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On primitive/Seifert-fibered constructions

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Abstract

J. Dean introduced the notion of primitive/Seifert-fibered knots and constructions to explain how Dehn surgeries on knots produce small Seifert fibered manifolds. We determine non-hyperbolic, primitive/Seifert-fibered knots, and show that for such knots any integral, small Seifert fibered surgery arises from a primitive/Seifert-fibered construction. We also show that any 2-orbifold $S^2(n_1, n_2, n_3)$ with a pair of indices coprime becomes the base orbifold of a Seifert fibered manifold obtained by Dehn surgery on a hyperbolic, primitive/Seifert-fibered knot in S^3.

1. *Introduction*

Let M be a Seifert fibered manifold. Identifying each fiber to a point, we obtain a surface which is naturally a 2-orbifold such that an exceptional fiber of index n corresponds to a cone point of index n. We call such an orbifold the *base orbifold* of M. A Seifert fibered manifold is *small* if its base orbifold is the 2-sphere with at most three cone points or the projective plane with at most one cone point.

In [**4**], Dean describes a new construction of knots with surgeries producing small Seifert fibered manifolds over the 2-sphere. We extend his idea slightly to construct Seifert fibered manifolds over the projective plane as well. (Eudave–Muñoz [**7**] makes the same extension.) Let K be a knot on the boundary of a genus two handlebody H. For convenience, we denote H with a 2-handle added along K by $H[K]$. We say that K is *primitive* with respect to H if $H[K]$ is a solid torus. Note that if K is primitive with respect to H, then K represents a generator of $\pi_1(H)$ and meets a properly embedded disc in H in a single point. We say that K is *Seifert-fibered* with respect to H if $H[K]$ is a Seifert fibered manifold over the disc with at most two cone points or

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the Möbius band with at most one cone point. (Dean [**4**] calls K Seifert-fibered only when the base orbifold of $H[K]$ is of the former type.) Now assume that K is a knot in a genus 2 Heegaard surface F for S^3; the closures H, H' of the components of $S^3 - F$ are genus 2 handlebodies. Then K is *primitive/Seifert-fibered* or in *primitive/Seifert-fibered position* if K is primitive with respect to H and Seifert-fibered with respect to H'. Similarly, we define K to be *primitive/primitive*, and so on. The notion of *primitive/primitive* knots was introduced and studied by Berge [**1**]. The isotopy class in $\partial N(K)$ of a component of $\partial N(K) \cap F$ is called the *surface slope* of K in F. We call the construction of a Seifert fibered manifold described in the lemma below a *primitive/Seifert-fibered construction*.

LEMMA 1·1. *Let K be a knot in primitive/Seifert-fibered position with surface slope γ. Then γ-Dehn surgery on K yields either a Seifert fibered manifold over S^2 or $\mathbb{R}P^2$, or a connected sum of lens spaces.*

Proof. Let H, H' be as above. By the definition of primitive/Seifert-fibered, the resulting manifold M of γ-surgery on K is $H[K] \cup H'[K]$, a union of a solid torus and a Seifert fibered manifold. If a regular fiber of $H'[K]$ is not a meridian of $H[K]$, then M is a Seifert fibered manifold over S^2 or $\mathbb{R}P^2$. So suppose not. Note that the Seifert fibered manifold $H'[K]$ has base space a disc or a Möbius band. In the former case M is homeomorphic to a connected sum of lens spaces, where lens spaces exclude S^3 and $S^2 \times S^1$ ([**17**]). In the latter case, M contains a non-separating 2-sphere. Thus $M \cong S^2 \times S^1$ by Property R ([**8**, corollary 8·3]).

A knot K is said to be a (p, q)-*cable* of a knot k if K is a simple loop in the boundary of $N(k)$ wrapping p times in the meridional direction and q times in the longitudinal direction.

By [**2**, **27** and **28**] a non-hyperbolic primitive/primitive knot is a torus knot or a $(2pq \pm 1, 2)$-cable of a (p, q)-torus knot. In fact, these knots are primitive/primitive (Section 4). We determine the non-hyperbolic primitive/Seifert-fibered knots as follows.

THEOREM 1·2. *Let K be a non-hyperbolic knot. Then K is a primitive/Seifert-fibered knot if and only if K is a torus knot or an $(spq \pm 1, s)$-cable of a (p, q)-torus knot. Furthermore, any integral small Seifert fibered surgery on such a knot K is realized by a primitive/Seifert-fibered construction.*

We prove the "only if" part in Sections 2 and 3. In Section 4, we first study primitive/Seifert-fibered positions of torus knots and cable knots, and then complete the proof of Theorem 1·2. In Section 5, we show that the twisted torus knots introduced in our previous paper [**20**] are primitive/Seifert-fibered. In Section 6, we study when twisted torus knots are hyperbolic by using Theorem 1·2. Consequently, we show that, in terms of base orbifolds, many small Seifert fibered manifolds are obtained by Dehn surgery on hyperbolic knots in S^3.

Let us denote a 2-orbifold which is topologically X with p cone points of indices $n_1, \ldots, n_p$ by $X(n_1, \ldots, n_p)$.

THEOREM 1·3. *Let $S^2(n_1, n_2, n_3)$ $(n_i \geqslant 2)$ be a 2-orbifold such that n_i and n_j are coprime for some i, j. Then there is a hyperbolic, tunnel number one (and thus strongly*

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invertible) *knot which admits a Dehn surgery producing a Seifert fibered manifold with base orbifold* $S^2(n_1, n_2, n_3)$.

This result was obtained by Dean in [**5**], except that he was not able to show that the knots in question were hyperbolic.

Dean [**4**] and Gordon [**10**] asked if every small Seifert fibered Dehn surgery can be explained by a primitive/Seifert-fibered construction. In [**19**] we gave two infinite families of non-strongly invertible knots each of which admits a Seifert fibered surgery not arising from a primitive/Seifert-fibered construction. For example, the $(-3, 3, 5)$-pretzel knot belongs to the families. Very recently Hyung–Jong Song has observed that the 1-surgery of the $(-3, 3, 3)$-pretzel knot is a Seifert fibered surgery, but does not arise from a primitive/Seifert-fibered construction. (In [**18**], Mattman had observed that 1-surgery on the $(-3, 3, 3)$-pretzel knot gives a Seifert fibered manifold.) In contrast with our examples, the $(-3, 3, 3)$-pretzel knot is strongly invertible; but it has tunnel number greater than one and cyclic period 2 like ours.

2. *Proof of the "only if" part of Theorem* 1·2

By Thurston's uniformization theorem, if a knot K is non-hyperbolic, then K is a torus knot or a satellite knot. The exterior of a satellite knot K contains a non-boundary-parallel incompressible torus; the solid torus in S^3 bounded by such a torus is called a *companion solid torus* of K.

Let K be a satellite knot in primitive/Seifert-fibered position. As noticed in [**4**, chapter 4], any knot which is primitive with respect to an unknotted genus two handlebody in S^3 has tunnel number 1, so that K has tunnel number 1. Let M be the result of the surgery on K along the surface slope. If M is reducible, then the positive solution to the cabling conjecture for satellite knots [**25**] implies that K is a cable knot. Since M is a connected sum of lens spaces (none of which is S^3 or $S^2 \times S^1$) by Lemma 1·1, [**14**, theorem 3] shows that K is a $(spq \pm 1, s)$-cable of a (p, q)-torus knot, as claimed in Theorem 1·2. Then we may assume that M is Seifert fibered over S^2 or $\mathbb{R}P^2$. If the base space of M is $\mathbb{R}P^2$, then by [**21**, theorem 1·3] K is a $(6p \pm 1, 3)$-cable of a $(p, 2)$-torus knot or a $(2pq \pm 1, 2)$-cable of a (p, q)-torus knot, as desired. Now assume for a contradiction that the base orbifold of M is S^2 with more than three cone points. Since the tunnel number of K is one, M has Heegaard genus two. It follows that the base orbifold of M is $S^2(2, 2, 2, 2n + 1)$ for some integer n ([**3**]). However, such a manifold cannot be obtained by Dehn surgery on a knot in S^3 because its first homology group is non-cyclic. Finally assume that the base orbifold of M is S^2 with at most three cone points. Then Proposition 2·1 below implies the desired result.

Proposition 2·1. *Let K be a satellite knot of tunnel number one. If a surgery on K yields a small Seifert fibered manifold, then K is an $(spq \pm 1, s)$-cable of a (p, q)-torus knot.*

3. *Proof of Proposition* 2·1

In [**20**, **21**], we introduced the operation called "modification" to study Seifert fibered surgeries on satellite knots. Let us recall the definition.

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Definition (modification). Let K be a satellite knot with a companion solid torus V. Let (M, L) be a preferred meridian-longitude pair of the knotted solid torus $V(\subset S^3)$. Paste a solid torus, say W, and V along their boundaries so that the boundary of a meridian disc of W represents $L + tM \in H_1(\partial V)$ for some $t \in \mathbb{Z}$; then the resulting 3-manifold is homeomorphic to S^3. We denote by K' the knot K in this new 3-sphere $V \cup W$. We call the new knot K' a *t-twisted modification* of K *with respect to* V.

Now we suppose that K is a satellite knot of tunnel number one. Such knots have been completely classified by Morimoto and Sakuma [**24**]. Their classification shows that if K is a satellite knot of tunnel number one, then the (unique) companion knot (i.e., the core of the companion solid torus of K) is a (p, q)-torus knot and the pq-modification K', together with a core of W, forms a 2-bridge link. Since each component of a 2-bridge link is a trivial knot in S^3, we have:

LEMMA 3·1. *If K is a satellite knot of tunnel number one, then the companion knot is a (p, q)-torus knot and the pq-twisted modification is unknotted in S^3.*

Let us assume further that $(K; \gamma)$ is a small Seifert fibered manifold. Since $(K; \gamma)$ cannot contain a separating incompressible torus, it follows from [**20**] that K is a 0- or 1-bridge braid in the companion solid torus V of K in the sense of Gabai [**9**]. The knot $K \subset V$ is called a 0-bridge braid if K can be isotoped to be a braid in V which lies entirely in ∂V; K is called a 1-bridge braid in $V = S^1 \times D^2$ if K is not a 0-bridge braid and can be isotoped to a knot of the form $\alpha \cup \beta$, where α, β are arcs such that $\alpha \subset \{\text{pt.}\} \times D^2$ and $\beta \subset \partial V$ and is transverse to each $\{\text{pt.}\} \times D^2$. If K is a 0-bridge braid in V, then by definition K is an (r, s)-cable of a (p, q)-torus knot for some p, q, r, s. Since the pq-modification of K is the $(r - spq, s)$-torus knot and unknotted in S^3 by Lemma 3·1, we obtain $r - spq = \pm 1$. Therefore K is an $(spq \pm 1, s)$-cable of a (p, q)-torus knot, as claimed in Proposition 2·1.

Assume for a contradiction that K is a 1-bridge braid in V. Then the pq-modification K' is a 1-bridge braid in an unknotted solid torus V' in S^3. The proposition below shows that K' is knotted in S^3. This contradicts Lemma 3·1.

PROPOSITION 3·2. *If k is a 1-bridge braid in an unknotted solid torus, then k is knotted in S^3.*

Proof. Let V_1 be an unknotted solid torus in S^3 and $k = \alpha \cup \beta$ a 1-bridge braid in V_1. Let D_1 be a meridian disc of V_1 disjoint from α such that $l = \partial D_1$ intersects β in w points in the same direction.

LEMMA 3·3. *There does not exist a simple loop $c \subset \partial V_1$ such that c intersects l in a single point and $c \cap \beta = \varnothing$.*

Proof of Lemma 3·3. Assume for a contradiction that there exists such a simple loop c. Let A be the annulus ∂V_1 cut along l. Then c becomes an arc c' connecting distinct components of ∂A, and β is cut into the union of $w - 1$ arcs connecting distinct components of ∂A and 2 arcs each of which has one end in $\text{int}A$. The arc c' is not adjacent to any arc of the first type, for otherwise k is disconnected. Thus in each side of c' an arc of the second type is adjacent to c'. This implies that k is a 0-bridge braid in V_1, a contradiction.

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Now let us assume for a contradiction that k is unknotted in S^3; we will find a simple loop c described in Lemma 3·3.

Let V_2 be the complementary solid torus $S^3 - \mathrm{int}V_1$, and β' a properly embedded arc in V_2 such that β and β' cobound a disc Δ in V_2. Then $H = V_2 - \mathrm{int}N(\beta')$ is a genus two handlebody. Let μ be a meridian of β'.

LEMMA 3·4. *The set $\{\mu, l\}$ is a primitive set for H, i.e., there exist dual discs D_μ, D_l in H such that $D_\mu \cap D_l = \varnothing$ and D_u meets v in δ_{uv} points, where $u, v \in \{\mu, l\}$.*

Proof. We denote the 3-ball $V_1 - \mathrm{int}N(D_1)$ by B, which contains the trivial arc α. Then we have:

$$\begin{aligned} S^3 - \mathrm{int}N(k) &\cong (V_1 - \mathrm{int}N(\alpha)) \cup (V_2 - \mathrm{int}N(\beta')) \\ &= ((B - \mathrm{int}N(\alpha)) \cup N(D_1)) \cup (V_2 - \mathrm{int}N(\beta')) \\ &\cong N(D_1) \cup (V_2 - \mathrm{int}N(\beta')) \\ &= H \cup N(D_1). \end{aligned}$$

Thus we have $H[l] = H \cup_l (2\text{-handle})$ is a solid torus. On the other hand, $H[\mu] \cong V_2$, a solid torus; $H[\mu][l]$ is a 3-ball. It then follows from [**12**, theorem 1] that $\{\mu,\ l\}$ is a primitive set for H.

Set $\Delta' = \Delta \cap H$, a properly embedded disc in H, Then $\partial\Delta'$ intersects μ in a single point and l in w points in the same direction.

LEMMA 3·5. *We can choose the dual disc D_l in Lemma 3·4 so that $D_l \cap \Delta' = \varnothing$.*

Proof. Since $\{\mu,\ l\}$ is a primitive set, we can take a separating disc D which splits H into two solid tori V_μ and V_l containing μ and l, respectively. Then we may assume that $D \cap \Delta'$ consists of properly embedded arcs in D. Let D' be an outermost disc of $D - \Delta'$ in D. Applying boundary compression to Δ' along the outermost disc D', we obtain two discs Δ'_1 and Δ'_2. Further we isotope Δ'_i $(i = 1, 2)$ slightly so that $\Delta'_i \cap \Delta' = \varnothing$. Without loss of generality, we may assume that $\Delta'_1 \cap l \neq \varnothing$; since Δ intersects l in the same direction, Δ'_1 also intersects l in the same direction. Note that $\Delta'_1 \cap D$ has fewer components than $\Delta' \cap D$. If $\Delta'_1 \cap D \neq \varnothing$, apply boundary compression to Δ'_1 along an outermost disc of $D - \Delta'_1$. After repeating this process, we have a disc $\overline{\Delta'}$ such that $\overline{\Delta'}$ intersects l non-trivially and in the same direction, $\overline{\Delta'} \cap D = \varnothing$ and $\overline{\Delta'} \cap \Delta' = \varnothing$. It follows that $\overline{\Delta'}$ is contained in the solid torus V_l, in which l is a longitude. Thus, $\overline{\Delta'}$ is a meridian disc of H_l, and thus intersects l in a single point. The proof is now completed by setting $D_l = \overline{\Delta'}$.

The disc D_l obtained by Lemma 3·5 is properly embedded in V_2 and disjoint from β. Then, ∂D_l contradicts Lemma 3·3. Hence k is knotted in S^3.

4. *Primitive/Seifert-fibered, cable knots*

Let H be a genus two handlebody standardly embedded in S^3, and H' the complementary genus two handlebody; let $F = \partial H$. We use the oriented disc system D_1, D_2 in H corresponding to generators a and b for $\pi_1(H)$ respectively, and D'_1, D'_2 in H' to generators x, y for $\pi_1(H')$ respectively, as pictured in Figure 4.1.

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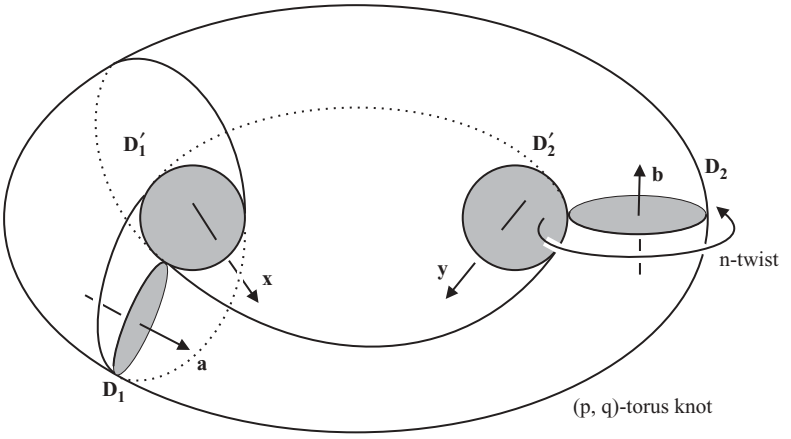


Fig. 4·1.

We first consider primitive/Seifert-fibered positions for torus knots.

Example 1 (torus knots). Let k be a (p, q)-torus knot on the genus two Heegaard surface F such that the intersection of k and each of D_2, D_2' is one point and the algebraic intersection numbers $k \cdot D_2$, $k \cdot D_2'$ are 1. See Figure 4·1. We write K_n for the knot obtained from k by adding n full twists along the meridian disc D_2. Then K_n represents $a^q b$ in $\pi_1(H)$ independent of n; hence it is primitive with respect to H. On the other hand, K_n represents $y^{-n+1}x^{-p}$ in $\pi_1(H')$; hence $H'[K_n]$ is a Seifert fibered manifold (cf. [**26**]). Thus K_n is primitive/Seifert-fibered. In particular, K_n is primitive/primitive for $n = 0, 2$. A simple computation shows that the surface slope of K_n in F is $pq + n - 1$.

We next show that:

LEMMA 4·1 (cable knots). *An $(spq + 1, s)$-cable (resp. $(spq - 1, s)$-cable) of a (p, q)-torus knot admits primitive/Seifert-fibered positions with surface slopes $s(spq + 1) \pm 1$ (resp. $s(spq - 1) \pm 1$).*

Proof. We take s parallel copies, L, of a (p, q)-torus knot k on F such that k meets D_2' in one point, $k \cdot D_2' = 1$, and $k \cap D_2 = \varnothing$ (Figure 4·2). Cut L along $\partial D_2'$, and re-connect after sliding along $\partial D_2'$ by $2\pi/s$, where the s points $\partial D_2' \cap L$ are regarded to be evenly spaced in $\partial D_2'$; see Figure 4·3. We denote the resulting knot by k. Let K_n be the knot obtained from k by adding n full twists along the meridian disc D_2. The knot K_n is an $(spq + 1, s)$-cable of a (p, q)-torus knot. The surface slope of K_n in F is $s(spq + 1) + n$.

Whatever the values of p and n are, K_n represents $a^{sq}b$ in $\pi_1(H)$; hence it is primitive with respect to H. On the other hand, K_n represents $y^{-n}(y^{-1}x^{-p})^s$ in $\pi_1(H')$. Let $n = -1$; then the automorphism of $\langle x, y\rangle$ which sends $y^{-1}x^{-p} \mapsto y$ and $x^{-1} \mapsto x$ sends K_{-1} to $x^p y^{s-1}$. This shows that K_{-1} is Seifert-fibered with respect to H' (see [**4**, 2.2]); the surface slope is $s(spq + 1) - 1$.

Let us show that an $(spq + 1, s)$-cable of a (p, q)-torus knot admits a primitive/Seifert-fibered position with surface slope $(spq + 1) + 1$. It suffices to see that K_1 is Seifert-fibered with respect to H'. The knot K_1 represents $y^{-1}(y^{-1}x^{-p})^s$

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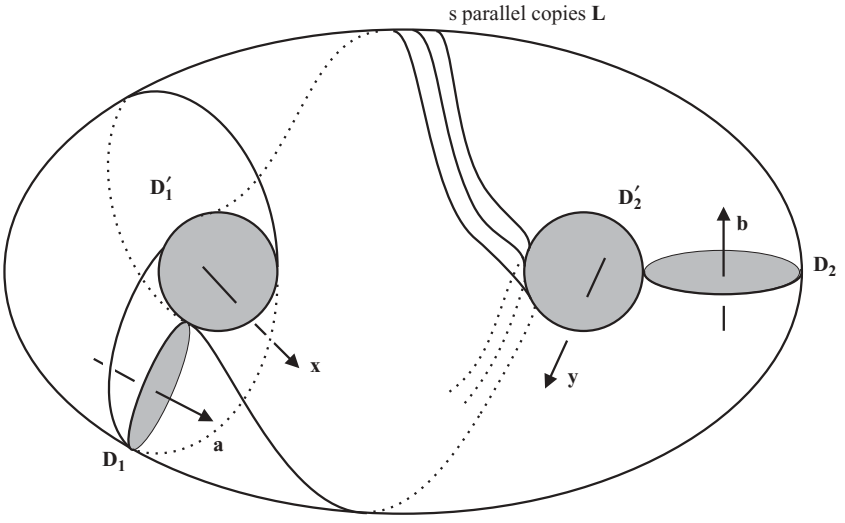


Fig. 4·2.

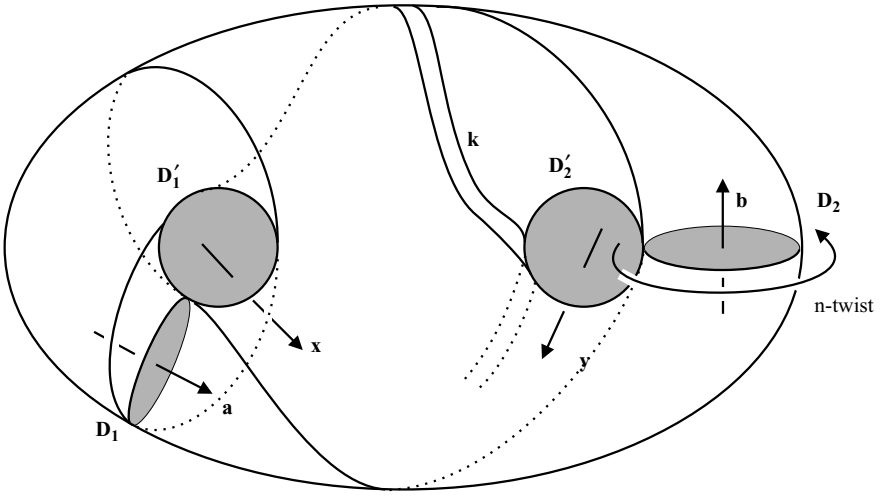


Fig. 4·3.

in $\pi_1(H')$, which is cyclically equivalent to $(y^{-1}x^{-p})^s y^{-1} = (y^{-1}x^{-p})^{s+1}x^p$. The automorphism of $\langle x, y\rangle$ which sends $y^{-1}x^{-p} \mapsto y$ and fixes x sends K_1 to $y^{s+1}x^p$, so that K_1 is Seifert-fibered with respect to H'.

When we start with a (p, q)-torus knot k in F such that $k \cdot D_2'$ equals -1 not 1, and apply a $-2\pi/s$ slide to s copies of k cut along $\partial D_2'$, the resulting knot K_n in F is an $(spq - 1, s)$-cable of a (p, q)-torus knot with surface slope $s(spq - 1) + n$. By the arguments as above, this K_n in F is in primitive/Seifert-fibered position.

Proof of Theorem 1·2. The "only if" part of the first statement of the theorem is proved in Sections 2 and 3. The "if" part of the first statement follows from Example 1 and Lemma 4·1.

To complete the proof, we need to prove the statement about small Seifert fibered surgeries. If an integral surgery on a (p, q)-torus knot yields a small Seifert fibered

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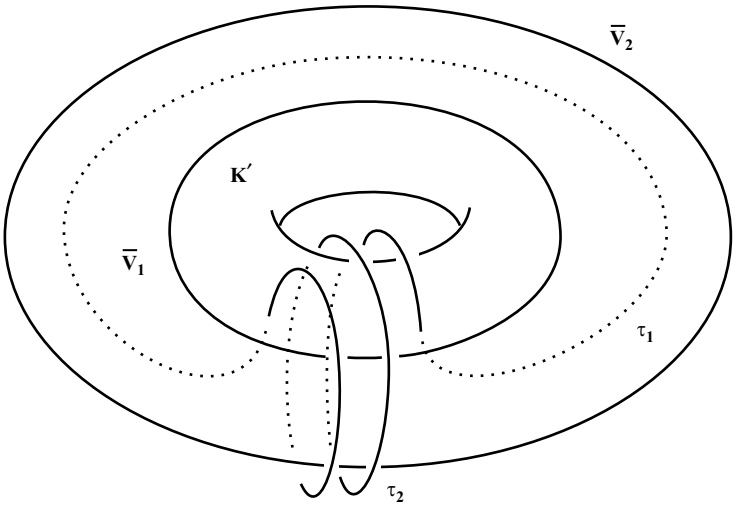


Fig. 4·4.

manifold, then the surgery coefficient is an arbitrary integer other than pq (see [**11**] or Claim 6·2 below). Example 1 shows that any such integer can be the surface slope of the torus knot in a primitive/Seifert-fibered position.

If an integral surgery on an $(spq \pm 1, s)$-cable of a (p, q)-torus knot yields a small Seifert fibered manifold, then the surgery coefficient m satisfies $|s(spq \pm 1) - m| = 1$ (see [**11**] or Claim 6·2). Lemma 4·1 shows that any such m can be the surface slope.

Finally we give an example of a primitive/Seifert-fibered construction producing a Seifert fibered manifold over $\mathbb{R}P^2$.

Example 2. Let K be a $(2pq \pm 1, 2)$-cable of a (p, q)-torus knot. Then $4pq$-surgery on K yields a Seifert fibered manifold with base orbifold $\mathbb{R}P^2(|p|, q)$ (see [**20**]). This toroidal Seifert fibered surgery can be explained by a primitive/Seifert-fibered construction.

Proof of Example 2. Let K' be the $(\pm 1, 2)$-cable of a standard solid torus in S^3, and τ the core of the complementary solid torus. Since the components of the link $K' \cup \tau$ are interchangeable after isotopy, we can isotope $K' \cup \tau$ into the following position (Figure 4·4). The knot K' is a preferred longitude of a standard solid torus V_1, $\tau_1 = \tau \cap V_1$ is a trivial arc wrapping around V_1 "almost" once, and $\tau_2 = \tau \cap V_2$ is a trivial arc wrapping around V_2 "almost" twice, where $V_2 = S^3 - \mathrm{int}V_1$.

Let $V = S^3 - \mathrm{int}N(\tau)$ with $K' \subset \mathrm{int}V$. Then ∂V is the union of the two annuli $\mathrm{Fr}_{V_i} N(\tau_i) = \partial N(\tau) \cap V_i$, where $i = 1, 2$. Let $\overline{V}_i = V_i - \mathrm{int}N(\tau)$; $\overline{V}_1$ and $\overline{V}_2$ are genus 2 handlebodies glued along $\partial V_i - \mathrm{int}N(\tau)$, a torus with 2 holes. We have $V = \overline{V}_1 \cup \overline{V}_2$. Let $\mu_i \subset \overline{V}_i$ be a core of $\mathrm{Fr}_{V_i} N(\tau_i)$; then μ_i is a preferred longitude of V.

On the other hand, we decompose the exterior of a (p, q)-torus knot $T_{p,q}$ as the union of two solid tori W_1 and W_2 such that $W_1 \cap W_2$ is an annulus whose core is a (p, q)-cable (resp. (q, p)-cable) of W_1 (resp. W_2). Let A_1 (resp. A_2) be the closure of $\partial W_1 - \partial W_2$ (resp. $\partial W_2 - \partial W_1$); then $\partial N(T_{p,q}) = A_1 \cup A_2$. Now take an orientation preserving homeomorphism $f: V \to N(T_{p,q})$ such that $f(\mathrm{Fr}_{V_i} N(\tau_i)) = A_i$ for $i = 1, 2$. Note that μ_i, a preferred longitude of V, is sent to a loop linking $T_{p,q}$ pq times. We

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then see that $f(K')$ is equivalent to K, the $(2pq \pm 1, 2)$-cable of $T_{p,q}$. We obtain the decomposition $S^3 = (W_1 \cup W_2) \cup (f(\overline{V}_1) \cup f(\overline{V}_2))$, where W_i and $f(\overline{V}_i)$ are pasted along $A_i = f(\mathrm{Fr}_{V_i} N(\tau_i))$. Set $H_i = W_i \cup f(\overline{V}_i)$ for $i = 1, 2$. Since a core of $\mathrm{Fr}_{V_i} N(\tau_i)$ is primitive with respect to $\overline{V}_i$, H_i is a handlebody of genus 2. Then $S^3 = H_1 \cup H_2$ is a genus 2 Heegaard decomposition such that ∂H_i contains K.

Attaching a 2-handle h^2 to $\overline{V}_1$ along K', we have a solid torus with μ_1 a longitude. Thus $H_1[K] = W_1 \cup (f(\overline{V}_1) \cup h^2)$ is a fibered solid torus with μ_1 a fiber such that the core is an exceptional fiber of index q. Attaching a 2-handle h^2 to V_2 along K', we obtain $S^2 \times S^1 - \mathrm{int}(B^3)$, where B^3 is a 3-ball. It follows that $\overline{V}_2[K'] = (V_2 \cup h^2) - \mathrm{int} N(\tau_2) \cong S^2 \times S^1 - \mathrm{int}(N(\tau_2) \cup B^3)$, the exterior of the $(\pm 1, 2)$-cable of a regular neighbourhood of a core of $S^2 \times S^1$. Hence, $\overline{V}_2[K']$ is a circle bundle over the Möbius band with μ_2 a fiber. Recall that W_2 is a fibered solid torus with μ_2 a fiber such that the core is an exceptional fiber of index $|p|$. It follows that $H_2[K] = W_2 \cup (f(\overline{V}_2) \cup h^2)$ is a Seifert fibered manifold over the Möbius band with one exceptional fiber of index $|p|$. This shows that K is in primitive/Seifert-fibered position with respect to the decomposition $S^3 = H_1 \cup H_2$.

Remark. Eudave–Muñoz [**7**] translates primitive/Seifert-fibered constructions to tangles and shows that there are infinite families of hyperbolic knots producing Seifert fibered manifolds over $\mathbb{R}P^2$ with two exceptional fibers by Dehn surgery. Gordon and Luecke also announced the construction of a hyperbolic knot with surgery producing a Seifert fibered manifold over $\mathbb{R}P^2$ [**15**].

5. *Primitive/Seifert-fibered, twisted torus knots*

Twisted torus knots are, loosely speaking, obtained by adding full twists to some parellel strands of torus knots. The definition of twisted torus knots given in our previous paper [**22**] is ambiguous. We first give a correct definition. Let W be a standardly embedded solid torus in S^3. Let $K_{p,q} (\subset \partial W)$ be a (p, q)-cable of W, where $q > |p| \geqslant 2$. Then take a $(-1, 1)$-cable of W intersecting $K_{p,q}$ in $p + q$ points of the same sign. Given an integer r with $0 \leqslant r \leqslant p+q$, let α be a subarc of the $(-1, 1)$-cable such that $\partial\alpha \cap K_{p,q} = \varnothing$, and $\alpha \cap K_{p,q}$ consists of r points. Note that two such α are isotopic to each other in ∂W so that $\partial\alpha$ keeps away from $K_{p,q}$ during the isotopy. Let $D = \alpha \times [-1, 1]$ be an embedded disc in S^3 such that D intersects ∂W transversely in $\alpha \times 0 = \alpha$. Let V be the unknotted solid torus $S^3 - \mathrm{int} N(\partial D)$ containing $K_{p,q}$ in its interior; a meridian disc of V intersects $K_{p,q}$ in r points.

Definition (twisted torus knot). Performing $-1/n$-Dehn surgery on $\tau = \partial D$ (or equivalently twisting V n times along a meridian disc), we obtain a new knot as the image of $K_{p,q}$ in the twisted solid torus V_n $(\subset S^3)$. The knot obtained in this manner is called a *twisted torus knot* $K(p, q, r, n)$, where p and q are coprime, $q > |p| \geqslant 2$, and $0 \leqslant r \leqslant p + q$. Note that $K(p, q, r, 0)$ and $K(p, q, 1, n)$ are (p, q)-torus knots for any r, n.

Remarks. (1) The arcs α in [**22**] are chosen with less restriction, so that the isotopy classes of α are not uniquely determined if r is specified (although otherwise stated in [**22**]). Hence the twisted torus knots $K(p, q, r, n)$ given in [**22**] are not uniquely determined.
(2) Dean [**5**] generalizes the notion of twisted torus knots.

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In [**20**, section 9], we studied surgery of V on $K_{p,q}$ for $r = p+q$. Lemma 9.1 of [**20**] implies the following; see also [**22**, section 6·3].

PROPOSITION 5·1. *$pq+(p+q)^2n$-surgery on $K(p,q,p+q,n)$ produces a Seifert fibered manifold with base orbifold $S^2(|p|,q,|n|)$.*

We reprove Proposition 5·1 in the course of proving the proposition below.

PROPOSITION 5·2.

(1) *A twisted torus knot $K(p,q,p+q,n)$ is primitive/Seifert-fibered with surface slope $pq+(p+q)^2n$.*
(2) *$K(p,q,p+q,n)$ is of tunnel number 1, and thus strongly invertible.*

Proof. The proof is essentially given in [**20**].

If $n=0$, then $K(p,q,p+q,0) \cong T_{p,q}$, which is primitive/Seifert-fibered (Example 1). Let us assume $n \neq 0$. Set $V_1 = W$ and $V_2 = S^3 - \mathrm{int}V_1$; V_i are solid tori. By the choice of τ, $H_1 = V_1 - \mathrm{int}N(\tau)$ and $H_2 = V_2 - \mathrm{int}N(\tau)$ are genus 2 handlebodies, and $K_{p,q}$ lies on $\partial V_i - \mathrm{int}N(\tau)$. Let A_i be the annulus $H_i \cap \partial N(\tau)$, whose core is a meridian of τ. Let V be the solid torus glued to $S^3 - \mathrm{int}N(\tau)$ to construct $(\tau; -1/n)$. Then, H_1 and V are pasted so that $A_1 (\subset \partial H_1)$ is identified with an annulus in ∂V whose core is the $(1,n)$-cable of V. Since a core of A_1 is primitive with respect to H_1, $H_1' = H_1 \cup V$ is also a handlebody of genus 2. We have $(\tau; -1/n) = H_1' \cup H_2$, a Heegaard decomposition of genus 2. The twisted torus knot $K(p,q,p+q,n)$ lies on the Heegaard surface $\partial H_1'$ as the image of $K_{p,q} \subset \partial H_1$. For simplicity, we denote the simple loop $K_{p,q} = K(p,q,p+q,n)$ by K. As shown in [**20**, section 9], $H_1[K]$ is a fibered solid torus over the base orbifold $D^2(q)$ with a core of A_1 a fiber, and $H_2[K]$ is a fibered solid torus over $D^2(|p|)$ with a core of A_2 a fiber. It follows that $H_1'[K] = H_1[K] \cup V$ is a Seifert fibered manifold over $D^2(q,|n|)$. Therefore $K(p,q,p+q,n)$ is primitive with respect to H_2 and Seifert-fibered with respect to H_1'. Since a meridian of τ is a regular fiber in $H_1'[K]$ and $H_2[K]$, surgery on K along the surface slope of $K \subset H_1'$ produces a Seifert fibered manifold with base orbifold $S^2(|p|,q,|n|)$, as stated in Proposition 5·1. The surface slope of $K_{p,q} = K(p,q,p+q,0)$ in ∂H_1 is pq, and the image of the slope after $-1/n$-surgery on τ is the surface slope of $K(p,q,p+q,n) \subset \partial H_1'$. The linking number of $K_{p,q}$ and τ is $p+q$, so that the surface slope of $K(p,q,p+q,n)$ is $pq+(p+q)^2n$. This completes the proof of assertion (1).

As mentioned in [**4**, 4·5], any knot which is primitive with respect to a genus two handlebody unknotted in S^3 has tunnel number one. Since $K(p,q,p+q,n)$ does not separate the genus two Heegaard surface ∂H_2, the following claim shows that $K(p,q,p+q,n)$ is strongly invertible.

CLAIM 5·3. *If K is a knot on a genus two Heegaard surface F of S^3 which does not separate F, then K is strongly invertible.*

Proof. Note that there is an involution on S^3 which restricts to a standard ("hyperelliptic") involution τ on the Heegaard surface F; the fixed point set $\mathrm{Fix}(\tau)$ consists of four points. Let C be a nonseparating circle on F intersecting $\mathrm{Fix}(\tau)$ in two points and invariant under τ. Then there is a homeomorphism h of F such that $h(C)$ is isotopic to K; moreover since the genus of F is two, we can choose h so that $h \circ \tau = \tau \circ h$. Thus $h(C)$ is invariant under the involution of S^3, and hence K is strongly invertible.

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6. *Base orbifolds*

Which orbifolds $S^2(n_1, n_2, n_3)$ are realized as the base orbifold of a Seifert fibered manifold obtained by Dehn surgery on a knot in S^3? For homological reason we have the following restriction:

PROPOSITION 6·1. *If $S^2(n_1, n_2, n_3)$ is the base orbifold of a Seifert fibered manifold obtained by Dehn surgery on a knot K in S^3, then the greatest common divisor (n_1, n_2, n_3) is* 1.

Proof. Suppose that $(K; r)$ is a Seifert fibered manifold with base orbifold $S^2(n_1, n_2, n_3)$. Then the fundamental group $\pi_1((K; r))$ has the presentation $\langle c_1, c_2, c_3, h \mid c_i h c_i^{-1} h^{-1} = 1, c^{n_i} = h^{m_i}, h^b = c_1 c_2 c_3 \rangle$. Add the relation $h = 1$ and abelianize; we then have the presentation $\langle c_1, c_2, c_3 \mid n_i c_i = 0, c_1 + c_2 + c_3 = 0 \rangle \cong \langle c_1, c_2 \mid n_1 c_1 = 0, n_2 c_2 = 0, n_3(c_1 + c_2) = 0 \rangle$. Assume for a contradiction that $(n_1, n_2, n_3) = k \geqslant 2$. By adding the relations $kc_1 = 0$ and $kc_2 = 0$, we have the group with presentation $\langle c_1, c_2 | kc_1 = 0, kc_2 = 0 \rangle$, isomorphic to $\mathbb{Z}_k \oplus \mathbb{Z}_k$. This contradicts the fact that $H_1((K; r))$ is cyclic.

If $(n_i, n_j) = 1$ for some i and j, then such an orbifold can be the base orbifold of a Seifert fibered manifold obtained by Dehn surgery on a torus knot (see [**23**], [**11**], or Claim 6·2). We then ask whether hyperbolic knots yield these base orbifolds by Dehn surgery. In [**5**], Dean gives knots which admit surgeries producing Seifert fibered manifolds with required base orbifolds, and also shows that the knots are not torus knots. Our purpose in this section is to prove:

THEOREM 1·3. *Let $S^2(n_1, n_2, n_3)$ $(n_i \geqslant 2)$ be a 2-orbifold such that n_i and n_j are coprime for some i, j. Then there is a tunnel number* 1 *(hence strongly invertible) hyperbolic knot in S^3 which admits a Dehn surgery producing a Seifert fibered manifold with base orbifold $S^2(n_1, n_2, n_3)$.*

Propositions 5·1 and 5·2 suggest that twisted torus knots $K = K(p, q, p + q, n)$ are candidates for the knots required in the theorem. In [**22**, claim 6·5], we have shown that K is hyperbolic if $p + q \geqslant 2$ and $|n| > 5$. To prove Theorem 1·3, we establish the hyperbolicity under weaker conditions. The proof is a long but elementary arithmetical computation. If K is not hyperbolic, then K, a primitive/Seifert-fibered knot, is a torus knot or a cable of a torus knot described in Theorem 1·2. Let us denote a (p, q)-torus knot and an (r, s)-cable of a (p, q)-torus knot by $T_{p,q}$ and $T_{p,q}(r, s)$, respectively. Note that $T_{p,q} \cong T_{-p,-q} \cong T_{q,p} \cong T_{-q,-p}$. It is known that:

CLAIM 6·2. *Assume that p and q satisfy $q > |p| \geqslant 2$.*

(1) *If $(T_{p,q}; m)$ is Seifert fibered, then $m \neq pq$ and $(T_{p,q}; m)$ has base orbifold $S^2(|p|, q, |pq - m|)$* ([**11**, **23**]).
(2) *If $(T_{p,q}(r, s); m)$ is a small Seifert fibered manifold, then $r = spq \pm 1$, $|rs - m| = 1$, and $(T_{p,q}(r, s); m) \cong (T_{p,q}; m/s^2)$, which has base orbifold $S^2(|p|, q, |pqs^2 - m|)$* ([**11**]).

In the rest of this section we assume that $q > p \geqslant 2$ unless otherwise indicated.

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CLAIM 6·3.

(1) *If $K(p,q,p+q,n)$ is a torus knot, then it is $T_{\varepsilon p,q}$, $T_{\varepsilon p,n}$, or $T_{\varepsilon q,n}$, where $\varepsilon = \pm 1$.*

(2) *If $K(p,q,p+q,n)$ is a satellite knot, then it is $T_{\varepsilon p,q}(\varepsilon spq \pm 1, s)$, $T_{\varepsilon p,n}(\varepsilon spn \pm 1, s)$, or $T_{\varepsilon q,n}(\varepsilon sqn \pm 1, s)$ for some integer s and $\varepsilon = \pm 1$. Consequently, $(K(p,q,p+q,n); pq+(p+q)^2 n) \cong (T_{\varepsilon p,q}; (pq+(p+q)^2 n)/s^2)$, $(T_{\varepsilon p,n}; (pq+(p+q)^2 n)/s^2)$, or $(T_{\varepsilon q,n}; (pq+(p+q)^2 n)/s^2)$.*

Proof. If a 3-manifold admits a Seifert fibration over S^2 with three cone points, then any such fibration gives the same triple of indices of cone points. The indices of the exceptional fibers of $(K(p,q,p+q,n); pq+(p+q)^2 n)$ are p, q and $|n|$ (Proposition 5·1). Referring to Theorem 1·2 and comparing the (unordered) triple $p, q, |n|$ with the triples of indices given in Claim 6·2, we obtain Claim 6·3.

LEMMA 6·4. *A twisted torus knot $K(p,q,p+q,n)$ cannot be a torus knot for $n \neq 0$.*

Remark. If $p < 0$, then a twisted torus knot $K(p,q,p+q,n)$ can be a torus knot. For example, $K(-2,5,3,1)$ is the unknot and $K(-2,5,3,2)$ is the $(2,5)$-torus knot.

Proof. Suppose for a contradiction that $K = K(p,q,p+q,n)$ is a torus knot. Then by Claim 6·3, there are six possibilities: $K = T_{\varepsilon p,q}$, $T_{\varepsilon p,n}$ or $T_{\varepsilon q,n}$ ($\varepsilon = \pm 1$). If $K = T_{\varepsilon p,q}$, then the $pq+(p+q)^2 n$-surgery on K yields a Seifert fibered manifold whose exceptional fibers have indices $p, q, |\varepsilon pq - (pq+(p+q)^2 n)|$ (Claim 6·2(1)); the last index has to equal $|n|$ by Proposition 5·1. We then obtain $n = \frac{(\varepsilon-1)pq}{(p+q)^2 \pm 1}$. Since $(p+q)^2 \pm 1 = p^2+q^2+2pq \pm 1 > 2pq \geqslant |(\varepsilon-1)pq| \geqslant 0$, the integer n has to be zero. If $K = T_{\varepsilon p,n}$, then $|\varepsilon pn - (pq+(p+q)^2 n)| = q$. Thus $n = \frac{q(p \pm 1)}{-p^2-q^2-2pq+\varepsilon p}$. Since $|-p^2-q^2-2pq+\varepsilon p| = p^2+q^2+2pq-\varepsilon p > 2pq > q(p \pm 1) > 0$, we have $0 < |n| < 1$. This is absurd. A similar argument leads to a contradiction also when $K = T_{\varepsilon q,n}$.

LEMMA 6·5. *A twisted torus knot $K(p,q,p+q,n)$ is a hyperbolic knot for $|n| > 3$.*

Proof. By Lemma 6·4, $K = K(p,q,p+q,n)$ is not a torus knot, so it is sufficient to show that K is not a satellite knot. Let $\tau, V, K_{p,q}$ be as in the definition of twisted torus knots. We denote $S^3 - \mathrm{int}N(\tau \cup K(p,q,p+q,0)) = V - \mathrm{int}N(K_{p,q})$ by M, which is hyperbolic by [**20**, claim 9·2]. Recall that $S^3 - \mathrm{int}N(K(p,q,p+q,n))$ is the result of $-1/n$-Dehn filling M along ∂V, and denote this manifold by $M(-1/n)$. If $n = 0$, $K(p,q,p+q,0)$ is a torus knot, so $M(1/0)$ contains an essential annulus. If $K(p,q,p+q,n)$ is a satellite knot, then $M(-1/n)$ contains an essential torus. Then [**13**, theorem 1·3] shows $|n| \leqslant 5$. Furthermore, [**16**, theorem 1·1] shows that if $|n| = 5$, then M is homeomorphic to the exterior of the Whitehead sister link, and that if $|n| = 4$, then M is homeomorphic to the exterior of the Whitehead link or the 2-bridge link parametrized by $3/10$. Recall that the linking number of τ and $K_{p,q}$ is $p+q (\geqslant 5)$. On the other hand, the linking number of the Whitehead sister link, the Whitehead link, and the 2-bridge link associated to $3/10$ are 5, 0, and 3, respectively. Thus the claim below, which asserts the invariance of linking numbers for link exteriors, shows that except for the case $p+q = 5$ and $|n| = 5$, $K(p,q,p+q,n)$ cannot be a satellite knot for $|n| > 3$.

CLAIM 6·6. *Let $L = K_1 \cup K_2$ and $L' = K_1' \cup K_2'$ be two links in S^3. If $S^3 - \mathrm{int}N(L)$ and $S^3 - \mathrm{int}N(L')$ are homeomorphic, then the linking number of K_1 and K_2 coincides with that of K_1' and K_2', up to sign.*

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Proof of Claim 6·6. Let w (resp. w') be the linking number of K_1 and K_2 (resp. K'_1 and K'_2). We denote a preferred meridian-longitude pair of K_i (resp. K'_i) by (μ_i, λ_i) (resp. (μ'_i, λ'_i)). Let $X = S^3 - \mathrm{int}N(K_1 \cup K_2)$ and $X' = S^3 - \mathrm{int}N(K'_1 \cup K'_2)$. For simplicity we denote the first homology classes represented by μ_i, λ_i (resp. μ'_i, λ'_i) in $H_1(X)$ (resp. $H_1(X')$) by the same symbols. Then $H_1(X) \cong \mathbb{Z}\langle\mu_1\rangle \oplus \mathbb{Z}\langle\mu_2\rangle$, $H_1(X') \cong \mathbb{Z}\langle\mu'_1\rangle \oplus \mathbb{Z}\langle\mu'_2\rangle$, and $\lambda_i = w\mu_{3-i}$ and $\lambda'_i = w'\mu'_{3-i}$ for $i = 1, 2$. We may assume, without loss of generality, that a homeomorphism $f : X \to X'$ satisfies $f(\partial N(K_i)) = \partial N(K'_i)$ for each i. Then, for some integers p, q, r, s with $|ps - qr| = 1$ we have:

$$\begin{cases} f_*(\mu_1) = p\mu'_1 + q\lambda'_1 = p\mu'_1 + qw'\mu'_2 \\ f_*(\lambda_1) = r\mu'_1 + s\lambda'_1 = r\mu'_1 + sw'\mu'_2. \end{cases}$$

On the other hand, $f_*(\lambda_1) = wf_*(\mu_2)$, so r and sw' are multiples of w. Since $|ps - qr| = 1$, $(r, s) = 1$. Thus we have w' is a multiple of w. Applying the same argument to f_*^{-1}, we see that w is a multiple of w'. Hence $w = \pm w'$, as desired.

We return to the proof of Lemma 6·5. Let us assume that $p + q = 5$. Since $q > p \geqslant 2$, it follows $p = 2$ and $q = 3$. First let $n = 5$. By Proposition 5·1, $(K(2, 3, 5, 5); 131)$ is a Seifert fibered manifold with base orbifold $S^2(2, 3, 5)$. Let us assume that $K = K(2, 3, 5, 5)$ is a satellite knot; then by Claim 6·3, $K = T_{2\varepsilon,3}(6\varepsilon s \pm 1, s)$, $T_{2\varepsilon,5}(10\varepsilon s \pm 1, s)$, or $T_{3\varepsilon,5}(15\varepsilon s \pm 1, s)$. It follows from Claim 6·3 that $(K; 131) \cong (T_{2\varepsilon,3}; 131/s^2)$, $(T_{2\varepsilon,5}; 131/s^2)$, or $(T_{3\varepsilon,5}; 131/s^2)$. For these Seifert fibered manifolds, the indices of the "third cone points" of their base orbifolds are $|6\varepsilon s^2 - 131|$, $|10\varepsilon s^2 - 131|$, and $|15\varepsilon s^2 - 131|$ by Claim 6·2; these numbers cannot be 5, 3, 2 for any integer s, respectively. This shows that $K(2, 3, 5, 5)$ is not a satellite knot, and hence a hyperbolic knot. Now let $n = -5$. Then $(K(2, 3, 5, -5); -119)$ is a Seifert fibered manifold with base orbifold $S^2(2, 3, 5)$. Applying arguments similar to those above, we conclude that $K(2, 3, 5, -5)$ is also a hyperbolic knot. This completes the proof.

Proof of Theorem 1·3. Let (n_1, n_2, n_3) $(n_i \geqslant 2)$ be any triple such that n_i and n_j are coprime for some i, j. Without loss of generality, we may assume that $(n_1, n_2) = 1$ and $n_1 < n_2$. If $n_3 > 3$, $K(n_1, n_2, n_1 + n_2, n_3)$ is hyperbolic by Lemma 6·5, and yields a Seifert fibered manifold with base orbifold $S^2(n_1, n_2, n_3)$ by Dehn surgery (Proposition 5·1).

Suppose that $n_3 \leqslant 3$. First assume that both n_1 and n_2 are also less than or equal to 3. Then the triples are $(2, 3, 2)$, $(2, 3, 3)$. We deal with these cases later. Then assume that at least one of n_1, n_2 is greater than 3, so that $n_2 > 3$. If $(n_1, n_3) = 1$, then $K(n_1, n_3, n_1 + n_3, n_2)$ is the required hyperbolic knot by Proposition 5·1. So suppose that $(n_1, n_3) \neq 1$. If $(n_2, n_3) \neq 1$, then since $n_3 = 2, 3$, both n_1 and n_2 are multiples of $n_3 \geqslant 2$. This contradicts the assumption $(n_1, n_2) = 1$. We hence obtain $(n_2, n_3) = 1$. If $n_1 > 3$, then $K(n_3, n_2, n_2 + n_3, n_1)$ is the required hyperbolic knot. Thus we are led to the case where $(n_1, n_3) \neq 1, n_2 > 3, n_3 \leqslant 3$, and $n_1 \leqslant 3$. This implies that $(n_1, n_2, n_3) = (2, x, 2),\ (3, x, 3)$, where $x > 3$.

The claim below takes care of these remaining triples, and completes the proof of Theorem 1·3.

Claim 6·7. *The twisted torus knots in the table below are hyperbolic, and yield the indicated triples as indices of cone points of base orbifolds by the indicated Dehn surgeries.*

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triple	twisted torus knot	surgery coefficient
$(2,2,3)$	$K(2,3,5,-2)$	-44
$(2,3,3)$	$K(2,3,5,3)$	81
$(2,2,x)$ $(3<x)$	$K(2,x,2+x,2)$	$2x+2(x+2)^2$
$(3,3,x)$ $(3<x\neq 8)$	$K(3,x,3+x,3)$	$3x+3(x+3)^2$
$(3,3,8)$	$K(3,8,11,-3)$	-339

Proof. By Lemma 6·4 the twisted torus knots given in the table are not torus knots. First we show that $K=K(2,3,5,-2)$ is in fact a hyperbolic knot. Assume not for a contradiction. Then K is a satellite knot, and thus by Claim 6·3 K is $T_{2\varepsilon,3}(6\varepsilon s\pm 1,s)$ and $(K;-44)\cong(T_{2\varepsilon,3};-44/s^2)$. Comparing the indices of cone points of the base orbifolds of $(K;-44)$ and $(T_{2\varepsilon,3};-44/s^2)$, we obtain $|6\varepsilon s^2+44|=2$ for some integer s. But this does not occur. Similar computations show that $K(2,3,5,3)$ and $K(3,8,11,-3)$ are hyperbolic knots.

Now we show that $K=K(2,x,2+x,2)$ $(x>3)$ is a hyperbolic knot. The proof is essentially the same as above. Assume for a contradiction that K is not hyperbolic. Then Lemma 6·4 and Claim 6·3 imply that K is $T_{2\varepsilon,x}(2\varepsilon xs\pm 1,s)$ for some $s\geqslant 2$, and that $(K;2x+2(2+x)^2)\cong(T_{2\varepsilon,x};(2x+2(2+x)^2)/s^2)$. Thus we have $|2\varepsilon xs^2-2x-2(2+x)^2|=2$. If $\varepsilon=-1$, then this equation gives $xs^2+x+(2+x)^2=1$, which has no positive integral solutions. If $\varepsilon=1$, the equation above implies $x^2-(s^2-5)x+3=0$ or $x^2-(s^2-5)x+5=0$. If $x=x_0>3$ and $s=s_0$ are integral solutions to $x^2-(s^2-5)x+3=0$, then x_0 is a solution to the integral quadratic equation $x^2-({s_0}^2-5)x+3=0$. Hence x_0 divides 3, a contradiction to the assumption $x_0>3$.

If $x^2-(s^2-5)x+5=0$, a similar argument shows that the only integral solution satisfying $x>3$ is $x=5$; then $s^2=11$, a contradiction. In the same manner we see that if $K(3,x,3+x,3)$ is not hyperbolic, then the only integral solution satisfying $3<x\neq 8$ is $x=10$ and then $s^2=18$, a contradiction.

In this section, our forcus is on base orbifolds $S^2(n_1,n_2,n_3)$ with two indices coprime. However, there is a Dehn surgery yielding a Seifert fibered manifold whose base orbifold $S^2(n_1,n_2,n_3)$ satisfies $(n_i,n_j)\neq 1$ for any i,j. Such an example can be found in [**6**] and [**4**]. We conclude this paper with the natural question below, which is also asked by Dean [**5**].

Question. Does each orbifold $S^2(n_1,n_2,n_3)$ with $(n_1,n_2,n_3)=1$ occur as the base orbifold of a Seifert fibered manifold obtained by Dehn surgery on a hyperbolic knot?

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