Viscous flows with time-dependent free boundaries in a non-planar Hele–Shaw cell

VLADIMIR M. ENTOV[1] and PAVEL I. ETINGOF[2]

[1] *Institute for Problems in Mechanics, Russian Academy of Sciences, prosp. Vernadskogo, 101, Moscow, Russia*
[2] *Harvard University, Department of Mathematics, Cambridge, MA 02138, USA*
(Email: etingof@math.harvard.edu)

(*Received in revised form 24 May 1996*)

The dynamics of blobs of viscous fluid in a non-planar Hele–Shaw cell is considered. The general approach developed by Richardson and some of the resulting analytic techniques are extended to flows in non-planar cells, including cells shaped as surfaces of revolution and helical surfaces. An example related to the development and coalescence of two initially separated blobs in a cell on a spherical surface is presented. Some applications to mathematically equivalent problems dealing with planar Hele–Shaw cells with a non-uniform gap and flows through porous media are also discussed.

1 Introduction

Viscous flows with a time-dependent free boundary in a non-planar Hele–Shaw cell having the shape of an arbitrary two-dimensional surface arise in the mathematical description of pressure moulding of molten metals and polymer materials (Richardson, 1972). For the case of a planar cell, a well developed theory of such flows exists. This theory originated from the papers by Polubarinova-Kochina (1945), Galin (1945), Vinogradov & Kufarev (1947) and Kufarev (1950), dealing mainly with mathematically equivalent flows in porous media, and was further developed by Richardson (1972, 1981) Howison *et al.* (1985) and Howison (1986). Richardson (1972, 1981) showed that these flows are completely integrable in a certain sense, and developed a method of deriving explicit solutions. In the case of a cell shaped after a two-dimensional surface, considered previously by Entov and Etingof (1991), Richardson's approach turns out to have a good modification which is most effective for surfaces of revolution and helical surfaces. In these two cases, the authors developed a special analytic method allowing one to describe the dynamics of the fluid domain explicitly. It also applies when the free boundary is not necessarily connected. This method also applies to time-dependent free boundary flows in a non-homogeneous porous layer or Hele–Shaw cell of a variable thickness. The present paper contains a detailed exposition of the statement and general theory of the problem, a few basic examples, and a brief description of generalizations and applications.

2 Statement of the problem

We will study viscous fluid flows in a non-polar Hele–Shaw cell – a narrow gap between two solid walls. We wish to model a gap of a uniform thickness b and shaped after a two-

https://doi.org/10.1017/S0956792596002938 Published online by Cambridge University Press

dimensional smooth surface Σ in three-dimensional space. (Of course, the shapes of the two walls, are, in general, different from each other, but this difference is negligible if the gap is sufficiently narrow.)

Let ξ and η be local coordinates on the surface σ, and let $\boldsymbol{N}(\xi, \eta)$ be the unit normal to Σ at ξ, η. Let $P(\xi, \eta)$ be the midpoint of the line segment along $\boldsymbol{N}(\xi, \eta)$ whose ends are the points on the two walls of the gap with coordinates (ξ, η). The length of this segment is b, so its midpoint is at the distance $b/2$ from its ends. Any point M within the cell can be described by coordinates (ξ, η, ζ) such that $P(\xi, \eta)$ and M lie on the same normal to Σ, and the signed distance between them is ζ. The solid walls of the cell are then prescribed by the condition $\zeta = \pm b/2$. Let $\boldsymbol{v}(\xi, \eta, \zeta)$ be the fluid velocity field in the cell. It satisfies the continuity equation

$$\operatorname{div} \boldsymbol{v} = 0.$$

Integrating this equation over a cylindrical domain $(\xi, \eta) \in \Delta, |\zeta| \leqslant b/2$, and applying the Gauss formula, taking into account the no-slip condition $\boldsymbol{v}(\xi, \eta, \pm b/2) = 0$, one obtains

$$\int_{-b/2}^{b/2} d\zeta \int_{\partial \Delta} \boldsymbol{v}_{\mathrm{h}}(\xi, \eta, \zeta)_n \, dl = 0,$$

where $\boldsymbol{v}_{\mathrm{h}}(\xi, \eta, \zeta)$ is the component of $\boldsymbol{v}$ parallel to the walls at the point (ξ, η).

Let $\boldsymbol{V}(\xi, \eta) = 1/b \int_{/b/2}^{b/2} \boldsymbol{v}_{\mathrm{h}}(\xi, \eta, \zeta) \, d\zeta$ be the *mean fluid velocity* in the gap, a vector defined on Σ and lying in the tangent plane to Σ at the point (ξ, η). Then from the previous equation, one deduces

$$\int_{\partial \Delta} \boldsymbol{V}_n \, dl = 0.$$

So that

$$\operatorname{div}_{\Sigma} \boldsymbol{V} = 0, \tag{2.1}$$

where $\operatorname{div}_{\Sigma}$ denotes the divergence operator of a vector field on the surface defined by

$$\operatorname{div}_{\Sigma} \boldsymbol{U} = \lim_{\operatorname{Area} \Delta \to 0} \frac{1}{\operatorname{Area}(\Delta)} \int_{\partial \Delta} \boldsymbol{U}_n \, dl.$$

For a sufficiently thin gap, the equation of motion becomes

$$\boldsymbol{V} = -\frac{b^2}{12\mu} \operatorname{grad}_{\Sigma} p, \tag{2.2}$$

where μ is the fluid viscosity, p is the pressure, and $\operatorname{grad}_{\Sigma}$ is the gradient operator corresponding to the surface Σ (it is defined by the condition $\operatorname{grad}_{\Sigma} f \cdot \boldsymbol{U} = \partial f / \partial \boldsymbol{U}$ for every function f and vector field $\boldsymbol{U}$ on the surface).

It is convenient to introduce the velocity potential

$$\Phi = \frac{b^2}{12\mu} p + C \tag{2.3}$$

so that

$$\boldsymbol{V} = \operatorname{grad}_{\Sigma} \Phi. \tag{2.4}$$

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Combining (1.1) and (1.4), one obtains

$$\Delta_\Sigma \Phi = 0, \tag{2.5}$$

where $\Delta_\Sigma = \mathrm{div}_\Sigma \mathrm{grad}_\Sigma$ is the Laplace–Beltrami operator on the surface Σ. It can be defined geometrically by

$$\Delta_\Sigma f(P) = \lim_{\mathrm{Area}(\Delta)\to 0} \frac{1}{\mathrm{Area}(\Delta)} \int_{\partial\Delta} \frac{\partial f}{\partial n} dl, \tag{2.6}$$

where f is a smooth function on Σ, Δ is a small region containing the point P, dl is the length element, and $\partial/\partial n$ is the normal derivative operator on $\partial\Delta$.

Now consider flows with time-dependent free boundary. Let the fluid occupy a simply connected bounded domain $D \subset \Sigma$ with a smooth boundary. We will study the evolution of this domain in time: $D = D(t)$ caused by injection of the fluid into the gap through a finite number of point sources, e.g. small openings in one of the walls. We suppose that outside of D the gap is filled by an inviscid fluid ('air'), so that the pressure is constant there. Let $P_1, P_2, \ldots, P_n \in \Sigma$ be the locations of the sources and $q_1, q_2, \ldots, q_n$ be the respective flow rates expressed as the rates of increase of the fluid domain area due to injection at $P_1, P_2, \ldots, P_n$, respectively. Then the velocity potential has logarithmic behaviour at $P_1, P_2, \ldots, P_n$:

$$\Phi(P, t) \sim \frac{q_j}{2\pi} \ln \rho(P, P_j), \quad P \to P_j, \tag{2.7}$$

$\rho(A, B)$ being the distance between A and B. Let us neglect the capillary effects on the boundary. This implies that at each time t the fluid pressure is constant along the boundary, so we can choose the constant C in (2.3) in such a way that

$$\Phi(P, t) = 0, \quad P \in \partial D. \tag{2.8}$$

Conditions (2.5), (2.7), (2.8) determine $\Phi(P, t)$ uniquely because they define a Dirichlet problem for Laplace's equation. Its solution can be expressed in terms of the Green's function $G_D(P, Q)$ of the domain D with a logarithmic singularity at $Q \in D$:

$$\Phi(P, t) = \sum_{j=1}^{n} q_j G_{D(t)}(P, P_j). \tag{2.9}$$

Equations (2.5), (2.7) can be expressed by one equation using the Dirac function $\delta(P, Q)$:

$$\Delta_\Sigma \Phi = \sum q_j \delta(P, P_j). \tag{2.10}$$

The boundary velocity, i.e. the velocity of the boundary along its normal, is expressed by the formula

$$W(P) = \frac{\partial \Phi(P, t)}{\partial n}, \quad P \in \partial D(t). \tag{2.11}$$

The problem to be solved is to find the family of domains $D(t)$ given the initial domain $D(0)$, the injection points P_j, and the rates q_j. We will refer to it as *the injection problem*.

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Remark It is obvious that all the ingredients of the statement of the problem are essentially independent of the external geometry of the surface Σ, and are determined by its Riemannian metric. Thus, if Σ' is a surface obtained by bending Σ, then the corresponding injection problems on Σ and Σ' are identical. For example, any injection problem on a surface of zero curvature reduces to the corresponding problem in the plane and hence admits a rich set of explicit solutions (Richardson, 1981).

3 Integrals of motion

The following theorem is a straightforward generalization of Richardson's result (Richardson, 1981).

Theorem 1 *Let $D(t)$ be a solution of the injection problem, and let u be a function on Σ such that $\Delta_\Sigma u = 0$. Then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{D(t)} u\,d\sigma = \sum_{j=1}^{n} q_j\,u(P_j), \tag{3.1}$$

$d\sigma$ being the area element on Σ.

Proof Differentiating the integral over the varying domain $D(t)$, one obtains

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{D(t)} u\,d\sigma = \int_{\partial D(t)} uW\,dl = \int_{\partial D(t)} u\frac{\partial \Phi}{\partial n}\,dl. \tag{3.2}$$

Since $\Phi = 0$ on ∂D, one can rewrite (3.2) as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{D(t)} u\,d\sigma = \int_{\partial D(t)} \left(u\frac{\partial \Phi}{\partial n} - \Phi\frac{\partial u}{\partial n}\right) dl. \tag{3.3}$$

Applying Green's formula to the right-hand side of (3.3), one gets

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{D(t)} u\,d\sigma = \int_{D(t)} (u\Delta_\Sigma \Phi - \Phi\Delta_\Sigma u)\,d\sigma. \tag{3.4}$$

But $\Delta_\Sigma u = 0$, and $\Delta_\Sigma \Phi$, by (2.10), is a linear combination of Dirac functions with poles at injection points, so the right-hand side of (3.4) is equal to

$$\sum_{j=1}^{n} q_j\,u(P_j),$$

and we arrive at (3.1).

Integrating (3.1), one gets

$$\int_{D(t)} u\,d\sigma = \int_{D(0)} u\,d\sigma + \sum_{j=1}^{n} q_j\,tu(P_j). \tag{3.5}$$

Thus, we know the integrals over $D(t)$ of all solutions of $\Delta_\Sigma u = 0$ for all times t. This means that we have reduced the injection problem to the problem of reconstruction of a domain from the integrals over it of all solutions of the equation $\Delta_\Sigma u = 0$; using the

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terminology of Richardson (1981), we will call these integrals the (generalized) moments of the domain. We will apply a complex variable technique to derive explicit solutions of the injection problem, with the help of Theorem 1.

First, we need to transfer the problem from the surface to the complex plane. Let U be a bounded simply connected domain on Σ containing $D(t)$ as long as t belongs to a time interval $[0, T)$, and let $z: U \to \boldsymbol{C}$ be a conformal (i.e. preserving angles between smooth curves), one-to-one mapping of U onto a domain in $\boldsymbol{C}$. Such a mapping always exists by the Riemann mapping theorem for surfaces, which says that any simply-connected smooth surface is conformally equivalent to the unit disk, the complex plane, or to the Riemann sphere (see Cartan, 1963).

It is convenient to preserve the notation $D(t)$ for the image of the fluid domain under the mapping z. Let us introduce real coordinates in $U: x = \operatorname{Re} z$, $y = \operatorname{Im} z$. The area element on Σ expresses in these coordinates as

$$d\sigma = g(x, y)\,\mathrm{d}x\,\mathrm{d}y. \tag{3.6}$$

Under a conformal mapping, solutions of the Laplace–Beltrami equation of the surface go to solutions of the Laplace's equation. Hence, identity (3.5) can be rewritten as

$$\int_{D(t)} v(x, y)\, g(x, y)\,\mathrm{d}x\,\mathrm{d}y = \int_{D(0)} v(x, y)\, g(x, y)\,\mathrm{d}x\,\mathrm{d}y + \sum_{j=1}^{n} q_j\, tv(x(P_j), y(P_j)), \tag{3.7}$$

$v(x, y)$ being an arbitrary harmonic function. We have come now to the problem of reconstruction of planar domain D from the values of its generalized moments $\int_D vg\,\mathrm{d}x\,\mathrm{d}y$, where $v(x, y)$ are all possible harmonic functions. This problem is equivalent to the problem of finding a planar domain D, given the (non-homogeneous) Cauchy transform of D:

$$h_D(w) = \frac{1}{\pi} \int_D \frac{g(x, y)\,\mathrm{d}x\,\mathrm{d}y}{w - z}, \quad z = x + iy, \quad w \notin D, \tag{3.8}$$

where w is a complex parameter. This function is holomorphic outside D and vanishes at infinity. In the next section, we describe a method of finding a conformal map of the unit disk onto D by a given function h_D.

Remark The vector field $\overline{h_D(w)}$ (the bar means conjugation) in the complement of D is equal to the gradient of the Newtonian potential of the domain D for the matter density distribution $g(x, y)$. So, the problem of reconstruction of D from h_D is equivalent to the inverse Newtonian potential problem for a given density distribution $g(x, y)$. This problem was studied in Novikov (1938). For a given function h_D, a solution D to this problem is not necessarily unique (Sakai, 1978), but the set of solutions is discrete in a proper sense – every smooth family of solutions must be constant (see Varchenko & Etingof, 1992).

4 Explicit solutions on surfaces of revolution

If Σ is a surface of revolution, then a conformal mapping z can be chosen in such a way that meridians and parallels on Σ go into radial lines and circles centred at the origin, respectively. In this case, g will be a function of $z\bar{z} = |z|^2$:

$$g(x, y) = J(z\bar{z}). \tag{4.1}$$

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Thus, for a domain D on Σ

$$h_D(w) = \frac{1}{2\pi i}\int_D \frac{J(z\bar{z})\,\mathrm{d}\bar{z}\,\mathrm{d}z}{w-z}. \tag{4.2}$$

Assume that the surface Σ is real analytic. Then J is an analytic function of y for $y > 0$. Let

$$I(x) = \int_0^x J(y)\,\mathrm{d}y. \tag{4.3}$$

Then, using Green's formula, from (4.2) one obtains:

$$h_D(w) = \frac{1}{2\pi i}\int_{\partial D} \frac{I(z\bar{z})}{z(w-z)}\mathrm{d}z. \tag{4.4}$$

Suppose that the boundary of D is a regular analytic curve. Let $f\colon K \to D$ be a conformal mapping of the unit disk onto D. Then f can be analytically continued to a neighbourhood of the closed unit disk. The following lemma is the main tool of our treatment.

Lemma *The function*

$$(\zeta) = \frac{I(f(\zeta)\overline{f(1/\bar{\zeta})})}{f(\zeta)} - h_D(f(\zeta))$$

which is analytic in a neighbourhood of the unit circle, can be analytically continued to a holomorphic function inside the unit disk.

Remark If Σ is a plane then obviously $J(x) = 1$ and $I(x) = x$, so $(\zeta) = \overline{f(1/\bar{\zeta})} - h_D(f(\zeta))$, and by the lemma it is holomorphic for $|\zeta| \leqslant 1$. This fact was found and used by Richardson (1981), and became the foundation of his method.

Proof By the Cauchy formula, (4.4) implies

$$\frac{1}{2\pi i}\int_{\partial D} \frac{I(z\bar{z})/z - h_D(z)}{w-z}\mathrm{d}z = 0, \quad w \notin D. \tag{4.5}$$

By the Sokhotsky–Plemelj theorem, it follows that the function $I(z\bar{z})/z - h_D(z)$, defined along ∂D, analytically continues to a holomorphic function in D. Substituting $f(\zeta)$ instead of z and taking into account that the analytic extension of $\overline{f(\zeta)}$ from the unit circles is $\overline{f(1/\bar{\zeta})}$, we get the statement of the lemma. □

Definition *We call a domain D algebraic if $h_D(w)$ is a rational function, and semialgebraic if $d/dw(wh_D(w))$ is rational. Below, a method is given for explicit reconstruction of algebraic and semialgebraic domains from their Cauchy transforms.*

Theorem 2 *The function $h_D(w)$ is rational if and only if $\theta(\zeta) = I(f(\zeta)\overline{f(1/\bar{\zeta})})$ is. Moreover,* deg $\theta = 2\deg(wh_D(w))$, deg *being the number of poles.*

Proof Since $wh_D(w)$ is holomorphic outside D and at infinity, it is rational if and only if it has no singularities inside D but poles of finite order, or, in other terms, if $f(\zeta)\,h_D(f\zeta))$

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has no singularities other than poles inside the unit disk. By the lemma, this occurs if and only if $\theta(\zeta)$ does not have any singularities there except poles. But $\theta(\zeta) = \overline{\theta(1/\bar{\zeta})}$, so if $\theta(\zeta)$ has no singularities but poles inside the disk, then all its singularities are poles: to each singularity of θ outside K there corresponds a singularity of the same type inside K, at the symmetric point. In this case, $\theta(\zeta)$ is rational, and $\deg\theta$ is obviously equal to $2\deg(wh_D(w))$. □

Theorem 3 *The function $d[wh_D(w)]/dw$ is rational if and only if $d/d\zeta\,(I(f(\zeta)\overline{f(1/\bar{\zeta})}))$ is. In this case the degree of the second function is twice the degree of the first one.*

The proof is analogous to the proof of Theorem 2. Theorems 2 and 3 show that if h_D is a function such that the derivative $d/dw\,(wh_D(w))$ is rational, then one knows $\theta(\zeta)$ up to a finite number of real parameters. Since $J(x) > 0$ for $x > 0$, $I(x)$ is an increasing function, and $f(\zeta)\overline{f(1/\bar{\zeta})}$ can be found from the equation

$$f(\zeta)\overline{f(1/\bar{\zeta})} = I^{-1}(\theta(\zeta)), \tag{4.6}$$

again up to a finite number of parameters.

Taking the logarithm of the two sides of (4.6), one finds

$$\ln f(\zeta) + \ln\overline{f(1/\bar{\zeta})} = \ln I^{-1}(\theta(\zeta)). \tag{4.7}$$

Now the conformal mapping $f(\zeta)$ can be easily found from the Cauchy integral formula. Indeed, if D does not contain 0, the first summand in (4.7) is holomorphic inside the unit circle, while the second summand is holomorphic outside it. Assume that $f(0) \in \boldsymbol{R}^+$ (this condition can always be satisfied by a suitable rotation of the complex coordinate). Then

$$\ln f(\zeta) = \frac{1}{2\pi i}\int_{|z|=1} \frac{\ln I^{-1}(\theta(z))}{z-\zeta}\,dz - \ln f(0), \tag{4.8}$$

or

$$f(\zeta) = \frac{1}{f(0)}\exp\left(\frac{1}{2\pi i}\int_{|z|=1} \frac{\ln I^{-1}(\theta(z))}{z-\zeta}\,dz\right). \tag{4.9}$$

If D does not contain 0, then (4.9) needs a slight modification. Let us suppose that $f(0) = 0$, $f'(0) \in \boldsymbol{R}^+$. We can rewrite (4.7) as

$$\ln\frac{f(\zeta)}{\zeta} + \ln(\overline{f(1/\bar{z})}\,\zeta) = \ln I^{-1}(\theta(\zeta)), \tag{4.10}$$

and then use Cauchy integral formula, similar to (4.9):

$$f(\zeta) = \frac{\zeta}{f'(0)}\exp\left(\frac{1}{2\pi i}\int_{|z|=1} \frac{\ln I^{-1}(\theta(z))}{z-\zeta}\,dz\right),$$

$$f'(0) = \exp\left(\frac{1}{4\pi i}\int_{|z|=1} \frac{\ln I^{-1}(\theta(z))}{z}\,dz\right). \tag{4.11}$$

Now the unknown parameters in the expression of $\theta(\zeta)$ can be found from the conditions provided by the lemma: the principal parts of the functions $h_D(f(\zeta))$ and $I(f(\zeta)\overline{f(1/\bar{\zeta})})/f(\zeta)$

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at their singular points cancel. It is seen from this analysis that the problem of reconstruction of a semialgebraic domain from its Cauchy transform reduces to a finite system of transcendental equations. Let us now return to the injection problem stated in §1.

Theorem 4

1. *If $D(0)$ is an algebraic domain then $D(t)$ is an algebraic domain for all $t\in[0,T)$.*

2. *If $D(0)$ is a semialgebraic domain then $D(t)$ is a semialgebraic domain for all $t\in[0,T)$.*

Proof Put in (3.7) $v(x,y)=1/\pi(w-z)$, $z=x+iy$, with w as a parameter. Then one gets

$$h_{D(t)}(w)=h_{D(0)}(w)+\sum_{j=1}^{n}\frac{q_j}{\pi(w-z_j)},\quad z_j=z(P_j),\ w\notin D \tag{4.12}$$

which immediately implies both statements of the theorem. □

Thus, we have obtained a class of explicit solutions of the injection problem, namely, of solutions starting with semialgebraic domains. Let us now consider some examples of such solutions.

Example 1 Σ is a plane. $J(x)=1$; $I(x)=x$. In the case of an algebraic initial domain $D(0)$, we obtain no new solution: for this case the explicit solutions were worked out long ago (see Polubarinova-Kochina, 1945; Galin, 1945; Vinogradov & Kufarev, 1947; Kufarev, 1950; Richardson, 1981). But if $D(0)$ is a semialgebraic but not an algebraic domain, we get essentially new exact solutions. For instance, consider the domain $D(0)$ with Cauchy transform

$$h_{D(0)}(w)=\frac{1}{w}\left[A+B\ln\frac{w+a}{w-a}\right]. \tag{4.13}$$

It can be shown that such a domain exists, is convex, and is unique in the class of convex domains for sufficiently large $|A/B|$. Let $D(t)$ be the result of injection of fluid into $D(0)$ at a rate q at $z=0$. Then by Theorem 1,

$$h_{D(t)}(w)=\frac{1}{w}\left[A+qt+B\ln\frac{w+a}{w-a}\right]. \tag{4.14}$$

The domains $D(t)$ are semialgebraic, and

$$wh_D(w)=A+qt+B\ln\frac{w+a}{w-a}.$$

This function has two logarithmic branch points at $w=\pm a$. Hence, by the above construction, the function θ for $D(t)$ has the form

$$\theta_t(\zeta)=A+qt+B\ln\frac{(\zeta+\alpha_t)(\zeta+1/\alpha_t)}{(\zeta-\alpha_t)(\zeta-1/\alpha_t)}, \tag{4.15}$$

where $f_t(\alpha_t)=a$, $f_t\colon K\to D(t)$ being the conformal mapping at a time t such that $f(0)=0$ and $f'(0)>0$.

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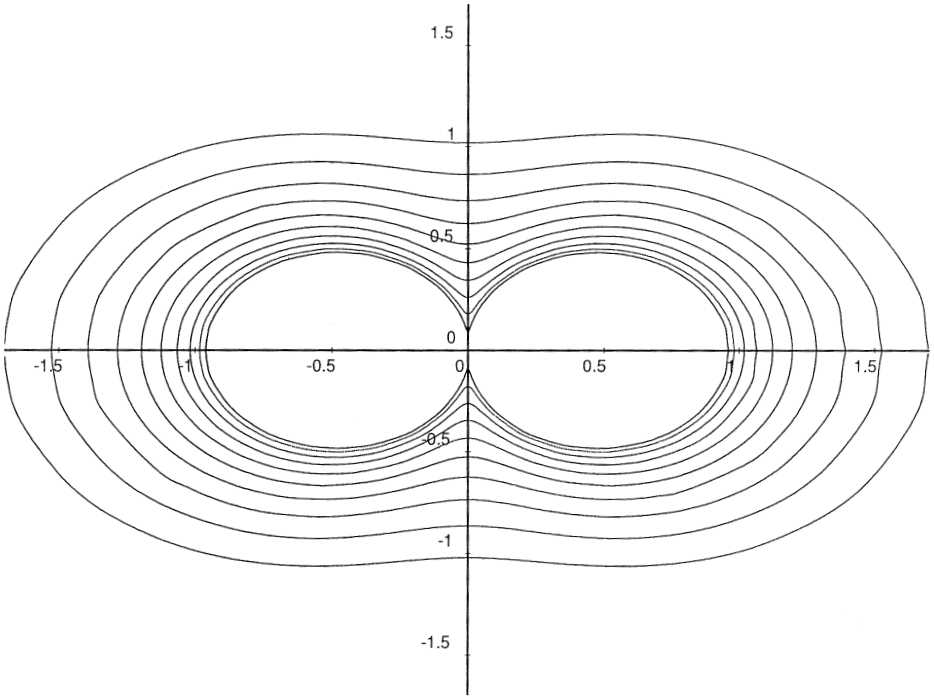


FIGURE 1. Evolution with time and coalescence of two blobs generated by injection at two points on a spherical Hele–Shaw cell.

Now we can apply formula (4.11) to derive the expression for $f_t(\zeta)$:

$$f_t(\zeta) = \zeta \exp\left(\frac{1}{4\pi i}\int_{|z|=1} \frac{z+\zeta}{z(z-\zeta)} \ln\left(A + qt + B\ln\frac{(z+\alpha_t)(z+\alpha_t^{-1})}{(z-\alpha_t)(z-\alpha_t^{-1})}\right) dz\right), \quad |\zeta| < 1. \quad (4.16)$$

In order to determine the boundary of $D(t)$, we need to be able to use this expression for $|\zeta| = 1$. In this case, by Plemelj's theorem, we must take the principal value of the singular integral (4.11) and subtract one half of the value of the integrand:

$$f_t(\zeta) = \zeta \exp\left(\frac{1}{4\pi i}\int_{|z|=1} \frac{z+\zeta}{z(z-\zeta)} - \ln\left(A + qt + B\ln\frac{(z+\alpha_t)(z+\alpha_t^{-1})}{(z-\alpha_t)(z-\alpha_t^{-1})}\right) dz \right.$$
$$\left. - \tfrac{1}{2}\ln\left(A + qt + B\ln\frac{(\zeta+\alpha_t)(\zeta+\alpha_t^{-1})}{(\zeta-\alpha_t)(\zeta-\alpha_t^{-1})}\right)\right).$$

This formula gives exactly the analytic continuation of the integral (4.16) to the circle $|\zeta| = 1$. So, we see that finding the shape of $D(t)$ reduces to solving an equation $f(\zeta, t, \alpha)|_{\zeta=\alpha} = a$, where f is given by an explicit integral formula.

Example 2 Σ is a sphere of radius R. The conformal coordinate z is given by the stereographical projection, $J(x) = 4R^4/(R^2+x)^2$, $I(x) = 4R^2 x/(R^2+x)$.

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Let us consider injection at the same rates at two distinct points of the sphere. The fluid domain $D(t)$ will then consist of two disks corresponding to two caps on the sphere bounded by circles centred at the injection points. It will hold until the two caps collide. After that, it is natural to suppose that Richardson's theorem holds for the further evolution, so that the Cauchy transform of the newborn connected fluid domain $D(t)$ in the appropriate complex coordinate is

$$h_{D(t)}(w) = \frac{qt}{\pi}\left(\frac{1}{w+a}+\frac{1}{w-a}\right), \quad a>0. \tag{4.17}$$

Then the function $\theta_t(\zeta)$ is rational and has the form

$$\theta_t(\zeta) = \frac{-A_t\zeta^2}{(\zeta-\alpha_t)(\zeta-1/\alpha_t)(\zeta+\alpha_t)(\zeta+1/\alpha_t)}, \quad \alpha_t>0. \tag{4.18}$$

Now we can write down the expression for $f_t(\zeta)$ even without using (4.11). Indeed,

$$I^{-1}(\theta_t(\zeta)) = \frac{-A_t R^2\zeta^2}{4R^2(\zeta^2-\alpha_t^2)(\zeta^2-1/\alpha_t^2)-A_t\zeta^2} = \frac{-A_t\zeta^2}{4(\zeta^2-\beta_t^2)(\zeta^2-1/\beta_t^2)}, \tag{4.19}$$

where β_t is defined by the conditions

$$\beta_t^2+\beta_t^{-2} = \alpha_t^2+\alpha_t^{-2}+A_t/4R^2, \quad \beta_t>1.$$

From this equation it immediately follows that

$$f_t(\zeta) = \frac{B_t\zeta}{\zeta^2-\beta_t^2}, \quad B_t = \tfrac{1}{2}\sqrt{(A_t/\beta_t)}, \tag{4.20}$$

and the equation $f(\alpha_t)=a$, together with (4.19) yields

$$B_t = \frac{a(\beta_t^4-1)}{\sqrt{[(1-a^2\beta_t^2R^{-2})(\beta_t^2-a^2R^{-2})]}}. \tag{4.21}$$

The dependence of β_t on time can be found by equating residues of $\theta_t(\zeta)$ and $h_{D(t)}(f_t(\zeta))$ at $\zeta=\alpha_t$.

The solution of the injection problem for $a=0.4$ and $R=1$ is displayed in Fig. 1.

Example 3 Σ is a catenoid, i.e. the surface obtained by rotating the catenary $y=b\cosh(x/b)$ around the x-axis. It has the equation $x_2^2+x_3^2=[b\cosh(x_1/b)]^2$, and it can be easily shown that

$$z = \frac{2(x_2+ix_3)}{b(1+e^{-2x/b})}$$

is a complex conformal coordinate on Σ. In this coordinate,

$$J(x) = \frac{4x^2}{b^2(1+x)^2}, \quad I(x) = \frac{x^2}{x+1}-2\ln(x+1).$$

Let us consider injection at the point P such that $z(P)=1$ (i.e. the point P is on the

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'bottleneck' of the catenoid), at a rate q. Then the Cauchy transform of the fluid domain $D(t)$ is

$$h_{D(t)}(z) = \frac{qt}{\pi(z-1)}, \tag{4.22}$$

and if we suppose that $f_t(0) = 1$, $f_t'(0) > 0$, then

$$\theta_t(\zeta) = C_t(\zeta + \zeta^{-1}) + B_t, \tag{4.23}$$

and by (4.9)

$$f(\zeta) = \exp\left(\frac{1}{2\pi i}\int_{|z|=1} \frac{\ln I^{-1}(C_t(\zeta+\zeta^{-1})+B_t)}{z-\zeta}\,dz\right), \quad |\zeta| < 1, \tag{4.24}$$

where B_t and C_t are found from the equations

$$f(0) = 1, \quad \frac{qt}{\pi f'(0)} = C_t. \tag{4.25}$$

5 Some generalizations and applications

5.1 Flows on helical surfaces

Flows on helical surfaces: $x_2 = r\cos\phi, x_3 = r\sin\phi,\ x_1 = t\phi + f(r)$, can be studied analogously to §4 because a conformal mapping with property (4.1) always exists (it maps helical curves $r = \text{const}$ onto circles $|z| = \text{const}$). However, this map is not one-to-one, and it is more convenient to use the coordinate $w = \ln z$, modifying all the formulas properly. For instance, instead of (4.1) we will have

$$g(x, y) = J(z + \bar{z}), \tag{5.1}$$

and so on.

5.2 Multiply-connected domains

The analysis of §4 helps to construct explicitly the solution of the evolution problem for a multiply connected initial fluid domain on a surface of revolution.

Let us associate to each domain $D \subset \Sigma$ with the boundary consisting of $g+1$ closed curves ($g > 0$) a closed Riemann surface Γ_D of genus g with an antiholomorphic involution $\tau\colon \Gamma_D \to \Gamma_D$ such that there exists a conformal mapping f of the fundamental domain Γ_D^+ of τ (half of Γ_D) onto D. This surface is unique for a given domain up to an isomorphism, and it is called the Schottky double of D. The Schottky double was used in Gustafsson (1983) to reconstruct a multiply connected domain in the plane from its moments. We can apply a similar technique in our case. Indeed, let us put $\theta(\zeta) = I(f(\zeta)\overline{f(\tau(\zeta))})$. Then similarly to the simply connected case, $I(f(\zeta)\overline{f(\tau(\zeta))}) - f(\zeta)\,h_D(f(\zeta))$ analytically continues to a holomorphic function in Γ_D^+. If h_D is rational then $\theta(\zeta)$ is meromorphic in Γ_D^+, and hence, by symmetry, in the whole Γ_D. This property determines $\theta(\zeta)$ uniquely up to a finite number of parameters, and f can be found from a counterpart of (4.9), with integration along the set of fixed points of τ, which reduces the problem to a finite system of transcendental equations. In particular, this generalization allows one to solve explicitly the injection problem for an initially empty cell having the shape of a surface of revolution or a helical surface, and any distribution of injection points and rates.

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5.3 Flows in a nonhomogeneous layer or Hele–Shaw cell of variable thickness

In this section we will treat two analogous problems:

Problem 1. Two-dimensional flows with a time-dependent free boundary in a nonhomogeneous porous layer of variable thickness.

Problem 2. Two-dimensional flows with a time-dependent free boundary in a Hele–Shaw cell of variable thickness.

In these two cases, the evolution of the fluid domain $D(t)$ is determined by the equations

$$\nabla\left(\frac{h(x,y)\,k(x,y)}{\mu}\nabla p(x,y,t)\right) = -\sum_{j=1}^{n}\frac{q_i}{m}\delta(x-x_j, y-y_j), \quad (x,y)\in D(t),$$

$$p(x,y,t) = 0, \quad (x,y)\in \partial D(t),$$

$$W(x,y) = -\frac{k(x,y)}{\mu}\frac{\partial p(x,y,t)}{\partial n}, \quad (x,y)\in \partial D(t), \tag{5.2}$$

where μ is the fluid viscosity, δ is the Dirac function, and W is the boundary velocity.

Here, in Problem 1, $k(x,y)$ is the medium permeability, $h(x,y)$ is the thickness of the layer, and m is the medium porosity. In Problem 2, $h(x,y)$ is the thickness of the cell, $k(x,y) = h(x,y)^2/12$, and $m = 1$.

It can be shown easily using Green's formula that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{D(t)} u(x,y)\,h(x,y)\,\mathrm{d}x\,\mathrm{d}y = \sum_{j=1}^{n}\frac{q_j}{m}u(x_j,y_j), \tag{5.3}$$

for any regular solution $u(x,y)$ of the equation $\nabla(kh\nabla u) = 0$ in a neighbourhood of $D(t)$. This system of relations is the counterpart of the Richardson integrals.

If $\sqrt{[k(x,y)\,h(x,y)]}$ is a harmonic function then a solution of $\nabla(kh\nabla u) = 0$ can be written as $u = (kh)^{-1/2}v$, $v(x,y)$ being any harmonic function. Hence, we can replace (5.3) by

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{\pi}\int_{D(t)}\frac{k(x,y)^{-1/2}\,h(x,y)^{1/2}\,\mathrm{d}x\,\mathrm{d}y}{w-z} = \sum_{j=1}^{n}\frac{q_j(k(x_j,y_j)\,h(x_j,y_j))^{-1/2}}{m\pi(w-x_j-iy_j)}, \tag{5.4}$$

which reduces the problem of finding the solution for a given initial domain $D(0)$ to reconstruction of D from its Cauchy transform

$$\frac{1}{\pi}\int_D \frac{k^{-1/2}h^{1/2}\,\mathrm{d}x\,\mathrm{d}y}{w-z}.$$

This is exactly the problem stated in §3, and in §4 we gave a method of solving it explicitly for $\sqrt{k/h} = f(x^2+y^2)$ or $\sqrt{k/h} = f(x)$ (cf. (5.1)). The solution for $k = Cx^2$, $h = 1$, a single injection point, and $D(0) = 0$ is given in Etingof (1990).

6 Discussion

A non-planar Hele–Shaw cell is an obvious simple hydrodynamic model of a variety of technologies of manufacturing thin-walled parts based on the filling of a thin cavity between two solid walls of a given shape with a molten metal or polymer. The above results,

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while applicable only to a limited class of idealized problems, may prove to be useful in analysing the generic properties of such flows, including motion of fluid fronts, creation and extinction of bubbles, development and evolution of singularities. On the other hand, they may serve as benchmarks for validation of numerous numerical techniques developed for treating these problems. Another interesting option is to use the generalized Richardson theory for developing new direct numerical techniques of computation of the boundary dynamics which would not involve solving the partial differential equations defining the flow field within the fluid blob.

Acknowledgements

The authors thank Professor Richard Beals for reading the manuscript and making valuable suggestions, and the referees for corrections and advice how to improve the paper. The study was partially supported by the Russian Foundation for Fundamental Studies, Grant no. 95-01-00448.

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