times\sigma')$, where $\sigma\times\sigma' = \mathrm{Ind}_{\Gamma_{\mathcal{U}}}^{\Gamma^2_{\mathcal{U}}\rtimes\mathbb{Z}/2\mathbb{Z}}(\sigma\boxtimes\sigma')$, if $\sigma\neq\sigma'$.

If $\rho=\rho'$, then we can extend $\rho\boxtimes\rho$ in two different ways to $\mathbb{G}(\mathbb{F}_q)$, denoted by $(\rho\times\rho)^{\pm}$ relative to the character of $\mathbb{Z}/2\mathbb{Z}$. The corresponding parameters in $\mathcal{M}(\widetilde{\Gamma}_{\mathcal{U}\boxtimes\mathcal{U}})$ are $((x,x),(\sigma\times\sigma)^{\pm})$, with the obvious notation.

For the second pure inner twist, the irreducible unipotent representations of $\mathbb{H}(\mathbb{F}_{q^2})$ are given by the same families $\mathcal{U}$ as for $\mathbb{H}(\mathbb{F}_q)$ and δ fixes each unipotent representation ρ of $\mathbb{H}(\mathbb{F}_{q^2})$. Let ρ be an irreducible $\mathbb{H}(\mathbb{F}_{q^2})$-representation in $\mathcal{U}$ parametrized by $\bar{x}_\rho = (x,\sigma)$, $x\in\Gamma_{\mathcal{U}}$, $\sigma\in\widehat{\Gamma}_{\mathcal{U}}$. Then it can be extended in two different ways $\rho^{\pm}$ to $\mathbb{H}(\mathbb{F}_{q^2})\rtimes\mathbb{Z}/2\mathbb{Z}$. The centralizer $\mathrm{Z}_{\widetilde{\Gamma}_{\mathcal{U}\boxtimes\mathcal{U}}}((x,1)\delta) = \langle\Gamma^{\Delta}_{\mathcal{U}},(x,1)\delta\rangle$ is isomorphic to the direct product $C_x := \langle(y,y)\mid y\neq x\in\Gamma_{\mathcal{U}}\rangle\times\langle(x,1)\delta\rangle$, since $\big((x,1)\delta\big)^2 = (x,x)$. Regard σ as a representation of the subgroup $\Gamma^{\Delta}_{\mathcal{U}}$. There are two ways $\sigma^{\pm}$ to extend it to C_x, coming from the short exact sequence $1\to\langle(x,x)\rangle\to\langle(x,1)\delta\rangle\to\mathbb{Z}/2\mathbb{Z}\to 1$. We attach $((x,1)\delta,\sigma^{\pm})\in\mathcal{M}(\widetilde{\Gamma}_{\mathcal{U}\boxtimes\mathcal{U}})$ to $\rho^{\pm}$. To fix a choice of $\pm$, we fix a choice of primitive ℓth

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root of unity ζ_ℓ for each ℓ. Then, if $\sigma((x,x)) = \zeta_k^j$, for some j, where k is the order of x, then $\sigma^+((x,1)\delta) = \zeta_{2k}^j$. For our applications, $\mathbb{H}$ will be a classical group and therefore, $\Gamma_{\mathcal{U}}$ a 2-group, hence x will have order $k \leqslant 2$.

Example 6.4. Let $\mathbb{G}^\circ = \mathsf{GL}_k^m$, $\mathbb{G}^{\circ\mathrm{Fr}_0} = \mathsf{GL}_k(\mathbb{F}_q)^m$ and $A = \mathbb{Z}/m\mathbb{Z}$ acting by cyclic permutations on the factors of $\mathbb{G}^\circ$. Then

$$\mathrm{InnT}^p\mathbb{G} = \{\mathbb{G}^{\circ\mathrm{Fr}_r} \rtimes \mathbb{Z}/m\mathbb{Z} = \mathsf{GL}_k(\mathbb{F}_{q^{m/d}})^d \rtimes \mathbb{Z}/m\mathbb{Z} \mid r \in \mathbb{Z}/m\mathbb{Z},\ d = \gcd(r,m)\}.$$

Each unipotent family of $\mathsf{GL}_k(\mathbb{F}_{q^{m/d}})^d$ is a singleton $\{\rho_1 \boxtimes \cdots \boxtimes \rho_d\}$ where $\rho_i \in \widehat{S}_k$, $1 \leqslant i \leqslant d$. Hence, we can ignore the difference between unipotent families and irreducible representations of symmetric groups. The irreducible representations of $S_k^d \rtimes \mathbb{Z}/m\mathbb{Z}$ are constructed by Mackey theory.

Start with a unipotent representation $\rho = \rho_1 \boxtimes \cdots \boxtimes \rho_m$ of $\mathsf{GL}_k(\mathbb{F}_q)^m \rtimes \mathbb{Z}/m\mathbb{Z}$ with stabilizer $\mathbb{Z}/c\mathbb{Z}$, $c|m$. This means that $\rho_i = \rho_{i+m/c}$ for all i (viewed mod m) and that $\mathbb{Z}/(m/c)\mathbb{Z}$ has no fixed points under the cyclic action on $\rho_1 \boxtimes \cdots \boxtimes \rho_{m/c}$. The corresponding unipotent family $\widetilde{\mathcal{U}}$ that we construct for $\mathrm{InnT}^p\mathbb{G}$ has

$$\widetilde{\Gamma}_{\mathcal{U}}^{\mathbb{Z}/m\mathbb{Z}} = \mathbb{Z}/c\mathbb{Z}.$$

The irreducible representations $\widetilde{\rho}$ of $\mathbb{G}^{\mathrm{Fr}_0}$ whose restriction to $\mathbb{G}^{\circ\mathrm{Fr}_0}$ contain ρ are in one-to-one correspondence to the characters of $\mathbb{Z}/c\mathbb{Z}$, hence they are parametrized in $\mathcal{M}(\widetilde{\Gamma}_{\mathcal{U}})$ by the pairs $(0,\sigma)$, $\sigma \in \widehat{\mathbb{Z}/c\mathbb{Z}}$.

For every $r \in \mathbb{Z}/m\mathbb{Z}$ such that m/c divides $d = \gcd(r,m)$, consider the representation of $\mathbb{G}^{\circ\mathrm{Fr}_r}$ given by $\rho^r = \rho_1 \boxtimes \cdots \boxtimes \rho_d$. The stabilizer of this representation in $\mathbb{Z}/m\mathbb{Z}$ is also $\mathbb{Z}/c\mathbb{Z}$. The irreducible representations $\widetilde{\rho}^r$ of $\mathbb{G}^{\mathrm{Fr}_r}$ whose restriction to $\mathbb{G}^{\circ\mathrm{Fr}_r}$ contain ρ^r are again in one-to-one correspondence to the characters of $\mathbb{Z}/c\mathbb{Z}$, and we parametrize them in $\mathcal{M}(\widetilde{\Gamma}_{\mathcal{U}})$ by the pairs $(rc/m,\sigma)$, $\sigma \in \widehat{\mathbb{Z}/c\mathbb{Z}}$. This completes the parametrization via $\mathcal{M}(\widetilde{\Gamma}_{\mathcal{U}}^{\mathbb{Z}/m\mathbb{Z}})$ of the unipotent representations for $\mathrm{InnT}^p\mathbb{G}$ corresponding to the family $\mathcal{U} = \{\rho\}$ in $\mathbb{G}^{\circ\mathrm{Fr}_0}$.

Example 6.5. Let $\mathbb{G} = \mathrm{O}_{2n}, \mathbb{G}^\circ = \mathsf{SO}_{2n}$, $n \geqslant 2$, $A = \mathbb{Z}/2\mathbb{Z} = \langle\delta\rangle$. There are two pure inner twists $\mathrm{InnT}^p\mathbb{G} = \{\mathrm{O}_{2n}^+(\mathbb{F}_q), \mathrm{O}_{2n}^-(\mathbb{F}_q)\}$. In this case, we use the parametrizations of [Luz84a, §§ 4.6, 4.18]. Recall that a symbol for type D_n is an array $\Lambda = \left(\begin{smallmatrix}\lambda_1 & \lambda_2 & \cdots & \lambda_b\\ \mu_1 & \mu_2 & \cdots & \mu_{b'}\end{smallmatrix}\right)$, $b + b' = 2m$, $0 \leqslant \lambda_1 < \cdots < \lambda_{b'}$, $0 \leqslant \mu_1 < \cdots < \mu_{b'}$, which is considered the same as the array where the rows are flipped. A symbol where $b = b' = m$ and $\lambda_1 \leqslant \mu_1 \leqslant \lambda_2 \leqslant \mu_2 \leqslant \ldots$, $\sum \lambda_i^2 + \lambda\mu_i^2 = n + m^2 - m$ is called special.

Let Z be a special symbol. In the case when $\lambda_i = \mu_i$ for all $1 \leqslant i \leqslant m$, one attaches to Z two unipotent $\mathbb{G}^{\circ\mathrm{Fr}_0}$-families, $\mathcal{U}' = \{\rho\}$ and $\mathcal{U}'' = \{\rho'\}$, each consisting of a single unipotent representation and with $\Gamma_{\mathcal{U}'} = \Gamma_{\mathcal{U}''} = \{1\}$. In this case, the action of δ flips the two families. Hence they give rise to a single family $\{\widetilde{\rho}\}$ for $\mathbb{G}^{F_0}$, $\widetilde{\rho}|_{\mathbb{G}^{\circ\mathrm{Fr}_0}} = \rho \oplus \rho'$, and $\widetilde{\Gamma}_{\mathcal{U}'} = \{1\}$.

Assume now that the two rows of the symbol Z are not identical, i.e. Z is nondegenerate in the sense of [Luz84a]. Then Z defines one unipotent family for $\mathbb{G}^{\circ\mathrm{Fr}_0}$ and one for $\mathbb{G}^{\circ\mathrm{Fr}_1}$. Each element of these families is stable under δ so it can be lifted to two different $\mathbb{G}^{\mathrm{Fr}_0}$, respectively $\mathbb{G}^{\mathrm{Fr}_1}$, representations.

The unipotent representations in the $\mathbb{G}^{\circ\mathrm{Fr}_0}$-family $\mathcal{U}_Z$ corresponding to Z are indexed by the set $\mathcal{M}_Z$ of symbols Λ such that $b - b' \equiv 0 \bmod 4$, $b + b' = 2m$. The unipotent representations in the $\mathbb{G}^{\circ\mathrm{Fr}_1}$-family $\mathcal{U}_Z^-$ corresponding to Z are indexed by the set $\mathcal{M}_Z^-$ of symbols Λ such that $b - b' \equiv 2 \bmod 4$, $b + b' = 2m$. Let Z_1 be the set of elements that appear as entries of Z only once. Let $2d = |Z_1|$. Define:

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– V_{Z_1} as the set of subsets $X \subseteq Z_1$ of even cardinality, with the structure of an $\mathbb{F}_2$-vector space with the sum given by the symmetric difference;
– V'_{Z_1} as the set of subsets $X \subseteq Z_1$ with the structure of an $\mathbb{F}_2$-vector space with the sum given by the symmetric difference, modulo the line spanned by Z_1 itself;
– $(V'_{Z_1})^+$ as the subspace of V'_{Z_1} where the elements are the subsets X of even cardinality;
– $(V'_{Z_1})^-$ as the subspace of V'_{Z_1} where the elements are the subsets X of odd cardinality.

Note that $(V'_{Z_1})^+$ is also the image of the projection of V_{Z_1} to V'_{Z_1}. The dimensions of V'_{Z_1} and V_{Z_1} over $\mathbb{F}_2$ are $2d-1$, while the dimension of (V'_{Z_1}) is $2d-2$. There is a nonsingular pairing

$$(\,,\,)\colon V'_{Z_1} \times V_{Z_1} \to \mathbb{F}_2, \quad (X_1, X_2) \mapsto |X_1 \cap X_2| \bmod 2. \tag{6.4}$$

This pairing restricts to a nonsingular symplectic $\mathbb{F}_2$-form of $(V'_{Z_1})^+$. If V_{Z_1} has the basis $e_1, e_2, \dots, e_{2d-1}$ as in [Luz84a], then $(V'_{Z_1})^+$ is spanned by the images $\bar{e}_1, \bar{e}_2, \dots, \bar{e}_{2d-1}$ modulo the relation $\bar{e}_1 + \bar{e}_3 + \cdots + \bar{e}_{2d-1} = 0$. Let

$$\bar{I}' = \text{subspace of } (V'_{Z_1})^+ \text{ spanned by } \bar{e}_1, \bar{e}_3, \dots, \bar{e}_{2d-1},$$
$$\bar{I}'' = \text{subspace of } (V'_{Z_1})^+ \text{ spanned by } \bar{e}_2, \bar{e}_4, \dots, \bar{e}_{2d-2};$$

they are maximal isotropic subspaces of $(V'_{Z_1})^+$ and $(V'_{Z_1})^+ = \bar{I}' \oplus \bar{I}''$. Then

$$\Gamma_{\mathcal{U}_Z} = \bar{I}'' \cong (\mathbb{Z}/2\mathbb{Z})^{d-1}.$$

As shown in [Luz84a, § 4.6], there is a natural bijection

$$\mathcal{M}_Z \leftrightarrow \mathcal{M}(\Gamma_{\mathcal{U}_Z}) \cong (V'_{Z_1})^+ = \bar{I}' \oplus \bar{I}''$$

(where $\bar{I}'$ is identified with the group of characters of $\Gamma_{\mathcal{U}_Z}$). Denote

$$\widetilde{\Gamma}_{\mathcal{U}_Z} = \{v \in V'_{Z_1} \mid (v, e_{2i}) = 0,\ 1 \leqslant i \leqslant d-1\} \cong (\mathbb{Z}/2\mathbb{Z})^d. \tag{6.5}$$

Clearly, $\Gamma_{\mathcal{U}_Z} \trianglelefteq \widetilde{\Gamma}_{\mathcal{U}_Z}$. As shown in [Luz84a, § 4.18], there is a natural bijection

$$\mathcal{M}_Z^- \leftrightarrow \overline{\mathcal{M}}(\Gamma_{\mathcal{U}_Z} \trianglelefteq \widetilde{\Gamma}_{\mathcal{U}_Z}) \cong (V'_{Z_1})^- \cong (\widetilde{\Gamma}_{\mathcal{U}_Z} \setminus \Gamma_{\mathcal{U}_Z}) \times \bar{I}'.$$

Let $\widetilde{\mathcal{U}}_Z$ be the set (family) of irreducible representations in $\mathrm{Irr}_{\mathrm{un}}\mathbb{G}^{\mathrm{Fr}_0} \sqcup \mathrm{Irr}_{\mathrm{un}}\mathbb{G}^{\mathrm{Fr}_1}$ whose restrictions to $\mathbb{G}^{\circ\mathrm{Fr}_0}$ (respectively, $\mathbb{G}^{\circ\mathrm{Fr}_1}$) are in $\mathcal{U}_Z$ (respectively, $\mathcal{U}_Z^-$). Since each unipotent representation in $\mathcal{U}_Z$ and $\mathcal{U}_Z^-$ extends in two different ways to the corresponding disconnected group, the parametrization above implies easily that there is natural bijection

$$\widetilde{\mathcal{U}}_Z \longleftrightarrow \mathcal{M}(\widetilde{\Gamma}_{\mathcal{U}_Z}). \tag{6.6}$$

Explicitly, let $\{\bar{f}_1, \bar{f}_2, \dots, \bar{f}_d\}$ be the spanning set of V'_{Z_1} subject to $\sum_{i=1}^{2d} \bar{f}_i = 0$, such that $\bar{e}_i = \bar{f}_i + \bar{f}_{i+1}$, $1 \leqslant i \leqslant 2d-1$. Then an $\mathbb{F}_2$-basis of $\widetilde{\Gamma}_{\mathcal{U}_Z}$ is given by $\{\bar{f}_1, \bar{e}_2, \bar{e}_4, \dots, \bar{e}_{2d}\}$. An irreducible character of $\Gamma_{\mathcal{U}_Z} = \langle \bar{e}_2, \bar{e}_4, \dots, \bar{e}_{2d} \rangle$ can be extended in two different ways to $\widetilde{\Gamma}_{\mathcal{U}_Z}$ by setting the character value on $\bar{f}_1$ to 1 or -1. The value 1 corresponds to the representations of the identity components of $\mathbb{G}^{\mathrm{Fr}_0}$, $\mathbb{G}^{\mathrm{Fr}_1}$ extended by letting δ act trivially, while the -1 value corresponds to those where δ acts by -1.

Example 6.6. Let $\mathbb{G}^\circ$ be of type A_{k-1}, $k \geqslant 3$ or E_6, and $A = \mathbb{Z}/2\mathbb{Z} = \langle \delta \rangle$ acting by the nontrivial automorphism of the Dynkin diagram. The nonsplit pure inner twist has $\mathbb{G}^{\circ\mathrm{Fr}_1}$ of type ${}^2A_{k-1}$ or ${}^2E_6$, respectively. By [Luz84a, § 4.19], every unipotent family $\mathcal{U}$ of $\mathbb{G}^\circ$ is fixed pointwise by A.

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Hence,

$$\widetilde{\Gamma}_{\mathcal{U}}^{\mathbb{Z}/2\mathbb{Z}} = \Gamma_U \times \mathbb{Z}/2\mathbb{Z}, \quad \text{for all } \mathcal{U}.$$

Each irreducible representation ρ of the split form $\mathbb{G}^{\circ\mathrm{Fr}_0}$ can be extended to $\mathbb{G}^{\mathrm{Fr}_0}$ in two different ways $\rho^{\pm}$ corresponding to the two characters of $\mathbb{Z}/2\mathbb{Z}$. If the parameter for ρ is $\bar{x}_\rho = (x, \sigma) \in \mathcal{M}(\Gamma_{\mathcal{U}})$, then the parameters for $\rho^{\pm}$ are $((x, 1), \sigma^{\pm})$ with the obvious notation.

Similarly, an irreducible representation ρ' of the nonsplit pure inner twist $\mathbb{G}^{\circ\mathrm{Fr}_1}$ can be extended to $\mathbb{G}^{\mathrm{Fr}_1}$ in two different ways $\rho'^{\pm}$. If the parameter for ρ' is $\bar{x}_\rho = (x', \sigma') \in \mathcal{M}(\Gamma_{\mathcal{U}})$, then the parameters for $\rho'^{\pm}$ are $((x', \delta), \sigma'^{\pm})$.

Example 6.7. Let $\mathbb{H}$ be of type D_k, $k \geqslant 2$ and $\mathbb{G}^\circ = \mathbb{H} \times \mathbb{H}$. Let $A = \langle \delta_1 \rangle \times \langle \delta_2 \rangle \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, where δ_1 acts by the nontrivial outer automorphism of the Dynkin diagram of type D_k, and δ_2 flips the $\mathbb{H}$-factors. This case is therefore a combination of Examples 6.5 and 6.3 and the parametrization of families for the pure inner twists of $\mathbb{G}$ follows from these examples, i.e. the same parametrizations as in Example 6.3 but constructed from the orthogonal families $\widetilde{U}_Z$ from Example 6.5.

Now for the same $\mathbb{H}$ and $\mathbb{G}^\circ$, suppose $A = \langle \delta \rangle = \mathbb{Z}/4\mathbb{Z}$. If s_1', s_1'' are the two commuting extremal reflections of the first $\mathbb{H} = D_k$ and s_2', s_2'' are the similar reflections for the second $\mathbb{H} = D_k$, then δ acts by the cyclic permutation:

$$\delta\colon \quad s_1' \mapsto s_2' \mapsto s_1'' \mapsto s_2'' \mapsto s_1'.$$

On all the other simple reflections of the two components of type D_k, δ acts by the obvious diagram flip (of order 2). To describe the pure inner twists, let Fr denote the Frobenius map of $\mathbb{H}$ whose fixed points is the nonsplit group of type ${}^2D_k$. Then

$$\mathrm{Fr}_1\colon \mathbb{H} \times \mathbb{H} \to \mathbb{H} \times \mathbb{H}, \quad \mathrm{Fr}_1(h_1, h_2) = (\mathrm{Fr}(h_2), h_1)$$

is a Frobenius automorphism and $\mathrm{Fr}_r = \mathrm{Fr}_1^r$, $r \in \mathbb{Z}/4\mathbb{Z}$ (see, e.g., [GM20, Example 1.4.23]). The identity components of the pure inner twists are the finite reductive groups of types:

$$\mathbb{G}^{\circ\mathrm{Fr}_0}\colon\ D_k \times D_k, \quad \mathbb{G}^{\circ\mathrm{Fr}_1}\colon\ {}^2D_k, \quad \mathbb{G}^{\circ\mathrm{Fr}_2}\colon\ {}^2D_k \times\ {}^2D_k, \quad \mathbb{G}^{\circ\mathrm{Fr}_3}\colon\ {}^2D_k.$$

If ρ_1, ρ_2 are two unipotent representations of D_k, the action of δ is

$$\delta(\rho_1, \rho_2) = (\rho_2', \rho_1), \quad \text{where } \rho_2' = \begin{cases} \rho_2, & \text{if the symbol of } \rho_2 \text{ is nondegenerate,} \\ \rho_2^-, & \text{otherwise,} \end{cases}$$

where ρ_2^- is unipotent representation parametrized by the other degenerate symbol with the same rows. See Example 6.5.

We start with a family $\mathcal{U}_1 \times \mathcal{U}_2$ of $\mathbb{G}^{\circ\mathrm{Fr}_0} = D_k \times D_k$. If $\mathcal{U}_2$ consists of a degenerate symbol, then the stabilizer in A is always 1, regardless of what $\mathcal{U}_1$ is. (Similarly if $\mathcal{U}_1$ is degenerate.) In this case,

$$\widetilde{\Gamma}_{\mathcal{U}_1 \times \mathcal{U}_2}^{\mathbb{Z}/4\mathbb{Z}} = \Gamma_{\mathcal{U}_1}.$$

(Recall that $\Gamma_{\mathcal{U}_2} = 1$ necessarily.) Since the stabilizer in $\mathbb{Z}/4\mathbb{Z}$ of each representation $\rho_1 \boxtimes \rho_2$, $\rho_1 \in \mathcal{U}_1$, $\rho_2 \in \mathcal{U}_2$ is also trivial in this case, it follows that there is a one-to-one correspondence between the representations $\mathrm{Ind}_{\mathbb{G}^{\circ\mathrm{Fr}_0}}^{\mathbb{G}^{\mathrm{Fr}_0}}(\rho_1 \boxtimes \rho_2)$ and $\rho_1 \in \mathcal{U}_1$, hence a parametrization by $\mathcal{M}(\Gamma_{\mathcal{U}_1})$ as expected.

For the rest of the example, assume that all families correspond to nondegenerate symbols. Let Z_1, Z_2 be two nondegenerate symbols of type D_k. Let $\mathcal{U}_1$,$\mathcal{U}_2$ be the corresponding families

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for D_k and $\mathcal{U}_1^-, \mathcal{U}_2^-$ the families for ${}^2D_k$. Suppose first that $Z_1 \neq Z_2$. Then the stabilizer in A is $\mathbb{Z}/2\mathbb{Z} = \langle \delta^2 \rangle$, hence the group for the pure inner twists is

$$\widetilde{\Gamma}^{\mathbb{Z}/4\mathbb{Z}}_{\mathcal{U}_1 \boxtimes \mathcal{U}_2} = \Gamma_{\mathcal{U}_1} \times \Gamma_{\mathcal{U}_2} \times \mathbb{Z}/2\mathbb{Z}.$$

If $\rho_1 \in \mathcal{U}_1$ and $\rho_2 \in \mathcal{U}_2$, the stabilizer in A of $\rho_1 \boxtimes \rho_2$ is also $\mathbb{Z}/2\mathbb{Z} = \langle \delta^2 \rangle$. By Mackey theory, we get two irreducible representations of $\mathbb{G}^{\mathrm{Fr}_0}$ by inducing $\rho_1 \boxtimes \rho_2$ twisted by the trivial or the sign character of $\mathbb{Z}/2\mathbb{Z}$. If $\bar{x}_{\rho_i} = (x_i, \sigma_i) \in \mathcal{M}(\Gamma_{\mathcal{U}_i})$, $i = 1, 2$, then the two induced representations are parametrized by $((x_1, x_2, 1), \sigma_1 \boxtimes \sigma_2 \boxtimes \tau) \in \mathcal{M}(\Gamma_{\mathcal{U}_1} \times \Gamma_{\mathcal{U}_2} \times \mathbb{Z}/2\mathbb{Z})$, where τ is the trivial or the sign character of $\mathbb{Z}/2\mathbb{Z}$.

If $\rho_1' \in \mathcal{U}_1^-$ and $\rho_2' \in \mathcal{U}_2^-$, the analysis is analogous. The difference is that $\bar{x}_{\rho_i'} = (x_i', \sigma_i') \in \overline{\mathcal{M}}(\Gamma_{\mathcal{U}_i} \trianglelefteq \widetilde{\Gamma}_{\mathcal{U}_i})$, $i = 1, 2$, where $x_i' \in \widetilde{\Gamma}_{\mathcal{U}_i} \setminus \Gamma_{\mathcal{U}_i}$, $\sigma_i \in \widehat{\Gamma}_{\mathcal{U}_i}$. Write $x_i = y_i \alpha$, $i = 1, 2$, $y_i \in \Gamma_{\mathcal{U}_i}$, where α is the nontrivial automorphism of the D_k diagram. Then the two unipotent representations of $\mathbb{G}^{\mathrm{Fr}_2}$ whose restriction to $\mathbb{G}^{\circ \mathrm{Fr}_2}$ contain $\rho_1' \boxtimes \rho_2'$ are parametrized by $((y_1, y_2, \delta^2), \sigma_1 \boxtimes \sigma_2 \boxtimes \tau) \in \mathcal{M}(\Gamma_{\mathcal{U}_1} \times \Gamma_{\mathcal{U}_2} \times \mathbb{Z}/2\mathbb{Z})$, where τ is the trivial or the sign character of $\mathbb{Z}/2\mathbb{Z}$.

Finally, if $Z_1 = Z_2 = Z$ with the families $\mathcal{U}$ of D_k and $\mathcal{U}^-$ of ${}^2D_k$, then the stabilizer of $\mathcal{U} \boxtimes \mathcal{U}$ is $A = \mathbb{Z}/4\mathbb{Z}$. In this case, set

$$\widetilde{\Gamma}^{\mathbb{Z}/4\mathbb{Z}}_{\mathcal{U} \boxtimes \mathcal{U}} = (\Gamma_{\mathcal{U}} \times \Gamma_{\mathcal{U}}) \rtimes \mathbb{Z}/4\mathbb{Z}. \tag{6.7}$$

All four pure inner twists $\mathbb{G}^{\mathrm{Fr}_r}$ contribute in this case. Each conjugacy class in $\widetilde{\Gamma}^{\mathbb{Z}/4\mathbb{Z}}_{\mathcal{U} \boxtimes \mathcal{U}}$ is represented by an element (x, y, r) with $x, y \in \Gamma_{\mathcal{U}}$ and $r \in \mathbb{Z}/4\mathbb{Z}$. It will correspond to a unipotent representation of $\mathbb{G}^{\mathrm{Fr}_r}$, for the same r.

If $r = 0$, then the conjugacy classes are given by $(x, x', 0) \sim (x', x, 0)$ and its stabilizer in $\widetilde{\Gamma}^{\mathbb{Z}/4\mathbb{Z}}_{\mathcal{U} \boxtimes \mathcal{U}}$ is $\Gamma^2_{\mathcal{U}} \times \langle \delta^2 \rangle$ if $x \neq x'$, or all of $\widetilde{\Gamma}^{\mathbb{Z}/4\mathbb{Z}}_{\mathcal{U} \boxtimes \mathcal{U}}$ if $x = x'$. If $\rho, \rho' \in \mathcal{U}$ with parameters $\bar{x}_\rho = (x, \sigma)$, $\bar{x}_{\rho'} = (x', \sigma')$, it is clear that there is a perfect matching between the induced representation coming from the Mackey construction and the parameters $((x, x', \bar{0}), \widetilde{\sigma}) \in \mathcal{M}(\widetilde{\Gamma}^{\mathbb{Z}/4\mathbb{Z}}_{\mathcal{U} \boxtimes \mathcal{U}})$, where $\widetilde{\sigma} \in \widehat{\mathrm{Z}_{\widetilde{\Gamma}^{\mathbb{Z}/4\mathbb{Z}}_{\mathcal{U} \boxtimes \mathcal{U}}}((x, x', \bar{0}))}$.

If $r = 2$, the conjugacy classes are given by $(x, x', \delta^2) \sim (x', x, \delta^2)$ and its stabilizer in $\widetilde{\Gamma}^{\mathbb{Z}/4\mathbb{Z}}_{\mathcal{U} \boxtimes \mathcal{U}}$ is $\Gamma^2_{\mathcal{U}} \times \langle \delta^2 \rangle$ if $x \neq x'$ or all $\widetilde{\Gamma}^{\mathbb{Z}/4\mathbb{Z}}_{\mathcal{U} \boxtimes \mathcal{U}}$ if $x = x'$. If $\rho, \rho' \in \mathcal{U}^-$ with parameters $\bar{x}^-_\rho = (x\alpha, \sigma)$, $\bar{x}^-_{\rho'} = (x'\alpha, \sigma')$, again there is a perfect matching between the induced representations coming from the Mackey construction and the parameters $((x, x', \delta^2), \widetilde{\sigma}) \in \mathcal{M}(\widetilde{\Gamma}^{\mathbb{Z}/4\mathbb{Z}}_{\mathcal{U} \boxtimes \mathcal{U}})$, where $\widetilde{\sigma} \in \widehat{\mathrm{Z}_{\widetilde{\Gamma}^{\mathbb{Z}/4\mathbb{Z}}_{\mathcal{U} \boxtimes \mathcal{U}}}((x, x', \bar{2}))}$.

If $r = 1$ or 3, the conjugacy classes are given by $(x, 1, \delta^r)$, cf. Example 6.3. The stabilizer in this case is $\langle \Gamma^\Delta_{\mathcal{U}}, \delta^2, (x, 1, \delta) \rangle$. Let ρ be a representation in the unipotent family $\mathcal{U}^-$ with parameter $\bar{x}_\rho = (x\alpha, \sigma)$, $\sigma \in \widehat{\Gamma}_{\mathcal{U}}$. It can be extended in four different ways to ${}^2D_k \rtimes \mathbb{Z}/4\mathbb{Z}$ corresponding to the characters of $\mathbb{Z}/4\mathbb{Z}$. Since x has order 2, the cyclic group

$$\langle (x, 1, \delta) \rangle = \langle (1, 1, 1), (x, 1, \delta), (x, x, \delta^2), (1, x, \delta^3) \rangle \cong \mathbb{Z}/4\mathbb{Z}.$$

Note that there is a short exact sequence (that does not split)

$$1 \longrightarrow \Gamma^\Delta_{\mathcal{U}} \longrightarrow \mathrm{Z}_{\widetilde{\Gamma}^{\mathbb{Z}/4\mathbb{Z}}_{\mathcal{U} \boxtimes \mathcal{U}}}((x, 1, \delta^r)) \longrightarrow \mathbb{Z}/4\mathbb{Z} \longrightarrow 1,$$

where the quotient $\mathbb{Z}/4\mathbb{Z}$ is generated by the image of $(x, 1, \delta)$. This means that $\sigma \in \widehat{\Gamma}_{\mathcal{U}}$, viewed as a representation of $\Gamma^\Delta_{\mathcal{U}}$ can be lifted in four different ways to $\mathrm{Z}_{\widetilde{\Gamma}^{\mathbb{Z}/4\mathbb{Z}}_{\mathcal{U} \boxtimes \mathcal{U}}}((x, 1, \delta^r))$: the first can lift σ in two different ways to $\sigma^\pm$, representations of $\Gamma^\Delta_{\mathcal{U}} \times \langle \delta^2 \rangle$ corresponding to the trivial and the sign character of $\langle \delta^2 \rangle$. Then, fixing a square roots $\zeta^\pm$ of $\sigma^\pm(x, x, \delta^2)$, one constructs lifts $\widetilde{\sigma}^i$ of

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σ, $0 \leqslant i \leqslant 3$, by setting

$$\widetilde{\sigma}^0((x,1,\delta)) = \zeta^+, \quad \widetilde{\sigma}^1((x,1,\delta)) = -\zeta^+, \quad \widetilde{\sigma}^2((x,1,\delta)) = \zeta^-, \quad \widetilde{\sigma}^3((x,1,\delta)) = -\zeta^-.$$

Note that $\{\pm\zeta^\pm\}$ is the set of fourth roots of 1, and this gives the desired parametrization.

Example 6.8. Let $\mathbb{G}^\circ$ be of type D_4 and $A = \mathbb{Z}/3\mathbb{Z} = \langle\delta\rangle$ acting on the Dynkin diagram by cyclically permuting the extremal nodes. The Weyl group $W(D_4)$ has 13 irreducible representations which we denote by bipartitions of 4, $\alpha \times \beta$ up to swapping α and β, except where $\alpha = \beta$, there are two nonisomorphic representations $\alpha \times \alpha^\pm$. There is one cuspidal unipotent representation ρ_c, and in total 14 unipotent representations of $\mathbb{G}^{\circ\mathrm{Fr}_0}$.

All families are singletons with associated finite group $\Gamma_{\mathcal{U}} = \{1\}$, except the family

$$\{(12) \times (1), (22) \times \emptyset, (11) \times (2), \rho_c\}$$

for which the finite group is $\Gamma_{\mathcal{U}} = \mathbb{Z}/2\mathbb{Z}$. This family and the following four singleton families:

$$\{(4) \times \emptyset\}, \quad \{(1111) \times \emptyset\}, \quad \{(3) \times (1)\}, \quad \{(111) \times (1)\}$$

are A-stable and, in fact, each element in the four-element family is A-stable. The remaining six unipotent (singleton) families form two A-orbits:

$$\{(13) \times \emptyset, (2) \times (2)^+, (2) \times (2)^-\} \quad \text{and} \quad \{(112) \times \emptyset, (11) \times (11)^+, (11) \times (11)^-\}.$$

According to our recipe, the groups $\widetilde{\Gamma}_{\mathcal{U}}^{\mathbb{Z}/3\mathbb{Z}}$ are:

- $\mathbb{Z}/3\mathbb{Z}$ corresponding to each of the four A-stable singleton families $\mathcal{U}$;
- $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ for the unique family with four elements;
- $\{1\}$ for each one of the two nontrivial A-orbits.

Hence, the right-hand side of Proposition 6.1 is $\mathcal{M}(\mathbb{Z}/3\mathbb{Z})^4 \sqcup \mathcal{M}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}) \sqcup \mathcal{M}(\{1\})^2$ which has $3^2 \times 4 + 6^2 + 1^2 \times 2 = 74$ elements.

The irreducible unipotent representations of the disconnected group $D_4 \rtimes \mathbb{Z}/3\mathbb{Z}$ are parametrized, via Mackey theory, by the elements $(x,\sigma) \in \mathcal{M}(\widetilde{\Gamma}_{\mathcal{U}}^{\mathbb{Z}/3\mathbb{Z}})$, where $x \in \Gamma_{\mathcal{U}}$ and $\mathcal{U}$ ranges over a set of representatives of the A-orbits of families of D_4. There are 26 such irreducible representations.

The other two A-forms corresponding to δ and δ^{-1} are both isomorphic to the finite group of type ${}^3D_4$. There are eight irreducible unipotent representations of ${}^3D_4$ each coming from one of γ-stable irreducible unipotent representations of D_4. By induction, there are $8 \times 3 = 24$ irreducible unipotent representations of ${}^3D_4 \rtimes \mathbb{Z}/3\mathbb{Z}$. The irreducible representations of the ${}^3D_4$ corresponding to δ are parametrized by $(x\delta,\sigma) \in \mathcal{M}(\widetilde{\Gamma}_{\mathcal{U}}^{\mathbb{Z}/3\mathbb{Z}})$, where $x \in \Gamma_{\mathcal{U}}$ and $\mathcal{U}$ ranges over the set of A-stable families. Similarly for δ^{-1}.

7. Maximal compact subgroups

We return to the setting of § 2, so $\mathbf{G}$ is a connected reductive group over F and $G = \mathbf{G}(F)$. In this section, we assume in addition that $\mathbf{G}$ is simple and F-split with maximal F-split torus $\mathbf{S}$. Let Π_G be a set of simple roots for $\mathbf{G}$ with respect to $\mathbf{S}$, and extend Π_G to a set of simple affine roots $\Pi_G^a = \Pi_G \cup \{\alpha_0\}$. Let $I \subset \mathbf{G}(\mathfrak{o}_F)$ be the corresponding Iwahori subgroup of G, with $\mathbf{S}(\mathfrak{o}_F) = \mathbf{S}(F) \cap I$. Let $\widetilde{W}_G = \mathrm{N}_G(\mathbf{S}(F))/\mathbf{S}(\mathfrak{o}_F)$ be the Iwahori–Weyl group. We have

$$G = \bigsqcup_{w \in \widetilde{W}_G} I\dot{w}I,$$

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where $\dot{w}$ denotes a choice of a lift in $N_G(\mathbf{S}(F))$ of $w \in \widetilde{W}_G$. The finite Weyl group is $W_G = \mathrm{N}_G(\mathbf{S}(F))/\mathbf{S}(F)$. Let W_G^a be the affine Weyl group generated by the simple reflections $\{s_i \mid i \in \Pi_G^a\}$. Then

$$\widetilde{W}_G = W_G^a \rtimes \Omega_G,$$

where Ω_G is a finite abelian group, the stabilizer in $\widetilde{W}_G$ of I.

Let $\max(G)$ denote the set of conjugacy classes of maximal compact open subgroups in G. To parametrize $\max(G)$, we define $S_{\max}(G)$ to be the set of Ω_G-orbits of pairs $(A, \mathcal{O})$, where A is a subgroup of Ω_G and $\mathcal{O}$ is an A-orbit in Π_G^a satisfying

$$\mathrm{Stab}_{\Omega_G}(\mathcal{O}) = A.$$

By [IM65, BT72], $\max(G)$ is parametrized by $S_{\max}(G)$. Explicitly, given $(A, \mathcal{O}) \in S_{\max}(G)$, we construct an element $K_{\mathcal{O}} \in \max(G)$ as follows: let $\widetilde{W}_{\mathcal{O}}$ be the finite subgroup of $\widetilde{W}_G$ generated by A and $\{s_i,\ i \in \Pi_G^a \setminus \mathcal{O}\}$. Set

$$K_{\mathcal{O}} = \bigsqcup_{w \in \widetilde{W}_{\mathcal{O}}} I\dot{w}I.$$

The map $(A, \mathcal{O}) \mapsto K_{\mathcal{O}}$ defines a bijection between $S_{\max}(G)$ and $\max(G)$. (Note that a pair $(A, \mathcal{O}) \in S_{\max}(G)$ is completely determined by $\mathcal{O}$, so the notation $K_{\mathcal{O}}$ is unambiguous.) In this notation, the maximal hyperspecial subgroup $\mathbf{G}(\mathfrak{o}_F)$ is $K_{\{\alpha_0\}}$, where α_0, as defined above, is the unique simple affine root in $\Pi_G^a \setminus \Pi_G$.

Given $(A, \mathcal{O}) \in S_{\max}(G)$, let $\widetilde{W}_{\mathcal{O}}^\circ$ be the normal subgroup of $\widetilde{W}_{\mathcal{O}}$ generated by $\{s_i,\ i \in \Pi_G^a \setminus \mathcal{O}\}$. Then $K_{\mathcal{O}}^\circ := \bigsqcup_{w \in \widetilde{W}_{\mathcal{O}}^\circ} I\dot{w}I$ is a parahoric subgroup of G, and we denote by $K_{\mathcal{O}}^+$ its pro-unipotent radical. There is a short exact sequence

$$1 \longrightarrow K_{\mathcal{O}}^\circ \longrightarrow K_{\mathcal{O}} \longrightarrow A \longrightarrow 1.$$

Set $\overline{K}_{\mathcal{O}}^\circ = K_{\mathcal{O}}^\circ/K_{\mathcal{O}}^+$ and $\overline{K}_{\mathcal{O}} = K_{\mathcal{O}}/K_{\mathcal{O}}^+$. Then $\overline{K}_{\mathcal{O}} = M_{\mathcal{O}}(k_F)$, $\overline{K}_{\mathcal{O}}^\circ = M_{\mathcal{O}}^\circ(k_F)$, for a reductive k_F-split group $M_{\mathcal{O}}$ with identity component $M_{\mathcal{O}}^\circ$ and $M_{\mathcal{O}}/M_{\mathcal{O}}^\circ \cong A$. Let $\mathrm{InnT}^p\overline{K}_{\mathcal{O}} \leftrightarrow A$ be the collection of pure inner twists of $M_{\mathcal{O}}$.

Now we consider pure inner twists of G. By § 2, $\mathrm{InnT}^pG \cong \mathrm{H}^1(F, \mathbf{G})$. If $\mathbf{G}_{\mathrm{sc}}$ is the simply connected cover of $\mathbf{G}$ and identifying Ω_G with the kernel of the surjection $\mathbf{G}_{\mathrm{sc}} \to \mathbf{G}$, then by [Kne65, Satz 2], $\mathrm{H}^1(F, \mathbf{G}) \cong \mathrm{H}^2(F, \Omega_G) \cong \Omega_G$, with the last equivalence because $\mathbf{G}$ is F-split. Given $x \in \Omega_G$, let $G_x \in \mathrm{InnT}^pG$ be the corresponding pure inner twist. If we denote the set of conjugacy classes of maximal compact open subgroups of G_x by $\max(G_x)$, then there is a one-to-one correspondence

$$\{(A, \mathcal{O}) \in S_{\max}(G) \mid x \in A\} \longleftrightarrow \max(G_x), \tag{7.1}$$

which we write as $(A, \mathcal{O}) \mapsto K_{x,\mathcal{O}}$. More precisely, we can realize $K_{x,\mathcal{O}}$ in the following way. We realize G_x as the subgroup of $\mathbf{G}(F_{\mathrm{un}})$ fixed under the $\mathrm{Gal}(F_{\mathrm{un}}/F)$-action corresponding to x. Then since $x \in A$, the parahoric of $\mathbf{G}(F_{\mathrm{un}})$ corresponding to $(A, \mathcal{O})$ as above is $\mathrm{Gal}(F_{\mathrm{un}}/F)$-stable. Let $K_{x,\mathcal{O}}^\circ \subset G_x$ be the Galois-fixed subgroup. By [BT72, 3.3.4 Proposition] (applied using [BT84, 5.2.12 Proposition]), the normalizer $K_{x,\mathcal{O}} := N_{G_x}(K_{x,\mathcal{O}}^\circ)$ is a maximal compact subgroup of G_x.

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Note that $\overline{K}_{x,\mathcal{O}} \in \mathrm{Inn}\mathrm{T}^p\overline{K}_{\mathcal{O}}$ is the pure inner twist given by $x \in A \cong \mathrm{H}^1(k_F, M_{\mathcal{O}}) \leftrightarrow \mathrm{Inn}\mathrm{T}^p\overline{K}_{\mathcal{O}}$ (see § 6). For fixed $(A, \mathcal{O}) \in S_{\max}(G)$, we have

$$\mathrm{Inn}\mathrm{T}^p\overline{K}_{\mathcal{O}} = \{\overline{K}_{x,\mathcal{O}} \mid x \in A\}. \tag{7.2}$$

Given $G' \in \mathrm{Inn}\mathrm{T}^p(G)$, for every maximal compact open subgroup $K' \in \max(G')$, define $R_{\mathrm{un}}(\overline{K}')$ to be the $\mathbb{C}$-span of $\mathrm{Irr}_{\mathrm{un}}\overline{K}'$, and let

$$\mathcal{C}(G)_{\mathrm{cpt,un}} = \bigoplus_{G' \in \mathrm{Inn}\mathrm{T}^p G} \bigoplus_{K' \in \max(G')} R_{\mathrm{un}}(\overline{K}'). \tag{7.3}$$

By the discussion above, we have

$$\mathcal{C}(G)_{\mathrm{cpt,un}} = \bigoplus_{x \in \Omega} \bigoplus_{\substack{(A,\mathcal{O}) \in S_{\max}(G) \\ \text{with } x \in A}} R_{\mathrm{un}}(\overline{K}_{x,\mathcal{O}}) \tag{7.4}$$

$$= \bigoplus_{(A,\mathcal{O}) \in S_{\max}(G)} \bigoplus_{x \in A} R_{\mathrm{un}}(\overline{K}_{x,\mathcal{O}}). \tag{7.5}$$

Note that for each $(A, \mathcal{O}) \in S_{\max}(G)$, by (7.2), $\bigoplus_{x \in A} R_{\mathrm{un}}(\overline{K}_{x,\mathcal{O}})$ has the involution $\mathrm{FT}_{\overline{K}_{\mathcal{O}}}$ given by (6.3). Putting together these involutions for all choices of $(A, \mathcal{O})$ gives the following definition.

Definition 7.1. Let $\mathrm{FT}_{\mathrm{cpt,un}} = \bigoplus_{(A,\mathcal{O}) \in S_{\max}(G)} \mathrm{FT}_{\overline{K}_{\mathcal{O}}}$ be the involution on $\mathcal{C}(G)_{\mathrm{cpt,un}}$ defined by using (6.3) and (7.2).

Note that $\mathrm{FT}_{\mathrm{cpt,un}}$ always preserves the space $R_{\mathrm{un}}(\overline{\mathbf{G}(\mathfrak{o}_F)})$, since $\mathbf{G}(\mathfrak{o}_F)$ corresponds to the pair $(A, \{\alpha_0\})$ with A trivial. In the case when $\mathbf{G}$ is simply connected, we have $\Omega_G = 1$ and $K_{\mathcal{O}} = K_{\mathcal{O}}^\circ$ for all $\mathcal{O}$, so $\mathrm{FT}_{\mathrm{cpt,un}}$ preserves the space $R_{\mathrm{un}}(\overline{K}_{\mathcal{O}})$ for all $K_{\mathcal{O}} \in \max(G)$. But, in general, it does not preserve $R_{\mathrm{un}}(\overline{K}_{x,\mathcal{O}})$ for every maximal compact open subgroup $K_{x,\mathcal{O}}$, which can be seen even in the case when $\mathbf{G} = \mathsf{PGL}_2$ (see Example 14.3).

Example 7.2. We list the type of groups $\overline{K}_{\mathcal{O}}$ in the case when $\mathbf{G}$ is adjoint. Since we are only interested in unipotent representations, only the Lie type of $\overline{K}_{\mathcal{O}}^\circ$ is important.

(i) If $\mathbf{G}$ is also simply connected, which is the case for types G_2, F_4 and E_8, the set $\max(G)$ is in one-to-one correspondence with the maximal subsets of Π_a (equivalently, $\mathcal{O}$ is a single vertex in Π_G^a).

(ii) If $\mathbf{G} = \mathsf{PGL}_n$, $\Omega_G = \mathbb{Z}/n\mathbb{Z}$ acting by cyclically permuting Π_G^a, then for every divisor m of n, we have an orbit $\mathcal{O}_m$ in Π_G^a with stabilizer $A = \mathbb{Z}/m\mathbb{Z}$ and

$$\overline{K}_{\mathcal{O}_m} = P(\mathsf{GL}_{n/m}^m) \rtimes \mathbb{Z}/m\mathbb{Z},$$

where the semidirect product is given by the permutation action. This case corresponds to Example 6.4.

(iii) If $\mathbf{G} = \mathsf{SO}_{2n+1}$, $\Omega_G = \mathbb{Z}/2\mathbb{Z}$, then $\overline{K}_{\mathcal{O}}$ is either SO_{2n+1} ($A = 1$) or $\mathsf{SO}_{2m+1} \times \mathrm{O}_{2(n-m)}$ ($A = \mathbb{Z}/2\mathbb{Z}$), for $0 \leqslant m < n$. This case corresponds to Example 6.5.

(iv) If $\mathbf{G} = \mathrm{PSp}_{2n}$, $\Omega = \mathbb{Z}/2\mathbb{Z}$, then $\overline{K}_{\mathcal{O}}$ is of type:
- $C_k \times C_\ell$ ($A = 1$) for $0 \leqslant k < \ell$, $k + \ell = n$;
- $(C_k \times C_k) \rtimes \mathbb{Z}/2\mathbb{Z}$ ($A = \mathbb{Z}/2\mathbb{Z}$), if $n = 2k$; or
- $(C_i \times C_i \times A_{n-1-2i}) \rtimes \mathbb{Z}/2$ ($A = \mathbb{Z}/2\mathbb{Z}$), for $0 \leqslant i < n/2$.

Here $\mathbb{Z}/2\mathbb{Z}$ acts by flipping the two type C factors and by the nontrivial diagram automorphism on the type A factor. These cases are covered by Examples 6.3 and 6.6.

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(v) If $\mathbf{G} = \mathrm{PSO}_{4m}$, $m \geqslant 2$, $\Omega_G = \langle \delta_1 \rangle \times \langle \delta_2 \rangle \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, then δ_1 acts by flipping Π_a horizontally and δ_2 by the vertical flip. For each subgroup $A \leqslant \Omega_G$, we give the possible $\overline{K}^\circ_{\mathcal{O}}$.
  (a) If $A = \Omega$, $\overline{K}^\circ_{\mathcal{O}}$ is of type $D_k \times D_k \times A_{2m-2k-1}$, $2 \leqslant k < m$, $D_m \times D_m$ or A_{2m-3}, where A acts on $D_\ell \times D_\ell$ as in Example 6.7, while on $A_{2m-2k-1}$ δ_1 acts trivially and δ_2 by the nontrivial diagram automorphism.
  (b) If $A = \langle \delta_1 \rangle$, $\overline{K}^\circ_{\mathcal{O}}$ is of type $D_k \times D_{2m-k}$, $2 \leqslant k < m$ or D_{2m-1}, where δ_1 acts by the nontrivial automorphism of type D_ℓ. These cases are covered by Example 6.5.
  (c) If $A = \langle \delta_2 \rangle$ or $A = \langle \delta_1 \delta_2 \rangle$, $\overline{K}^\circ_{\mathcal{O}}$ is of type A_{2m-1} with the diagram automorphism action as in Example 6.6.
  (d) If $A = 1$, then $\overline{K}^\circ_{\mathcal{O}}$ is the hyperspecial subgroup of type D_{2m-1}.

(vi) If $\mathbf{G} = \mathrm{PSO}_{4m+2}$, $m \geqslant 2$, then $\Omega_G = \langle \delta \rangle \cong \mathbb{Z}/4\mathbb{Z}$, and we have the following.
  (a) If $A = \Omega_G$, $\overline{K}^\circ_{\mathcal{O}}$ is of type $D_k \times D_k \times A_{2m-2k}$, $2 \leqslant k \leqslant m$ or A_{2m-2}, where δ acts on $D_\ell \times D_\ell$ as in Example 6.7, while on A_{2m-2k}, δ acts by the nontrivial diagram automorphism.
  (b) If $A = \langle \delta^2 \rangle$, $\overline{K}^\circ_{\mathcal{O}}$ is of type D_{2m} or $D_k \times D_{2m-k+1}$, $2 \leqslant k \leqslant m$, where δ^2 acts on each factor by the nontrivial automorphism of type D_ℓ, as in Example 6.5.
  (c) If $A = 1$, then $\overline{K}^\circ_{\mathcal{O}}$ is the hyperspecial subgroup of type D_{2m}.

(vii) Let $\mathbf{G}$ be of type E_6 with $\Omega_G = \langle \delta \rangle \cong \mathbb{Z}/3\mathbb{Z}$.
  (a) If $A = 1$, then $\overline{K}^\circ_{\mathcal{O}}$ is either of type E_6 or $A_5 \times A_1$.
  (b) If $A = \Omega_G$, then $\overline{K}^\circ_{\mathcal{O}}$ is of type: A_2^3 with δ acting by permutation, as in Example 6.4; $A_1^3 \times A_1$, where δ permutes the first three factors and it fixes the last one; or D_4, where δ acts as in Example 6.8.

(viii) Let $\mathbf{G}$ be of type E_7 with $\Omega_G = \langle \delta \rangle \cong \mathbb{Z}/2\mathbb{Z}$.
  (a) If $A = 1$, then $\overline{K}^\circ_{\mathcal{O}}$ is of type E_7, $D_6 \times A_1$ or $A_5 \times A_2$.
  (b) If $A = \Omega_G$, then $\overline{K}^\circ_{\mathcal{O}}$ is of type: E_6 with δ acting by the nontrivial diagram automorphism as in Example 6.6; $D_4 \times A_1^2$ with δ acting by an order-2 diagram automorphism of D_4 (Example 6.5) and by flipping the two A_1; $A_2^2 \times A_2$, flipping the first A_2 and acting trivially on the third A_2; $A_3^2 \times A_1$, with the flip on A_3^2 and trivial action on A_1; or A_7 with the nontrivial diagram automorphism (see Examples 6.3 and 6.6).

8. Elliptic pairs

8.1 Finite groups

Suppose H is a finite group. Given a finite-dimensional H-representation (δ, V_δ) over $\mathbb{C}$ and functions $f, f' \colon H \to \mathbb{C}$, define

$$(f, f')^\delta_{\mathrm{ell}} = \frac{1}{|H|} \sum_{h \in H} \det_{V_\delta}(1 - \delta(h)) f(h^{-1}) f'(h). \tag{8.1}$$

For $\rho, \rho' \in R(H)$, set

$$(\rho, \rho')^\delta_{\mathrm{ell}} = (\chi_\rho, \chi_{\rho'})^\delta_{\mathrm{ell}},$$

where χ_ρ, $\chi_{\rho'}$ denote the corresponding characters. The basic facts about $(\,,\,)^\delta_H$ can be found in [Ree01, § 2]. An element $h \in H$ is called (δ-)elliptic if $V_\delta^{\delta(h)} = 0$. The set H_{ell} of elliptic elements of H is obviously closed under conjugation by H. Let $H \backslash H_{\mathrm{ell}}$ denote the set of elliptic conjugacy classes. Fix (δ, V_δ), and let $\overline{R}(H)$ be the quotient of $R(H)$ by the radical of the form $(\,,\,)^\delta_{\mathrm{ell}}$. As in [Ree01, § 2], there is a natural identification of $\overline{R}(H)$ with the space of class functions of H supported on H_{ell}.

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For every $h \in H$, let $\mathbf{1}_h$ denote the characteristic function of the conjugacy class of h. Clearly, $\{\mathbf{1}_h \mid h \in H\backslash H_{\mathrm{ell}}\}$ is an orthogonal basis of $\overline{R}(H)$ with respect to the elliptic pairing $(\,,\,)^{\delta}_{\mathrm{ell}}$.

Suppose that, in addition, we are given an automorphism $\theta : H \to H$. Let $\langle\theta\rangle$ denote the cyclic group generated by θ and $H' = H \rtimes \langle\theta\rangle$. Given a finite-dimensional complex H'-representation (δ, V_δ) and functions $f, f' : H' \to \mathbb{C}$, define

$$(f, f')^{\delta}_{\theta-\mathrm{ell}} = \frac{1}{|H|} \sum_{h\in H} \det_{V_\delta}(1 - \delta(\theta h)) f((\theta h)^{-1}) f'(\theta h). \tag{8.2}$$

For $\rho, \rho' \in R(H')$, set

$$(\rho, \rho')^{\delta}_{\theta-\mathrm{ell}} = (\chi_\rho, \chi_{\rho'})^{\delta}_{\theta-\mathrm{ell}},$$

where χ_ρ, $\chi_{\rho'}$ denote the corresponding characters. Note that if the representation δ is understood, we may write $(\ ,\ )^{H}_{\mathrm{ell}}$ for $(\ ,\ )^{\delta}_{\mathrm{ell}}$, and similarly for $(\ ,\ )^{H}_{\theta-\mathrm{ell}}$.

8.2 Complex reductive groups

Let $\mathcal{G}$ be a possibly disconnected complex reductive group with identity component $\mathcal{G}^\circ$. If $x \in \mathcal{G}$ is given, fix a Borel subgroup B_x of $\mathrm{Z}_{\mathcal{G}}(x)^\circ$ and a maximal torus T_x in B_x. Let $\mathfrak{t}_x$ be the Lie algebra of T_x. As in [Wal07, § 2] (see also [Ree01, § 3.2]), we define a complex representation $(\delta_x, \mathfrak{t}_x)$ of $A_{\mathcal{G}}(x)$ as follows. Every $z \in \mathrm{Z}_{\mathcal{G}}(x)$ acts on $\mathrm{Z}_{\mathcal{G}}(x)^\circ$ by the adjoint action, denote by α_z the resulting automorphism of $\mathrm{Z}_{\mathcal{G}}(x)^\circ$. There exists $y \in \mathrm{Z}_{\mathcal{G}}(x)^\circ$ such that $\alpha_z \circ \mathrm{Ad}(y)$ preserves B_x and T_x. This means that $\alpha_z \circ \mathrm{Ad}(y)$ defines an automorphism of the cocharacter lattice $X_*(T_x)$ in $\mathrm{Z}_{\mathcal{G}}(x)^\circ$, and therefore a linear isomorphism of $\mathfrak{t}_x$ denoted $\delta_x(z)$. If $\bar{z} \in A_{\mathcal{G}}(x)$, let $z \in Z_{\mathcal{G}}(x)$ be a representative and set $\delta_x(\bar{z}) := \delta_x(z)$. This construction gives a representation of $A_{\mathcal{G}}(x)$. We consider the elliptic theory of the finite group $A_{\mathcal{G}}(x)$ with respect to the representation δ_x.

An element $g \in \mathcal{G}$ is called *elliptic* if the centralizer $\mathrm{Z}_{\mathcal{G}}(g)$ contains no nontrivial torus.

8.3 Definitions

Suppose Γ is a (possibly disconnected) complex reductive group with identity component Γ°. Extending the definition in § 5.1, we define the sets (cf. [Ciu20, Definition 1.1])

$$\begin{aligned} \mathcal{Y}(\Gamma) &= \{(s,h) \in \Gamma \times \Gamma \mid s, h \text{ semisimple},\ sh = hs\}, \\ \mathcal{Y}(\Gamma)_{\mathrm{ell}} &= \{(s,h) \in \Gamma \times \Gamma \mid s, h \text{ semisimple},\ sh = hs,\ \mathrm{Z}_\Gamma(s,h) \text{ is finite}\}. \end{aligned} \tag{8.3}$$

Here $\mathrm{Z}_\Gamma(s,h) = \mathrm{Z}_\Gamma(s) \cap \mathrm{Z}_\Gamma(h)$ and the finiteness condition is equivalent to saying that no nontrivial torus in Γ centralizes both s and h. We refer to elements of $\mathcal{Y}(\Gamma)_{\mathrm{ell}}$ as *elliptic pairs.* Note that the condition in $\mathcal{Y}(\Gamma)_{\mathrm{ell}}$ is equivalent to saying that h is elliptic in $\mathrm{Z}_\Gamma(s)$ or equivalently s is elliptic in $\mathrm{Z}_\Gamma(h)$.

The sets $\mathcal{Y}(\Gamma), \mathcal{Y}(\Gamma)_{\mathrm{ell}}$ have Γ-actions via conjugation: $g \cdot (s,h) = (gsg^{-1}, ghg^{-1})$. They also have a natural Γ-equivariant involution given by the flip

$$(s,h) \mapsto (h,s).$$

Let $\Gamma\backslash\mathcal{Y}(\Gamma)$, $\Gamma\backslash\mathcal{Y}(\Gamma)_{\mathrm{ell}}$ be the sets of Γ-orbits, and given $(s,h) \in \mathcal{Y}(\Gamma)$, write $[(s,h)]$ for the corresponding orbit in $\Gamma\backslash\mathcal{Y}(\Gamma)$. Then we get an involution

$$\mathsf{flip}\colon \Gamma\backslash\mathcal{Y}(\Gamma) \to \Gamma\backslash\mathcal{Y}(\Gamma), \mathsf{flip}([(s,h)]) = [(h,s)],$$

which preserves $\Gamma\backslash\mathcal{Y}(\Gamma)_{\mathrm{ell}}$.

Lemma 8.1. *The set $\Gamma\backslash\mathcal{Y}(\Gamma)_{\mathrm{ell}}$ is finite.*

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Proof. Suppose $(s,h) \in \mathcal{Y}(\Gamma)_{\mathrm{ell}}$. The cyclic group $\langle s \rangle$ is in $\mathrm{Z}_\Gamma(s,h)$, hence s has finite order. Moreover, s must be isolated in Γ in the sense that $\mathrm{Z}_\Gamma(s)$ does not contain a nontrivial central torus. The classification of isolated semisimple automorphisms of Γ is well-known [Ste68, DM18], in particular, there are finitely many automorphisms up to inner conjugation. □

In the next lemma, we relate elliptic pairs in $\mathrm{Z}_\Gamma(s)$ to elements of $A_\Gamma(s)$ that are elliptic with respect to the action described in § 8.2.

Lemma 8.2. *Fix $s \in \Gamma$ semisimple. The projection map $\mathrm{Z}_\Gamma(s) \to A_\Gamma(s)$, $h \mapsto \bar{h}$, induces a bijection between $\mathrm{Z}_\Gamma(s)$-orbits of elliptic pairs (s,h) and the elliptic conjugacy classes in $A_\Gamma(s)$.*

Proof. We need a result from the theory of semisimple automorphisms of reductive groups, e.g. [Som98, Proposition 9]: if x, y are semisimple elements in a reductive group $\mathcal{G}$ such that their images in the group of components $\mathcal{G}/\mathcal{G}^\circ$ are in the same conjugacy class, and S is a maximal torus in $\mathrm{Z}_\mathcal{G}(x)$, then there exist $g \in \mathcal{G}$ and $s \in S$ such that $gyg^{-1} = xs$.

We apply this to $\mathcal{G} = \mathrm{Z}_\Gamma(s)$ (a reductive group). Suppose h, h' are semisimple elements such that (s,h) and (s,h') are elliptic pairs. The elliptic condition implies that the maximal torus in $\mathrm{Z}_\mathcal{G}(h)$ is trivial, hence $s = 1$ in the relation above, and h and h' are $\mathcal{G}$-conjugate. This implies that if $\overline{h} = \overline{h'}$, then $[(s,h)] = [(s,h')]$.

It remains to show that (s,h) is an elliptic pair if and only if $\bar{h}$ is elliptic in $A_\Gamma(s)$. This is just a matter of checking the definitions in the case $\mathcal{G} = \mathrm{Z}_\Gamma(s)$. Given the semisimple element $h \in \mathcal{G}$, choose a maximal torus T_s in $\mathcal{G}$ that is normalized by h. Then h is not elliptic if and only if there exists a nontrivial torus $S \subset T_s$ that centralizes h, equivalently if and only if $\delta_s(\bar{h})$ fixes a nonzero element of $\mathfrak{t}_s$, i.e. if $\overline{h}$ is not elliptic in $A_\Gamma(s)$. □

For every $(s,h) \in \mathcal{Y}(\Gamma)$, define

$$\Pi(s,h) = \sum_{\phi \in \widehat{A_\Gamma(s)}} \phi(h)\phi \in R(A_\Gamma(s)), \tag{8.4}$$

and let $\overline{\Pi}(s,h)$ denote the image in $\overline{R}(A_\Gamma(s))$. Here $\phi(h)$ is interpreted as $\phi(\bar{h})$ where $\bar{h}$ is the image of h in $A_\Gamma(s)$. Let $\mathbb{C}[\mathcal{Y}(\Gamma)_{\mathrm{ell}}]^\Gamma$ denote the Γ-invariant functions on $\mathcal{Y}(\Gamma)_{\mathrm{ell}}$; this space can be identified with $\mathbb{C}[\Gamma\backslash\mathcal{Y}(\Gamma)_{\mathrm{ell}}]$. Let $\mathbf{1}_{[(s,h)]}$ denote the characteristic function of the Γ-orbit of (s,h).

Proposition 8.3. *The correspondence $\mathbf{1}_{[(s,h)]} \mapsto \overline{\Pi}(s,h)$ induces an isomorphism*

$$\mathbb{C}[\mathcal{Y}(\Gamma)_{\mathrm{ell}}]^\Gamma \cong \bigoplus_{s \in \mathcal{C}(\Gamma)_{\mathrm{ss}}} \overline{R}(A_\Gamma(s)).$$

Proof. In light of Lemma 8.2, the only thing left is to remark that the elements $\overline{\Pi}(s,h)$ form a basis of $\overline{R}(A_\Gamma(s))$ as h ranges over a set of representatives of $\mathrm{Z}_\Gamma(s)$-conjugacy classes such that (s,h) is an elliptic pair. It is elementary that in $R(A_\Gamma(s))$,

$$\overline{\Pi}(s,h) = |\mathrm{Z}_{A_\Gamma(s)}(\overline{h})|\ \mathbf{1}_{\overline{h}^{-1}},$$

and the claim follows. □

We say that $\Gamma_M \subset \Gamma$ is a *Levi subgroup* if there exists a torus $S \subset \Gamma^\circ$ such that $\Gamma_M = \mathrm{Z}_\Gamma(S)$. If a pair (s,h) is in $\mathcal{Y}(\Gamma_M)$, denote by $\Pi^{\Gamma_M}(s,h)$ the combination defined analogously to (8.4).

Lemma 8.4. *Suppose $s \in \Gamma_M$ is semisimple.*

(i) *The inclusion $\mathrm{Z}_{\Gamma_M}(s) \to \mathrm{Z}_\Gamma(s)$ induces an inclusion $A_{\Gamma_M}(s) \to A_\Gamma(s)$.*

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(ii) *For every* $(s,h)\in\mathcal{Y}(\Gamma_M)$, $\mathrm{Ind}_{A_{\Gamma_M}(s)}^{A_\Gamma(s)}\Pi^{\Gamma_M}(s,h)=\Pi(s,h)$.

Proof. (i) This is a well-known argument. We need to show that $\mathrm{Z}_{\Gamma_M}(s)\cap\mathrm{Z}_\Gamma(s)^\circ$ is connected and, hence, in $\mathrm{Z}_{\Gamma_M}(s)^\circ$. But $\mathrm{Z}_{\Gamma_M}(s)\cap\mathrm{Z}_\Gamma(s)^\circ=\Gamma_M\cap\mathrm{Z}_\Gamma(s)^\circ=\mathrm{Z}_\Gamma(S)\cap\mathrm{Z}_\Gamma(s)^\circ=\mathrm{Z}_{\mathrm{Z}_\Gamma(s)^\circ}(S)$, which is connected since the centralizer of any torus in a connected reductive group is connected.

(ii) This is elementary using that $\phi(\overline{h})=\sum_{\psi\in\widehat{A_{\Gamma_M}(s)}}\langle\phi,\psi\rangle_{A_{\Gamma_M}(s)}\psi(\overline{h})$ for every $\phi\in\widehat{A_\Gamma(s)}$, by restriction of characters. □

Lemma 8.5. *Let* $(s,h)\in\mathcal{Y}(\Gamma)$ *be given and suppose* S *is a maximal torus in* $\mathrm{Z}_\Gamma(s,h)^\circ$. *Set* $\Gamma_M=\mathrm{Z}_\Gamma(S)$. *Then* $\mathrm{Z}_{\Gamma_M}(s,h)^\circ=\mathrm{Z}^\circ_{\Gamma_M}$, *i.e.* (s,h) *is an elliptic pair in* $\Gamma_M/\mathrm{Z}^\circ_{\Gamma_M}$.

Proof. Let S_1 be a torus in $\mathrm{Z}_{\Gamma_M}(s,h)^\circ$. Then $S_1\subset\mathrm{Z}_\Gamma(s,h)^\circ$ and since it commutes with S which is maximal in $\mathrm{Z}_\Gamma(s,h)^\circ$, it follows that $S_1\subset S\subset\mathrm{Z}^\circ_{\Gamma_M}$. □

Remark 8.6. Our main application will be to consider $\Gamma=\Gamma_u$, the reductive part of the centralizer of a unipotent element u in the Langlands dual group $G^\vee$, while Γ_M will be the centralizer of u in a Levi subgroup $M^\vee$.

8.4 Elliptic pairs in Γ°

In applications, we will often encounter the situation where the group Γ is connected. For this reason, it is useful to have a precise description of the elliptic pairs in Γ°. Suppose $s\in\Gamma^\circ$ a semisimple element. Let T be a maximal torus of Γ containing s and let Φ be the system of roots of T in Γ° and $W(\Gamma^\circ)$ the Weyl group of T in Γ°. If $\alpha\in\Phi$, let X_α be the corresponding one-parameter unipotent subgroup in Γ°. For each $w\in W(\Gamma^\circ)$, we fix a representative $\dot{w}$ of w in $\mathrm{N}_{\Gamma^\circ}(T)$. Recall [Car93, Theorem 3.5.3]

$$\begin{aligned} Z_{\Gamma^\circ}(s)^\circ &= \langle T, X_\alpha \mid \alpha(s)=1,\ \alpha\in\Phi\rangle,\\ Z_{\Gamma^\circ}(s) &= \langle T, X_\alpha, \dot{w} \mid \alpha(s)=1,\ \alpha\in\Phi,\ wsw^{-1}=s,\ w\in W(\Gamma^\circ)\rangle. \end{aligned} \tag{8.5}$$

We say that $w\in W(\Gamma^\circ)$ is *elliptic* if T^w is finite, equivalently if $\mathfrak{t}^w=0$, where $\mathfrak{t}$ is the Lie algebra of T.

Proposition 8.7. *With the notation as above,*

$$\Gamma^\circ\backslash\mathcal{Y}(\Gamma^\circ)_{\mathrm{ell}}\leftrightarrow W(\Gamma^\circ)\backslash\{(s,w)\mid s\in T\ \textit{is regular},\ w\in W(\Gamma^\circ)\ \textit{is elliptic},\ s\in T^w\}.$$

Proof. Since we are considering Γ°-orbits of pairs $(s,h)\in\mathcal{Y}(\Gamma^\circ)_{\mathrm{ell}}$, we may assume that $s\in T$ (in a fixed $W(\Gamma^\circ)$-orbit in T) and h is in a semisimple conjugacy class of $Z_{\Gamma^\circ}(s)$. If $h\in Z_{\Gamma^\circ}(s)^\circ$, since $Z_{\Gamma^\circ}(s)^\circ$ is reductive [Car93, Theorem 3.5.4], h is contained in a maximal torus of $Z_{\Gamma^\circ}(s)^\circ$, hence (s,h) is not an elliptic pair. This means that h must be in $Z_{\Gamma^\circ}(s)\setminus Z_{\Gamma^\circ}(s)^\circ$. By (8.5), we can assume that $h=\dot{w}$ for some $w\in W(\Gamma^\circ)$ such that $s\in T^w$. It is clear that $Z_{\Gamma^\circ}(s,\dot{w})\supseteq T^w$, which means that w is necessarily elliptic if $(s,\dot{w})$ is an elliptic pair. Suppose s is not regular. Then there exists $\alpha\in\Phi$ such that $\alpha(s)=1$. Let $O_w=\{\alpha,w(\alpha),w^2(\alpha),\dots,w^{n-1}(\alpha)\}$, where n is the order of w. Then in the Lie algebra of Γ, there exists an appropriate sum of root vectors $e=\sum_{\beta\in O_w}e_\beta$ that is invariant under $\mathrm{Ad}(\dot{w})$ and, therefore, $Z_{\Gamma^\circ}(s,\dot{w})$ contains the one-parameter subgroup for e and it is infinite.

Conversely, suppose $(s,\dot{w})$ is such that s is regular and w is elliptic. By (8.5), $Z_{\Gamma^\circ}(s)=W(\Gamma^\circ)_sT$, where $W(\Gamma^\circ)_s=\{w_1\in W(\Gamma^\circ)\mid w_1sw_1^{-1}=s\}$. Then $Z_{\Gamma^\circ}(s,\dot{w})$ is finite if and only if $\mathrm{Ad}(\dot{w})$ has no nonzero fixed points on the Lie algebra $\mathfrak{z}_{\Gamma^\circ}(s)$. But $\mathfrak{z}_{\Gamma^\circ}(s)=\mathfrak{t}$, so this is equivalent to w being elliptic. □

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Remark 8.8. If Γ is connected and simply connected, then $\mathcal{Y}(\Gamma)_{\mathrm{ell}} = \emptyset$. This is because in that case, for every regular semisimple $s \in T$, $Z_\Gamma(s) = Z_\Gamma(s)^\circ = T$, a maximal torus.

9. The dual nonabelian Fourier transform

Let $\mathbf{G}$ be a connected semisimple algebraic F-group and $G = \mathbf{G}(F)$. Let $\mathfrak{R}_{\mathrm{un}}(G)$ denote the category of smooth unipotent representations of G. If $V, V' \in \mathrm{Irr}_{\mathrm{un}}(G)$, let

$$\mathrm{EP}_G(V, V') = \sum_{i \geqslant 0} (-1)^i \dim \mathrm{Ext}^i(V, V'), \tag{9.1}$$

where $\mathrm{Ext}^i(V, V')$ are calculated in the category $\mathfrak{R}(G)$ of all smooth G-representations [SS97] or, equivalently since $\mathfrak{R}_{\mathrm{un}}(G)$ is a direct summand of $\mathfrak{R}(G)$, in the category $\mathfrak{R}_{\mathrm{un}}(G)$. We remark that this is a finite sum by Bernstein's result on the finiteness of the cohomological dimension of G. Extend, as we may, $\mathrm{EP}_G(\,,\,)$ as a Hermitian pairing on $R_{\mathrm{un}}(G)$ (as defined in § 4.3). Let $\overline{R}_{\mathrm{un}}(G)$ denote the quotient of $R_{\mathrm{un}}(G)$ by the radical of EP_G.

Let $R^{\mathrm{temp}}_{\mathrm{un}}(G)$ be the subspace spanned by the irreducible unipotent tempered representations and let $\overline{R}^{\mathrm{temp}}_{\mathrm{un}}(G)$ be the image of $R^{\mathrm{temp}}_{\mathrm{un}}(G)$ in $\overline{R}_{\mathrm{un}}(G)$. As is well-known [SS97, Ree02], as a consequence of the (parabolic induction) Langlands classification:

$$\overline{R}^{\mathrm{temp}}_{\mathrm{un}}(G) = \overline{R}_{\mathrm{un}}(G). \tag{9.2}$$

Let $\mathfrak{B}_{\mathrm{un}}(G)$ denote the unipotent Bernstein center so that $R_{\mathrm{un}}(G) = \bigoplus_{\mathfrak{s} \in \mathfrak{B}_{\mathrm{un}}(G)} R(G)^{\mathfrak{s}}$, where $R(G)^{\mathfrak{s}}$ is the $\mathbb{C}$-span of irreducible objects in the subcategory $\mathfrak{R}(G)^{\mathfrak{s}}$ (defined in § 4.2). Since there are no nontrivial extensions between objects in different Bernstein components, we have an EP-orthogonal decomposition:

$$\overline{R}_{\mathrm{un}}(G) = \bigoplus_{\mathfrak{s} \in \mathfrak{B}_{\mathrm{un}}(G)} \overline{R}(G)^{\mathfrak{s}}.$$

With the same notation for a pure inner twist G' of G, we get

$$\bigoplus_{G' \in \mathrm{Inn T}^p(G)} \overline{R}_{\mathrm{un}}(G') = \bigoplus_{G' \in \mathrm{Inn T}^p(G)} \bigoplus_{\mathfrak{s} \in \mathfrak{B}_{\mathrm{un}}(G')} \overline{R}(G')^{\mathfrak{s}}. \tag{9.3}$$

Recall the unipotent Langlands correspondence in the form (4.18). Given a semisimple element $s \in G^\vee$ and a unipotent element $u \in \mathcal{G}^p_s$, apply the definitions of § 8.2 to $u \in \mathcal{G}^p_s$ to obtain a representation δ^s_u of $A_{\mathcal{G}^p_s}(u)$ on the Cartan subalgebra $\mathfrak{t}^s_u$ in the Lie algebra of $\mathrm{Z}_{\mathcal{G}^p_s}(u)$. Let $(\,,\,)^{\delta^s_u}_{\mathrm{ell}}$ be the elliptic inner product on $R(A_{\mathcal{G}^p_s}(u))$ and let $\overline{R}(A_{\mathcal{G}^p_s}(u))$ be the elliptic quotient by the radical of the form. One expects the following correspondence to hold.

Conjecture 9.1. The unipotent Langlands correspondence (4.18) induces an isometric isomorphism

$$\overline{\mathsf{LLC}^p}_{\mathrm{un}} : \bigoplus_{s \in \mathcal{C}(G^\vee)_{\mathsf{ss}}} \bigoplus_{u \in \mathcal{C}(\mathcal{G}^p_s)_{\mathrm{un}}} \overline{R}(A_{\mathcal{G}^p_s}(u)) \longrightarrow \bigoplus_{G' \in \mathrm{Inn T}^p(G)} \overline{R}_{\mathrm{un}}(G'), \tag{9.4}$$

where the spaces on the left are endowed with the elliptic inner products $(\,,\,)^{\delta^s_u}_{\mathrm{ell}}$, while the spaces on the right have the Euler–Poincaré pairings $\mathrm{EP}_{G'}$.

Here $\mathcal{C}(G^\vee)_{\mathsf{ss}}$ and $\mathcal{C}(\mathcal{G}_s)_{\mathrm{un}}$ refer to conjugacy classes of semisimple and unipotent elements, as defined in § 4.3.

Remark 9.2. In [Ree02], Reeder proves that this elliptic correspondence holds in the case of irreducible representations with Iwahori-fixed vectors of a split adjoint group. In § 11, Theorem 11.1,

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we prove Conjecture 9.1 in the form (9.4) for a semisimple adjoint F-split group $\mathbf{G}$. In § 11.4, we explain how this result could be extended to arbitrary isogenies and, in particular, in Corollary 11.13 we prove it in the Iwahori-fixed case for an arbitrary F-split group $\mathbf{G}$. As a concrete example, in Proposition 13.6, we also illustrate the result with a direct proof for $\mathbf{G} = \mathsf{SL}_n$.

Remark 9.3. Given $s \in G^\vee$ as above and $u \in \mathcal{G}_s$, it also makes sense to define a representation of $A_{\mathcal{G}_s}(u)$ similarly to δ_u^s. It would be natural to expect that the LLC for inner twists as described in (4.16) also induces an isometric isometry

$$\overline{\mathsf{LLC}}_{\mathrm{un}}\colon \bigoplus_{s\in\mathcal{C}(G^\vee)_{\mathsf{ss}}} \bigoplus_{u\in\mathcal{C}(\mathcal{G}_s)_{\mathrm{un}}} \overline{R}(A_{\mathcal{G}_s}(u)) \longrightarrow \bigoplus_{G'\in \mathrm{InnT}(G)} \overline{R}_{\mathrm{un}}(G'), \tag{9.5}$$

though we will not consider a conjecture of this form in this paper.

Remark 9.4. One can formulate Conjecture 9.1 without restricting to the unipotent case. In general, the expected Langlands correspondence should induce an isometric isomorphism

$$\overline{\mathsf{LLC}^p}\colon \bigoplus_{\varphi} \overline{R}(A_\varphi) \longrightarrow \bigoplus_{G'\in \mathrm{InnT}^p(G)} \overline{R}(G'), \tag{9.6}$$

where φ ranges over the $G^\vee$-conjugacy classes of L-parameters $\varphi : W_F' \to {}^LG$ (equivalently, tempered L-parameters), and $A_\varphi = \pi_0(\mathrm{Z}_{G^\vee}(\varphi))$ (cf. § 4.3). The elliptic theory of the finite group A_φ is taken with respect to the action on a Cartan subalgebra of $\mathrm{Z}_{G^\vee}(\varphi)$ as before. This formulation is, of course, related to Arthur's ideas on elliptic representations: in [Art93, Corollary 6.3], Arthur proved the equality of Kazhdan's elliptic pairing between irreducible tempered representations with the elliptic pairing of the corresponding irreducible characters of the Knapp–Stein R-groups. Later, Opdam and Solleveld [OS13, Theorems 6.5 and 7.3] extended this work to all admissible representations using the homological Euler–Poincaré pairing. Moreover, Arthur [Art89, § 7] expected an identification between the R-groups and the geometric A-groups and, in fact, Reeder [Ree02, § 8], as part of his proof of the elliptic correspondence, proved this matching in the Iwahori case.

9.1 The elliptic Fourier transform: the split case

Suppose G is the split F-form. In order to apply the ideas in § 8.3, we rephrase the left-hand side of (9.4). Since Frob acts trivially on $G^\vee$, in this situation we have $\mathcal{G}_s^p = \mathrm{Z}_{G^\vee}(s)$ and, hence,

$$\bigoplus_{s\in\mathcal{C}(G^\vee)_{\mathsf{ss}}} \bigoplus_{u\in\mathcal{C}(\mathcal{G}_s^p)_{\mathrm{un}}} \overline{R}(A_{\mathcal{G}_s^p}(u)) = \bigoplus_{s\in\mathcal{C}(G^\vee)_{\mathsf{ss}}} \bigoplus_{u\in\mathcal{C}(\mathrm{Z}_{G^\vee}(s))_{\mathrm{un}}} \overline{R}(A_{G^\vee}(su)),$$

which can be written as

$$\bigoplus_{u\in\mathcal{C}(G^\vee)_{\mathrm{un}}} \bigoplus_{s\in\mathbb{C}(\Gamma_u)_{\mathsf{ss}}} \overline{R}(A_{\Gamma_u}(s)) = \bigoplus_{u\in\mathcal{C}(G^\vee)_{\mathrm{un}}} \mathbb{C}[\mathcal{Y}(\Gamma_u)_{\mathrm{ell}}]^{\Gamma_u},$$

via Proposition 8.3, where Γ_u is the reductive part of $\mathrm{Z}_{G^\vee}(u)$, as before. For simplicity, we define

$$\mathcal{R}^p_{\mathrm{un,ell}}(G) = \bigoplus_{G'\in \mathrm{InnT}^p(G)} \overline{R}_{\mathrm{un}}(G'), \tag{9.7}$$

endowed with the Euler–Poincaré pairing $\mathrm{EP} = \bigoplus_{G'} \mathrm{EP}_{G'}$. Hence, the elliptic unipotent LLC for pure inner twists of a split group can be viewed as the isomorphism

$$\overline{\mathsf{LLC}^p}_{\mathrm{un}}\colon \bigoplus_{u\in\mathcal{C}(G^\vee)_{\mathrm{un}}} \mathbb{C}[\mathcal{Y}(\Gamma_u)_{\mathrm{ell}}]^{\Gamma_u} \longrightarrow \mathcal{R}^p_{\mathrm{un,ell}}(G). \tag{9.8}$$

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For every class of elliptic pairs $[(s,h)] \in \Gamma_u \backslash \mathcal{Y}(\Gamma_u)_{\mathrm{ell}}$, define the virtual combination (cf. [Wal18, Ciu20]):

$$\Pi(u,s,h) = \sum_{\phi \in \widehat{A_{\Gamma_u}(s)}} \phi(h)\, \pi(s,u,\phi). \tag{9.9}$$

Regard $\Pi(u,s,h)$ (or rather its image) as an element in $\mathcal{R}^p_{\mathrm{un,ell}}(G)$. As before $\phi(h) = \phi(\bar{h})$, where $\bar{h}$ is the image of h in $A_{\Gamma_u}(s)$.

Lemma 9.5. *With notation as above and setting $A_x = A_{G^\vee}(x)$, we have:*

(i) $\mathrm{EP}(\Pi(u,s,h), \Pi(u',s',h')) = 0$ *if $x = su$ and $x' = s'u'$ are not $G^\vee$-conjugate;*
(ii)

$$\begin{aligned}\mathrm{EP}(\Pi(u,s,h), \Pi(u,s,h')) &= (|\mathrm{Z}_{A_x}(\bar{h})|\mathbf{1}_{\bar{h}^{-1}}, |\mathrm{Z}_{A_x}(\bar{h}')|\mathbf{1}_{\bar{h}'^{-1}})^{A_x}_{\mathrm{ell}} \\ &= \begin{cases} |Z_{A_x}(\bar{h})| \det_{\mathfrak{t}_x}(1-\bar{h}^{-1}), & \text{if } h, h' \text{ are conjugate,} \\ 0, & \text{otherwise.} \end{cases}\end{aligned}$$

Hence, the combinations $\{\Pi(u,s,h)\}$ define an orthogonal basis of $\mathcal{R}^p_{\mathrm{un,ell}}(G)$.

Proof. This is a straightforward consequence of Theorem 11.1 (and an elementary calculation for the last equality in part (ii)). □

Definition 9.6 (Cf. [Ciu20, Wal18]). The (dual) *elliptic nonabelian Fourier transform* is the involutive linear map $\mathrm{FT}^\vee_{\mathrm{ell}} \colon \mathcal{R}^p_{\mathrm{un,ell}}(G) \to \mathcal{R}^p_{\mathrm{un,ell}}(G)$, defined by

$$\mathrm{FT}^\vee_{\mathrm{ell}}(\Pi(u,s,h)) = \Pi(u,h,s), \quad (s,h) \in \Gamma_u \backslash \mathcal{Y}(\Gamma_u)_{\mathrm{ell}},\ u \in G^\vee \text{ unipotent.}$$

Note that $\mathrm{FT}^\vee_{\mathrm{ell}}$ is the just the image under $\overline{\mathsf{LLC}^p}_{\mathrm{un}}$ of the canonical involution of $\bigoplus_{s,u} \overline{R}(A_{su})$

$$[u,s,h] \mapsto [u,h,s], \quad \text{where } [u,s,h] := \sum_{\psi \in \widehat{A}_{su}} \psi(h)\psi.$$

For every $G' \in \mathrm{Inn}\mathrm{T}^p(G)$, and $K'_{\mathcal{O}} \in \max(G')$ consider the restriction map

$$\mathrm{res}_{K'_{\mathcal{O}}} \colon \mathrm{Irr}_{\mathrm{un}} G' \to R_{\mathrm{un}}(\overline{K}_{\mathcal{O}}),\ V \mapsto V^{K'^+_{\mathcal{O}}}. \tag{9.10}$$

We define a linear map $\mathrm{res}_{\mathrm{cpt,un}} \colon \bigoplus_{G' \in \mathrm{Inn}\mathrm{T}^p(G)} R_{\mathrm{un}}(G') \to \mathcal{C}(G)_{\mathrm{cpt,un}}$ by setting

$$\mathrm{res}_{\mathrm{cpt,un}}(V) = \sum_{K'_{\mathcal{O}} \in \max(G')} \mathrm{res}_{K'_{\mathcal{O}}}(V)$$

for all $G' \in \mathrm{Inn}\mathrm{T}^p(G)$ and $V \in \mathrm{Irr}_{\mathrm{un}}(G')$. With notation as in § 7, for each $(A,\mathcal{O}) \in S_{\max}(G)$, we let $\mathrm{proj}_{\mathcal{O}}$ be the projection map $\mathcal{C}(G)_{\mathrm{cpt,un}} \to \bigoplus_{x \in A} R_{\mathrm{un}}(\overline{K}_{x,\mathcal{O}})$ with respect to the decomposition (7.5), and let $\mathrm{res}_{\mathcal{O}} = \mathrm{proj}_{\mathcal{O}} \circ \mathrm{res}_{\mathrm{cpt,un}}$. We have

$$\mathrm{res}_{\mathrm{cpt,un}} = \bigoplus_{(A,\mathcal{O}) \in S_{\max}(G)} \mathrm{res}_{\mathcal{O}}\,. \tag{9.11}$$

We can now formulate the conjecture for elliptic representations.

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Conjecture 9.7. Let G be a simple F-split group. Consider the following diagram.

$$\begin{array}{ccc} \mathcal{R}^p_{\mathrm{un,ell}}(G) & \xrightarrow{\mathrm{FT}^\vee_{\mathrm{ell}}} & \mathcal{R}^p_{\mathrm{un,ell}}(G) \\ \Big\downarrow{\scriptstyle \mathrm{res}_{\mathrm{cpt,un}}} & & \Big\downarrow{\scriptstyle \mathrm{res}_{\mathrm{cpt,un}}} \\ \mathcal{C}(G)_{\mathrm{cpt,un}} & \xrightarrow[\mathrm{FT}_{\mathrm{cpt,un}}]{} & \mathcal{C}(G)_{\mathrm{cpt,un}} \end{array}$$

For every unipotent element $u \in G^\vee$, elliptic pair $(s,h) \in \mathcal{Y}(\Gamma_u)_{\mathrm{ell}}$ and maximal compact open subgroup $K_\mathcal{O}$ of G, there exists a root of unity $\zeta = \zeta(u,s,h,\mathcal{O})$ such that

$$\mathrm{res}_\mathcal{O}(\Pi(u,h,s)) = \zeta \cdot (\mathrm{FT}_{\mathrm{cpt,un}} \circ \mathrm{res}_\mathcal{O})(\Pi(u,s,h)).$$

Remark 9.8. If $K_\mathcal{O}$ is the maximal hyperspecial compact subgroup of G, so that in particular $\mathrm{res}_\mathcal{O} = \mathrm{res}_{K_\mathcal{O}}$, we expect that the only roots of unity ζ that appear are the well-known $\Delta(\bar{x}_\rho) \in \{\pm 1\}$ (see [Luz84a, § 6.7]) for certain families of unipotent representations of the finite groups of types E_7 and E_8. But for other maximal compact subgroups, Proposition 13.9 shows that in SL_n for example, new roots of unity can appear.

We remark that extra roots of unity already appear in relation with the nonabelian Fourier transform for finite reductive groups, although we do not know if this is a related issue. In that setting there are three bases of the Grothendieck group of unipotent characters:

(1) the irreducible characters;
(2) the 'almost characters' which, by definition, are the image of the basis of irreducible characters under Lusztig's nonabelian Fourier transform; and
(3) the traces of the Frobenius on unipotent character sheaves.

Lusztig's conjecture states that each element of the basis (2) equals an element of the basis (3) times a root of unity. Determining these roots of unity is a difficult question, see Shoji [Sho95] for classical groups, also Hetz's recent thesis [He23] for progress on the exceptional groups.

Remark 9.9. Note that the definition of the linear combinations $\Pi(u,s,h)$ and, thus, the definition of $\mathrm{FT}^\vee_{\mathrm{ell}}$, depends on the (in general, non-canonical) map $\mathsf{LLC}^p_{\mathrm{un}}$. It is an interesting question to understand how Conjecture 9.7 depends on $\mathsf{LLC}^p_{\mathrm{un}}$ and, more specifically, what choices one must make in constructing $\mathsf{LLC}^p_{\mathrm{un}}$ for the conjecture to be true. In the cases of the conjecture proved below, these subtleties do not arise.

9.2 Regular unipotent elements

In § 13, we will verify this conjecture completely when $G = \mathsf{SL}_n$ and PGL_n, but here we illustrate it in the case when u is a regular unipotent element.

Proposition 9.10. *Let $u_r \in G^\vee$ be a regular unipotent element. Then*

$$\mathrm{res}_{\mathrm{cpt,un}}(\Pi(u_r,h,s)) = \mathrm{FT}_{\mathrm{cpt,un}} \circ \mathrm{res}_{\mathrm{cpt,un}}(\Pi(u_r,s,h))$$

for all $(s,h) \in \mathcal{Y}(\Gamma_{u_r})$. In particular, Conjecture 9.7 holds with trivial roots of unity.

Proof. In this case $\Gamma_{u_r} = \mathrm{Z}_{G^\vee}$ and every pair (s,h) in $\mathcal{Y}(\mathrm{Z}_{G^\vee})$ is elliptic. Write the natural identification

$$\Omega_G \xrightarrow{\sim} \widehat{\mathrm{Z}_{G^\vee}}$$

as $x \mapsto \phi_x$. Then for $(s,h) \in \mathcal{Y}(\Gamma_{u_r})$, we have $\Pi(u_r,s,h) = \sum_{x \in \Omega} \phi_x(h)\pi(s,u_r,\phi_x)$. Note that $\pi(1,u_r,\phi_x)$ is the Steinberg representation St_{G_x} of G_x, so $\pi(s,u_r,\phi_x) \simeq \mathrm{St}_{G_x} \otimes \chi_s$, where χ_s is

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the weakly unramified character corresponding to s under (4.3). This follows from the fact that $\mathsf{LLC}_{\mathrm{un}}$ is equivariant for the action of weakly unramified characters, cf. [Sol23a, Theorem 1(b)].

For the rest of the proof, we fix $(A, \mathcal{O}) \in S_{\max}(G)$. Then given $s \in \mathrm{Z}_{G^\vee}$, the character χ_s is trivial on the parahoric $K^\circ_{x,\mathcal{O}}$ so defines a character, call it σ_s, of $K_{x,\mathcal{O}}/K^\circ_{x,\mathcal{O}} = \overline{K}_{x,\mathcal{O}}/\overline{K}^\circ_{x,\mathcal{O}}$. We have

$$\mathrm{res}_{\mathcal{O}}\, \pi(s, u_r, \phi_x) = \begin{cases} 0, & \text{if } x \notin A, \\ \mathrm{St}_{\overline{K}^\circ_{x,\mathcal{O}}} \rtimes \sigma_s, & \text{if } x \in A, \end{cases}$$

where $\mathrm{St}_{\overline{K}^\circ_{x,\mathcal{O}}}$ is the Steinberg character of the finite group $\overline{K}^\circ_{\mathcal{O},x}$. Note that for every $x \in A$,

$$\phi_x(h) = \sigma_h(x) \quad \text{for all } h \in \mathrm{Z}_{G^\vee}.$$

Thus,

$$\mathrm{res}_{\mathcal{O}}\, \Pi(u_r, s, h) = \sum_{x \in A} \sigma_h(x) \mathrm{St}_{\overline{K}^\circ_{x,\mathcal{O}}} \rtimes \sigma_s. \tag{9.12}$$

With notation as in § 5, let $\mathcal{U}_{\mathcal{O},\mathrm{St}} = \{\mathrm{St}_{K^\circ_{\mathcal{O}}}\}$ be the Steinberg family in $\mathrm{Irr}_{\mathrm{un}}(\overline{K}_{\mathcal{O}})$, and let $\widetilde{\mathcal{U}}_{\mathcal{O},\mathrm{St}} \subset \bigcup_{x \in A} \mathrm{Irr}_{\mathrm{un}}(\overline{K}_{x,\mathcal{O}})$ be the family parametrized by $\widetilde{\Gamma}^A_{\widetilde{\mathcal{U}}_{\mathcal{O},\mathrm{St}}} = A$ under the bijection of Proposition 6.1. Then by (9.12), $\mathrm{res}_{\mathcal{O}}\, \Pi(u_r, s, h)$ corresponds to $\Pi_{\widetilde{\mathcal{U}}_{\mathcal{O},\mathrm{St}}}(\sigma_s, \sigma_h)$ defined as in (5.13). The claim then follows from Lemma 5.1. □

10. A definition of the elliptic Fourier transform à la Lusztig

We present an alternative definition of the elliptic nonabelian Fourier transform (cf. Definition 9.6) along the lines of Lusztig's pairing defined in [Lus14, § 1]. Retain the notation from § 8. In particular, Γ is a complex reductive group, not necessarily connected, and $\mathcal{Y}(\Gamma)_{\mathrm{ell}}$ is the set of elliptic semisimple pairs in Γ. Let Σ be the set of semisimple elements of Γ. Extending the definition in § 5.1, we let

$$\mathcal{M}(\Gamma) = \{(x, \sigma) \mid x \in \Sigma,\ \sigma \in \widehat{A_\Gamma(x)}\},$$

modulo the equivalence relation given by conjugation by Γ.

For any two semisimple elements $x, y \in \Gamma$, define the set

$$\mathcal{A}_{x,y} = \{z \in \Gamma \mid zxz^{-1} \in Z_\Gamma(y)\} \tag{10.1}$$

with an action of $Z_\Gamma(x)^\circ \times Z_\Gamma(y)^\circ$ by $(\gamma, \gamma') \cdot z = \gamma' z \gamma^{-1}$. Let ${}^0\mathcal{A}_{x,y}$ denote the set of orbits. By [Lus14, Lemma 1.2], this is a finite set. It is clear that $\mathcal{A}_{y,x} = \mathcal{A}^{-1}_{x,y}$.

Consider the subset

$$\mathcal{A}^{\mathrm{ell}}_{x,y} = \{z \in \Gamma \mid (zxz^{-1}, y) \in \mathcal{Y}(\Gamma)_{\mathrm{ell}}\}, \tag{10.2}$$

and let ${}^0\mathcal{A}^{\mathrm{ell}}_{x,y}$ be the corresponding finite set of $Z_\Gamma(x)^\circ \times Z_\Gamma(y)^\circ$-orbits. Following [Lus14, § 1.3], we suppose $\kappa : \Sigma \times \Sigma \to \mathbb{R}$ is a nonnegative function satisfying

$$\kappa(x', y') = \kappa(y', x'), \quad \kappa(\gamma x' \gamma^{-1}, \gamma' y' (\gamma')^{-1}) = \kappa(x', y'), \quad \kappa(\zeta x', \zeta' y') = \kappa(x', y'),$$

for all $x', y' \in \Sigma, \gamma, \gamma' \in \Gamma$, $\zeta, \zeta' \in \mathrm{Z}_\Gamma$. The following definition is an elliptic analogue of [Lus14, § 1.3 (a)].

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Definition 10.1. For $(x,\sigma),(y,\tau)\in\mathcal{M}(\Gamma)$, set

$$\{(x,\sigma),(y,\tau)\}_{\mathrm{ell}}=\kappa(x,y)\sum_{z\in{}^0\mathcal{A}^{\mathrm{ell}}_{x,y}}\tau(zx^{-1}z^{-1})\sigma(z^{-1}yz).$$

It is immediate that $\{(x,\sigma),(y,\tau)\}_{\mathrm{ell}}=\overline{\{(y,\tau),(x,\sigma)\}}_{\mathrm{ell}}$ for all $(x,\sigma),(y,\tau)\in\mathcal{M}(\Gamma)$. Moreover, if $x\in\Sigma$ is such that x does not belong to any elliptic pair, then it is clear that (x,σ) is in the radical of $\{\,,\}_{\mathrm{ell}}$ for all σ.

Let $\mathbf{V}=\mathbb{C}[\mathcal{M}(\Gamma)]$ denote the $\mathbb{C}$-span of $\mathcal{M}(\Gamma)$, and given $(x,\sigma)\in\mathcal{M}(\Gamma)$, write $[(x,\sigma)]$ for its image in $\mathbf{V}$. Then $\{,\}_{\mathrm{ell}}$ extends to a Hermitian pairing on $\mathbf{V}$. Similarly to (5.13), for every pair of commuting elements $x,y\in\Sigma$, denote

$$\Pi(x,y)=\sum_{\sigma\in\widehat{A_\Gamma(x)}}\sigma(y^{-1})[(x,\sigma)]\in\mathbf{V}. \tag{10.3}$$

Let $\bar{y}$ denote the image of y in $A_\Gamma(x)$. Let $C_{\mathrm{Z}(x)}(y)$ denote the conjugacy class of y in $\mathrm{Z}_\Gamma(x)$, and $C_{A(x)}(\bar{y})$ the conjugacy class of $\bar{y}$ in $A_\Gamma(x)$. Since x and y commute, we have that $\mathcal{A}_{x,y}$ contains the set $\mathrm{Z}_\Gamma(y)\mathrm{Z}_\Gamma(x)$, and the action of $\mathrm{Z}_\Gamma(x)^\circ\times\mathrm{Z}_\Gamma(y)^\circ$ on $\mathcal{A}_{x,y}$ restricts to an action on $\mathrm{Z}_\Gamma(y)\mathrm{Z}_\Gamma(x)$. Let ${}^0\mathcal{B}_{x,y}$ denote a set of orbit representatives for this restricted action.

Lemma 10.2. *For every $(t,\tau)\in\mathcal{M}(\Gamma)$,*

$$\{\Pi(x,y),(t,\tau)\}_{\mathrm{ell}}=\begin{cases}\kappa(x,y)|\mathrm{Z}_{A(x)}(\bar{y})|\sum_{z\in{}^0\mathcal{B}_{x,y}}\tau(zx^{-1}z^{-1}), & \text{if } t=y \text{ and } (x,y)\in\mathcal{Y}(\Gamma)_{\mathrm{ell}},\\ 0, & \text{otherwise.}\end{cases}$$

Proof. This is similar to the calculation for the nonabelian Fourier transform in the case of finite reductive groups. We compute

$$\begin{aligned}\{\Pi(x,y),(t,\tau)\}_{\mathrm{ell}}&=\sum_{\sigma\in\widehat{A_\Gamma(x)}}\sigma(y^{-1})\{(x,\sigma),(t,\tau)\}_{\mathrm{ell}}\\&=\kappa(x,t)\sum_{z\in{}^0\mathcal{A}^{\mathrm{ell}}_{x,t}}\tau(zx^{-1}z^{-1})\Big(\sum_{\sigma\in\widehat{A_\Gamma(x)}}\sigma(y^{-1})\sigma(z^{-1}tz)\Big).\end{aligned}$$

Fix $z\in{}^0\mathcal{A}^{\mathrm{ell}}_{x,t}$. By the orthogonality relations for characters, $\sum_{\sigma\in\widehat{A_\Gamma(x)}}\sigma(y^{-1})\sigma(z^{-1}tz)=0$ unless the image of $z^{-1}tz$ in $A_\Gamma(x)$ is conjugate to $\bar{y}$, in which case it equals $|\mathrm{Z}_{A(x)}(\bar{y})|$. In addition, $(x,z^{-1}tz)\in\mathcal{Y}(\Gamma)_{\mathrm{ell}}$, which implies by Lemma 8.2 that the image of $z^{-1}tz$ in $A_\Gamma(x)$ is in an elliptic class, so $\bar{y}$ is elliptic in $A_\Gamma(x)$, and so again by Lemma 8.2, $(x,y)\in\mathcal{Y}(\Gamma)_{\mathrm{ell}}$. Moreover, by the same result, the image of $z^{-1}tz$ in $A_\Gamma(x)$ is conjugate to $\bar{y}$ if and only if $z^{-1}tz$ is conjugate to y in $\mathrm{Z}_\Gamma(x)$. In particular, this means that t is conjugate to y in Γ. This proves that $\{\Pi(x,y),(t,\tau)\}_{\mathrm{ell}}=0$ unless $t=y$ and $(x,y)\in\mathcal{Y}(\Gamma)_{\mathrm{ell}}$. (Since $\mathcal{M}(\Gamma)$ consists of Γ-orbits, we may identify $t=y$.)

To complete the proof, assume $(x,y)\in\mathcal{Y}(\Gamma)_{\mathrm{ell}}$. We have

$$\{\Pi(x,y),(y,\tau)\}_{\mathrm{ell}}=\kappa(x,y)|\mathrm{Z}_{A(x)}(\bar{y})|\sum_{\substack{z\in{}^0\mathcal{A}^{\mathrm{ell}}_{x,y}\\ z^{-1}yz\in C_{Z(x)}(y)}}\tau(zx^{-1}z^{-1}).$$

We analyze the index of summation. Note that $z^{-1}yz\in C_{\mathrm{Z}(x)}(y)$ is equivalent to $z\in\mathrm{Z}_\Gamma(y)\mathrm{Z}_\Gamma(x)$. Since (x,y) is an elliptic pair, it is also automatic that (zxz^{-1},y) is for any $z\in\mathrm{Z}_\Gamma(y)\mathrm{Z}_\Gamma(x)$, hence z ranges over representatives of $\mathrm{Z}_\Gamma(x)^\circ\times\mathrm{Z}_\Gamma(y)^\circ$-orbits in $\mathrm{Z}_\Gamma(y)\mathrm{Z}_\Gamma(x)$. □

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Proposition 10.3. *For* $(x,y)\in\mathcal{Y}(\Gamma)_{\mathrm{ell}}$:

(a) $\displaystyle\sum_{\tau\in\widehat{A_\Gamma(y)}}\{\Pi(x,y),(y,\tau)\}_{\mathrm{ell}}[(y,\tau)]=\kappa(x,y)|\mathrm{Z}_{A(x)}(\bar{y})|\,|{}^0\mathcal{B}_{x,y}|\cdot\Pi(y,x)$ *in* $\mathbf{V}$;

(b) $\displaystyle\sum_{h\in A_\Gamma(y)}\{\Pi(x,y),\Pi(y,h)\}_{\mathrm{ell}}\Pi(y,h)=\kappa(x,y)|A_\Gamma(y)||\mathrm{Z}_{A(x)}(\bar{y})|\,|{}^0\mathcal{B}_{x,y}|\cdot\Pi(y,x)$.

Proof. (a) Denote by $\tau^{z^{-1}}\in\widehat{A_\Gamma(y)}$ the twist of τ by z^{-1}, so $\tau^{z^{-1}}(a)=\tau(zaz^{-1})$. Applying the previous lemma, we get

$$\begin{aligned}&\frac{1}{|\kappa(x,y)||\mathrm{Z}_{A(x)}(\bar{y})|}\sum_{\tau\in\widehat{A_\Gamma(y)}}\{\Pi(x,y),(y,\tau)\}_{\mathrm{ell}}[(y,\tau)]\\&=\sum_{z\in{}^0\mathcal{B}_{x,y}}\sum_{\tau\in\widehat{A_\Gamma(y)}}\tau^{z^{-1}}(x^{-1})[(y,\tau)]=\sum_{z\in{}^0\mathcal{B}_{x,y}}\sum_{\tau\in\widehat{A_\Gamma(y)}}\tau^{z^{-1}}(x^{-1})[(z^{-1}yz,\tau^{z^{-1}})]\\&=\sum_{z\in{}^0\mathcal{B}_{x,y}}\sum_{\tau'\in\widehat{A_\Gamma(z^{-1}yz)}}\tau'(x^{-1})[(z^{-1}yz,\tau')]=\sum_{z\in{}^0\mathcal{B}_{x,y}}\Pi(z^{-1}yz,x)=|{}^0\mathcal{B}_{x,y}|\;\Pi(y,x),\end{aligned}$$

since $z^{-1}yz$ is conjugate to y in $\mathrm{Z}_\Gamma(x)$.

(b) This is immediate from part (a) using $\sum_{h\in A_\Gamma(y)}\tau(h^{-1})\tau'(h)=|A_\Gamma(y)|\delta_{\tau,\tau'}$, $\tau,\tau'\in\widehat{A_\Gamma(y)}$. □

Consider the set

$$B_{\mathbf{V}}=\{|C_{A(y)}(\bar{h})|^{1/2}\Pi(y,h)\mid(y,h)\in\Gamma\backslash\mathcal{Y}(\Gamma)_{\mathrm{ell}}\}.\tag{10.4}$$

By Lemma 10.2, $B_{\mathbf{V}}$ spans $\bar{\mathbf{V}}_{\mathrm{ell}}:=\mathbf{V}/\ker\{\,,\,\}_{\mathrm{ell}}$, where $\ker\{\,,\,\}_{\mathrm{ell}}$ denotes the radical of the pairing. Define

$$\mathcal{F}'_{\mathrm{ell}}:\mathbf{V}\to\mathbf{V},\quad\mathcal{F}'_{\mathrm{ell}}(v)=\sum_{b\in B_{\mathbf{V}}}\{v,b\}b.\tag{10.5}$$

Clearly, $\mathcal{F}'_{\mathrm{ell}}$ descends to a linear map on $\bar{\mathbf{V}}_{\mathrm{ell}}$.

Corollary 10.4. *For every* $(x,y)\in\mathcal{Y}(\Gamma)_{\mathrm{ell}}$,

$$\mathcal{F}'_{\mathrm{ell}}(\Pi(x,y))=\kappa(x,y)|A_\Gamma(y)||\mathrm{Z}_{A(x)}(\bar{y})|\,|{}^0\mathcal{B}_{x,y}|\cdot\Pi(y,x).$$

Proof. From Lemma 10.2, we have $\mathcal{F}'_{\mathrm{ell}}(\Pi(x,y))=\sum_h|C_{A(y)}(\bar{h})|\{\Pi(x,y),\Pi(y,h)\}_{\mathrm{ell}}[\Pi(y,h)]$, where the sum is over a set of representatives h of the conjugacy classes of elliptic semisimple elements in $\mathrm{Z}_\Gamma(y)$, equivalently, elliptic conjugacy classes in $A_\Gamma(y)$. Thus, we may rewrite

$$\mathcal{F}'_{\mathrm{ell}}(\Pi(x,y))=\sum_{h\in A_\Gamma(y)_{\mathrm{ell}}}\{\Pi(x,y),\Pi(y,h)\}_{\mathrm{ell}}\Pi(y,h)=\sum_{h\in A_\Gamma(y)}\{\Pi(x,y),\Pi(y,h)\}_{\mathrm{ell}}\Pi(y,h),$$

using Lemma 10.2 again. Then the claim follows from Proposition 10.3. □

In other words, $\mathcal{F}'_{\mathrm{ell}}$ acts, up to a scalar multiple, by the flip on each $\Pi(x,y)$. If we set

$$\kappa(x,y)=\frac{|C_{A(x)}(\bar{y})|^{1/2}|C_{A(y)}(\bar{x})|^{1/2}}{|A_\Gamma(x)||A_\Gamma(y)||{}^0\mathcal{B}_{x,y}|},\tag{10.6}$$

then

$$\mathcal{F}'^{\,2}_{\mathrm{ell}}=\mathrm{Id}.\tag{10.7}$$

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Remark 10.5. When we specialize to $\Gamma = \Gamma_u$ for a unipotent element $u \in \Gamma$, we see from Lemma 9.5 that $B_{\mathbf{V}}$ is in fact an orthogonal basis of $\bar{\mathbf{V}}_{\mathrm{ell}}$ with respect to the Euler–Poincaré pairing.

11. Elliptic unipotent representations

The main result of this section is as follows.

Theorem 11.1. *Suppose G is a semisimple split F-group of adjoint type. Then Conjecture 9.1 holds for all pure inner twists of G.*

Remark 11.2. In § 11.4, we explain how Theorem 11.1 can be extended to other isogenies. See, in particular, Corollary 11.13.

The strategy of the proof is as follows. As explained in § 4, the set of unipotent enhanced Langlands parameters $\Phi_{\mathrm{e,un}}(G')$, where G' is an inner twist of G, decomposes into a disjoint union $\Phi_{\mathrm{e,un}}(G') = \bigsqcup_{\mathfrak{s}^\vee \in \mathfrak{B}^\vee_{\mathrm{un}}(G')} \Phi_{\mathrm{e}}(G')^{\mathfrak{s}^\vee}$. Consequently, there is a decomposition

$$R(\Phi_{\mathrm{e,un}}(G')) = \bigoplus_{\mathfrak{s}^\vee \in \mathfrak{B}^\vee_{\mathrm{un}}(G')} R(\Phi_{\mathrm{e}}(G')^{\mathfrak{s}^\vee}),$$

where $R(\Phi_{\mathrm{e,un}}(G'))$ and $\Phi_{\mathrm{e}}(G')^{\mathfrak{s}^\vee}$ are defined analogously to $R(\Phi_{\mathrm{e,un}}({}^L G))$ (see (4.16)). In [Lus95] (for adjoint simple groups) and later in [AMS17] (for arbitrary groups), an affine Hecke algebra with possibly unequal parameters $\mathcal{H}(\mathfrak{s}^\vee)$ is constructed such that there is a bijection

$$\mathrm{Irr}\ \mathcal{H}(\mathfrak{s}^\vee) \longleftrightarrow \Phi_{\mathrm{e}}(G')^{\mathfrak{s}^\vee}, \tag{11.1}$$

which induces a linear isomorphism

$$R(\mathcal{H}(\mathfrak{s}^\vee)) \cong R(\Phi_{\mathrm{e}}(G')^{\mathfrak{s}^\vee}).$$

We need to study the elliptic space $\overline{R}(\mathcal{H}(\mathfrak{s}^\vee))$. The important fact for the elliptic theory is that $\mathcal{H}(\mathfrak{s}^\vee)$ is a deformation of an extended affine Weyl group $\widetilde{W}_{\mathfrak{s}^\vee} = W_{\mathfrak{s}^\vee} \ltimes X^*(T_{\mathfrak{s}^\vee})$, where $T_{\mathfrak{s}^\vee} = \Phi_{\mathrm{e}}(L')^{\mathfrak{s}^\vee_{L'}}$ for L' a Levi subgroup of G' that corresponds to $\mathfrak{s}^\vee$. This allows us to use the results of [OS09] to further reduce to $\overline{R}(\mathcal{H}(\mathfrak{s}^\vee)) \cong \overline{R}(\widetilde{W}_{\mathfrak{s}^\vee})$. Moreover, the latter space is equivalent to a direct sum of elliptic spaces for certain finite groups

$$\overline{R}(\widetilde{W}_{\mathfrak{s}^\vee}) \cong \bigoplus_{s \in W_{\mathfrak{s}^\vee} \backslash T_{\mathfrak{s}^\vee}} \overline{R}(\mathrm{Z}_{W_{\mathfrak{s}^\vee}}(s)).$$

We then use results of [Wal07] and the generalized Springer correspondence to relate the spaces $\overline{R}(\mathrm{Z}_{W_{\mathfrak{s}^\vee}})(s)$ to the relevant spaces of Langlands parameters (for the various unipotent elements) in $\Phi_{\mathrm{e,un}}(G')^{\mathfrak{s}^\vee}$.

Finally, by [Lus95, Sol23a] for each $\mathfrak{s}^\vee \in \mathfrak{B}^\vee_{\mathrm{un}}(G')$, there exists $\mathfrak{s} \in \mathfrak{B}_{\mathrm{un}}(G')$ that is sent to $\mathfrak{s}^\vee$ by the LLC, and then the Hecke algebra $\mathcal{H}(\mathfrak{s})$ for $\mathfrak{s}$ is isomorphic to $\mathcal{H}(\mathfrak{s}^\vee)$. The fact that the elliptic space for the representations in the block $\mathfrak{R}(G')^{\mathfrak{s}}$ is naturally isomorphic to $\overline{R}(\mathcal{H}(\mathfrak{s}))$ is immediate by the exactness of the equivalence of categories between $\mathfrak{R}(G')^{\mathfrak{s}}$ and $\mathcal{H}(\mathfrak{s})$-modules.

11.1 Euler–Poincaré pairings for affine Hecke algebras

We begin by recalling several known facts about elliptic theory for affine Weyl groups and affine Hecke algebras. The main reference is [OS09] (see also [CO15]). The notation in this section is self contained and independent of the previous sections. For applications, the root datum in this section will be specialized to the root datum of the Langlands dual group $G^\vee$, as well as to the root data for the affine Hecke algebras $\mathcal{H}(\mathfrak{s}^\vee)$ that occur on the dual side of the LLC.

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Let $\mathcal{R} = (X, R, X^\vee, R^\vee, \Pi)$ be a based root datum. Here $X, X^\vee$ are lattices in perfect duality $\langle\ ,\ \rangle\colon X \times X^\vee \to \mathbb{Z}$, $R \subset X \setminus \{0\}$ and $R^\vee \subset X^\vee \setminus \{0\}$ are the finite sets of roots and coroots, respectively, and $\Pi \subset R$ is a basis of simple roots. Let W be the finite Weyl group with set of generators $S = \{s_\alpha : \alpha \in \Pi\}$. Set $\widetilde{W} = W \ltimes X$, the (dual) extended affine Weyl group, and $W^a = W \ltimes Q$, the (dual) affine Weyl group, where Q is the root lattice of R. Then W^a is normal in $\widetilde{W}$ and $\Omega := \widetilde{W}/W^a \cong X/Q$ is an abelian group. We assume that $\mathcal{R}$ is semisimple, which means that Ω is a finite group.

The set $R^a = R^\vee \times \mathbb{Z} \subset X^\vee \times \mathbb{Z}$ is the set of affine roots. (Note that $\widetilde{W}$ is the extended affine Weyl group of a split p-adic group G with root datum dual to $\mathcal{R}$, and R^a is the set of affine roots for G.) A basis of simple affine roots is given by $\Pi^a = (\Pi^\vee \times \{0\}) \cup \{(\gamma^\vee, 1) : \gamma^\vee \in R^\vee \text{ minimal}\}$. For every affine root $\mathbf{a} = (\alpha^\vee, n)$, let $s_{\mathbf{a}}\colon X \to X$ denote the reflection $s_{\mathbf{a}}(x) = x - ((x, \alpha^\vee) + n)\alpha$. The affine Weyl group W^a has a set of generators $S^a = \{s_{\mathbf{a}} \mid \mathbf{a} \in \Pi^a\}$. Given $J \subset S^a$, let W_J be the subgroup of W^a generated by $\{s_{\mathbf{a}} \mid \mathbf{a} \in J\}$. Let $l\colon \widetilde{W} \to \mathbb{Z}$ be the length function.

Set $E = X \otimes_{\mathbb{Z}} \mathbb{C}$, so the discussion regarding the elliptic theory of W and E from the previous sections applies. We denote a typical element of $\widetilde{W}$ by wt_x, where $w \in W$ and $x \in X$. The extended affine Weyl group $\widetilde{W}$ acts on E via $(wt_x) \cdot v = w \cdot v + x$, $v \in E$.

An element $wt_x \in \widetilde{W}$ is called elliptic if $w \in W$ is elliptic (with respect to the action on E). For basic facts about elliptic theory for $\widetilde{W}$, see [OS09, §§ 3.1, 3.2]. There are finitely many elliptic conjugacy classes in $\widetilde{W}$ (and in W^a). The following fact is well-known (see for example [CO15, Lemma 5.4]).

Lemma 11.3. *Suppose C is an elliptic conjugacy class in W^a. Then there exists one and only one maximal $J \subsetneq S^a$ such that $C \cap W_J \neq \emptyset$, and in this case $C \cap W_J$ forms a single elliptic W_J-conjugacy class.*

Let $R(\widetilde{W})$ be the Grothendieck group of $\widetilde{W}$-mod (the category of finite-dimensional modules). Define the Euler–Poincaré pairing of $\widetilde{W}$ by

$$\langle V_1, V_2\rangle_{\mathrm{EP}}^{\widetilde{W}} = \sum_{i \geqslant 0} (-1)^i \dim \mathrm{Ext}^i_{\widetilde{W}}(V_1, V_2), \quad V_1, V_2 \text{ finite-dimensional } \widetilde{W}\text{-modules.} \tag{11.2}$$

Set $\overline{R}(\widetilde{W}) = R(\widetilde{W})/\mathrm{rad}\langle\ ,\ \rangle_{\mathrm{EP}}^{\widetilde{W}}$. By [OS09, Theorem 3.3], the Euler–Poincaré pairing for $\widetilde{W}$ can also be expressed as an elliptic integral. More precisely, define the conjugation-invariant elliptic measure μ_{ell} on $\widetilde{W}$ by setting $\mu_{\mathrm{ell}} = 0$ on nonelliptic conjugacy classes, and for an elliptic conjugacy class C such that $v \in E$ is an isolated fixed point for some element of C, set

$$\mu_{\mathrm{ell}}(C) = \frac{|Z_{\widetilde{W}}(v) \cap C|}{|Z_{\widetilde{W}}(v)|};$$

here $Z_{\widetilde{W}}(v)$ is the isotropy group of v in $\widetilde{W}$. Then

$$\langle V_1, V_2\rangle_{\mathrm{EP}}^{\widetilde{W}} = (\chi_{V_1}, \chi_{V_2})_{\mathrm{ell}}^{\widetilde{W}} := \int_{\widetilde{W}} \chi_{V_1} \overline{\chi_{V_2}}\, d\mu_{\mathrm{ell}}, \quad V_1, V_2 \in \widetilde{W}\text{-mod}, \tag{11.3}$$

where χ_{V_1}, χ_{V_2} are the characters of V_1 and V_2.

Set $T = \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{C}^\times)$. Then W acts on T. For every $s \in T$, set $W_s = \{w \in W : w \cdot s = s\}$. One considers the elliptic theory of the finite group W_s acting on the cotangent space $\mathfrak{t}_s^*$ of T at s. By Clifford theory, the induction map

$$\mathrm{Ind}_s\colon W_s\text{-mod} \to \widetilde{W}\text{-mod}, \quad \mathrm{Ind}_s(V_1) := \mathrm{Ind}_{W_s \ltimes X}^{\widetilde{W}}(V_1 \otimes s)$$

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maps irreducible modules to irreducible modules. By [OS09, Theorem 3.2], the map

$$\bigoplus_{s\in W\backslash T} \mathrm{Ind}_s \colon \bigoplus_{s\in W\backslash T} \overline{R}(W_s)_{\mathbb{C}} \to \overline{R}(\widetilde{W})_{\mathbb{C}} \tag{11.4}$$

is an isomorphism of metric spaces, in particular,

$$\langle \mathrm{Ind}_s V_1, \mathrm{Ind}_s V_2\rangle_{\mathrm{EP}}^{\widetilde{W}} = (V_1, V_2)_{\mathrm{ell}}^{W_s}, \quad V_1, V_2 \in W_s\text{-mod}. \tag{11.5}$$

A space $\overline{R}(W_s)_{\mathbb{C}}$ in the left-hand side of (11.4) is nonzero if and only if s is an isolated element of T, more precisely $s \in T_{\mathrm{iso}}$, where

$$T_{\mathrm{iso}} = \{s \in T^\vee : w \cdot s = s \text{ for some elliptic } w \in W\}.$$

Example 11.4. Let $\mathcal{R}$ be the root datum of $\mathsf{PGL}_n(\mathbb{C})$. We may take T to be the maximal diagonal torus of $\mathsf{PGL}_n(\mathbb{C})$, with $X = X^*(T)$ the group of characters and $X^\vee = X_*(T)$ the group of cocharacters. In this case, $Q = X$ and

$$\widetilde{W} = W^a = \langle s_i,\ 0 \leqslant i \leqslant n-1 \mid (s_i s_j)^{m(i,j)} = 1,\ 0 \leqslant i, j \leqslant n-1\rangle,$$

where $m(i,i) = 1$, $m(i,j) = 2$ if $1 < |i-j| < n-1$, and $m(i,j) = 3$ if $|i-j| = 1$ or $|i-j| = n-1$, when $n \geqslant 3$. This is the extended affine Weyl group associated to the p-adic group SL_n. If $n = 2$, then $\widetilde{W} = W^a = \langle s_0, s_1 \mid s_0^2 = s_1^2 = 1\rangle$.

With this notation, the finite Weyl group is $W = \langle s_1, \dots, s_{n-1}\rangle \subset W^a$. For every $0 \leqslant i \leqslant n-1$, let $W_i = \langle s_0, s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_{n-1}\rangle \subset W^a$. These are the maximal (finite) parabolic subgroups of W^a. In particular, $W_0 = W$ and $W_i \cong S_n$ for all i. The space $E = \mathfrak{t}^* \cong \mathbb{C}^{n-1}$ and each W_i acts on E by the reflection representation. Therefore, there exists a unique elliptic W_i-conjugacy class represented by the Coxeter element $w_i = s_0 s_1 \cdots s_{i-1} s_{i+1} \cdots s_{n-1}$. Thus, by Lemma 11.3, there are exactly n elliptic conjugacy classes in W^a each determined by the condition that it meets W_i in the conjugacy class of w_i, $0 \leqslant i \leqslant n-1$. In particular, $\dim \overline{R}(\widetilde{W})_{\mathbb{C}} = n$ in this case.

On the other hand, by (11.4), we need to consider W-orbits in T_{iso}. Since there is only one elliptic conjugacy class in W, every W-orbit in T_{iso} is represented by an element of

$$T^{s_1 s_2 \cdots s_{n-1}} = \{\Delta_n(z) \mid z \in \mu_n\}, \quad \Delta_n(z) = \mathrm{diag}(1, z, z^2, \dots, z^{n-1}) \in T, \quad \mu_n = \{z \mid z^n = 1\},$$

as noted previously. Two elements $\Delta_n(z)$ and $\Delta_n(z')$ of $T^{s_1 s_2 \cdots s_{n-1}}$ are W-conjugate if and only if z and z' have the same order. Fix a primitive nth root ζ of 1. This means that (11.4) becomes, in this case,

$$\bigoplus_{d|n} \overline{R}(W_{\Delta_n(\zeta^{n/d})})_{\mathbb{C}} \cong \overline{R}(\widetilde{W})_{\mathbb{C}}.$$

If $z = \zeta^m$, where $n = dm$, then $\Delta_n(z)$ is W-conjugate to $\mathrm{diag}(\underbrace{1, \dots, 1}_{m}, \underbrace{z, \dots, z}_{m}, \dots, \underbrace{z^{d-1}, \dots, z^{d-1}}_{m})$.

Hence, one calculates

$$W_{\Delta_n(z)} \cong S_m^d \rtimes C_d \quad \text{and} \quad \mathfrak{t}^*_{\Delta_n(z)} = \left\{(a_1, \dots, a_n) \,\middle|\, \sum_{i=1}^n a_i = 0\right\}. \tag{11.6}$$

The action is the natural permutation action. There are $\varphi(d)$ elliptic conjugacy classes, represented by $(w_m, 1, \dots, 1) \cdot x_d$, where w_m is a fixed m-cycle (Coxeter element) of S_m and x_d is one of the $\varphi(d)$ generators of C_d (see Lemma 13.5). Again $\sum_{d|n} \varphi(d) = n$.

Let $\vec{\mathbf{q}} = \{\mathbf{q}(s) \mid s \in S^a\}$ be a set of invertible, commuting indeterminates such that $\mathbf{q}(s) = \mathbf{q}(s')$ whenever s, s' are W^a-conjugate. Let $\Lambda = \mathbb{C}[\mathbf{q}(s), \mathbf{q}(s)^{-1} \mid s \in S^a]$.

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The generic affine Hecke algebra $\mathcal{H}(\mathcal{R}, \vec{\mathbf{q}})$ associated to the root datum $\mathcal{R}$ and the set of indeterminates $\vec{\mathbf{q}}$ is the unique associative, unital Λ-algebra with basis $\{T_w : w \in \widetilde{W}\}$ and relations:

(i) $T_w T_{w'} = T_{ww'}$, for all $w, w' \in W$ such that $l(ww') = l(w) + l(w')$;
(ii) $(T_s - \mathbf{q}(s)^2)(T_s + 1) = 0$ for all $s \in S^a$.

Fix a real number $q > 1$. Given a W^a-invariant function $m\colon S^a \to \mathbb{R}$, we may define a homomorphism $\lambda_m\colon \Lambda \to \mathbb{C}$, $\mathbf{q}(s) = q^{m(s)}$. Let $\mathbb{C}_{\lambda_m}$ be the one-dimensional complex module on which Λ acts by λ_m. Consider the specialized affine Hecke $\mathbb{C}$-algebra

$$\mathcal{H}(\mathcal{R}, q, m) = \mathcal{H}(\mathcal{R}, \vec{\mathbf{q}}) \otimes_\Lambda \mathbb{C}_{\lambda_m}. \tag{11.7}$$

Example 11.5. Let $\mathcal{R}$ be the root datum of $\mathsf{PGL}_n(\mathbb{C})$. If $n = 2$, the generic affine Hecke algebra has two indeterminates $\mathbf{q}(s_0)$ and $\mathbf{q}(s_1)$ and it is generated by $T_0 = T_{s_0}$, $T_1 = T_{s_1}$ subject only to the quadratic relations

$$(T_i - \mathbf{q}(s_i)^2)(T_i + 1) = 0, \quad i = 0, 1.$$

If $n \geqslant 3$, all the simple reflections are W^a-conjugate. There is only one indeterminate $\mathbf{q}$ such that the affine Hecke algebra is generated by $\{T_i = T_{s_i}, 0 \leqslant i \leqslant n-1\}$ subject to the relations:

(i) $T_i T_j = T_j T_i$, $1 < |i - j| < n - 1$;
(ii) $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$, $0 \leqslant i \leqslant n-1$; $T_0 T_{n-1} T_0 = T_{n-1} T_0 T_{n-1}$;
(iii) $(T_i - \mathbf{q}^2)(T_i + 1) = 0$.

Let $\mathcal{H} = \mathcal{H}(\mathcal{R}, q, m)$ for simplicity of notation. If V_1, V_2 are two finite-dimensional $\mathcal{H}$-modules, define the Euler–Poincaré pairing [OS09, § 3.4]:

$$\mathrm{EP}_{\mathcal{H}}(V_1, V_2) = \sum_{i \geqslant 0} (-1)^i \dim \mathrm{Ext}^i_{\mathcal{H}}(V_1, V_2). \tag{11.8}$$

This is a finite sum since $\mathcal{H}$ has finite cohomological dimension [OS09, Proposition 2.4]. The pairing $\mathrm{EP}_{\mathcal{H}}$ is symmetric and positive semidefinite. It extends to a Hermitian positive-semidefinite pairing on the complexified Grothendieck group $R(\mathcal{H})_{\mathbb{C}}$ of finite-dimensional $\mathcal{H}$-modules. We wish to compare the Euler–Poincaré pairings for $\mathcal{H}(\mathcal{R}, q, m)$ and $\mathcal{H}(\mathcal{R}, q^\epsilon, m)$, where $\epsilon \in [0, 1]$. Suppose we have a family of maps

$$\sigma_\epsilon\colon \mathcal{H}(\mathcal{R}, q, m)\text{-mod} \to \mathcal{H}(\mathcal{R}, q^\epsilon, m)\text{-mod}, \quad \sigma_\epsilon(\pi, V) = (\pi_\epsilon, V), \tag{11.9}$$

such that:

(a) for every $w \in \widetilde{W}$ and every (π, V), the assignment $\epsilon \mapsto \pi(\epsilon)(T_w)$ is a continuous map $[0, 1] \to \mathrm{End}(V)$.

Then [OS09, Theorem 3.5] shows that

$$\mathrm{EP}_{\mathcal{H}(\mathcal{R}, q, m)}(V_1, V_2) = \mathrm{EP}_{\mathcal{H}(\mathcal{R}, q^\epsilon, m)}(\sigma_\epsilon(V_1), \sigma_\epsilon(V_2)), \quad \text{for all } \epsilon \in [0, 1].$$

In particular, note that $\mathcal{H}(\mathcal{R}, q^0, m) = \mathbb{C}[\widetilde{W}]$, meaning that

$$\mathrm{EP}_{\mathcal{H}(\mathcal{R}, q, m)}(V_1, V_2) = \langle \sigma_0(V_1), \sigma_0(V_2) \rangle^{\widetilde{W}}_{\mathrm{EP}}. \tag{11.10}$$

Using [OS09, Theorem 1.7] or alternatively, for the affine Hecke algebras that occur for unipotent representations of p-adic groups, via the geometric constructions of [KL87, Lus95, Lus02], we know that scaling maps σ_ϵ as above exist and in addition, they also behave well with respect to harmonic analysis:

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(b) for every $\epsilon \in [0,1]$, V is unitary (respectively, tempered) if and only if $\sigma_\epsilon(V)$ is unitary (respectively, tempered);
(c) for every $\epsilon \in (0,1]$, V is discrete series if and only if $\sigma_\epsilon(V)$ is discrete series.

Denoting by $\overline{R}(\mathcal{H})_{\mathbb{C}}$ the quotient of $R(\mathcal{H})_{\mathbb{C}}$ by the radical of $\mathrm{EP}_{\mathcal{H}}$, it follows [OS09, Proposition 3.9] that the scaling map σ_0 induces an injective isometric map

$$\sigma_0 \colon \overline{R}(\mathcal{H})_{\mathbb{C}} \to \overline{R}(\widetilde{W})_{\mathbb{C}} \cong \bigoplus_{s \in T/W} \overline{R}(W_s)_{\mathbb{C}}. \tag{11.11}$$

In fact, this map is also an isomorphism, for example via [CH17, Theorem 8.1].

11.2 Elliptic inner products for Weyl groups (after Waldspurger [Wal07])

Let $\mathcal{G} = \mathcal{G}^\circ$ be a complex connected reductive group and $\theta \colon \mathcal{G} \to \mathcal{G}$ a quasi-semisimple automorphism of $\mathcal{G}$ of finite order.

As in § 3, we let $\mathbf{I}$ be the set of pairs $(U, \mathcal{E})$ where U is a unipotent conjugacy class in $\mathcal{G}$ and $\mathcal{E}$ is an irreducible $\mathcal{G}$-equivariant local system on U. The automorphism θ acts on $\mathbf{I}$ via $(U, \mathcal{E}) \mapsto (\theta(U), (\theta^{-1})^*(\mathcal{E}))$. Let $\mathbf{I}^\theta$ denote the fixed points of this action and suppose $(U, \mathcal{E}) \in \mathbf{I}^\theta$. If we fix $u \in U$, there exists $x \in \mathcal{G}$ such that $\mathrm{Ad}(x) \circ \theta(u) = u$, hence $\mathrm{Ad}(x) \circ \theta$ preserves $\mathrm{Z}_{\mathcal{G}}(u)$ and, hence, it defines an automorphism of A_u, denoted θ_u. As explained in [Wal07, p. 612], if $\phi \in \widehat{A}_u$ corresponds to the local system $\mathcal{E}$, the fact that $(\theta^{-1})^*(\mathcal{E}) \cong \mathcal{E}$ is equivalent to the condition that ϕ extends to a representation $\widetilde{\phi}$ of $A_u \rtimes \langle \theta_u \rangle$.

Fix a Borel subgroup B_u of $\mathrm{Z}_{\mathcal{G}}(u)^\circ$ and a maximal torus T_u in B_u. Let $\mathfrak{t}_u$ be the complex Lie algebra of T_u. Define a complex representation $(\delta_u, \mathfrak{t}_u)$ of $A_u \rtimes \langle \theta_u \rangle$, extending the previous definition for the action of A_u. Similarly to § 8.2, since $\mathrm{Z}_{\mathcal{G}}(u)$ acts on $\mathrm{Z}_{\mathcal{G}}(u)^\circ$ by conjugation and θ_u acts on $\mathrm{Z}_{\mathcal{G}}(u)^\circ$ as above, every element $z \in \mathrm{Z}_{\mathcal{G}}(u) \rtimes \langle \theta_u \rangle$ acts on $\mathrm{Z}_{\mathcal{G}}(u)^\circ$ via an automorphism α_z. There exists $y \in \mathrm{Z}_{\mathcal{G}}(u)^\circ$ such that $\alpha_z \circ \mathrm{Ad}(y)$ preserves B_u and T_u. This means that $\alpha_z \circ \mathrm{Ad}(y)$ defines an automorphism of the cocharacter lattice $X_*(T_u)$ that also preserves the sublattice $X_*(Z_{\mathcal{G}}^\circ)$, and therefore a linear isomorphism $\delta_u(z)$ of $\mathfrak{t}_u$. If $\bar{z} \in A_u \rtimes \langle \theta_u \rangle$, set $\delta_u(\bar{z}) := \delta_u(z)$, where z is a lift of $\bar{z}$ in $\mathrm{Z}_{\mathcal{G}}(u) \rtimes \langle \theta_u \rangle$. This defines a representation of $A_u \rtimes \langle \theta_u \rangle$.

Suppose $(U, \mathcal{E})$ and $(U', \mathcal{E}')$ are two elements of $\mathbf{I}^\theta$ represented by (u, ϕ) and (u', ϕ'), respectively. Define

$$(\widetilde{\phi}, \widetilde{\phi}')_{\theta-\mathrm{ell}} = \begin{cases} (\widetilde{\phi}, \widetilde{\phi}')^{\delta_u}_{\theta-\mathrm{ell}}, & \text{if } U = U', \\ 0, & \text{if } U \neq U'. \end{cases} \tag{11.12}$$

This is the θ-elliptic pairing on $\bigoplus_U R(A_u \rtimes \langle \theta_u \rangle)$.

The relation between this elliptic pairing and the generalized Springer correspondence [Lus84b] is explained in [Wal07, § 3]. The automorphism θ acts naturally on all of the objects involved in the definition of the Springer correspondence. As discussed in [Wal07, § 3], this leads to an action of $W_j \rtimes \langle \theta \rangle$ on $\mathfrak{z}_{\mathcal{M}}$, the Lie algebra of $\mathrm{Z}_{\mathcal{M}}$ (notation as in § 3), and to a θ-generalized Springer correspondence $\nu \colon \mathbf{I}^\theta \to \widetilde{\mathbf{J}}^\theta$. For every $(j, \rho) \in \widetilde{\mathbf{J}}^\theta$, let $\widetilde{\rho}$ denote the extension of ρ to a representation of $W_j \rtimes \langle \theta \rangle$ as in [Wal07, § 3].

Let $i = (U, \mathcal{E})$, $i' = (U', \mathcal{E}')$ be two elements of $\mathbf{I}^\theta$, and $\nu(i) = (j, \rho)$, $\nu(i') = (j', \rho')$. For every $m \in \mathbb{Z}$, the constructible sheaf $\mathcal{H}^{2m+a_{U'}}(A_{j',\rho'})|_U$ decomposes as a direct sum of G-equivariant local systems on U. As in [Wal07], setting

$$H^m_{i,i'} = \mathrm{Hom}(\mathcal{E}, \mathcal{H}^{2m+a_{U'}}(A_{j',\rho'})|_U),$$

the automorphism θ defines a linear map $\theta^m_{i,i'} : H^m_{i,i'} \to H^m_{i,i'}$. In particular,

$$H^m_{i,i} = 0 \quad \text{if } m \neq 0 \quad \text{and} \quad \dim H^\circ_{i,i} = 1.$$

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We may arrange the construction so that $\theta^{\circ}_{i,i}$ is the identity map. Moreover, it is clear that $H^m_{i,i'} \neq 0$ for some m only if $U \subset \overline{U'}$. (Recall that the restriction of $A_{j',\rho'}$ to the set of unipotent elements of $\mathcal{G}$ is supported on $\overline{U'}$.)

Define the virtual representation of $W_j \rtimes \langle\theta\rangle$

$$\widetilde{\boldsymbol{\rho}} = \sum_{\rho' \in \widehat{W}_j^{\theta}} P_{j,\rho,\rho'} \widetilde{\rho'}, \quad \text{where } P_{j,\rho,\rho'} = \sum_{m \in \mathbb{Z}} \operatorname{tr}(\theta^m_{i,i'}). \tag{11.13}$$

In this virtual combination, $P_{j,\rho,\rho} = 1$, and $P_{j,\rho,\rho'} \neq 0$ implies that $U \subset \overline{U'}$ if $(U, \mathcal{E}) = \nu^{-1}(j,\rho)$ and $(U', \mathcal{E}') = \nu^{-1}(j,\rho')$.

Example 11.6. When θ is the trivial automorphism of $\mathcal{G}$ and $j = j_0$ (the case of the classical Springer correspondence), $\widetilde{\boldsymbol{\rho}}$ can be identified with the reducible W-representation on the ϕ-isotypic component ($\phi \in \widehat{A}_u$ corresponding to $\mathcal{E}$) of the total cohomology of the Springer fiber of u.

Consider the θ-elliptic pairing $(\,,\,)^{W_j}_{\theta-\mathrm{ell}}$ on $\bigoplus_{j \in \mathbf{J}^{\theta}} R(W_j \rtimes \langle\theta\rangle)$, defined on each summand via the action of $W_j \rtimes \langle\theta\rangle$ on $\mathfrak{z}_{\mathcal{M}}$ and extended orthogonally to the direct sum.

Theorem 11.7 ([Wal07, Théorème p. 616]). *Let $i = (U, \mathcal{E})$, $i' = (U', \mathcal{E}')$ be two elements of $\mathbf{I}^{\theta}$, and $\nu(i) = (j,\rho)$, $\nu(i') = (j',\rho')$. Let (u,ϕ), (u',ϕ'), $\phi \in \widehat{A}_u$ and $\phi' \in \widehat{A}_{u'}$ be representatives for i, i', respectively. Then*

$$(\widetilde{\phi}, \widetilde{\phi'})_{\theta-\mathrm{ell}} = (\widetilde{\boldsymbol{\rho}}, \widetilde{\boldsymbol{\rho}'})^{W_j}_{\theta-\mathrm{ell}}.$$

The equality in the theorem does not depend on the choices involved in the construction.

11.3 The proof of Theorem 11.1: the case of adjoint groups

In this subsection, suppose that G is a simple F-split group of adjoint type. This means that $G^{\vee}$ is simply connected, hence, for every $s \in T^{\vee}$, $\mathrm{Z}_{G^{\vee}}(s)$ is connected. We may apply Theorem 11.7 to

$$\mathcal{G} = \mathrm{Z}_{G^{\vee}}(s) \quad \text{and} \quad \theta \text{ the trivial automorphism.}$$

Let $\mathbf{I}^s = \mathbf{I}^{\mathrm{Z}_{G^{\vee}}(s)}$, $\mathbf{J}^s = \mathbf{J}^{\mathrm{Z}_{G^{\vee}}(s)}$, and $\widetilde{\mathbf{J}}^s = \widetilde{\mathbf{J}}^{\mathrm{Z}_{G^{\vee}}(s)}$, so that the generalized Springer correspondence for $\mathrm{Z}_{G^{\vee}}(s)$ is the map

$$\nu_s \colon \mathbf{I}^s \to \widetilde{\mathbf{J}}^s, \quad (U, \mathcal{E}) \mapsto (j, \rho),$$

and

$$\widetilde{\boldsymbol{\rho}} = \sum_{\rho' \in \widehat{W}_j} P_{j,\rho,\rho'} \rho', \quad \text{where } P_{j,\rho,\rho'} = \sum_{m \in \mathbb{Z}} \dim \mathrm{Hom}(\mathcal{E}, \mathcal{H}^{2m+a_{U'}}(A_{j',\rho'})|_U).$$

For convenience, let us also define

$$\widetilde{\nu}_s \colon \mathbf{I}^s \to \widetilde{\mathbf{J}}^s, \quad (U, \mathcal{E}) \mapsto (j, \widetilde{\boldsymbol{\rho}}). \tag{11.14}$$

Recall that for every semisimple element $s \in G^{\vee}$, $\mathcal{G}^p_s = \mathrm{Z}_{G^{\vee}}(s)$.

Proposition 11.8. *Suppose G is simple F-split group of adjoint type. The maps $\widetilde{\nu}_s$ from (11.14) induce an isometric isomorphism*

$$\bigoplus_{s \in \mathcal{C}(G^{\vee})_{\mathrm{ss}}} \bigoplus_{u \in \mathcal{C}(\mathcal{G}^p_s)_{\mathrm{un}}} \overline{R}(A_{\mathcal{G}^p_s}(u)) \cong \bigoplus_{s \in \mathcal{C}(G^{\vee})_{\mathrm{ss}}} \bigoplus_{j \in \mathbf{J}^s} \overline{R}(W_j), \quad (\phi, \phi')^{\delta^s_u}_{\mathrm{ell}} = (\widetilde{\nu}_s(\phi), \widetilde{\nu}_s(\phi'))^{W_j}_{\mathrm{ell}}.$$

Proof. This is immediate from Theorem 11.7 applied to each $\mathcal{G}^p_s$. □

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11.4 Extending Theorem 11.1

In order to extend the results to the case when G is simple F-split but not adjoint, we first need some results about Mackey induction. We follow a construction from [CH16, § 4.2]. Suppose H' is a finite group, H a normal subgroup of H', and $H'/H = \mathfrak{R}$ is abelian. The groups $H', \mathfrak{R}$ act on $\widehat{H}$. For every H-character χ, and $\gamma \in \mathfrak{R}$, denote by ${}^{\gamma}\chi$ the H-character ${}^{\gamma}\chi(h) = \chi(\gamma^{-1}h\gamma)$ (it does not depend on the choice of coset representative γ).

If $\sigma \in \widehat{H}$, let $\mathfrak{R}_\sigma$ and H'_σ denote the corresponding isotropy groups of σ. For each $\gamma \in \mathfrak{R}_\sigma$, fix an isomorphism $\phi_\gamma \colon {}^{\gamma}\sigma \to \sigma$ and define the twisted trace as $\mathrm{tr}_\gamma(\sigma)(h) = \mathrm{tr}(\sigma(h) \circ \phi_\gamma)$, $h \in H$. The choices of ϕ_γ (each unique up to scalar) define a factor set, or a 2-cocyle, $\beta_\sigma : \mathfrak{R}_\sigma \times \mathfrak{R}_\sigma \to \mathbb{C}^\times$.

Remark 11.9. We assume that the action of $\mathfrak{R}$ can be normalized so that β_σ is trivial. This is the case, for example, when $\mathfrak{R}$ is cyclic.

If τ is a (virtual) $\mathfrak{R}_\sigma$-representation, we may form the Mackey induced (virtual) H'-representation

$$\sigma \rtimes \tau = \mathrm{Ind}_{H'_\sigma}^{H'}(\sigma \otimes \tau).$$

If τ is an irreducible $\mathfrak{R}_\sigma$-representation, then $\sigma \rtimes \tau$ is an irreducible H'-representation. In fact, $\widehat{H'} = \{\sigma \rtimes \tau \mid \sigma \in \mathfrak{R}\backslash\widehat{H},\ \tau \in \widehat{\mathfrak{R}}_\sigma\}$.

Given $\gamma \in \mathfrak{R}$, if $\gamma \in \mathfrak{R}_\sigma$, define $\tau_{\sigma,\gamma}$ to be the virtual $\mathfrak{R}_\sigma$-representation whose character is the delta function on γ. Then $\{\sigma \rtimes \tau_{\sigma,\gamma} \mid \sigma \in \mathfrak{R}\backslash\widehat{H},\ \gamma \in \mathfrak{R}\}$ is a basis of $R(H')$. As in [CH16, Lemma 4.2.2],

$$\chi_{\sigma\rtimes\tau_{\sigma,\gamma}}(h) = \begin{cases} 0, & \text{if } h \notin H\gamma, \\ \sum_{\gamma' \in \mathfrak{R}/\mathfrak{R}_\sigma} {}^{\gamma'}(\mathrm{tr}_\gamma(\sigma))(h\gamma^{-1}), & \text{if } h \in H\gamma. \end{cases} \tag{11.15}$$

Note that

$$H'/H'_\sigma \cong \mathfrak{R}/\mathfrak{R}_\sigma$$

indexes the $\mathfrak{R}$-orbit (equivalently, the H'-orbit) of $\sigma \in \widehat{H}$. Suppose H' is endowed with a representation δ and we define the corresponding elliptic pairing $(\ ,\ )_{\mathrm{ell}}^{H'} = (\ ,\ )_{\mathrm{ell}}^{\delta}$.

From now on, assume that $H' = H \rtimes \mathfrak{R}$, so that each $\gamma \in \mathfrak{R}$ acts on H by automorphisms of H. (If H is abelian, which is often the case for component groups, this assumption is not necessary.) Then we may define the twisted elliptic pairing $(\ ,\ )_{\gamma-\mathrm{ell}}^{H}$ (for each $\gamma \in \mathfrak{R}$).

The existence of the intertwiner ϕ_γ ($\gamma \in \mathfrak{R}_\sigma$) is equivalent to the existence of an extension of σ to a representation of $H \rtimes \langle\gamma\rangle$, by setting $\sigma(h\gamma) = \sigma(h) \circ \phi_\gamma$, so $\mathrm{tr}_\gamma(\sigma)(h) = \mathrm{tr}\sigma(h\gamma)$, $h \in H$. This is implicit in the following lemma.

Lemma 11.10. *For every $\gamma_1, \gamma_2 \in \mathfrak{R}$ and every $\sigma_1, \sigma_2 \in \widehat{H}$, the H'-elliptic pairing is given by*

$$(\sigma_1 \rtimes \tau_{\sigma_1,\gamma_1}, \sigma_2 \rtimes \tau_{\sigma_2,\gamma_2})_{\mathrm{ell}}^{H'} = \begin{cases} \dfrac{1}{|\mathfrak{R}|} \sum_{\gamma' \in \mathfrak{R}/\mathfrak{R}_{\sigma_1}, \gamma'' \in \mathfrak{R}/\mathfrak{R}_{\sigma_2}} ({}^{\gamma'}\sigma_1, {}^{\gamma''}\sigma_2)_{\gamma_1-\mathrm{ell}}^{H}, & \text{if } \gamma_1 = \gamma_2, \\ 0, & \text{if } \gamma_1 \neq \gamma_2. \end{cases}$$

Proof. The orthogonality of the two characters when $\gamma_1 \neq \gamma_2$ follows at once since the first is supported on $\gamma_1 H$ and the second on $\gamma_2 H$. The first formula follows from (11.15) by the definition of the elliptic pairing. □

Lemma 11.10 allows us to extend the proof of Theorem 11.7 to the case when $\mathcal{G}'$ is disconnected as long as the following holds.

($\star$) The cocycles $\sharp_j$ that occur in the disconnected Springer correspondence (3.5) can be trivialized.

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With the notation from § 3, set

$$A_u(j) = \mathrm{Z}_{\mathfrak{R}_j \mathcal{G}^\circ}(u)/\mathrm{Z}_{\mathcal{G}}(u)^\circ.$$

This is a normal subgroup of A_u containing $A_{\mathcal{G}^\circ}(u)$.

In our case $\mathcal{G}$ is not an arbitrary disconnected reductive group, but rather $\mathcal{G} = Z_{G^\vee}(s)$, for some semisimple element $s \in T^\vee$. Let $W^\vee$ be the Weyl group of $T^\vee$ in $G^\vee$. If we fix $B(s)$ a Borel subgroup of $\mathcal{G}^\circ$ with $T^\vee \subset B(s)$, and denote by $\Phi^+(\mathcal{G}^\circ)$ the positive roots of $T^\vee$ in $\mathcal{G}^\circ$, then $\mathcal{A} = \mathcal{G}/\mathcal{G}^\circ$ can be identified with

$$\mathcal{A} = \{w \in W^\vee \mid w(\Phi^+(\mathcal{G}^\circ)) = \Phi^+(\mathcal{G}^\circ)\}, \tag{11.16}$$

see, for example, [Bon05, § 1]. With this identification, $\mathcal{A}$ acts on $T^\vee$ and $\Phi^+(\mathcal{G}^\circ)$, hence by automorphisms of the root datum of $\mathcal{G}^\circ$. We will use this 'global' action on $\mathcal{G}^\circ$ in the proof of the next result in order to construct the extensions to the appropriate semidirect products and apply Waldspurger's result to θ-elliptic pairings.

Proposition 11.11. *Retain the notation from § 3 and suppose that $(\star)$ holds. Let $\nu : \mathbf{I}^{\mathcal{G}} \to \widetilde{\mathbf{J}}^{\mathcal{G}}$ be the generalized Springer correspondence (3.5). Let $i = (U, \mathcal{E})$, $i' = (U', \mathcal{E}')$ be two elements of $\mathbf{I}^{\mathcal{G}}$, and $\nu(i) = (j, \rho)$, $\nu(i') = (j, \rho')$. Let (u, ϕ), (u', ϕ'), $\phi \in \widehat{A}_u$, $\phi' \in \widehat{A}_{u'}$ be representatives for i, i', respectively. Then:*

(i) *if $U \neq U'$, $(\phi, \phi')_{\mathrm{ell}}^{A_u} = 0 = (\boldsymbol{\rho}, \boldsymbol{\rho}')_{\mathrm{ell}}^{W_j}$;*
(ii) *if $U = U'$, $(\phi, \phi')_{\mathrm{ell}}^{A_u(j)} = (\boldsymbol{\rho}, \boldsymbol{\rho}')_{\mathrm{ell}}^{W_j}$.*

Proof. Suppose $j = j'$, otherwise the claim is true by definition. Let $\rho_1^\circ, \rho_2^\circ \in \widehat{W_j^\circ}$ and suppose that they have unipotent supports U_1°, U_2°, respectively, in the connected generalized Springer correspondence such that $\mathcal{G} \cdot U_1^\circ \neq \mathcal{G} \cdot U_2^\circ$. Let $\boldsymbol{\rho}_i^\circ$, $i = 1, 2$ be the corresponding reducible Springer representations, as in § 11.2. We assume that they all have appropriate twisted extensions as in § 11.2 and drop $\widetilde{\ }$ from the notation. Using Lemma 11.10 applied to $H = W_j^\circ$, $H' = W_j$, $\mathfrak{R} = \mathfrak{R}_j$, for every $\gamma \in (\mathfrak{R}_j)_{\rho_1^\circ} \cap (\mathfrak{R}_j)_{\rho_2^\circ}$,

$$(\boldsymbol{\rho}_1^\circ \rtimes \tau_{\rho_1^\circ, \gamma}, \boldsymbol{\rho}_2^\circ \rtimes \tau_{\rho_1^\circ, \gamma})_{\mathrm{ell}}^{W_j} = \frac{1}{|\mathfrak{R}_j|} \sum_{\gamma', \gamma''} ({}^{\gamma'}\boldsymbol{\rho}_1^\circ, {}^{\gamma''}\boldsymbol{\rho}_2^\circ)_{\gamma-\mathrm{ell}}^{W_j^\circ} = 0,$$

by Theorem 11.7. We used implicitly here that the stabilizers in $\mathfrak{R}_j$ of ρ_i° and $\boldsymbol{\rho}_i^\circ$ are the same. In conjunction with the second claim in Lemma 11.10, this implies that $(\boldsymbol{\rho}_1^\circ \rtimes \tau_1, \boldsymbol{\rho}_2^\circ \rtimes \tau_2)_{\mathrm{ell}}^{W_j} = 0$ for all $\tau_i \in \widehat{(\mathfrak{R}_j)_{\rho_i^\circ}}$, $i = 1, 2$. Hence, $(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2)_{\mathrm{ell}}^{W_j} = 0$ whenever ρ_1, ρ_2 have distinct unipotent (disconnected) Springer support, which proves the first part of the claim.

Now assume that $u = u'$ and $\phi, \phi' \in \widehat{A}_u$. Suppose ρ_i° occurs in the restriction of ρ_i to W_j° and that $\phi_i^\circ \in \widehat{A}_{\mathcal{G}^\circ}(u)$ (which necessarily occurs in the restriction of ϕ_i) corresponds to ρ_i° in the connected generalized Springer correspondence. We observe that there is a natural injection

$$\mathfrak{R}_j = W_j/W_j^\circ \simeq \mathrm{N}_{\mathcal{G}}(j)/\mathrm{N}_{\mathcal{G}^\circ}(\mathcal{M}^\circ) \hookrightarrow \mathcal{G}/\mathcal{G}^\circ = \mathcal{A}. \tag{11.17}$$

Hence, every $\gamma \in \mathfrak{R}_j$ can be regarded as an automorphism of $\mathcal{G}^\circ$ via (11.16) and, in particular, Theorem 11.7 can be applied with γ in place of θ. We wish to compare $(\boldsymbol{\rho}_1^\circ \rtimes \tau_{\rho_1^\circ, \gamma_1}, \boldsymbol{\rho}_2^\circ \rtimes \tau_{\rho_2^\circ, \gamma_2})_{\mathrm{ell}}^{W_j}$ and $(\phi_1^\circ \rtimes \tau_{\phi_1^\circ, \gamma_1}, \phi_2^\circ \rtimes \tau_{\phi_2^\circ, \gamma_2})_{\mathrm{ell}}^{A_u}$. By [AMS18, Lemma 4.4],

$$(\mathfrak{R}_j)_{\rho_i^\circ} \cong (W_j)_{\rho_i^\circ}/W_j \cong (A_u)_{\phi_i^\circ}/A_{\mathcal{G}^\circ}(u), \quad i = 1, 2,$$

which implies that there is an identification between γ_i, $i = 1, 2$ for the ρ_i° and for the ϕ_i° in the setting of Lemma 11.10. Hence, if $\gamma_1 \neq \gamma_2$, both elliptic products are zero.

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Suppose $\gamma_1 = \gamma_2 = \gamma \in (\mathfrak{R}_j)_{\rho_1^\circ} \cap (\mathfrak{R}_j)_{\rho_2^\circ} = ((A_u)_{\phi_1^\circ} \cap (A_u)_{\phi_2^\circ})/A_{\mathcal{G}^\circ}(u)$. To simplify the formulas, set $n_i = |(\mathfrak{R}_j)_{\rho_i^\circ}| = |(A_u)_{\phi_i^\circ}/A_{\mathcal{G}^\circ}(u)|$, $i = 1, 2$. Then, first by Lemma 11.10 and second by Theorem 11.7,

$$\begin{aligned} n_1 n_2 |\mathfrak{R}_j| (\boldsymbol{\rho}_1^\circ \rtimes \tau_{\rho_1^\circ,\gamma}, \boldsymbol{\rho}_2^\circ \rtimes \tau_{\rho_2^\circ,\gamma})_{\mathrm{ell}}^{W_j} &= \sum_{\gamma',\gamma'' \in \mathfrak{R}_j} ({}^{\gamma'}\boldsymbol{\rho}_1^\circ, {}^{\gamma''}\boldsymbol{\rho}_2^\circ)_{\gamma-\mathrm{ell}}^{W_j^\circ} \\ &= \sum_{u_0 \in \mathfrak{R}_j \mathcal{G}^\circ \cdot u/\mathcal{G}^\circ} \sum_{\substack{\gamma',\gamma'' \in \mathfrak{R}_j \\ \mathcal{G}^\circ \cdot ({}^{\gamma'}u_0) = \mathcal{G}^\circ \cdot ({}^{\gamma''}u_0) = \mathcal{G}^\circ \cdot u_0}} ({}^{\gamma'}\phi_1^\circ, {}^{\gamma''}\phi_2^\circ)_{\gamma-\mathrm{ell}}^{A_{\mathcal{G}^\circ}(u_0)}. \end{aligned}$$

The first sum is over the representatives of the $\mathcal{G}^\circ$-orbits that are conjugate to $\mathcal{G}^\circ \cdot u$ via $\mathfrak{R}_j$. Since the corresponding summand for two different $\mathcal{G}^\circ$-orbits $u_0, u_0' \in \mathcal{G} \cdot u$ are equal (as they are related by an outer automorphism of $\mathcal{G}^\circ$), it follows that

$$n_1 n_2 \frac{|\mathfrak{R}_j|}{N_u} (\boldsymbol{\rho}_1^\circ \rtimes \tau_{\rho_1^\circ,\gamma}, \boldsymbol{\rho}_2^\circ \rtimes \tau_{\rho_2^\circ,\gamma})_{\mathrm{ell}}^{W_j} = \sum_{\substack{\gamma',\gamma'' \in \mathfrak{R}_j \\ \mathcal{G}^\circ \cdot ({}^{\gamma'}u) = \mathcal{G}^\circ \cdot ({}^{\gamma''}u) = \mathcal{G}^\circ \cdot u}} ({}^{\gamma'}\phi_1^\circ, {}^{\gamma''}\phi_2^\circ)_{\gamma-\mathrm{ell}}^{A_{\mathcal{G}^\circ}(u)}, \tag{11.18}$$

where N_u is the number of $\mathcal{G}^\circ$-conjugacy classes in $\mathfrak{R}_j \mathcal{G}^\circ \cdot u$. It is easy to see (using orbit-stabilizer counting) that $|\mathfrak{R}_j|/N_u = |\mathrm{Z}_{\mathfrak{R}_j \mathcal{G}^\circ}(u)/\mathrm{Z}_{\mathcal{G}^\circ}(u)|$. Moreover, an element $\gamma \in \mathfrak{R}_j$ has the property that $\mathcal{G}^\circ \cdot ({}^{\gamma}u) = \mathcal{G}^\circ \cdot u$ if and only if $\gamma \in \mathrm{Z}_{\mathcal{G}}(u) \bmod \mathcal{G}^\circ$. Hence, (11.18) becomes

$$n_1 n_2 |\mathrm{Z}_{\mathfrak{R}_j \mathcal{G}^\circ}(u)/\mathrm{Z}_{\mathcal{G}^\circ}(u)| (\boldsymbol{\rho}_1^\circ \rtimes \tau_{\rho_1^\circ,\gamma}, \boldsymbol{\rho}_2^\circ \rtimes \tau_{\rho_2^\circ,\gamma})_{\mathrm{ell}}^{W_j} = \sum_{\gamma',\gamma'' \in \mathrm{Z}_{\mathfrak{R}_j \mathcal{G}^\circ}(u)/\mathrm{Z}_{\mathcal{G}^\circ}(u)} ({}^{\gamma'}\phi_1^\circ, {}^{\gamma''}\phi_2^\circ)_{\gamma-\mathrm{ell}}^{A_{\mathcal{G}^\circ}(u)}. \tag{11.19}$$

On the other hand, applying Lemma 11.10 to $A_u(j)$, we get

$$n_1 n_2 |A_u(j)/A_{\mathcal{G}^\circ}(u)| (\phi_1^\circ \rtimes \tau_{\phi_1^\circ,\gamma}, \phi_2^\circ \rtimes \tau_{\phi_2^\circ,\gamma})_{\mathrm{ell}}^{A_u(j)} = \sum_{r',r'' \in A_u(j)/A_{\mathcal{G}^\circ}(u)} ({}^{r'}\phi_1^\circ, {}^{r''}\phi_2^\circ)_{\gamma-\mathrm{ell}}^{A_{\mathcal{G}^\circ}(u)}. \tag{11.20}$$

Note that

$$A_u(j)/A_{\mathcal{G}^\circ}(u) \cong \mathrm{Z}_{\mathfrak{R}_j \mathcal{G}^\circ}(u)/\mathrm{Z}_{\mathcal{G}^\circ}(u) \hookrightarrow \mathcal{G}/\mathcal{G}^\circ.$$

The claim follows by comparing (11.19) and (11.20). □

Remark 11.12. (i) In our case, $\mathcal{G} = \mathrm{Z}_{G^\vee}(s)$ for a semisimple element s, and hence all of the groups W_j° that occur in our setting are finite Weyl groups. This means that ($\star$) holds by [ABPS17a, Proposition 4.3].

(ii) If $\mathrm{Z}_{\mathfrak{R}_j \mathcal{G}^\circ}(u)/\mathrm{Z}_{\mathcal{G}^\circ}(u) = \mathrm{Z}_{\mathcal{G}}(u)/\mathrm{Z}_{\mathcal{G}^\circ}(u)$, it follows from Proposition 11.11(ii) that, in fact, $(\phi, \phi')_{\mathrm{ell}}^{A_u} = (\boldsymbol{\rho}, \boldsymbol{\rho}')_{\mathrm{ell}}^{W_j}$. For example, when $j = j_0$ is the cuspidal datum associated to the trivial local system on the maximal torus of $\mathcal{G}^\circ$, the $\mathfrak{R}_{j_0} = \mathcal{G}/\mathcal{G}^\circ$, hence this condition holds automatically.

Corollary 11.13. *Let G is a semisimple split F-group. Then Conjecture 9.1 holds for all Iwahori-spherical representations of the pure inner twists of G, in the sense of (9.4).*

Proof. By Proposition 11.11 and Remark 11.12(ii), the claim follows just as for adjoint groups. □

12. $\mathsf{Sp}_4(F)$

As a useful example, we present the case $G = \mathsf{Sp}_4(F)$. First, there are six unipotent representations of the finite group $\mathsf{Sp}_4(\mathbb{F}_q)$: five in bijection with irreducible representations of the finite

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Weyl group of type C_2 and one cuspidal representation θ. Using Lusztig's notation for the irreducible representations of the Weyl group of type B/C, there are three families $\mathcal{F}$ of unipotent representations with associated finite groups Γ as follows:

- $\Gamma = \{1\}$, $\mathcal{F} = \{2 \times \emptyset\}$;
- $\Gamma = \{1\}$, $\mathcal{F} = \{0 \times 11\}$;
- $\Gamma = \mathbb{Z}/2\mathbb{Z}$, $\mathcal{F} = \{1 \times 1, 11 \times \emptyset, \emptyset \times 2, \theta\}$ with associated parameters, in order, $M(\Gamma) = \{(1, \mathbf{1}), (-1, \mathbf{1}), (1, \epsilon), (-1, \epsilon)\}$.

For the $\mathbb{Z}/2\mathbb{Z}$-family, the stable combinations are

$$\begin{aligned} \sigma(1,1) &= 1 \times 1 + \emptyset \times 2, \quad \sigma(-1,1) = 11 \times \emptyset + \theta, \\ \sigma(1,-1) &= 1 \times 1 - \emptyset \times 2, \quad \sigma(-1,-1) = 11 \times \emptyset - \theta, \end{aligned} \tag{12.1}$$

and Lusztig's Fourier transform acts by the flip $\sigma(x,y) \mapsto \sigma(y,x)$. For the singleton families, the Fourier transform is the identity.

Next, we consider the p-adic group $\mathsf{Sp}_4(F)$: the unipotent representations are parameterized by data in the dual group $G^\vee = \mathsf{SO}_5(\mathbb{C})$. In particular, the list of unipotent classes u and their attached groups Γ_u is as follows.

u	Γ_u
(5)	1
(311)	$\mathsf{S}(\mathsf{O}_1 \times \mathsf{O}_2) \cong \mathsf{O}_2$
(221)	Sp_2
(1^5)	SO_5

The interesting case is $u = (311)$. Write

$$\Gamma_u = \langle z, \delta \mid z \in \mathbb{C}^\times,\ \delta^2 = 1,\ \delta z \delta^{-1} = z^{-1} \rangle.$$

Then $\mathrm{Z}_{\Gamma_u}(\pm\delta) = A_{\Gamma_u}(\pm\delta) = \{\pm 1, \pm\delta\} \cong C_2 \times C_2$ and $A_{\Gamma_u}(\pm 1) = \{1, \delta\} \cong C_2$. There are six conjugacy classes of elliptic pairs:

$$[(\pm 1, \delta)], \quad [(\delta, \pm 1)], \quad [(\delta, \pm\delta)], \tag{12.2}$$

and the flip acts as

$$\mathsf{flip}([(\pm 1, \delta)]) = [(\delta, \pm 1)], \quad \mathsf{flip}([(\delta, \pm\delta)]) = [(\delta, \pm\delta)]. \tag{12.3}$$

There are three conjugacy classes of isolated semisimple elements in $T^\vee = \{(a,b) \mid a, b \in \mathbb{C}^\times\}$ in $\mathsf{SO}_5(\mathbb{C})$. In this notation, the Weyl group $W(B_2)$ acts on T by flips and inverses. The representatives of the three classes are:

- $s_0 = (1,1)$, $\mathrm{Z}_{G^\vee}(s_0) = \mathsf{SO}_5$;
- $s_1 = (-1,1)$, $\mathrm{Z}_{G^\vee}(s_1) = \mathsf{S}(\mathsf{O}_2 \times \mathsf{O}_3)$;
- $s_2 = (-1,-1)$, $\mathrm{Z}_{G^\vee}(s_2) = \mathsf{S}(\mathsf{O}_1 \times \mathsf{O}_4) \cong \mathsf{O}_4$.

All three s_0, s_1, s_2 occur in $\Gamma_u = \mathsf{O}_2$ and in the notation above for $\mathsf{O}_2 = \langle z, \delta \rangle$, they are

$$s_0 \leftrightarrow 1 \in \mathsf{O}_2, \quad s_1 \leftrightarrow -1 \in \mathsf{O}_2, \quad s_2 \leftrightarrow \delta \in \mathsf{O}_2.$$

Consequently, there are eight elliptic tempered representations of the form $\pi(s_i, u, \phi)$, $i = 0, 1, 2$, $u = (311)$: six are Iwahori-spherical and two are supercuspidal. Out of these, four are discrete

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Table 1. Elliptic $\mathsf{Sp}_4(F)$-representations attached to $u = (311) \in \mathsf{SO}_5$.

$\pi(u, s, \phi)$	$K_0 \to \mathsf{Sp}_4(\mathbb{F}_q)$	$K_1 \to \mathsf{Sp}_2(\mathbb{F}_q)^2$	$K_2 \to \mathsf{Sp}_4(\mathbb{F}_q)$
$(s_0, \mathbf{1})$	$1 \times 1 + \emptyset \times 11$	$\mathbf{1} \boxtimes \epsilon + \epsilon \boxtimes \mathbf{1} + \epsilon \boxtimes \epsilon$	$1 \times 1 + \emptyset \times 11$
(s_0, ϵ)	$\emptyset \times 2$	$\epsilon \boxtimes \epsilon$	$\emptyset \times 2$
$(s_1, \mathbf{1})$	$\emptyset \times 2 + \emptyset \times 11$	$\mathbf{1} \boxtimes \epsilon + \epsilon \boxtimes \epsilon$	1×1
(s_1, ϵ)	1×1	$\epsilon \boxtimes \mathbf{1} + \epsilon \boxtimes \epsilon$	$\emptyset \times 2 + \emptyset \times 11$
$(s_2, \mathbf{1} \boxtimes \mathbf{1})$	$\emptyset \times 11$	$\mathbf{1} \boxtimes \epsilon$	$11 \times \emptyset$
$(s_2, \epsilon \boxtimes \mathbf{1})$	θ_{K_0}	0	0
$(s_2, \mathbf{1} \boxtimes \epsilon)$	0	0	θ_{K_2}
$(s_2, \epsilon \boxtimes \epsilon)$	$11 \times \emptyset$	$\epsilon \boxtimes \mathbf{1}$	$\emptyset \times 11$

Table 2. Elliptic $\mathsf{Sp}_4(F)$ stable combinations attached to $u = (311) \in \mathsf{SO}_5$.

(s, h)	K_0	K_1	K_2
$(1, \delta)$	$(1 \times 1 - \emptyset \times 2) + \emptyset \times 11$	$\mathbf{1} \boxtimes \epsilon + \epsilon \boxtimes \mathbf{1}$	$(1 \times 1 - \emptyset \times 2) + \emptyset \times 11$
$(\delta, 1)$	$(11 \times \emptyset + \theta_{K_0}) + \emptyset \times 11$	$\mathbf{1} \boxtimes \epsilon + \epsilon \boxtimes \mathbf{1}$	$(11 \times \emptyset + \theta_{K_2}) + \emptyset \times 11$
$(-1, \delta)$	$(-1 \times 1 + \emptyset \times 2) + \emptyset \times 11$	$\mathbf{1} \boxtimes \epsilon - \epsilon \boxtimes \mathbf{1}$	$(1 \times 1 - \emptyset \times 2) - \emptyset \times 11$
$(\delta, -1)$	$(-11 \times \emptyset - \theta_{K_0}) + \emptyset \times 11$	$\mathbf{1} \boxtimes \epsilon - \epsilon \boxtimes \mathbf{1}$	$(11 \times \emptyset + \theta_{K_2}) - \emptyset \times 11$
(δ, δ)	$(-11 \times \emptyset + \theta_{K_0}) + \emptyset \times 11$	$\mathbf{1} \boxtimes \epsilon - \epsilon \boxtimes \mathbf{1}$	$(11 \times \emptyset - \theta_{K_2}) - \emptyset \times 11$
$(\delta, -\delta)$	$(11 \times \emptyset - \theta_{K_0}) + \emptyset \times 11$	$\mathbf{1} \boxtimes \epsilon + \epsilon \boxtimes \mathbf{1}$	$(11 \times \emptyset - \theta_{K_2}) + \emptyset \times 11$

series representations, all those for $s_2 = \delta$. The parahoric restrictions are given in Table 1. We computed them using the same method as in [Ree00, (6.2)], but since in our case $G^\vee$ is not simply connected, we also need to involve the Mackey induction for graded affine Hecke algebras attached to disconnected groups.

The corresponding stable combinations are

$$\begin{aligned}
\Pi(u, 1, \delta) &= \pi(u, s_0, \mathbf{1}) - \pi(u, s_0, \epsilon),\\
\Pi(u, -1, \delta) &= \pi(u, s_1, \mathbf{1}) - \pi(u, s_1, \epsilon),\\
\Pi(u, \delta, 1) &= \pi(u, s_2, \mathbf{1} \boxtimes \mathbf{1}) + \pi(u, s_2, \mathbf{1} \boxtimes \epsilon) + \pi(u, s_2, \epsilon \boxtimes \mathbf{1}) + \pi(u, s_2, \epsilon \boxtimes \epsilon),\\
\Pi(u, \delta, -1) &= \pi(u, s_2, \mathbf{1} \boxtimes \mathbf{1}) + \pi(u, s_2, \mathbf{1} \boxtimes \epsilon) - \pi(u, s_2, \epsilon \boxtimes \mathbf{1}) - \pi(u, s_2, \epsilon \boxtimes \epsilon),\\
\Pi(u, \delta, \delta) &= \pi(u, s_2, \mathbf{1} \boxtimes \mathbf{1}) - \pi(u, s_2, \mathbf{1} \boxtimes \epsilon) + \pi(u, s_2, \epsilon \boxtimes \mathbf{1}) - \pi(u, s_2, \epsilon \boxtimes \epsilon),\\
\Pi(u, \delta, -\delta) &= \pi(u, s_2, \mathbf{1} \boxtimes \mathbf{1}) - \pi(u, s_2, \mathbf{1} \boxtimes \epsilon) - \pi(u, s_2, \epsilon \boxtimes \mathbf{1}) + \pi(u, s_2, \epsilon \boxtimes \epsilon).
\end{aligned}$$

The corresponding parahoric restrictions are in Table 2. Here the column labelled K_i contains $\mathrm{res}_{K_i}(\Pi(u, s, h))$.

One can easily verify by inspection using Table 2 that the conjecture holds in this case.

13. $\mathsf{SL}_n(F)$

13.1 Elliptic pairs for $G^\vee = \mathsf{PGL}_n(\mathbb{C})$

Consider the case $G^\vee = \mathsf{PGL}_n(\mathbb{C})$. Let Z denote the centre of $\mathsf{GL}_n(\mathbb{C})$. In the Weyl group of type A_{n-1} ($W = S_n$), denote by $\dot{w}_n$ the permutation matrix corresponding to the n-cycle $(1, 2, 3, \ldots, n)$. For every nth root z of 1, let

$$\Delta_n(z) = \mathrm{diag}(1, z, z^2, \ldots, z^{n-1})Z \in \mathsf{PGL}_n(\mathbb{C}).$$

Fix ζ_n a primitive nth root of 1 and set $s_n = \Delta_n(\zeta_n)$. Note that $\dot{w}_n$ and s_n commute in $\mathsf{PGL}_n(\mathbb{C})$.

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Lemma 13.1. *Suppose* $\Gamma = \mathsf{PGL}_n(\mathbb{C})$. *Then*

$$\mathcal{Y}(\Gamma)_{\mathrm{ell}} = \bigsqcup_{k\in(\mathbb{Z}/n\mathbb{Z})^\times} \Gamma\cdot(s_n, \dot{w}_n^k).$$

In particular, there are $\varphi(n)$ Γ*-orbits in* $\mathcal{Y}(\Gamma)_{\mathrm{ell}}$. *The flip* $(s,h)\to(h,s)$ *induces the following map on* Γ*-orbits in* $\mathcal{Y}(\Gamma)_{\mathrm{ell}}$*:*

$$\mathsf{flip}\colon ([(s_n,\dot{w}_n^k)]) \to [(s_n,\dot{w}_n^{-k})],\quad k\in(\mathbb{Z}/n\mathbb{Z})^\times.$$

Proof. Let T be the diagonal torus in Γ. By Proposition 8.7, the only possible elliptic pairs are conjugate to $(s,\dot{w})$ where w is elliptic and s is regular such that $s\in T^w$. If the group is semisimple of type A_{n-1}, then the only elliptic elements of the Weyl group are the n-cycles. We may assume that $\dot{w}=\dot{w}_n$. It is easy to see that

$$T^{\dot{w}_n} = \{\Delta_n(z) \mid z^n = 1\}.$$

Since $s\in T^{\dot{w}_n}$ needs to be regular, it follows that the corresponding z must be a primitive root of 1.

Now fix $s=s_n$. Every other $\Delta(\zeta')$ with ζ' a primitive nth root is conjugate in Γ to s_n. The centralizer is $\mathrm{Z}_\Gamma(s) = \langle T, \dot{w}_n^k \mid k\in\mathbb{Z}/n\mathbb{Z}\rangle \cong T\rtimes\mathbb{Z}/n\mathbb{Z}$. This means that $\dot{w}_n^i$ is conjugate to $\dot{w}_n^j$ in $\mathrm{Z}_\Gamma(s)$ if and only if $i=j$. On the other hand, $\dot{w}_n^k$ is elliptic if and only if $k\in(\mathbb{Z}/n\mathbb{Z})^\times$, hence the claim follows.

For the claim about the Fourier transform, let $x\in\mathsf{GL}_n(\mathbb{C})$ be such that $x^{-1}\dot{w}_n x = s_n$, where x is the matrix corresponding to a basis of eigenvectors of $\dot{w}_n$. Then a calculation shows that

$$x^{-1}s_n x = \dot{w}_n^{-1} \text{ in } \mathsf{PGL}_n(\mathbb{C}).$$

From this,

$$\mathsf{flip}([(s_n,\dot{w}_n^k)]) = [(w_n^k, s_n)] = [(xs_n^k x^{-1}, s_n)] = [(s_n^k, x^{-1}s_n x)] = [(s_n^k, \dot{w}_n^{-1})].$$

Finally, let p be a permutation matrix such that $p^{-1}s_n^k p = s_n$ (this exists since k is coprime to n). This has the effect $p^{-1}\dot{w}_n p = \dot{w}_n^k$, hence $[(s_n^k,\dot{w}_n^{-1})] = [(s_n,\dot{w}_n^{-k})]$. □

Now let u be a unipotent element in $\mathsf{PGL}_n(\mathbb{C})$. Via the Jordan canonical form, u is parameterized by a partition λ of n, where we write $\lambda = (\underbrace{1,\dots,1}_{r_1},\underbrace{2,\dots,2}_{r_2},\dots,\underbrace{\ell,\dots,\ell}_{r_\ell})$. As is well-known (see, for example, [CM93, Theorem 6.1.3])

$$\Gamma_u = \left(\prod_{i=1}^{\ell}\mathsf{GL}_{r_i}(\mathbb{C})^i_\Delta\right)\Big/ Z, \tag{13.1}$$

where H^i_Δ means H embedded diagonally into the product of i copies of H. In particular, Γ_u is connected. Let T_r denote the diagonal torus in GL_r, Z_r the center of GL_r, $\bar{T}_r = T_r/Z_r$ and $W_r\cong S_r$ the Weyl group. A maximal torus in Γ_u is $T_u = \prod_{i=1}^{\ell}(T_{r_i})^i_\Delta/Z$ and the Weyl group is $W_u = \prod_{i=1}^{\ell}(W_{r_i})^i_\Delta$.

Let $w=\prod_{i=1}^{\ell}(w_i)^i_\Delta\in W_u$, $w_i\in W_{r_i}$ be given. We need T_u^w to be finite. The morphism

$$\pi\colon T_u \twoheadrightarrow \prod_{i=1}^{\ell}(\bar{T}_{r_i})^i_\Delta,\ (t_i)_i \bmod Z \mapsto (t_i \bmod Z_i)$$

is surjective and W_u-equivariant. Since $(\bar{1},\dots,\bar{1},(\bar{T}_{r_i}^{w_i})^i_\Delta,\bar{1},\dots,\bar{1})\subset T_u^w$ for each i, it follows that w_i is elliptic for PGL_{r_i}, hence, each w_i is an r_i-cycle.

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Proposition 13.2. *For $u \in \mathsf{PGL}_n(\mathbb{C})$, $\mathcal{Y}(\Gamma_u)_{\mathrm{ell}} \neq \emptyset$ if and only if the partition λ corresponding to u is rectangular, i.e. $\lambda = (\underbrace{i, \dots, i}_{r_i})$ for some i. In this case,*

$$\Gamma_u = \mathsf{GL}_{r_i}(\mathbb{C})^i_\Delta / Z \cong \mathsf{PGL}_{r_i}(\mathbb{C}),$$

so $\mathcal{Y}(\Gamma_u)_{\mathrm{ell}}$ is as in Lemma 13.1.

Proof. Let $w \in W_u$ be elliptic as above. We pass to the Lie algebra $\mathfrak{t}_u = \mathfrak{s}(\bigoplus(\mathfrak{t}_{r_i})^i_\Delta)$; here $\mathfrak{s}$ denotes the traceless matrices. Since $\mathfrak{t}_{r_i}^{w_i} = \mathbb{C} \cdot \mathrm{Id}_{r_i}$, we see that

$$\mathfrak{t}_u^w = \left\{ (a_1 \mathrm{Id}_{r_1}, a_2 \mathrm{Id}_{r_2}, a_2 \mathrm{Id}_{r_2}, \dots, \underbrace{a_i \mathrm{Id}_{r_i}, \dots, a_i \mathrm{Id}_{r_i}}_{i}, \dots) \;\middle|\; \sum_{i=1}^{\ell} i a_i = 0 \right\}.$$

The element w is elliptic if and only if $\mathfrak{t}_u^w = 0$. From the condition $\sum_{i=1}^{\ell} i a_i = 0$, we that this can only happen if there exists a unique i such that $r_i \neq 0$. □

Corollary 13.3. *The number of orbits of elliptic pairs for $\mathsf{PGL}_n(\mathbb{C})$ is*

$$\sum_{u \text{ unipotent class}} |\Gamma_u \backslash \mathcal{Y}(\Gamma_u)_{\mathrm{ell}}| = n.$$

Proof. From Proposition 13.2, the only unipotent classes that contribute are the rectangular ones, which are in one-to-one correspondence with divisors d of n. For the unipotent class $u = (\underbrace{n/d, \dots, n/d}_{d})$, Lemma 13.1 states that there are $\varphi(d)$ orbits of elliptic pairs. Hence, the total number is $\sum_{d|n} \varphi(d) = n$. □

13.2 Elliptic unipotent representations of $\mathsf{SL}_n(F)$

It is instructive to make explicit the elliptic correspondence (Conjecture 9.1) for $G = \mathsf{SL}_n(F)$. Let $K_0 = \mathsf{SL}_n(\mathfrak{o}_F)$ and let $I \subset K_0$ be an Iwahori subgroup. Let $\mathcal{H}(G, I) = \{f \in C_c^\infty(G) \mid f(i_1 g i_2) = f(g), \text{ for all } i_1, i_2 \in I\}$ be the Iwahori–Hecke algebra (under convolution with respect to a fixed Haar measure). The algebra $\mathcal{H}(G, I)$ is naturally isomorphic to the affine Hecke algebra $\mathcal{H} = \mathcal{H}(\mathcal{R}, \sqrt{q}, 1)$, where q is the order of the residue field of F and $\mathcal{R}$ is the root datum for $\mathsf{PGL}_n(\mathbb{C})$.

Every irreducible unipotent G-representation has nonzero fixed vectors under I, in other words, $\mathfrak{R}_{\mathrm{un}}(G) = \mathfrak{R}_I(G)$, where $\mathfrak{R}_I(G)$ is the category of smooth representations generated by their I-fixed vectors. The functor

$$m_I \colon \mathfrak{R}_{\mathrm{un}}(G) = \mathfrak{R}_I(G) \to \mathcal{H}(G, I)\text{-mod}, \quad V \mapsto V^I,$$

is an equivalence of categories. The Langlands parameterization in this case can be read off the Kazhdan–Lusztig classification of irreducible modules for $\mathcal{H}(G, I)$ extended to this setting in [Ree02]:

$$\mathrm{Irr}_{\mathrm{un}} \mathsf{SL}_n(F) \leftrightarrow \mathrm{Irr}\ \mathcal{H}(G, I) \leftrightarrow \mathsf{PGL}_n(\mathbb{C}) \backslash \{(x, \phi) \mid x \in \mathsf{PGL}_n(\mathbb{C}), \phi \in \widehat{A}_x\}. \tag{13.2}$$

The exact functor m_I induces an isomorphism

$$\mathrm{Ext}^i_G(V, V') = \mathrm{Ext}^i_{\mathcal{H}(G,I)}(V^I, V'^I), \quad \text{for all } i,$$

and therefore $\mathrm{EP}_G(V, V') = \mathrm{EP}_{\mathcal{H}(G,I)}(V^I, V'^I)$ for all $V, V' \in \mathrm{Irr}\ \mathfrak{R}_I(G)$. Since EP_G and $\mathrm{EP}_{\mathcal{H}(G,I)}$ are additive, they extend to pairings on the Grothendieck groups of finite-length representations. Let $R_I(G)_{\mathbb{C}}$ be the $\mathbb{C}$-span of $\mathrm{Irr}\ \mathfrak{R}_I(G)$ and $\overline{R}_I(G)_{\mathbb{C}}$ the quotient by the radical of EP_G. Define $\overline{R}(\mathcal{H}(G, I))_{\mathbb{C}}$ similarly. Thus, we have the following.

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Lemma 13.4. *The equivalence of categories m_I gives an isomorphism $\overline{m}_I\colon \overline{R}_I(G)_{\mathbb{C}} \to \overline{R}(\mathcal{H}(G,I))_{\mathbb{C}}$ which is isometric with respect to EP_G and $\mathrm{EP}_{\mathcal{H}(G,I)}$.*

The elliptic theory of affine Hecke algebras is well understood [OS09], and we reviewed the basic facts in § 11.1. In particular, via (11.10) and (11.4)), we get that

$$\overline{R}_I(G)_{\mathbb{C}} \cong \overline{R}(\widetilde{W})_{\mathbb{C}} \cong \bigoplus_{s\in W\backslash T^\vee_{\mathrm{iso}}} \overline{R}(W_s)_{\mathbb{C}}, \tag{13.3}$$

where $\widetilde{W} = W^a$, W, $T^\vee$, W_s are as in Example 11.4. If $n = dm$, we consider $s_{d\times m} := \mathrm{diag}(\underbrace{1,\dots,1}_{m},\underbrace{\zeta_d,\dots,\zeta_d}_{m},\dots,\underbrace{\zeta_d^{d-1},\dots,\zeta_d^{d-1}}_{m})$, where ζ_d is a primitive dth root of 1. This is S_n-conjugate to $\Delta_n(\zeta_d)$. In that case,

$$\mathrm{Z}_{G^\vee}(s_{d\times m}) = \mathrm{P}(\mathsf{GL}_m(\mathbb{C})^d) \rtimes \mathbb{Z}/d\mathbb{Z},$$

which has component group $A_{G^\vee}(s_{d\times m}) = \mathbb{Z}/d\mathbb{Z}$. The Lie algebra of the maximal (diagonal) torus in $\mathrm{Z}_{G^\vee}(s_{d\times m})$ is

$$\mathfrak{t}^\vee_{d\times m} = \left\{ \underline{x} = (x_1,\dots,x_n) \;\middle|\; \sum_i x_i = 0 \right\},$$

on which $W_{s_{d\times m}}$ acts in the standard way: break $(x_1,\dots,x_n)$ into m-tuples

$$\underline{y}_i = (x_{(i-1)m+1}, x_{(i-1)m+2},\dots,x_{im}), \quad 1 \leqslant i \leqslant d.$$

Then the ith S_m acts by the natural permutation action on $\underline{y}_i$, whereas $\mathbb{Z}/d\mathbb{Z}$ permutes cyclically $(\underline{y}_1,\dots,\underline{y}_d)$. We consider the elliptic theory of $W_{s_{d\times m}}$ on $\mathfrak{t}^\vee_{d\times m}$ with respect to this action.

Lemma 13.5. *There are $\varphi(d)$ elliptic conjugacy classes of $W_{s_{d\times m}}$ acting on $\mathfrak{t}^\vee_{d\times m}$ with representatives $g_\xi = (w_m, 1, 1,\dots,1)\xi$, where ξ ranges over the elements of order d in $\mathbb{Z}/d\mathbb{Z}$, and w_m is a fixed m-cycle in S_m.*

Proof. We first show that each such element is elliptic. Without loss of generality, suppose that ξ acts as the standard cycle $(1,2,\dots,d)$ and $w_m = (1,2,\dots,m)$. Then $g_\xi \cdot \underline{x} = \underline{x}$ implies $x_1 = x_{(d-1)m+1} = x_{(d-2)m+1} = \cdots = x_{m+1}$ which then is equal to x_m (because of the effect of w_m), then with $x_{dm} = x_{(d-1)m} = \cdots = x_{2m}$, which then is equal to x_{m-1}, etc. It follows that all coordinates x_i are the same and since the sum of the coordinates is 0, we get that there are no nonzero fixed points.

Second, no two g_ξ are conjugate. This is clear because if ξ, ξ' are distinct in $\mathbb{Z}/d\mathbb{Z}$, then $x\xi$ and $x'\xi'$ are in different conjugacy classes for all x, x' in S_m^d.

It remains to show that these are all the elliptic conjugacy classes. Let $x\xi$ be an element with $x = (\sigma_1,\dots,\sigma_d) \in S_m^d$ and $\xi \in \mathbb{Z}/d\mathbb{Z}$. If ξ does not have order d, then there exists points $\underline{x} = (\underline{y}_1,\dots,\underline{y}_d) \in \mathfrak{t}^\vee_{d\times m}$ (here, as above, each $\underline{y}_i$ is an m-tuple) fixed under the action of ξ such that not all y_i are equal. This means, in particular, that there exists j such that $\underline{y}_j = (x_{(j-1)m+1},\dots,x_{jm})$ is arbitrary and $\sum_{l=(j-1)m+1}^{jm} x_l \neq 0$. But then every $\sigma_j \in S_m$ has a nonzero fixed point $\underline{y}_j$, for example taking all of the entries of $\underline{y}_j$ to be equal, and therefore $x\xi$ is not elliptic.

This means that necessarily ξ has order d. We claim that the conjugacy classes of $x\xi$ are in one-to-one correspondence with conjugacy classes of S_m via the correspondence

$$w \in S_m \mapsto (w, 1,\dots,1)\xi \in S_m^d \rtimes \mathbb{Z}/d\mathbb{Z}.$$

Without loss of generality, suppose ξ acts by shifting the indices $i \to i+1 \bmod d$. We show that every element $x\xi$, $x = (\sigma_1,\dots,\sigma_d)$, is conjugate to an element of the form $(w,1,\dots,1)\xi$. This is

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equivalent to the existence of permutations $z_1, \dots, z_d \in S_m$ such that

$$\sigma_1 = z_1 w z_2^{-1}, \quad \sigma_2 = z_2 z_3^{-1}, \dots, \sigma_{d-1} = z_{d-1} z_d^{-1}, \quad \sigma_d = z_d z_1^{-1}.$$

This can be solved easily, by taking $z_1 = 1$, then $z_d = \sigma_d$, $z_{d-1} = \sigma_{d-1}\sigma_d, \dots$, $z_2 = \sigma_2\sigma_3 \dots \sigma_d$ and $w = \sigma_1\sigma_2 \dots \sigma_d$.

A similar calculation shows that $(w, 1, \dots, 1)\xi$ and $(w', 1, \dots, 1)\xi$ are conjugate if and only if w, w' are conjugate in S_m. (If $w' = zwz^{-1}$, then $(w', 1, \dots, 1)\xi$ and $(w, 1, \dots, 1)\xi$ are conjugate via $(z, z, \dots, z)$.)

Finally, if an element $(w, 1, \dots, 1)\xi$, ξ of order d, is elliptic, then w is elliptic in S_m, otherwise if $\underline{y}$ is a fixed point of w, $(\underline{y}, \dots, \underline{y})$ is a fixed point of $(w, 1, \dots, 1)\xi$. This concludes the proof. □

On the other hand, we have unipotent classes u in $\mathrm{P}(\mathsf{GL}_m(\mathbb{C})^d)$ and we need to look at the elliptic theory of $A_{G^\vee}(s_{d\times m}u)$ on the Lie algebra of the maximal torus in $\mathrm{Z}_{\mathrm{P}(\mathsf{GL}_m(\mathbb{C})^d)\rtimes\mathbb{Z}/d\mathbb{Z}}(u)$. Let $u = u_{d\times m}$ be the unipotent element given by the principal Jordan normal form on each of the GL_m-blocks. Then the reductive part of the centralizer is

$$\mathrm{Z}_{G^\vee}(s_{d\times m}u_{d\times m})^{\mathrm{red}} = \mathrm{P}(\mathrm{Z}_{\mathsf{GL}_m(\mathbb{C})^d}) \rtimes \mathbb{Z}/d\mathbb{Z},$$

hence $A_{G^\vee}(s_{d\times m}u_{d\times m}) = \mathbb{Z}/d\mathbb{Z}$ and this acts on the Cartan subalgebra

$$\mathfrak{t}^\vee(s_{d\times m}u_{d\times m}) = \left\{ (z_1\mathrm{Id}_m, \dots, z_d\mathrm{Id}_m) \;\middle|\; \sum_i z_i = 0 \right\}.$$

In particular, $\overline{R}(A_{s_{d\times m}u_{d\times m}})_{\mathbb{C}}$ has dimension $\varphi(d)$ and can be identified with the class functions on the elements of order d in $\mathbb{Z}/d\mathbb{Z}$. Thus, in the case of $\mathsf{SL}_n(F)$, the elliptic correspondence for unipotent representations takes the following very concrete form.

Proposition 13.6. *Let $G = \mathsf{SL}_n(F)$. The local Langlands correspondence for unipotent representations induces an isometric isomorphism*

$$\overline{\mathsf{LLC}^p}_{\mathrm{un}} \colon \bigoplus_{d|n} \overline{R}(A_{x_{d\times m}}) \longrightarrow \overline{R}_{\mathrm{un}}(\mathsf{SL}_n(F)), \quad \phi \mapsto \pi(x_{d\times m}, \phi),$$

where $x_{d\times m} = s_{d\times m}u_{d\times m} \in \mathsf{PGL}_n(\mathbb{C})$ is as above and $A_{x_{d\times m}} = \mathbb{Z}/d\mathbb{Z}$.

The connection with the elliptic pairs for $G^\vee = \mathsf{PGL}_n(\mathbb{C})$ from Proposition 13.2 is

$$\bigoplus_{u\in\mathcal{C}(\mathsf{PGL}_n(\mathbb{C}))_{\mathrm{un}}} \mathbb{C}[\mathcal{Y}(\Gamma_u)_{\mathrm{ell}}]^{\Gamma_u} = \bigoplus_{d|n} \mathbb{C}[\mathcal{Y}(\Gamma_{u_{d\times m}})_{\mathrm{ell}}]^{\Gamma_{u_{d\times m}}} \cong \bigoplus_{d|n} \overline{R}(A_{x_{d\times m}}).$$

Remark 13.7. Note that by taking dimensions in Proposition 13.6, we recover the well-known formula $\sum_{d|n}\varphi(d) = n$, where, as above, φ denotes the Euler phi function.

13.3 The elliptic Fourier transform for $\mathsf{SL}_n(F)$

The results so far imply that we have an equivalence

$$\overline{R}_{\mathrm{un}}(\mathsf{SL}_n(F))_{\mathbb{C}} \cong \overline{R}(\mathcal{H})_{\mathbb{C}} \cong \overline{R}(\widetilde{W})_{\mathbb{C}} \cong \bigoplus_{u\in\mathcal{C}(\mathsf{PGL}_n(\mathbb{C}))_{\mathrm{un}}} \Gamma_u\backslash\mathcal{Y}(\Gamma_u)_{\mathrm{ell}}.$$

The spaces involved are all n-dimensional and we describe the basis of $\overline{R}_{\mathrm{un}}(G)_{\mathbb{C}}$ given by the virtual characters $\Pi(u, s, h)$.

First, consider the two extremes. At one extreme, we have the regular unipotent class u_{reg}. Then $\Gamma_{u_{\mathrm{reg}}} = \{1\}$ and $\pi(u_{\mathrm{reg}}, 1, 1) = \mathrm{St}$. At the other, end, $u = 1$, $\Gamma_1 = \mathsf{PGL}_n(\mathbb{C})$, and there are $\varphi(n)$ orbits of elliptic pairs $(s_n, \dot{w}_n^k)$, $k \in (\mathbb{Z}/n\mathbb{Z})^\times$, as in Lemma 13.1. The component group is

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$A_{s_n} = \langle \dot{w}_n \rangle \cong \mathbb{Z}/n\mathbb{Z}$. Let $\pi(1, s_n)$ denote the tempered unramified principal series of G with Satake parameter $s_n \in W\backslash T^\vee$. Since $W_{s_n} = A_{s_n} = \mathbb{Z}/n\mathbb{Z}$, the theory of (analytic) R-groups provides a well-known decomposition

$$\pi(1, s_n) = \bigoplus_{\phi \in \widehat{A}_{s_n}} \pi(1, s_n, \phi),$$

where each $\pi(1, s_n, \phi)$ is an irreducible tempered G-representation. Identifying $\widehat{\mathbb{Z}/n\mathbb{Z}}$ with $\mathbb{Z}/n\mathbb{Z}$ (via a choice ζ_n of primitive nth root of unity), we get

$$\Pi(1, s_n, \dot{w}_n^k) = \sum_{\ell \in \mathbb{Z}/n\mathbb{Z}} \zeta_n^{\ell k}\, \pi(1, s_n, \phi_\ell), \quad k \in (\mathbb{Z}/n\mathbb{Z})^\times, \tag{13.4}$$

where $\phi_\ell(\zeta_n) = \zeta_n^\ell$. Moreover, as an $\mathcal{H}$-module, $\pi(1, s_n, \phi_\ell)^I$ is the (unique) irreducible tempered $\mathcal{H}$-module with central character $W \cdot s_n$ such that

$$\sigma_0(\pi(1, s_n, \phi_\ell)^I) = \mathrm{Ind}_{\mathbb{Z}/n\mathbb{Z} \ltimes X}^{S_n \ltimes X}(\phi_\ell \otimes s_n).$$

Now, more generally, by Proposition 13.2, $\mathcal{Y}(\Gamma_u)_{\mathrm{ell}} \neq \emptyset$ if and only if $u = u_{d\times m}$ is labelled by a rectangular partition $(\underbrace{m, \dots, m}_{d})$ of n. In this case $\Gamma_u = \mathsf{PGL}_d(\mathbb{C})$. Recall $s_{d\times m}$ and $x_{d\times m} = s_{d\times m} u_{d\times m}$. Consider the parabolically induced tempered G-representation

$$\pi(u_{d\times m}, s_{d\times m}) = \mathrm{Ind}_{P_{d\times m}}^{\mathsf{SL}_n(F)}((\mathrm{St}_m \otimes \mathbb{C}_1) \boxtimes (\mathrm{St}_m \otimes \mathbb{C}_{\zeta_d}) \boxtimes \cdots \boxtimes (\mathrm{St}_m \otimes \mathbb{C}_{\zeta_d^{d-1}})),$$

where $P_{d\times m}$ is the block-upper-triangular parabolic subgroup with Levi subgroup $M_{d\times m} = S(\mathsf{GL}_m(F)^d)$, St_m is the Steinberg representation of $\mathsf{GL}_m(F)$, and $\mathbb{C}_z$ is the unramified character of $\mathsf{GL}_m(F)$ corresponding to the semisimple element $z\mathrm{Id}_m$ in the dual complex group $\mathsf{GL}_m(\mathbb{C})$. The R-group in this case is $\mathbb{Z}/d\mathbb{Z}$ which coincides with $A_{x_{d\times m}}$. We have a decomposition into irreducible tempered G-representations:

$$\pi(u_{d\times m}, s_{d\times m}) = \bigoplus_{\phi \in \widehat{A}_{x_{d\times m}}} \pi(u_{d\times m}, s_{d\times m}, \phi).$$

Taking I-fixed vectors and the deformation σ_0, we have

$$\sigma_0((\pi_{d\times m}, s_{d\times m}, \phi_\ell)^I) = \mathrm{Ind}_{W_{s_{d\times m}} \ltimes X}^{S_n \ltimes X}(\mathrm{sgn}_{d\times m} \phi_\ell \otimes s_{d\times m}),$$

where recall that $W_{s_{d\times m}} = S_m^d \rtimes \mathbb{Z}/d\mathbb{Z}$ and $\mathrm{sgn}_{d\times m}$ is the sign character of S_m^d.

Define

$$\Pi(u_{d\times m}, s_{d\times m}, \dot{w}_{d\times m}^k) = \sum_{\ell \in \mathbb{Z}/d\mathbb{Z}} \zeta_d^{\ell k}\, \pi(u_{d\times m}, s_{d\times m}, \phi_\ell), \quad k \in (\mathbb{Z}/d\mathbb{Z})^\times. \tag{13.5}$$

The elliptic Fourier transform in this case is

$$\mathrm{FT}_{\mathrm{ell}}^\vee(\Pi(u_{d\times m}, s_{d\times m}, \dot{w}_{d\times m}^k)) = \Pi(u_{d\times m}, s_{d\times m}, \dot{w}_{d\times m}^{-k}), \quad k \in (\mathbb{Z}/d\mathbb{Z})^\times. \tag{13.6}$$

The maximal compact open subgroups of $\mathsf{SL}_n(F)$ are maximal parahoric subgroups K_i, one for each vertex i of the affine Dynkin diagram. With this notation, $K_0 = \mathsf{SL}_n(\mathfrak{o}_F)$. Moreover, $\mathrm{Inn}\mathrm{T}^p\overline{K}_i = \{\overline{K}_i\}$. All K_i are isomorphic to K_0 (conjugate in $\mathsf{GL}_n(F)$), hence for all i, the non-abelian Fourier transform of $\overline{K}_i$ is the identity. Let $W_i \cong S_n$ denote the finite parahoric subgroup of W^a corresponding to K_i, so that $W_0 = W$. The isomorphism $W_i \cong W$ is given by the map

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$s_j \mapsto s_{(j-i) \bmod n}$. By Mackey induction/restriction

$$\mathrm{Ind}_{W_{s_{d\times m}} \ltimes X}^{S_n \ltimes X}(\mathrm{sgn}_{d\times m}\phi_\ell \otimes s_{d\times m})|_{W_i} \cong \mathrm{Ind}_{(W_{s_{d\times m}})_i}^{W_i}(\mathrm{sgn}_{d\times m}\phi_\ell \otimes s_{d\times m}), \tag{13.7}$$

where $(W_{s_{d\times m}})_i = (W_{s_{d\times m}} \ltimes X) \cap W_i$. Let $\gamma = \epsilon_1 - \epsilon_n$ be the highest root of type A_{n-1}, in the standard coordinates, so that $s_0 = s_\gamma t_\gamma$, denoting by $t_\gamma \in X \subset W^a$ the corresponding translation. Then one can see that $W_{s_{d\times m}} \cong (W_{s_{d\times m}})_i$ is given by sending

$$s_j \mapsto \begin{cases} s_j, & \text{if } j \neq i \\ s_i t_{\epsilon_i - \epsilon_{i+1}}, & \text{if } j = i, \end{cases} \quad 1 \leqslant j < n.$$

Lemma 13.8. *For every $0 \leqslant i < n$,*

$$\mathrm{Ind}_{(W_{s_{d\times m}})_i}^{W_i}(\mathrm{sgn}_{d\times m}\phi_\ell \otimes s_{d\times m}) \cong \mathrm{Ind}_{W_{s_{d\times m}}}^{S_n}(\mathrm{sgn}_{d\times m}\phi_{\ell+\lfloor i/m \rfloor}).$$

Proof. In light of the observation before the statement of the lemma, we only need to trace how the inducing character changes on the generator corresponding to i. Denote by $(S_m^d)_i$ the image of S_m^d inside $(W_{s_{d\times m}})_i$, and similarly for $(\mathbb{Z}/d\mathbb{Z})_i$. If s_i is a generator of $(S_m^d)_i$, then the value of the character $s_{d\times m}$ on $t_{\epsilon_i - \epsilon_{i+1}}$ is 1. On the other hand, if s_i is not a generator of $(S_m^d)_i$, then there is also no change. This means that the inducing character on the $(S_m^d)_i$ is still $\mathrm{sgn}_{d\times m}$.

The generator ξ of $\mathbb{Z}/d\mathbb{Z}$ is, in cycle notation, a product of the disjoint cycles $(l, m+l, 2m+l, \ldots, (d-1)m+l)$, where l ranges from 0 to $m-1$ (when $l = 0$, we mean the cycle $(m, 2m, \ldots, dm)$). Then the simple reflection i contributes to the cycle l for $i = jm + l$, $j = \lfloor i/m \rfloor$. In $(\mathbb{Z}/d\mathbb{Z})_i$, we then get a $\theta_{\epsilon_i - \epsilon_{i+1}}$, which we need to move to the end of the product of cycles, and we get that the image $(\xi)_i$ in $(\mathbb{Z}/d\mathbb{Z})_i$ is $(\xi)_i = \xi t_{\epsilon_i - \epsilon_{(d-1)m+l}}$. The character $s_{d\times m}$ acts on $t_{\epsilon_i - \epsilon_{(d-1)m+l}}$ by ζ_d^j, which means that $\phi_\ell s_{d\times m}$ acts on $(\xi)_i$ by $\zeta_d^{\ell+j}$, which proves the claim. □

Proposition 13.9. *Conjecture 9.7 holds for $G = \mathsf{SL}_n(F)$. More precisely, for each $0 \leqslant i < n$,*

$$\mathrm{res}_{K_i} \circ \mathrm{FT}_{\mathrm{ell}}^\vee(\Pi(u_{d\times m}, s_{d\times m}, \dot{w}_{d\times m}^k)) = \zeta_d^{2k\lfloor i/m \rfloor}\mathrm{FT}_{\mathrm{cpt,un}} \circ \mathrm{res}_{K_i}(\Pi(u_{d\times m}, s_{d\times m}, \dot{w}_{d\times m}^k)).$$

Proof. To verify Conjecture 9.7, given (13.6), it is sufficient to compare the restrictions to W_i of $\sigma_0(\mathrm{res}_{K_i}(\Pi(u_{d\times m}, s_{d\times m}, \dot{w}_{d\times m}^k)^I))$ and $\sigma_0(\mathrm{res}_{K_i}(\Pi(u_{d\times m}, s_{d\times m}, \dot{w}_{d\times m}^{-k})^I)$ as virtual W_i-characters. For this, we apply (13.7) and Lemma 13.8 with $j = \lfloor i/m \rfloor$ and get

$$\begin{aligned} \sigma_0(\mathrm{res}_{K_i}(\Pi(u_{d\times m}, s_{d\times m}, \dot{w}_{d\times m}^k)^I)) &\cong \sum_{\ell \in \mathbb{Z}/d} \zeta_d^{k\ell}\mathrm{Ind}_{W_{s_{d\times m}}}^{S_n}(\mathrm{sgn}_{d\times m}\phi_{\ell+j}) \\ &= \zeta_d^{-kj} \sum_\ell \zeta_d^{k\ell}\mathrm{Ind}_{W_{s_{d\times m}}}^{S_n}(\mathrm{sgn}_{d\times m}\phi_\ell). \end{aligned}$$

On the other hand,

$$\begin{aligned} \sigma_0(\mathrm{res}_{K_i}(\Pi(u_{d\times m}, s_{d\times m}, \dot{w}_{d\times m}^{-k})^I)) &\cong \sum_{\ell \in \mathbb{Z}/d} \zeta_d^{-k\ell}\mathrm{Ind}_{W_{s_{d\times m}}}^{S_n}(\mathrm{sgn}_{d\times m}\phi_{\ell+j}) \\ &= \sum_{\ell \in \mathbb{Z}/d} \zeta_d^{k\ell}\mathrm{Ind}_{W_{s_{d\times m}}}^{S_n}(\mathrm{sgn}_{d\times m}\phi_{-\ell+j}) \\ &= \sum_{\ell \in \mathbb{Z}/d} \zeta_d^{k\ell}\mathrm{Ind}_{W_{s_{d\times m}}}^{S_n}(\mathrm{sgn}_{d\times m}\phi_{\ell-j}) \\ &= \zeta_d^{kj} \sum_\ell \zeta_d^{k\ell}\mathrm{Ind}_{W_{s_{d\times m}}}^{S_n}(\mathrm{sgn}_{d\times m}\phi_\ell). \end{aligned}$$

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Here we have used that

$$\mathrm{Ind}_{W_{s_{d\times m}}}^{S_n}(\mathrm{sgn}_{d\times m}\phi_{\ell'}) \cong \mathrm{Ind}_{W_{s_{d\times m}}}^{S_n}(\mathrm{sgn}_{d\times m}\phi_{-\ell'}),$$

because $\phi_{-\ell'}$ is the $\mathbb{Z}/d\mathbb{Z}$-representation contragredient to $\phi_{\ell'}$. This implies that the two sides are contragredient to each other, but all irreducible S_n-representations are self-contragredient. □

14. $\mathbf{PGL_n(F)}$

Now suppose $\mathbf{G} = \mathsf{PGL}_n$. The dual group $G^\vee$ is $\mathsf{SL}_n(\mathbb{C})$. Each unipotent element $u \in G^\vee$ corresponds to a partition λ_u of n, and if $\lambda_u = \lambda = (\underbrace{1,\dots,1}_{r_1},\underbrace{2,\dots,2}_{r_2},\dots,\underbrace{\ell,\dots,\ell}_{r_\ell})$, then

$$\Gamma_u \simeq \left\{(x_1,\dots,x_\ell) \in \prod_{i=1}^{\ell}\mathsf{GL}_{r_i}(\mathbb{C}) \;\middle|\; \prod_{i=1}^{\ell}\det(x_i)^i = 1\right\}. \tag{14.1}$$

Lemma 14.1. *The group Γ_u contains elliptic pairs if and only if u is regular unipotent. In this case $\mathcal{Y}(\Gamma_u) = \mathcal{Y}(\Gamma_u)_{\mathrm{ell}} = \{(s,h) \mid s,h \in \mathrm{Z}_{\mathsf{SL}_n(\mathbb{C})}\}$.*

Proof. The proof is very similar to that of Proposition 13.2. Note that if u is not rectangular, then Γ_u has infinite center: for example, with notation as in (14.1), given $t \in \mathbb{C}^\times$, the element $(t\mathrm{Id}_{r_1}, t\mathrm{Id}_{r_2},\dots,t\mathrm{Id}_{r_{\ell-1}}, t^{r_\ell - n/\ell}\mathrm{Id}_{r_\ell}) \in \mathrm{Z}_{\Gamma_u}$. Thus, if Γ_u contains an elliptic pair, then λ_u is of the form $(k,k,\dots,k)$ for some k dividing n, and $\Gamma_u \simeq \{x \in \mathsf{GL}_{n/k}(\mathbb{C}) \mid \det(x)^k = 1\}$. Explicitly, we can think of Γ_u as a split extension

$$1 \to \mathsf{SL}_{n/k}(\mathbb{C}) \to \Gamma_u \to \mu_k \to 1,$$

where the first inclusion is the natural one, and the map to μ_k is given by the determinant. Now, given semisimple elements $s,h \in \Gamma_u$ such that $sh = hs$, there exists $g \in \mathsf{SL}_{n/k}(\mathbb{C})$ such that gsg^{-1}, ghg^{-1} are both diagonal in $\mathsf{GL}_{n/k}(\mathbb{C})$. Thus, a maximal torus of $\mathsf{SL}_{n/k}(\mathbb{C})$ centralizes both s and h, and if (s,h) is an elliptic pair, we must have $k = n$. □

We can now easily prove Conjecture 9.7 in this case.

Theorem 14.2. *Conjecture 9.7 holds when $\mathbf{G} = \mathsf{PGL}_n$. More precisely, when $\mathbf{G} = \mathsf{PGL}_n$, we have*

$$\mathrm{res}_{\mathrm{cpt,un}} \circ \mathrm{FT}_{\mathrm{ell}}^\vee = \mathrm{FT}_{\mathrm{cpt,un}} \circ \mathrm{res}_{\mathrm{cpt,un}}\,.$$

Proof. Using Lemma 14.1, the proof of the theorem reduces to Proposition 9.10. □

To illustrate the theorem, we explicitly describe the case when $\mathbf{G} = \mathsf{PGL}_2$. Note that even this low-rank example shows that certain choices were necessary in our setup: to relate $\mathrm{FT}_{\mathrm{ell}}^\vee$ to a finite Fourier transform for the non-split inner twist of G, first, we must consider maximal compact subgroups instead of just parahorics (otherwise the restrictions of $\Pi(u,1,1)$ and $\Pi(u,-1,1)$ would be the same, though $\mathrm{FT}_{\mathrm{ell}}^\vee$ fixes the first but not the second); second, $\mathrm{FT}_{\mathrm{cpt,un}}$ must mix subspaces corresponding to distinct inner twists to give a well-defined linear map.

Example 14.3. Now let $\mathbf{G} = \mathsf{PGL}_2$. Then G has a unique non-split inner twist G', which we can describe explicitly as follows: let

$$D = \left\{ \begin{pmatrix} a & \varpi b \\ \overline{b} & \overline{a} \end{pmatrix} \;\middle|\; a,b \in F_{(2)} \right\},$$

where $F_{(2)}$ is the degree-two unramified extension of F. Then D is a four-dimensional division algebra over F, and we can take $G' := D^\times/F^\times$.

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Let χ_0 be the unramified character of $F^\times$ given by $\varpi \mapsto -1$. Then the nontrivial weakly unramified character of G (respectively, G') is given by $\chi := \chi_0 \circ \det$ (respectively, $\chi' := \chi_0 \circ \det$). Let St_G denote the Steinberg representation of G (and similarly for $\mathrm{St}_{G'}$, which is the trivial representation of G'), and let $u \in \mathsf{SL}_2(\mathbb{C})$ be regular unipotent. Then the virtual representations corresponding to our four elliptic pairs are

$$\begin{aligned}\Pi(u,1,1) &= \mathrm{St}_G + \mathrm{St}_{G'},\\ \Pi(u,1,-1) &= \mathrm{St}_G - \mathrm{St}_{G'},\\ \Pi(u,-1,1) &= (\mathrm{St}_G \otimes \chi) + (\mathrm{St}_{G'} \otimes \chi'),\\ \Pi(u,-1,-1) &= (\mathrm{St}_G \otimes \chi) - (\mathrm{St}_{G'} \otimes \chi').\end{aligned}$$

The involution $\mathrm{FT}^\vee_{\mathrm{ell}}$ switches $\Pi(u,1,-1)$ and $\Pi(u,-1,1)$, and fixes the other two sums.

Let I be the Iwahori subgroup of G given by

$$I = \left\{ \begin{pmatrix} a & \varpi b \\ c & d \end{pmatrix} \,\middle|\, a,d \in \mathfrak{o}_F^\times, b,c \in \mathfrak{o}_F \right\}.$$

With notation as in § 7, we have $\Omega_G \simeq \mathbb{Z}/2\mathbb{Z}$. The set $S_{\max}(G)$ contains two elements $(A,\mathcal{O})$: one corresponding to $A = \Omega_G$ and one corresponding to A trivial. Thus, the group G has two conjugacy classes of maximal compact open subgroups: the maximal parahoric subgroup $K_0 := \mathsf{PGL}_2(\mathfrak{o}_F)$ (which corresponds to A trivial) and $K_1 := \mathrm{N}_G(I)$ (which corresponds to $A = \Omega_G$). Note that K_1 contains I with index 2: it is generated by I and $\sigma := \left(\begin{smallmatrix} 0 & \varpi \\ 1 & 0 \end{smallmatrix}\right)$. The reductive quotients are given by $\overline{K}_0 \simeq \mathsf{PGL}_2(\mathbb{F}_q)$ and $\overline{K}_1 \simeq \mathbb{F}_q^\times \rtimes \mathbb{Z}/2\mathbb{Z}$. Note that χ is trivial on K_0 and on I, but $\chi(\sigma) = -1$, so χ induces the sign character on the component group of $\overline{K}_1$.

In the notation of § 7, we have $G' = G_x$, where x is the nontrivial element of Ω_G. Thus, G' has a unique conjugacy class of maximal compact subgroups, corresponding to the element $(A,\mathcal{O})$ of $S_{\max}(G)$ with $A = \Omega_G$. Explicitly, G' itself is compact. The only parahoric subgroup of G' is $I' := \mathfrak{o}_D^\times/\mathfrak{o}_F^\times$, where $\mathfrak{o}_D$ is the ring of integers of D, and this parahoric is normal in G'. The group G' is generated by I' and σ (defined as above), which again has order 2 in G'. The reductive quotient $\overline{G'} \simeq (\mathbb{F}_{q^2}^\times/\mathbb{F}_q^\times) \rtimes \mathbb{Z}/2\mathbb{Z}$. Note that as above χ' is trivial on I' but takes the value -1 on σ, so χ' factors through the sign character of the component group of $\overline{G'}$.

The space $\mathcal{C}(G)_{\mathrm{cpt,un}}$ is given by

$$\mathcal{C}(G)_{\mathrm{cpt,un}} = R_{\mathrm{un}}(\overline{K}_0) \oplus R_{\mathrm{un}}(\overline{K}_1) \oplus R_{\mathrm{un}}(\overline{G'}),$$

and the map $\mathrm{res}_{\mathrm{cpt,un}}$ is defined on the virtual representations above by

$$\begin{aligned}\Pi(u,1,1) &\mapsto \mathrm{St}_{K_0} + \mathrm{St}_I + \mathrm{St}_{I'},\\ \Pi(u,1,-1) &\mapsto \mathrm{St}_{K_0} + \mathrm{St}_I - \mathrm{St}_{I'},\\ \Pi(u,-1,1) &\mapsto \mathrm{St}_{K_0} + (\mathrm{St}_I \otimes \mathrm{sgn}) + (\mathrm{St}_{I'} \otimes \mathrm{sgn}),\\ \Pi(u,-1,-1) &\mapsto \mathrm{St}_{K_0} + (\mathrm{St}_I \otimes \mathrm{sgn}) - (\mathrm{St}_{I'} \otimes \mathrm{sgn}),\end{aligned}$$

where, as in the proof of Proposition 9.10, St_K denotes the Steinberg representation of $\overline{K}$, and sgn denotes the sign representation of the relevant component group.

If $(A,\mathcal{O}) \in S_{\max}(G)$ with A trivial, then $\mathrm{res}_{\mathcal{O}}(\Pi(u,s,h)) = \mathrm{St}_{K_0}$ for all elliptic pairs (s,h). In this case, $\mathrm{FT}_{\mathrm{cpt,un}}$ restricts to the identity map on $R_{\mathrm{un}}(\overline{K}_0)$.

Now suppose $(A,\mathcal{O}) \in S_{\max}(G)$ with $A = \Omega_G$. Then $\mathrm{res}_{\mathcal{O}}$ is given by projection onto $R_{\mathrm{un}}(\overline{K}_1) \oplus R_{\mathrm{un}}(\overline{G'})$. In the notation of § 6, and with $\mathcal{U}$ the (one-element) family consisting of the Steinberg representation of $\overline{K}_1$, we have $\widetilde{\Gamma}^A_{\mathcal{U}} = A$, so $\mathcal{M}(\widetilde{\Gamma}^A_{\mathcal{U}})$ consists of four elements:

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$(1, \mathrm{triv}), (1, \mathrm{sign}), (x, \mathrm{triv}), (x, \mathrm{sgn})$ (where, as above, x is the nontrivial element of Ω_G). These correspond to the following elements of $R_{\mathrm{un}}(\overline{K}_1) \oplus R_{\mathrm{un}}(\overline{G}')$:

$$\begin{aligned}(1, \mathrm{triv}) &\longleftrightarrow \mathrm{St}_I,\\ (1, \mathrm{sgn}) &\longleftrightarrow \mathrm{St}_I \otimes \mathrm{sgn},\\ (x, \mathrm{triv}) &\longleftrightarrow \mathrm{St}_{I'},\\ (x, \mathrm{sgn}) &\longleftrightarrow \mathrm{St}_{I'} \otimes \mathrm{sgn}.\end{aligned}$$

With notation as in (5.13), we have

$$\begin{aligned}\mathrm{res}_{\mathcal{O}}(\Pi(u, 1, 1)) &= \Pi_{\widetilde{\mathcal{U}}}(\mathrm{triv}, \mathrm{triv}),\\ \mathrm{res}_{\mathcal{O}}(\Pi(u, 1, -1)) &= \Pi_{\widetilde{\mathcal{U}}}(\mathrm{triv}, \mathrm{sgn}),\\ \mathrm{res}_{\mathcal{O}}(\Pi(u, -1, 1)) &= \Pi_{\widetilde{\mathcal{U}}}(\mathrm{sgn}, \mathrm{triv}),\\ \mathrm{res}_{\mathcal{O}}(\Pi(u, -1, -1)) &= \Pi_{\widetilde{\mathcal{U}}}(\mathrm{sgn}, \mathrm{sgn}),\end{aligned}$$

where $\widetilde{\mathcal{U}}$ is the family indexed by $\Gamma_{\mathcal{U}}^A$. Thus the proof of Proposition 9.10, and Conjecture 9.7, may be easily verified.

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Conflicts of interest

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