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A note on simplex spaces

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Let X be a real Banach space which is partially ordered by a closed cone C. Let X^* denote the Banach dual space, and let

$$C^* = \{f \in X^* : f(x) \geqslant 0, \forall\, x \in C\}.$$

X is called a *simplex space* (2) if $(X^*, C^*, \|\,.\,\|)$ is an AL-space. We proved in (3) that X is a simplex space if and only if X is an approximate order-unit-normed space and has the Riesz decomposition property. When the given norm on X is an order-unit norm, some representation theorems are obtained by Effros ((2), theorem 4·8) and Størmer ((6), theorem 4·3). In this note, we shall study the representation problem for more general simplex spaces. One of our aims is to prove that X is equivalent to $C_0(\Omega)$ if and only if X has the Urysohn property (for definitions, see below).

Definitions and preliminary results. Let X be a simplex space.

Let $\Sigma = \{x \in X : \|x\| \leqslant 1\}$ and $\Sigma^* = \{f \in X^* : \|f\| \leqslant 1\}$. The *state space* of X is the set $\mathfrak{S}(X) = \{f \in C^* : \|f\| = 1\}$ with the relative w^*-topology. The *pure state space* of X is the set $\mathfrak{P}(X)$ of all extreme points in $\mathfrak{S}(X)$ with the relative w^*-topology. Note that $\Sigma^{*+} = \Sigma^* \cap C^*$ is w^*-compact and convex. It follows from the Krein–Milman theorem that Σ^{*+} is the w^*-closed convex hull of the extreme points $\partial e(\Sigma^{*+})$ of Σ^{*+}. We prove that

$$\partial e(\Sigma^{*+}) = \{0\} \cup \mathfrak{P}(X). \tag{1}$$

It is obvious that $0 \in \partial e(\Sigma^{*+})$. Let $\mu \in \mathfrak{P}(X)$. Suppose that

$$\mu = \alpha\mu_1 + (1-\alpha)\mu_2,$$

with $\mu_1, \mu_2 \in \Sigma^{*+}$ and $0 < \alpha < 1$. Then

$$\|\mu_1\| \leqslant 1, \quad \|\mu_2\| \leqslant 1$$

and

$$1 = \|\mu\| \leqslant \alpha\|\mu_1\| + (1-\alpha)\|\mu_2\|.$$

It follows that $\|\mu_1\| = 1 = \|\mu_2\|$. Thus $\mu_1, \mu_2 \in \mathfrak{S}(X)$. Hence, since μ is extreme in $\mathfrak{S}(X)$, $\mu_1 = \mu = \mu_2$. This shows that $\mu \in \partial e(\Sigma^{*+})$. We have shown that $\{0\} \cup \mathfrak{P}(X) \subseteq \partial e(\Sigma^{*+})$. On the other hand, it is easily seen that each non-zero element of $\partial e(\Sigma^{*+})$ is of norm one. Hence $\partial e(\Sigma^{*+}) \subseteq \{0\} \cup \mathfrak{P}(X)$. Thus (1) is proved.

Let Y and F be subsets of X and X^* respectively. The *annihilator* of Y is denoted by $Y^\perp$ and is defined by

$$Y^\perp = \{f \in X^* : f(y) = 0, \forall\, y \in Y\};$$

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the *subannihilator* of F is denoted by $F_\perp$ and is defined by

$$F_\perp = \{x \in X, f(y) = 0, \forall f \in F\}.$$

An order ideal I is called an *ideal* if $I = I^+ - I^+$, where $I^+ = I \cap C$. If X is a vector lattice then an order ideal is an ideal if and only if it is a sublattice. An ideal M is said to be a *maximal* if $M \neq X$ and M coincides with any proper ideal containing it. The set of all maximal and closed ideals in X is denoted by $\max X$. If $Y \subseteq X$, the *hull* of Y is denoted by $h(Y)$ and is defined by

$$h(Y) = \{M \in \max X \colon Y \subseteq M\}.$$

Note that $\max X = h(\phi)$ and $\phi = h(X)$.

Effros has shown that each hull is the hull of a closed ideal in X. Also, all hulls constitute the closed sets of a topology τ for $\max X$ (cf. (2)). The topological space $(\max X, \tau)$ is called the *structure space* of X.

Urysohn's property. Let Ω be a locally compact Hausdorff space, and let $C_0(\Omega)$ be the partially ordered Banach space of all continuous real-valued functions on Ω which vanish at infinity. Let $\Omega' = \Omega \cup \{\infty\}$ be the one point compactification of Ω (cf. (5)) and let $C_\infty(\Omega')$ be the subspace of $C(\Omega')$ consisting of all functions f with $f(\infty) = 0$. It is evident that we may identify $C_0(\Omega)$ with $C_\infty(\Omega')$.

Let $\Delta \subseteq \Omega$. Let

$$U = \{u \in C_0(\Omega) \colon u(\omega) = 0, \forall \omega \in \Delta\},$$

and let

$$I = \{\omega \in \Omega \colon u(\omega) = 0, \forall u \in U\}.$$

Certainly $\Delta \subseteq I$. We prove that $I \subseteq \Delta$ provided that Δ is closed in Ω. In fact, let $\omega_0 \in \Omega \backslash \Delta$. Then ω_0 and $\Delta \cup \{\infty\}$ are disjoint closed subsets of the compact Hausdorff space Ω'. By Urysohn's lemma, there exists u in $C(\Omega')$ such that:

(i) $0 \leqslant u(\omega) \leqslant 1$ $(\omega \in \Omega')$,

(ii) $u(\omega) = 0$ $(\omega \in \Delta \cup \{\infty\})$ and

(iii) $u(\omega_0) = 1$.

Hence $u \in U$ and $\omega_0 \notin I$. This shows that $I \subseteq \Delta$. We have shown that

$$\Delta = I, \tag{2}$$

whenever Δ is a closed subset of Ω.

As Ω may be identified with the pure state space $\mathfrak{P}(C_0(\Omega))$, the above discussion suggests the following definition.

DEFINITION 1. Let X be a simplex space and let $\mathfrak{P}(X)$ be the pure state space of X. X is said to have the *Urysohn property* if

$$\nabla = \nabla_\perp^+ \cap \mathfrak{P}(X)$$

whenever ∇ is a closed subset of $\mathfrak{P}(X)$.

LEMMA 1. *Let Ω be a locally compact Hausdorff space. Then $C_0(\Omega)$ has the Urysohn property.*

Proof. Let $[C_0(\Omega)]^*$ denote the Banach dual space of $C_0(\Omega)$, and let

$$K = \{\mu \in [C_0(\Omega)]^* \colon \mu \geqslant 0, \|\mu\| \leqslant 1\}.$$

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Let $\mathfrak{S}$ and $\mathfrak{P}$ be the state space and the pure state space of $C_0(\Omega)$ respectively. It is known that $\mathfrak{P}$ and Ω are homeomorphic (cf. (7), Lemmas V. 8·6 and V. 8·7).

Now, by (2), it is not difficult to verify that

$$\nabla = \nabla_{\perp}^{\perp} \cap \mathfrak{P}$$

whenever ∇ is a closed subset of $\mathfrak{P}$. Thus lemma 1 is proved.

In order to prove the converse result of lemma 1, we need the following theorem due to Effros:

THEOREM 1. *Let X be a simplex space with the pure state space $\mathfrak{P}(X)$ and the structure space $(\max X, \tau)$. Then*

(i) *the map $\pi\colon \mathfrak{P}(X) \to \max X$ defined by*

$$\pi(f) = \{x \in X\colon f(x) = 0\} \quad (f \in \mathfrak{P}(X)),$$

is a one-to-one continuous map of $\mathfrak{P}(X)$ onto $\max X$.

(ii) *if $x \in X$ and $\alpha > 0$ then the set*

$$C = \{M \in \max X\colon |\pi^{-1}(M)(x)| \geqslant \alpha\}$$

is τ-compact.

Proof. See (2).

We now prove our main result:

THEOREM 2. *Let X be a simplex space. Then the following statements are equivalent:*

(i) *X has the Urysohn property;*

(ii) *π is a homeomorphism of the pure state space $\mathfrak{P}(X)$ onto the structure space $(\max X, \tau)$;*

(iii) *$\partial e(\Sigma^{*+})$ is w^*-compact;*

(iv) *X is isometrically order-isomorphic to $C_0(\Omega)$ for some locally compact Hausdorff space Ω.*

Proof. (i) $\Rightarrow$ (ii). Let ∇ be a closed subset of $\mathfrak{P}(X)$. Then, by (i), $\nabla = \nabla_{\perp}^{\perp} \cap \mathfrak{P}(X)$. It follows easily that

$$\pi(\nabla) = \{M \in \max X\colon \nabla_{\perp} \subseteq M\} = h(\nabla_{\perp}).$$

Therefore $\pi(\nabla)$ is closed in the structure topology. Hence, since π is one-to-one, π is an open mapping. Combining this with Theorem 1, we see that π is a homeomorphism.

(ii) $\Rightarrow$ (iii). Let $\{G_\lambda, \lambda \in \Lambda\}$ be a family of w^*-open sets such that $\bigcup_{\lambda \in \Lambda} G_\lambda \supseteq \partial e(\Sigma^{*+})$. Since $0 \in \partial e(\Sigma^{*+})$, there exists λ_0 in Λ such that $0 \in G_{\lambda_0}$. Let $x_1, \ldots, x_n$ be elements in X such that

$$0 \in W \equiv \{f \in X^*\colon |f(x_i)| < 1,\ i = 1, 2, \ldots, n\} \subseteq G_{\lambda_0}.$$

By Theorem 1,

$$C_i = \{M \in \max X\colon |\pi^{-1}(M)(x_i)| \geqslant 1\}$$

are compact subsets of the structure space $(\max X, \tau)$. Hence, it follows from (ii) that

$$\pi^{-1}(C_i) = \{f \in \mathfrak{P}(X)\colon |f(x_i)| \geqslant 1\}$$

are compact sets in $\mathfrak{P}(X)$. Consequently $\bigcup_{i=1}^{n} \pi^{-1}(C_i)$ is w^*-compact. Hence there exist

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$\lambda_1, \lambda_2, \ldots, \lambda_m$ in Λ such that $\bigcup_{j=1}^{m} G_{\lambda_j} \supseteq \bigcup_{i=1}^{n} \pi^{-1}(C_i)$. Therefore, since $\partial e(\Sigma^{*+}) = \mathfrak{P} \cup \{0\}$, we have that

$$\bigcup_{j=0}^{m} G_{\lambda_j} \supseteq W \cup \bigcup_{i=1}^{n} \pi^{-1}(C_i) \supseteq \partial e(\Sigma^{*+}).$$

This shows that $\partial e(\Sigma^{*+})$ is w^*-compact.

(iii) $\Rightarrow$ (iv). Let $\mathfrak{P} = \partial e(\Sigma^{*+})\backslash\{0\}$. Then, with the relative w^*-topology, $\mathfrak{P}$ is a locally compact Hausdorff space. It follows from a theorem of Bauer (cf. (1)) that the space $A_0(\Sigma^{*+})$ of all real-valued continuous affine functions on Σ^{*+} vanishing at the origin is isometrically order-isomorphic to $C_0(\mathfrak{P})$. By a result of Effros, the simplex space X is isometrically order-isomorphic to $A_0(\Sigma^{*+})$, and hence to $C_0(\mathfrak{P})$ (cf. (4), theorem 8).

The implication (iv) $\Rightarrow$ (i) is proved in Lemma 1, and the proof of the theorem is complete.

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