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RESEARCH ARTICLE

Compactifications and measures for rational maps

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Abstract

We study extensions of the measure of maximal entropy to suitable compactifications of the parameter space and the moduli space of rational maps acting on the Riemann sphere. For parameter space, we consider a space which resolves the discontinuity of the iterate map. We show that the measure of maximal entropy extends continuously to this resolution space. For moduli space, we consider a space which resolves the discontinuity of the iterate map acting on its geometric invariant theory compactification. We show that the measure of maximal entropy, barycentered and modulo rotations, also extends continuously to this resolution, answering positively a question posed by DeMarco. A main ingredient is a description of limiting dynamics for some sequences.

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1. Introduction

The aim of this paper is to study iterations of rational maps in one complex variable near *degeneration*. The emphasis is on a further exploration of the interplay between *iterate maps* and *measures of maximal entropy*, initiated by DeMarco [2, 3] and continued by DeMarco and Faber [4], Favre [8], Favre and Gong [9] and Okuyama [15, 16].

The space of rational functions $f \in \mathbb{C}(z)$ of degree $d \geq 2$, denoted by Rat_d, is naturally identified with a Zariski open subset of $\mathbb{P}^{2d+1}$. This identification furnishes Rat_d with the compactification $\overline{\mathrm{Rat}}_d \equiv \mathbb{P}^{2d+1}$. Elements of the codimension one algebraic set $\partial\mathrm{Rat}_d := \overline{\mathrm{Rat}}_d \setminus \mathrm{Rat}_d$ are called *degenerate rational maps of degree d*. See §2.2.

For $n \geq 2$, the *n-th iterate map* $\Phi_n : \mathrm{Rat}_d \to \mathrm{Rat}_{d^n}$, sending f to its n-th iterate $f^{\circ n}$, extends to a rational map between the corresponding projective spaces $\overline{\mathrm{Rat}}_d$ and $\overline{\mathrm{Rat}}_{d^n}$. According to DeMarco [2], the map Φ_n has a nonempty indeterminacy locus $I(d) \subset \partial\mathrm{Rat}_d$ which is independent of n. In particular, Φ_n fails to have a continuous extension at any point of $I(d)$. To resolve this situation, it is convenient to consider the embedding

$$\begin{array}{rccc} \Phi : & \mathrm{Rat}_d & \hookrightarrow & \Pi_{n\geq 1}\overline{\mathrm{Rat}}_{d^n} \\ & f & \mapsto & (f, f^{\circ 2}, \cdots). \end{array}$$

This yields a larger compactification of Rat_d:

$$\widehat{\mathrm{Rat}}_d := \mathrm{closure}(\Phi(\mathrm{Rat}_d)) \subset \Pi_{n\geq 1}\overline{\mathrm{Rat}}_{d^n}.$$

Trivially, every iterate map Φ_n extends continuously to $\widehat{\mathrm{Rat}}_d$ as the projection to the n-th coordinate.

On the other hand, every rational map $f \in \mathrm{Rat}_d$ has a unique measure of maximal entropy μ_f, see [10, 12]. This measure is also characterized as the unique probability measure such that

$$\frac{1}{d^n}(f^{\circ n})^*\mu \to \mu_f,$$

for all probability measures μ (see §2.7) having no mass on the exceptional set of f. The support of μ_f is the Julia set of f, denoted by $\mathcal{J}(f)$. Moreover, $\mu_f \in M^1(\overline{\mathbb{C}})$ depends continuously on $f \in \mathrm{Rat}_d$, where $M^1(\overline{\mathbb{C}})$ denotes the space of probability measures endowed with the weak*-topology, see [13].

Given $g \in \partial\mathrm{Rat}_d$, the pull-back operator on (nonexceptional) measures extends continuously to g. More precisely, the *pull-back* g^* can be defined by the property that, for all nonexceptional probability measures μ for g, as $f \in \mathrm{Rat}_d$ converges to g, we have that $f^*\mu$ converges to $g^*\mu$, (see §2.6).

Our first result, extending DeMarco's and DeMarco-Faber's results ([2, Theorem 0.1] and [4, Theorem A]), is about the limiting behavior at $\partial\mathrm{Rat}_d$ of the measure of maximal entropy:

Theorem A. *For any* $\mathbf{g} = (g_n)_{n\geq 1} \in \widehat{\mathrm{Rat}}_d$, *there exists a probability measure* $\mu_{\mathbf{g}} \in M^1(\overline{\mathbb{C}})$ *such that if* $\mu \in M^1(\overline{\mathbb{C}})$ *is nonexceptional for* $\mathbf{g}$, *then as* $n \to \infty$,

$$\frac{1}{d^n}(g_n)^*\mu \to \mu_{\mathbf{g}}.$$

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Moreover, the map

$$\begin{array}{ccc} \widehat{\mathrm{Rat}}_d & \to & M^1(\overline{\mathbb{C}}) \\ \mathbf{g} & \mapsto & \mu_{\mathbf{g}} \end{array}$$

is continuous.

Note that $\overline{\mathrm{Rat}}_d \setminus I(d)$ can be identified with a subset of $\widehat{\mathrm{Rat}}_d$ via the continuous extension of $\Phi : \mathrm{Rat}_d \to \widehat{\mathrm{Rat}}_d$. After this identification, the previous theorem for $\mathbf{g} \in \overline{\mathrm{Rat}}_d \setminus I(d)$ was established by DeMarco in [2, Theorem 0.1].

The limiting behavior of measures at the indeterminacy locus $I(d)$ was further studied by DeMarco and Faber in [4]. They showed that any limit of the measures μ_f as $\mathrm{Rat}_d \ni f \to g \in \partial\mathrm{Rat}_d$ is purely atomic, see [4, Theorem A]. We build on their techniques to prove Theorem A.

Degeneration along holomorphic families can be studied with the aid of non-Archimedean dynamics. A holomorphic family of maps $f_t \in \overline{\mathrm{Rat}}_d$, parameterized by t in a neighborhood of $0 \in \mathbb{C}$, is *degenerate* if $f_t \in \partial\mathrm{Rat}_d$ only for $t = 0$. DeMarco and Faber showed that along any degenerate holomorphic family the measures of maximal entropy converge, see [4, Theorem B]. Moreover, they proved that the limit measure is closely related to an equidistribution measure associated to a non-Archimedean dynamical system. Related approaches studying the limit of measures and the corresponding Lyapunov exponents were implemented by Favre [8], Favre-Cong [9], and Okuyama [15, 16]. It is worth mentioning that, for every degenerate family f_t, there exists $\mathbf{g} = (g_n) \in \widehat{\mathrm{Rat}}_d$ such that $f_t^{\circ n} \to g_n$, as $t \to 0$, for all n. Thus, convergence of the measure of maximal entropy along degenerate holomorphic families can be deduced from Theorem A. Although the proof of Theorem A profits from non-Archimedean dynamics intuition, it does not rely on non-Archimedean tools.

There is a corresponding discussion in the *moduli space*, which was also initiated by DeMarco, in [3]. Consider the quotient $\mathrm{rat}_d := \mathrm{Rat}_d/\mathrm{PSL}(2,\mathbb{C})$ of Rat_d that identifies Möbius conjugate rational maps. Following Silverman [17], its GIT-compactification $\overline{\mathrm{rat}}_d$ is a projective variety and the boundary $\partial\mathrm{rat}_d := \overline{\mathrm{rat}}_d \setminus \mathrm{rat}_d$ consists of the GIT-conjugacy classes of semistable maps in $\overline{\mathrm{Rat}}_d$. For $n \geq 1$, the n-th iterate map $\Psi_n : \mathrm{rat}_d \to \mathrm{rat}_{d^n}$, sending $[f]$ to $[f^{\circ n}]$, extends to a rational map $\overline{\mathrm{rat}}_d \dashrightarrow \overline{\mathrm{rat}}_{d^n}$ with non trivial indeterminacy locus that depends on n, see [3, Theorem 5.1] and [11, Theorem A]. To resolve this situation, as above, we consider

$$\begin{array}{rccc} \Psi : & \mathrm{rat}_d & \hookrightarrow & \Pi_{n\geq 1}\overline{\mathrm{rat}}_{d^n} \\ & [f] & \mapsto & ([f], [f^{\circ 2}], \cdots), \end{array}$$

and let

$$\widehat{\mathrm{rat}}_d := \mathrm{closure}(\Psi(\mathrm{rat}_d)) \subset \Pi_{n\geq 1}\overline{\mathrm{rat}}_{d^n}.$$

Iterate maps Ψ_n extend continuously to $\widehat{\mathrm{rat}}_d$ as the projection to the corresponding coordinate.

In order to consider measures for elements in the moduli space, following DeMarco [3], it is convenient to work with *conformally barycentered probability measures*, see §5. The space of conformally barycentered probability measures modulo the push-forward action by $\mathrm{SO}(3)$ is locally compact. We denote by $\overline{M}^1_{dm}(\overline{\mathbb{C}})$ its one point (= $[\infty]$) compactification.

For any measure $\mu \in M^1(\overline{\mathbb{C}})$ with no atoms of weight $\geq 1/2$, there exists a Möbius transformation such that $\nu = A_*\mu$ is conformally barycentered. Moreover, ν is unique up to push-forward by an element in $\mathrm{SO}(3)$. Thus, we have a well defined class $[\mu] := [\nu] \in \overline{M}^1_{dm}(\overline{\mathbb{C}})$. If μ has an atom of weight $\geq 1/2$, we let $[\mu] := [\infty] \in \overline{M}^1_{dm}(\overline{\mathbb{C}})$. Since the measure of maximal entropy μ_f of any rational map f is atom free, there exists a continuous map

$$\begin{array}{ccc} \mathrm{rat}_d & \to & \overline{M}^1_{dm}(\overline{\mathbb{C}}) \\ f & \mapsto & [\mu_f]. \end{array}$$

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We show this map extends continuously to $\widehat{\mathrm{rat}}_d$, answering positively the question posed by DeMarco in [3, Section 10]:

Theorem B. *For any* $[\mathbf{g}] = ([g_n])_{n\geq 1} \in \widehat{\mathrm{rat}}_d$, *there exists* $x_{[\mathbf{g}]} \in \overline{M}^1_{dm}(\overline{\mathbb{C}})$ *such that if* $\mu \in M^1(\overline{\mathbb{C}})$ *is nonexceptional for* $(g_n)_{n\geq 1}$, *then as* $n \to \infty$,

$$\left[\frac{1}{d^n}(g_n)^*\mu\right] \to x_{[\mathbf{g}]}.$$

Moreover, the map

$$\begin{array}{ccc} \widehat{\mathrm{rat}}_d & \to & \overline{M}^1_{dm}(\overline{\mathbb{C}}) \\ [\mathbf{g}] & \mapsto & x_{[\mathbf{g}]} \end{array}$$

is continuous.

The space $\widehat{\mathrm{rat}}_d$ coincides with the compactification of rat_d constructed by taking the inverse limit of the closure of rat_d in $\Pi_{i=1}^n \overline{\mathrm{rat}}_{d^i}$ under the iterate map $\mathrm{rat}_d \to \Pi_{i=1}^n \overline{\mathrm{rat}}_{d^i}$. On the other hand, by considering the closure of the graph of the map $\mathrm{rat}_d \to \overline{M}^1_{dm}(\overline{\mathbb{C}})$ in $\overline{\mathrm{rat}}_d \times \overline{M}^1_{dm}(\overline{\mathbb{C}})$, DeMarco constructed a compactification $\overline{\overline{\mathrm{rat}}}_d$ of rat_d and established that, in the quadratic case, the compactifications $\widehat{\mathrm{rat}}_2$ and $\overline{\overline{\mathrm{rat}}}_2$ are canonically homeomorphic [3, Theorem 1.1]. However, for higher degrees, the compactifications $\widehat{\mathrm{rat}}_d$ and $\overline{\overline{\mathrm{rat}}}_d$ are not homeomorphic and, in particular, there cannot exist a continuous map $\overline{\overline{\mathrm{rat}}}_d \to \widehat{\mathrm{rat}}_d$ which restricts to the identity on rat_d, see [3, Section 10]. Our Theorem B immediately implies that there exists a continuous map $\widehat{\mathrm{rat}}_d \to \overline{\overline{\mathrm{rat}}}_d$ which restricts to the identity on rat_d. In other words, the GIT limits of the iterates of an unbounded sequence in rat_d determine the limit of the barycentered measures of maximal entropy.

Along a degenerating sequence of rational maps $\{f_k\}$, the choice of coordinates in the domain and in the range become of central importance. The key to Theorems A and B is to analyze the case in which an iterate of f_k uniformly converges to a nondegenerate rational map, modulo changes of coordinates in the domain and the range. We obtain a "Structure Theorem" that describes the limiting dynamics, in this case, see §3. The Structure Theorem yields, as a byproduct of independent interest, Theorem C, stated below.

Recall that a *polynomial-like map of degree d* is a triple (U, V, ψ), where U and V are open topological discs contained in $\overline{\mathbb{C}}$ with U relatively compact in V and $\psi : U \to V$ is a proper holomorphic map of degree d. Any polynomial-like map of degree d is quasi-conformally equivalent to a degree d polynomial, see [6, Theorem 1].

Theorem C. *Let* $\{f_k\}$ *be a sequence in* Rat_d *such that* $\{[f_k]\}$ *converges to an element in* $\partial\mathrm{rat}_d$. *Assume there exist* $\{A_k\}$ *and* $\{B_k\}$ *such that* $B_k \circ f_k^{\circ 3} \circ A_k^{-1}$ *converges in* Rat_{d^3}, *as* $k \to \infty$. *Then for any sufficiently large k, there exist* U_k *and* V_k *such that* (U_k, V_k, f_k) *is a polynomial-like map of degree d. Moreover,* $\mathcal{J}(f_k) \subset U_k$ *and* $\overline{\mathbb{C}} \setminus U_k$ *is contained in a completely invariant attracting Fatou component of* f_k.

In our context, the assumption that $f_k^{\circ 3}$ converges to a nondegenerate map, modulo changes of coordinates in the domain and range, is sharp for the Structue Theorem to hold (see Remark 3.4). However, we do not know if 3 can be replaced by 2 in the statement of Theorem C.

Overview of the Proofs of Theorems A–C

We now outline the main ideas behind the proofs of our principal results.

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To prove Theorem A, given (g_n) in $\widehat{\mathrm{Rat}}_d \setminus \mathrm{Rat}_d$ and a nonexceptional probability measure μ, define

$$\lambda_n := \frac{1}{d^n}(g_n)^*\mu.$$

Consider a sequence $\{f_k\}$ of degree d rational maps such that $\{f_k^{\circ n}\}_k$ converges to the degenerate map $g_n \in \partial\mathrm{Rat}_{d^n}$ for all n. Let μ_k be the equilibrium measure of f_k and assume, after passing to a subsequence, that $\mu_k \to \mu_\infty$.

Theorem A claims that $\{\lambda_n\}$ also converges to μ_∞. We may further assume that $[f_k] \subset \mathrm{rat}_d$ diverge in moduli space, since the nondivergent case is not difficult to understand (see §4.3).

Away from a finite subset of $\overline{\mathbb{C}}$, called the *holes* of g_n, the sequence $\{f_k^{\circ n}\}_k$ converges locally uniformly to a rational map $\tilde{g}_n$ of degree at most $d^n - 1$, called the *reduction* of g_n (see §2.1). The reduction $\tilde{g}_n$ may be constant, and in general one has $\deg \tilde{g}_n = o(d^n)$. Degenerate maps g_n with constant reduction encode little information about the limiting dynamics of $\{f_k\}$ and constitute one of the main difficulties in the study of measure convergence (cf. [2]).

A key idea is to regard the spheres in the domain and range of $f_k^{\circ n}$ as distinct spheres $\overline{\mathbb{C}}_0$ and $\overline{\mathbb{C}}_n$, and to introduce a dynamically meaningful normalization in the range. Indeed, a basic principle of degenerate rational dynamics applies: after passing to a subsequence, there exists a choice of coordinates $A_{n,k}$ in the image sphere such that $A_{n,k} \circ f_k^{\circ n}$ converges to a map $\varphi_n \in \overline{\mathrm{Rat}}_{d^n}$ having *nonconstant reduction* $\tilde{\varphi}_n : \overline{\mathbb{C}}_0 \to \overline{\mathbb{C}}_n$ (Proposition 2.1). The limit φ_n is unique modulo post-composition by a Möbius transformation. If the equilibrium measures μ_k converge to μ_∞ and $(A_{n,k})_*\mu_k$ converges to ν_n, then by DeMarco and Faber [4] (see Proposition 2.10) we have

$$\mu_\infty = \frac{1}{d^n}(\varphi_n)^*\nu_n. \tag{1.1}$$

At this point, the behavior of $\deg \tilde{\varphi}_n$ becomes decisive. We distinguish two cases: $\deg \tilde{\varphi}_n = o(d^n)$ or $\deg \tilde{\varphi}_n \neq o(d^n)$. In the former case, the techniques developed by DeMarco and Faber apply. More precisely, the right-hand side of (1.1) differs from λ_n by a measure of mass at most $\deg \tilde{\varphi}_n/d^n$ (see Proposition 4.3[1]). Therefore $\lambda_n \to \mu_\infty$.

Thus, the proof of Theorem A reduces to the remaining case $\deg \tilde{\varphi}_n \neq o(d^n)$, which requires a new ingredient: the Structure Theorem 3.1. To discuss it, we need to introduce *fully ramified times*.

The maps $\tilde{\varphi}_n : \overline{\mathbb{C}}_0 \to \overline{\mathbb{C}}_{n+1}$ inherit a natural decomposition from iteration. For each n, there exists a rational map $\tilde{\varphi}_{n,n+1} : \overline{\mathbb{C}}_n \to \overline{\mathbb{C}}_{n+1}$ of degree at most d such that $\tilde{\varphi}_{n+1} = \tilde{\varphi}_{n,n+1} \circ \tilde{\varphi}_n$. When $d = \deg \tilde{\varphi}_{n,n+1}$, we say that n is a *fully ramified time*. In the case $\deg \tilde{\varphi}_n \neq o(d^n)$, all sufficiently large times are fully ramified.

The Structure Theorem asserts that the presence of just 3 consecutive fully ramified times, say $m, m+1$ and $m+2$, imposes severe restrictions on the limiting dynamics. More precisely, it forces the spheres $\overline{\mathbb{C}}_m, \overline{\mathbb{C}}_{m+1}, \overline{\mathbb{C}}_{m+2}, \overline{\mathbb{C}}_{m+3}, \ldots$, which a priori form a tree of bouquets of spheres (see Appendix A), to organize into a linear concatenation obtained by identifying $0 \in \overline{\mathbb{C}}_n$ with $\infty \in \overline{\mathbb{C}}_{n+1}$ for all $n \geq m$, in suitable coordinates. Moreover, $\tilde{\varphi}_{n,n+1}$ is a polynomial for all $n > m$, and a monomial whenever both n and $n+1$ are fully ramified times.

In the proof of Theorem A, the Structure Theorem implies that the Julia sets of f_k converge to $\infty \in \overline{\mathbb{C}}_n$ for all n sufficiently large. In particular, $(A_{n,k})_*\mu_k \to \delta_\infty$ (see Corollary 3.15), which provides the key input in §4.4 to complete the argument in the case $\deg \tilde{\varphi}_n \neq o(d^n)$.

The location of the Julia sets of f_k is clarified by extracting polynomial-like maps. The mechanism enabling this extraction is provided by Proposition 3.7, which is also used in the proof of the Structure Theorem to rule out certain configurations of the initial spheres $\overline{\mathbb{C}}_m, \overline{\mathbb{C}}_{m+1}, \overline{\mathbb{C}}_{m+2}$.

In this language, and modulo conjugacy, the hypothesis of Theorem C is that the sequence $\{f_k\}$ has fully ramified times $m = 0, 1, 2$. Its proof relies on Lemmas 3.5 and 3.6, which determine the

[1]In the notation of Proposition 4.3, $\lambda_n = \eta_{g_n}/d^n$ and $\mu_\infty = \nu$.

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configuration of spheres and transition maps associated to an arbitrary number, at least 3, of consecutive fully ramified times. Although to prove Theorem C we only need to study the initial spheres $\overline{\mathbb{C}}_1, \overline{\mathbb{C}}_2, \overline{\mathbb{C}}_3$, the lemmas are proved in greater generality without raising the level of difficulty.

The proof of Theorem B is also based on the Structure Theorem but is more subtle, since the arguments take place in quotient spaces formed by equivalence classes of maps and measures. Let $\{[f_k]\} \subset \mathrm{rat}_d$ be such that $[f_k^{\circ n}]$ converges to $[g_n] \in \partial \mathrm{rat}_{d^n}$ for all $n \geq 1$. When the sequence $\{[\mu_k]\}$, where μ_k is the equilibrium measure of f_k, is bounded away from infinity, Theorem A applies almost directly. Otherwise, the setting of Theorem B leads to a family of sequences $\{f_{n,k}\} \subset \mathrm{Rat}_d$, one for each iterate n. That is, for each n there exists a sequence of coordinate changes $\{B_{n,k}\}$ such that $f_{n,k}^{\circ n}$ converges to g_n, where

$$f_{n,k} := B_{n,k} \circ f_k \circ B_{n,k}^{-1}.$$

Passing to subsequences, for each n there exists $\{A_{n,k}\}$ such that $A_{n,k} \circ f_{n,k}^{\circ n} \to \psi_n$, where $\psi_n \in \overline{\mathrm{Rat}}_{d^n}$ has nonconstant reduction $\tilde{\psi}_n$. At this stage, the basic distinction now arises between the cases $\deg \tilde{\psi}_n \neq o(d^n)$ and $\deg \tilde{\psi}_n = o(d^n)$. In the former case, there exist arbitrarily large n for which $\{f_{n,k}\}$ exhibits 3 consecutive fully ramified times, and the Structure Theorem applies. In the latter case, a more delicate analysis is required, using the associated bouquet of trees of spheres described in Appendix A.

Outline

The paper is organized as follows. In §2, we discuss the basic properties of degenerate rational maps including their action on measures via pull-back. In §3, based on the trees of bouquets of spheres from Appendix A, we establish our Structure Theorem for the limiting dynamics of "fully ramified sequences." As a byproduct, we obtain Theorem C. We prove Theorem A in §4 and Theorem B in §5. In Appendix A, we discuss maps acting on trees of bouquet of spheres. This is closely related to the work by Arfeux [1] regarding maps acting on trees of spheres.

2. Degenerate rational maps

A degenerate rational map $f \in \partial \mathrm{Rat}_d$ has a well defined action outside a finite set of points, called the *holes* of f. The action, known as the *reduction* $\tilde{f}$ of f, is a lower degree rational map, possibly constant. Intuitively, the holes "map," under f, onto $\overline{\mathbb{C}}$ with a degree, called its *depth*. The reduction $\tilde{f}$, the set of holes $\mathrm{Hole}(f)$ and their depth $d_h(f)$, completely prescribe the degenerate map f. We discuss these definitions from [2] in §2.1. The parameter space of rational maps, degenerate or not, is discussed in §2.2. A well known formula for the depth of the holes of compositions is written in §2.3. As rational maps converge to a degenerate map, their action is better understood by changing coordinates, or scales, at the domain and the range. In §2.4, we consider the space of rational maps, possibly degenerate but with nonconstant reduction, modulo post-composition by Möbius maps. It is a compact space. In §2.5, we present a formula for the depth of holes in the setting of §2.4 as an analogue of a formula obtained by the authors in [11], and also furnish a formula to compute the depth counting preimages. In §2.6, we recall the definitions of the pull-back of measures by a rational map and, following DeMarco [2], by a degenerate rational map. We show that the degenerate pull-back is the continuous extension of the pull-back (Proposition 2.9). In §2.7, we summarize some general properties of the measures of maximal entropy. Continuity of the pull-back of measures yields a key result established by DeMarco and Faber (see [4, Theorem 2.4]), which provides us with an equation satisfied by suitable limits of the measures of maximal entropy.

2.1. Holes and reduction

Although we mostly work in nonhomogeneous coordinates, to define reductions, holes and depths, homogeneous coordinates are convenient. A *rational map* $f : \mathbb{P}^1 \to \mathbb{P}^1$ *of degree* $d \geq 2$ can be written as

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$$f : [z : w] \mapsto [P(z, w) : Q(z, w)],$$

where P, Q are relatively prime degree d homogeneous polynomials. For concreteness, one may write

$$\begin{aligned} P(z, w) &= a_0 w^d + a_1 z w^{d-1} + \cdots + a_d z^d, \\ Q(z, w) &= b_0 w^d + b_1 z w^{d-1} + \cdots + b_d z^d. \end{aligned} \tag{2.1}$$

Then $f = [P : Q]$ is a degree d rational map if and only if P, Q are relatively prime.

A degree d *degenerate rational map* f is an expression $[P : Q]$ such that P and Q are not relatively prime. Consistent with the notation, for all $\lambda \in \mathbb{C} \setminus \{0\}$, we regard $[P : Q]$ and $[\lambda P : \lambda Q]$ as the same map. Let H_f be a common factor of P and Q of maximal degree. Then, there exist polynomials $\tilde{P}$, $\tilde{Q}$, such that

$$P = H_f \tilde{P} \quad \text{and} \quad Q = H_f \tilde{Q}.$$

We say that

$$\mathrm{Hole}(f) := \{h \in \mathbb{P}^1 : H_f(h) = 0\}$$

is the *set of holes of* f. The *depth* $d_h(f)$ of a hole h is the multiplicity of h as a root of H_f; that is, modulo multiplication by a constant,

$$H_f = \prod_{h \in \mathrm{Hole}(f)} L_h^{d_h(f)},$$

where L_h is a homogeneous linear form vanishing at h; moreover, for notational convenience, we let $d_h(f) := 0$ whenever $h \in \mathbb{P}^1 \setminus \mathrm{Hole}(f)$. We say that

$$\tilde{f} := [\tilde{P} : \tilde{Q}]$$

is the *reduction of* f. Note that $\tilde{f}$ is a well defined self-map of $\mathbb{P}^1$. We write $f = H_f \cdot \tilde{f}$.

Given a nondegenerate rational map $\tilde{f}$ of degree $\deg \tilde{f} < d$ and a set of holes $\mathrm{Hole}(f)$ with prescribed depths such that the total number of holes, counted with depths, is $d - \deg \tilde{f}$, there exists a unique degree d degenerate map f compatible with this information.

For notational simplicity, we mostly work with nonhomogeneous coordinates, after identification of $\mathbb{P}^1$ with $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$. Thus, we write $f(z) = P(z, 1)/Q(z, 1)$ or simply $f(z) = P(z)/Q(z)$.

2.2. Spaces of rational maps

The space of possibly degenerate rational maps $\overline{\mathrm{Rat}}_d$ is the $(2d + 1)$-dimensional complex projective space $\mathbb{P}^{2d+1}$. Its elements are identified with expressions of the form $f = [P : Q]$ by

$$\begin{array}{ccc} \overline{\mathrm{Rat}}_d & \leftrightarrow & \mathbb{P}^{2d+1} \\ f = [P : Q] & \leftrightarrow & [a_0 : \cdots : a_d : b_0 : \cdots : b_d], \end{array}$$

where P, Q are polynomials of the form (2.1).

The degree d homogeneous resultant of P, Q is a homogeneous polynomial in the coefficients. Its vanishing locus $\partial\mathrm{Rat}_d := \{\mathrm{Res} = 0\}$ is the set of degenerate maps in $\overline{\mathrm{Rat}}_d$. Hence, the Zariski open subset $\mathrm{Rat}_d := \overline{\mathrm{Rat}}_d \setminus \partial\mathrm{Rat}_d$ is the space of rational maps of degree d. Convergence in $\mathrm{Rat}_d \subset \mathbb{P}^{2d+1}$ coincides with uniform convergence of rational maps.

Möbius transformations form the space Rat_1 of degree 1 rational maps that is naturally identified with $\mathrm{PSL}(2, \mathbb{C})$. Each element of $\partial\mathrm{Rat}_1$ is a degenerate Möbius transformation: a map with constant reduction and one hole, counted with depth.

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2.3. Composition of degenerate maps

Given $d, d' \geq 1$, consider the map

$$\begin{aligned} \circ : \mathrm{Rat}_d \times \mathrm{Rat}_{d'} &\to \mathrm{Rat}_{dd'} \\ (f, g) &\mapsto f \circ g. \end{aligned}$$

It extends to a rational map from the projective variety $\overline{\mathrm{Rat}}_d \times \overline{\mathrm{Rat}}_{d'}$ into $\overline{\mathrm{Rat}}_{dd'}$ with indeterminacy locus

$$I(d, d') := \{(f, g) : \tilde{g} \equiv h \in \mathrm{Hole}(f)\}.$$

Following [3, Lemma 2.6], for all $(f, g) \in \overline{\mathrm{Rat}}_d \times \overline{\mathrm{Rat}}_{d'} \setminus I(d, d')$, we have that $f \circ g$ is the unique element of $\overline{\mathrm{Rat}}_{dd'}$ with reduction $\tilde{f} \circ \tilde{g}$ and holes $h \in \tilde{g}^{-1}(\mathrm{Hole}(f)) \cup \mathrm{Hole}(g)$ with depths

$$d_h(f \circ g) = d_h(g) \cdot d + \deg_h(\tilde{g}) \cdot d_{\tilde{g}(h)}(f). \tag{2.2}$$

2.4. Left classes of rational maps

To deal with degenerations towards maps with constant reduction, the key is the following "scaling" result:

Proposition 2.1 [4, Lemma 2.1]. *Let $\{f_k\}$ be a sequence in Rat_d. Then there exists a subsequence $\{f_{k_i}\}$ and a sequence $\{A_i\}$ in $\mathrm{PSL}(2, \mathbb{C})$ such that*

$$A_i \circ f_{k_i} \to \varphi \in \overline{\mathrm{Rat}}_d,$$

and $\tilde{\varphi}$ is not constant. Moreover, if $\{B_i\}$ is another such sequence, then $\{B_i^{-1} \circ A_i\}$ converges in $\mathrm{PSL}(2, \mathbb{C})$.

This result may be understood as the compactness of certain orbit space. Let us denote the maps with nonconstant reduction by

$$\mathrm{Rat}_d^* := \{f \in \overline{\mathrm{Rat}}_d : \deg \tilde{f} \neq 0\}.$$

Then post-composition by $\mathrm{PSL}(2, \mathbb{C})$ leaves Rat_d^* invariant. An equivalent formulation of the above proposition is the following:

Corollary 2.2. *Each $\mathrm{PSL}(2, \mathbb{C})$-orbit, under post-composition, is closed in Rat_d^*. The quotient topological space $[\![\mathrm{Rat}_d^*]\!] := \mathrm{PSL}(2, \mathbb{C}) \backslash \mathrm{Rat}_d^*$ is compact.*

Remark 2.3. In fact, Rat_d^* is the set of GIT-stable elements of $\overline{\mathrm{Rat}}_d$, under post-composition by $\mathrm{PSL}(2, \mathbb{C})$. There are no strictly semi-stable elements for this action. Hence, $[\![\mathrm{Rat}_d^*]\!]$ is a geometric quotient which is, naturally, a projective variety.

We denote the $\mathrm{PSL}(2, \mathbb{C})$-orbit of $f \in \mathrm{Rat}_d^*$ under the post-composition action by

$$[\![f]\!] \in [\![\mathrm{Rat}_d^*]\!].$$

Let $\{f_k\} \subset \mathrm{Rat}_d$ be a sequence such that $[\![f_k^{\circ n}]\!]$ converges to $[\![\varphi_n]\!]$ in $[\![\mathrm{Rat}_{d^n}^*]\!]$ for each $n \geq 1$. Omitting the dependence on the choice of φ_n, we say that $\{A_{n,k} : n, k \geq 1\} \subset \mathrm{Rat}_1$ is a *collection of scalings for* $\{f_k\}$ if for all $n \geq 1$, as $k \to \infty$,

$$A_{n,k} \circ f_k^{\circ n} \to \varphi_n.$$

The following result is similar to [4, Lemma 2.5].

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Lemma 2.4. *Assume that* $[\![f_k^{\circ n}]\!]$ *converges to* $[\![\varphi_n]\!]$ *in* $[\![\mathrm{Rat}_{d^n}^*]\!]$ *for each* $n \geq 1$. *Then for all* $n \geq 1$, *there exist* $\varphi_{n,n+1} \in \mathrm{Rat}_d^*$ *such that*

$$\varphi_{n+1} = \varphi_{n,n+1} \circ \varphi_n.$$

Moreover, if $\{A_{n,k}\}$ *is a collection of scalings for* $\{f_k\}$, *then*

$$A_{n+1,k} \circ f_k \circ A_{n,k}^{-1} \to \varphi_{n,n+1}.$$

Proof. For any $n \geq 1$, let $\{A_{n,k}\} \subset \mathrm{Rat}_1$ be a collection of scalings for $\{f_k\}$. Modulo passing to a subsequence, by Proposition 2.1 applied to $\{f_k \circ A_{n,k}^{-1}\}$, there exists $\{B_{n+1,k}\}$ such that, as $k \to \infty$,

$$B_{n+1,k} \circ f_k \circ A_{n,k}^{-1} \to \psi_{n,n+1},$$

for some $\psi_{n,n+1} \in \mathrm{Rat}_d^*$. Continuity of composition (§2.3) implies that $B_{n+1,k} \circ f_k^{\circ n+1} \to \psi_{n,n+1} \circ \varphi_n$. Again by Proposition 2.1, we conclude that $A_{n+1,k} \circ B_{n+1,k}^{-1} \to M_{n+1}$, for some Möbius transformation $M_{n+1} \in \mathrm{Rat}_1$. Let $\varphi_{n,n+1} := M_{n+1} \circ \psi_{n,n+1}$. Thus, a map $\varphi_{n,n+1} \in \overline{\mathrm{Rat}}_d$ as in the statement of the Lemma exists.

Note that any accumulation point $\varphi \in \overline{\mathrm{Rat}}_d$ of $A_{n+1,k} \circ f_k \circ A_{n,k}^{-1}$ satisfies that $\varphi \circ \varphi_n = \varphi_{n+1}$. To finish the proof, it suffices to show that φ is uniquely determined by this property, that is, $\varphi = \varphi_{n,n+1}$. Indeed, surjectivity of $\tilde{\varphi}_n$ yields that $\tilde{\varphi} = \tilde{\varphi}_{n,n+1}$. Moreover, the depth $d_h(\varphi)$, for any $h \in \overline{\mathbb{C}}$, is completely determined by the equation

$$d_h(\varphi_{n+1}) = d_h(\varphi_n) \cdot d + \deg_h \tilde{\varphi}_n \cdot d_{\tilde{\varphi}_n(h)}(\varphi).$$

Therefore, $\varphi = \varphi_{n,n+1}$. □

2.5. Depth formulas

The following result is the complex analytic analogue of [11, Lemma 2.9].

Lemma 2.5. *Let* $\{f_k\} \subset \mathrm{Rat}_d$ *be a sequence converging to* $g \in \partial\mathrm{Rat}_d$ *with constant reduction. Assume that* $[\![f_k]\!] \to [\![\varphi]\!]$. *Then there exists* $a \in \overline{\mathbb{C}}$ *such that*

$$\mathrm{Hole}(g) = \mathrm{Hole}(\varphi) \cup \tilde{\varphi}^{-1}(a).$$

Moreover,

$$d_h(g) = \begin{cases} d_h(\varphi), & \text{if } h \notin \tilde{\varphi}^{-1}(a), \\ d_h(\varphi) + \deg_h \tilde{\varphi}, & \text{otherwise.} \end{cases}$$

Furthermore, if $A_k \circ f_k \to \varphi$ *in* Rat_d^*, *then* $A_k \to A \in \partial\mathrm{Rat}_1$ *with* $\tilde{A} = a$ *and* $\mathrm{Hole}(A) = \{c\}$ *where* $\tilde{g} \equiv c$.

Proof. Consider a sequence $\{A_k\} \subset \mathrm{Rat}_1$ such that $A_k \circ f_k \to \varphi$. Passing to a subsequence, suppose $A_{k_i} \to A \in \partial\mathrm{Rat}_1$ and write $a := \tilde{A}$. Then $\mathrm{Hole}(A) = \{c\}$ where $\tilde{g} \equiv c$; for otherwise $A_{k_i} \circ f_{k_i}(z)$ would converge to a outside $\mathrm{Hole}(\varphi)$, and hence $\tilde{\varphi}$ would be constant, contrary to our hypothesis.

It follows that $A_{k_i}^{-1}$ converges to some $B \in \partial\mathrm{Rat}_1$ with reduction c and hole a, that is, $H_B(z) = z - a$. Hence,

$$A_{k_i}^{-1} \circ (A_{k_i} \circ f_{k_i}) \to (H_B \circ \tilde{\varphi}) H_\varphi \cdot c = g.$$

Thus, the first two assertions hold. Since a is independent of the subsequence, the last assertion also holds. □

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Given $g \in \partial \mathrm{Rat}_d$, we define the *algebraically exceptional set* $Ex_g \subset \overline{\mathbb{C}}$ by

$$Ex_g := \begin{cases} \{\tilde{g}\} & \text{if } \deg \tilde{g} = 0, \\ \emptyset & \text{if } \deg \tilde{g} \geq 1. \end{cases}$$

The next lemma provides a way to compute depths by counting preimages of nonexceptional points. It states precisely the sense in which holes "map onto $\overline{\mathbb{C}}$" by a degenerate map (c.f. [2, Lemmas 4.5 and 4.6]).

Lemma 2.6. *Let $\{f_k\} \subset \mathrm{Rat}_d$ be a sequence converging to $g \in \partial \mathrm{Rat}_d$ and assume that $[\![f_k]\!] \to [\![\varphi]\!]$. Pick $z_0 \in \overline{\mathbb{C}}$. Then for any $w \in \overline{\mathbb{C}}$, there exists an arbitrarily small neighborhood U of w, such that the following hold for all sufficiently large k:*

1. *If $z_0 \notin Ex_g$, then, counting with multiplicities,*

$$\#(f_k^{-1}(z_0) \cap U) = \begin{cases} d_w(g) & \text{if } \tilde{g}(w) \neq z_0, \\ d_w(g) + \deg_w \tilde{g} & \text{if } \tilde{g}(w) = z_0. \end{cases}$$

2. *Counting with multiplicities,*

$$\#(f_k^{-1}(z_0) \cap U) \leq d_w(\varphi) + \deg_w \tilde{\varphi} \leq d_w(g) + \deg_w \tilde{\varphi}.$$

Proof. Let $P, Q \in \mathbb{C}[z]$ be two degree d polynomials such that $g = P/Q$. Let H be a common divisor of P and Q with maximal degree. Then $P = H\tilde{P}$, $Q = H\tilde{Q}$ and $\tilde{g} = \tilde{P}/\tilde{Q}$ for some relatively prime polynomials $\tilde{P}, \tilde{Q}$. If $z_0 \notin Ex_g$, we have that $\tilde{g} \not\equiv z_0$. Hence, the multiplicity of w as a solution of

$$0 = P - z_0 Q = H(\tilde{P} - z_0 \tilde{Q})$$

is $d_w(g)$ if $\tilde{g}(w) \neq z_0$, and $d_w(g) + \deg_w \tilde{g}$ if $\tilde{g}(w) = z_0$. Thus writing $f_k = P_k/Q_k$ and observing that the number of solutions of $P_k - z_0 Q_k$ in a small neighborhood of w converge to the multiplicity of w as a solution of $P - z_0 Q = 0$, we obtain statement (1).

In view of the above, for statement (2), it suffices to consider the case where $\deg \tilde{g} = 0$. Let $\{A_k\} \subset \mathrm{Rat}_1$ such that $A_k \circ f_k \to \varphi$. Passing to a subsequence in k, we can assume that $A_k(z_0) \to a$. Then given a small neighborhood U of w, we have that, for all sufficiently large k,

$$\#(f_k^{-1}(z_0) \cap U) = \#((A_k \circ f_k)^{-1}(A_k(z_0)) \cap U) = \begin{cases} d_w(\varphi) + \deg_w \tilde{\varphi} & \text{if } a = \tilde{\varphi}(w), \\ d_w(\varphi) & \text{if } a \neq \tilde{\varphi}(w), \end{cases}$$

counted with multiplicities. Then statement (2) follows from Lemma 2.5. □

2.6. *Pull-back of measures*

Given a continuous function $\varphi : \overline{\mathbb{C}} \to \mathbb{R}$ and a rational map $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, the *push-forward* of φ by f is the continuous function:

$$f_*\varphi(z) := \begin{cases} \sum_{w \in f^{-1}(z)} \deg_w f \cdot \varphi(w) & \text{if } \deg f \geq 1, \\ 0 & \text{if } \deg f = 0. \end{cases}$$

Clearly, f_* is a nonnegative bounded linear operator on the space of continuous function with the sup-norm. Its dual is the *pull-back* f^*, whose action on a (positive Radon) measure μ in $\overline{\mathbb{C}}$ is determined by the property that

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$$\int \varphi(w) df^*\mu(w) = \int f_*\varphi(z) d\mu(z),$$

for all continuous functions φ. Taking $\varphi \equiv 1$, yields $f^*\mu(\overline{\mathbb{C}}) = \deg f \cdot \mu(\overline{\mathbb{C}})$. All measures considered here are nonnegative Radon measures.

For any $g \in \partial \mathrm{Rat}_d$, we can obtain a natural measure of total mass $d - \deg \tilde{g}$ induced by the depths of its holes:

Definition 2.7. Given $g \in \partial \mathrm{Rat}_d$, we say that the measure

$$\eta_g := \sum_{h \in \mathrm{Hole}(g)} d_h(g)\delta_h$$

is the *depth measure* of g.

Consider $g \in \overline{\mathrm{Rat}}_d$. If $g \in \partial \mathrm{Rat}_d$, recall from §2.5 the algebraically exceptional set $Ex_g \subset \overline{\mathbb{C}}$ for g. If $g \in \mathrm{Rat}_d$, then the dynamically exceptional set E_g of g, is the one formed by all finite grand orbits, under g. We say that a measure μ in $\overline{\mathbb{C}}$ is *nonexceptional* for $g \in \overline{\mathrm{Rat}}_d$ if $\mu(Ex_g) = 0$ when $g \in \partial \mathrm{Rat}_d$ and $\mu(E_g) = 0$ when $g \in \mathrm{Rat}_d$. Moreover, given an element in $\Pi_{i=1}^{\infty} \overline{\mathrm{Rat}}_{s_i}$, we say a measure μ in $\overline{\mathbb{C}}$ is nonexceptional for this element if μ is nonexceptional for each coordinate. Following DeMarco [2, Section 3], we introduce the pull-back of measures by g.

Definition 2.8. Given $g \in \partial \mathrm{Rat}_d$ and a nonexceptional measure μ for g, we say that

$$g^*\mu = \tilde{g}^*\mu + \mu(\overline{\mathbb{C}}) \cdot \eta_g$$

is the *pull-back of* μ under g.

Given $g \in \partial \mathrm{Rat}_d$ and a continuous function $\varphi : \overline{\mathbb{C}} \to \mathbb{R}$, we define the *push-forward of* φ *by* g

$$g_*\varphi := \tilde{g}_*\varphi + \sum_{h \in \mathrm{Hole}(g)} d_h(g)\varphi(h);$$

then for any a nonexceptional measure μ for g, one obtains that

$$\int \varphi dg^*\mu = \int g_*\varphi d\mu.$$

Although not explicitly discussed in DeMarco [2] and DeMarco-Faber [4], a key fact is that the pull-back defined above is the continuous extension of the pull-back of (nonexceptional) measures:

Proposition 2.9. *Let $g \in \partial \mathrm{Rat}_d$ and μ be a nonexceptional measure for g. Then $f^*\nu$ converges to $g^*\mu$, as $f \in \mathrm{Rat}_d$ converges to g and ν converges to μ.*

Proof. Let $W \subset \overline{\mathbb{C}}$ be a small neighborhood of the point in Ex_g if $Ex_g \neq \emptyset$; and set $W = \emptyset$ if $Ex_g = \emptyset$. Consider $z \in \overline{\mathbb{C}} \setminus W$. As $\mathrm{Rat}_d \ni f \to g$, the set $f^{-1}(z)$ converges to $\mathrm{Hole}(g) \cup \tilde{g}^{-1}(z)$; each point is counted with multiplicity. More precisely, given $w \in \mathrm{Hole}(g) \cup \tilde{g}^{-1}(z)$, let $m_w := d_w(f)$ if $w \notin \tilde{g}^{-1}(z)$, and $m_w := d_w(f) + \deg_w \tilde{g}$ otherwise. By Lemma 2.6 (1), given any small neighborhood of w, for all f sufficiently close to g, this neighborhood contains m_w preimages of z under f, counted with multiplicities.

Now, consider a nonnegative continuous test function $\varphi : \overline{\mathbb{C}} \to \mathbb{R}$. From the previous paragraph, as $f \to g$, we have that $f_*\varphi : \overline{\mathbb{C}} \setminus W \to \mathbb{R}$ pointwise converges to the continuous function $g_*\varphi : \overline{\mathbb{C}} \setminus W \to \mathbb{R}$. Since $\overline{\mathbb{C}} \setminus W$ is compact, the convergence is, in fact, uniform.

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Given $\varepsilon > 0$, assume that W is sufficiently small with $\mu(\partial W) = 0$ and ν is sufficiently close to μ so that both $\mu(W)$ and $\nu(W)$ are less than ε. Then

$$\begin{aligned}\left|\int_{\overline{\mathbb{C}}} \varphi(f^* d\nu - g^* d\mu)\right| &= \left|\int_{\overline{\mathbb{C}}} f_*\varphi d\nu - \int_{\overline{\mathbb{C}}} g_*\varphi d\mu\right| \\ &= \left|\int_{\overline{\mathbb{C}}\setminus W} f_*\varphi d\nu - \int_{\overline{\mathbb{C}}\setminus W} g_*\varphi d\mu + \int_W f_*\varphi d\nu - \int_W g_*\varphi d\mu\right| \\ &\le \left|\int_{\overline{\mathbb{C}}\setminus W} f_*\varphi d\nu - \int_{\overline{\mathbb{C}}\setminus W} g_*\varphi d\mu\right| + \left|\int_W f_*\varphi d\nu\right| + \left|\int_W g_*\varphi d\mu\right| \\ &\le \left|\int_{\overline{\mathbb{C}}\setminus W} f_*\varphi d\nu - \int_{\overline{\mathbb{C}}\setminus W} f_*\varphi d\mu\right| + \left|\int_{\overline{\mathbb{C}}\setminus W} (f_*\varphi - g_*\varphi) d\mu\right| + d\|\varphi\|_\infty 2\varepsilon,\end{aligned}$$

where the last inequality follows from the fact that $\|F_*\varphi\|_\infty \le d\|\varphi\|_\infty$ for any $F \in \overline{\mathrm{Rat}}_d$. The convergence of ν to μ and the uniform convergence of $f_*\varphi$ to $g_*\varphi$ in $\overline{\mathbb{C}} \setminus W$, as $f \to g$, yield that $f^*\nu \to g^*\mu$ as $(f, \nu) \to (g, \mu)$. □

2.7. *Measures of maximal entropy*

Recall that $M^1(\overline{\mathbb{C}})$ denotes the space of probability measures in $\overline{\mathbb{C}}$ endowed with the weak* topology. Given a rational map $f \in \mathrm{Rat}_d$, there exists a unique probability measure μ_f such that $f^*\mu_f = d\mu_f$ and $\mu_f(\mathcal{E}_f) = 0$ where $\mathcal{E}_f$ is the exceptional set for f (i.e., the elements of $\overline{\mathbb{C}}$ with finite grand orbit). Moreover, for any $\mu \in M^1(\overline{\mathbb{C}})$ satisfying $\mu(\mathcal{E}_f) = 0$,

$$\frac{1}{d^n}(f^{\circ n})^*\mu \to \mu_f.$$

Furthermore, μ_f is supported on the Julia set $\mathcal{J}(f)$ of f and is the unique measure of maximal entropy for f. This measure μ_f varies continuously with f. See [10, 12, 13].

The fundamental property of an accumulation point μ of measures μ_f as f approaches a degenerate map was formulated by DeMarco and Faber [4, Section 2.4] in terms of pairs of probability measures (μ, ν) that take into account the changes of scales involved. More precisely, let (μ, ν) be a pair of probability measures on $\overline{\mathbb{C}}$. For a sequence $\{A_k\} \subset \mathrm{Rat}_1$, we say that a sequence $\{\mu_k\} \subset M^1(\overline{\mathbb{C}})$ *converges* $\{A_k\}$*-weakly to* (μ, ν) if μ_k converges to μ and $(A_k)_*\mu_k$ converges to ν.

Proposition 2.10 [4, Theorem 2.4]. *Let* $\{f_k\}$ *be a sequence in* Rat_d *and let* $\{A_k\}$ *be a sequence in* Rat_1 *such that* $A_k \circ f_k$ *converges to a map* $\varphi \in \mathrm{Rat}_d^*$. *Assume that the* $\{A_k\}$*-weak limit of* $\{\mu_{f_k}\}$ *is* (μ, ν). *Then*

$$\mu = \frac{1}{d}\varphi^*\nu.$$

Proof. For all $k \ge 1$,

$$(A_k \circ f_k)^*(A_k)_*\mu_k = d \cdot \mu_{f_k}.$$

Continuity of the pull-back at (φ, ν) (Proposition 2.9) yields $\varphi^*\nu = d \cdot \mu$. □

3. Limiting dynamics of fully ramified sequences

Recall from §2.4 that Rat_d^* denotes the space of degree d rational maps with nonconstant reduction. Möbius transformations act by post-composition on Rat_d^*. Its orbit space, denoted by $[\![\mathrm{Rat}_d^*]\!]$, is a compact Hausdorff space, see Corollary 2.2.

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Here we consider sequences $\{f_k\} \subset \mathrm{Rat}_d$ such that $\{[\![f_k^{\circ n}]\!]\}$ converges to $[\![\varphi_n]\!] \in [\![\mathrm{Rat}_{d^n}^*]\!]$, for all n. From Lemma 2.4, there exists $\varphi_{n,n+1} \in \mathrm{Rat}_d^*$ such that $\varphi_{n+1} = \varphi_{n,n+1} \circ \varphi_n$, for all n. We say that n_0 is a *fully ramified time* if φ_{n_0,n_0+1} is a nondegenerate map of degree d.

In this section, we analyze the limiting dynamics for sequences having several successive fully ramified times. Our analysis is based on the trees of bouquet of spheres discussed in Appendix A and closely related to the work by Arfeux [1]. We establish the Structure Theorem 3.1 for sequences $\{f_k\}$ diverging in moduli space rat_d and possessing at least 3 successive fully ramified times. We also deduce Corollaries 3.15 and 3.16 which are crucial in our proofs of Theorems A and B. Moreover, we obtain Theorem C as a byproduct of the Structure Theorem 3.1.

Recall that we should regard $\tilde{\varphi}_{n,n+1}$ as a map between two spheres: $\overline{\mathbb{C}}_n$ and $\overline{\mathbb{C}}_{n+1}$. Assuming that for some $m \geq 0$, the times $m, m+1$ and $m+2$ are fully ramified, the Structure Theorem asserts that the spheres $\overline{\mathbb{C}}_m, \overline{\mathbb{C}}_{m+1}, \ldots$ are organized in a linear concatenation and describes the maps $\tilde{\varphi}_{n,n+1}$ for $n \geq m$. The main tools are the results from Appendix A and Proposition 3.7. The general idea is to combine the simple connectivity, given by Proposition A.1, with the formula for the multiplicities of the critical points of $\tilde{\varphi}_{n,n+1}$, given by Proposition A.3, to establish the linear concatenation for the initial segment of fully ramified times. Then, to show that the linear concatenation respects the order on the integers, we rely on the extraction of polynomial-like maps, given by Proposition 3.7. Finally, to extend our description beyond fully ramified times, we count preimages with the aid of Proposition A.2.

3.1. Statement of the Structure Theorem

Let $\{f_k\} \subset \mathrm{Rat}_d$ be such that $[\![f_k^{\circ n}]\!]$ converges to $[\![\varphi_n]\!]$ in $[\![\mathrm{Rat}_{d^n}^*]\!]$ for each $n \geq 1$. Recall from §2.4 that $\{A_{n,k} : n, k \geq 1\} \subset \mathrm{Rat}_1$ is a collection of scalings for $\{f_k\}$ if for any $n \geq 1$, as $k \to \infty$,

$$A_{n,k} \circ f_k^{\circ n} \to \varphi_n.$$

Such a collection $\{A_{n,k}\}$ has *convergent changes of coordinates* if for all $n \neq n'$,

$$A_{n',k} \circ A_{n,k}^{-1} \to A_{(n,n')} \in \overline{\mathrm{Rat}_1}.$$

When clear from context, $A_{(n,n')} \in \overline{\mathrm{Rat}_1}$ will be the limit of the corresponding coordinate changes $A_{n',k} \circ A_{n,k}^{-1}$ and, if $A_{(n,n')} \in \partial\mathrm{Rat}_1$, write

$$a_{n,n'} := \tilde{A}_{(n,n')} \in \overline{\mathbb{C}}.$$

In the case that $A_{(n,n')} \in \partial\mathrm{Rat}_1$ for all $n \neq n'$, we say that $\{A_{n,k} : n, k \geq 1\}$ is a collection of *independent scalings*. Note that $a_{n',n}$ is the unique hole of $A_{(n,n')}$. By definition, independent collections of scalings have convergent changes of coordinates.

Let $m \geq 1$ and consider a collection $\{A_{n,k} : n \geq m\}$ of independent scalings, where we abuse of notation and omit writing $k \geq 1$ and, sometimes, simply write $\{A_{n,k}\}$. In view of Appendix A, given $\ell \geq m$, the collection $\{A_{n,k} : m \leq n \leq \ell\}$ is, in a certain sense, a tree of bouquets of spheres. More precisely, for each integer $n \geq 1$, consider a copy $\overline{\mathbb{C}}_n$ of the Riemann sphere. Let $\sim$ be the equivalence relation in $\overline{\mathbb{C}}_m \sqcup \cdots \sqcup \overline{\mathbb{C}}_\ell$ that identifies $a_{n,n'} \in \overline{\mathbb{C}}_{n'}$ with $a_{n',n} \in \overline{\mathbb{C}}_n$ if and only if

$$a_{n,n''} = a_{n',n''}$$

for all $n'' \neq n, n'$ such that $m \leq n'' \leq \ell$. All other $\sim$-classes are trivial. Let

$$\mathcal{S}_\ell := \overline{\mathbb{C}}_m \sqcup \cdots \sqcup \overline{\mathbb{C}}_\ell / \sim .$$

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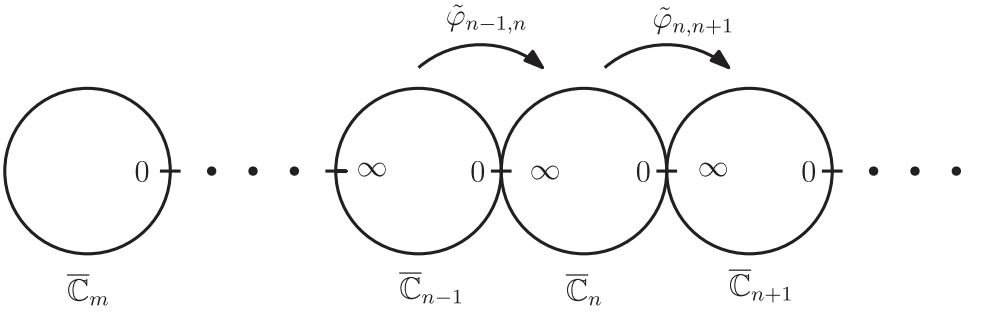


Figure 1. *Structure Theorem.*

By Proposition A.1, the space $\mathcal{S}_\ell$ is simply connected. For each n, we have the retraction

$$\rho_{\ell,n} : \mathcal{S}_\ell \to \overline{\mathbb{C}}_n$$

which maps $\overline{\mathbb{C}}_{n'}$ onto $a_{n',n}$, for all $n' \neq n$. It is convenient to identify $\overline{\mathbb{C}}_n$ with the corresponding subset of $\mathcal{S}_\ell$.

Collapsing $\overline{\mathbb{C}}_{\ell+1} \subset \mathcal{S}_{\ell+1}$ to a point produces a continuous map $\pi_{\ell+1} : \mathcal{S}_{\ell+1} \to \mathcal{S}_\ell$. The image of $\overline{\mathbb{C}}_{\ell+1}$ is the common point in $\mathcal{S}_\ell$ of all the spheres $\overline{\mathbb{C}}_j$ that intersect $\overline{\mathbb{C}}_{\ell+1}$ in $\mathcal{S}_{\ell+1}$. We say that the inverse limit of $(\mathcal{S}_\ell, \pi_\ell)$ is the *space $\mathcal{S}$ associated to the collection of scalings* $\{A_{n,k} : n \geq m\}$.

In view of Lemma 2.4, $A_{n+1,k} \circ f_k \circ A_{n,k}^{-1}$ converges to a map $\varphi_{n,n+1} \in \mathrm{Rat}_d^*$. Recall that we regard $\tilde{\varphi}_{n,n+1}$ as a map from $\overline{\mathbb{C}}_n$ onto $\overline{\mathbb{C}}_{n+1}$, and also regard $A_{n,k}$ as a map from $\overline{\mathbb{C}}$ onto $\overline{\mathbb{C}}_n$. We will show that, under certain conditions, the maps $\tilde{\varphi}_{n,n+1} : \overline{\mathbb{C}}_n \to \overline{\mathbb{C}}_{n+1}$ produce a well defined map from $\mathcal{S}$ to $\mathcal{S}$.

Theorem 3.1 (Structure Theorem). *Let $\{f_k\} \subset \mathrm{Rat}_d$ be a sequence such that $[\![f_k^{\circ n}]\!] \to [\![\varphi_n]\!] \in [\![\mathrm{Rat}_{d^n}^*]\!]$ for each $n \geq 1$, and $[f_k] \to [g] \in \partial \mathrm{rat}_d$, as $k \to \infty$. Let $\{A_{n,k}\}$ be a collection of scalings for $\{f_k\}$ having convergent changes of coordinates.*

Assume that there exists $m \geq 1$ such that $m, m+1$ and $m+2$ are fully ramified times. Then $\{A_{n,k} : n \geq m\}$ is a collection of independent scalings and the following statements hold:

1. *The space $\mathcal{S}$ associated to $\{A_{n,k}\}$ is the linear concatenation of $\overline{\mathbb{C}}_m, \overline{\mathbb{C}}_{m+1}, \cdots$. That is, in suitable coordinates, employed for the rest of the statement, $\mathcal{S}_\ell$ is obtained by identifying $0 \in \overline{\mathbb{C}}_n$ to $\infty \in \overline{\mathbb{C}}_{n+1}$, for all $m \leq n < \ell$ (see Figure 1).*
2. *The map*

$$\begin{aligned} \mathcal{F} : \mathcal{S} &\to \mathcal{S} \\ z &\mapsto \tilde{\varphi}_{n,n+1}(z) \in \overline{\mathbb{C}}_{n+1} \text{ if } z \in \overline{\mathbb{C}}_n, \end{aligned}$$

   is well defined and continuous.
3. *For all $n > m$,*

$$\deg_\infty \tilde{\varphi}_{n,n+1} = \deg \tilde{\varphi}_{n,n+1} = \deg_0 \tilde{\varphi}_{n-1,n}.$$

4. *If $n > m$ and $n+1$ is a fully ramified time, then $\tilde{\varphi}_{n,n+1}$ is a monomial of degree d.*
5. *If $n > m$ is not a fully ramified time, then $z = \infty$ is the unique hole of $\varphi_{n,n+1}$.*

Continuity of $\mathcal{F}$ is equivalent, once (1) is established, to $\tilde{\varphi}_{n,n+1}(\infty) = \infty$ and $\tilde{\varphi}_{n-1,n}(0) = 0$ for $n > m$. Statement (3) yields that $\deg \tilde{\varphi}_{n,n+1}$ is a nonincreasing function of $n > m$. Moreover, from statements (1)-(3), $\tilde{\varphi}_{n,n+1}$ is a polynomial for all $n > m$.

The map $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ should be regarded as a limiting action of the sequence $\{f_k\}$. In general, let us assume that $\{A_{n,k} : n \geq m\}$ is a collection of independent scalings for a sequence $\{f_k\}$. For any $\ell \geq m$, let

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$$J_\ell := \{n : m \le n \le \ell\}$$

and consider the embedding

$$\begin{aligned} \mathcal{A}_{J_\ell,k} : \overline{\mathbb{C}} &\hookrightarrow \overline{\mathbb{C}}^{J_\ell} \\ z &\mapsto (A_{n,k}(z))_{n\in J_\ell}. \end{aligned}$$

From Appendix A, the space $\mathcal{S}_\ell$ is naturally identified with the Hausdorff limit of the embedded copies $\mathcal{A}_{J_\ell,k}(\overline{\mathbb{C}})$ of $\overline{\mathbb{C}}$. Similarly, for $\ell+1$, the space $\mathcal{S}_{\ell+1}$ is the limit of $\mathcal{A}_{J_{\ell+1},k}(\overline{\mathbb{C}})$. Then, $f_k : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ acts on the embedded sphere as

$$\begin{aligned} F_k : \mathcal{A}_{J_\ell,k}(\overline{\mathbb{C}}) &\to \mathcal{A}_{J_{\ell+1},k}(\overline{\mathbb{C}}) \\ w &\mapsto \mathcal{A}_{J_{\ell+1},k} \circ f_k \circ \mathcal{A}^{-1}_{J_\ell,k}(w). \end{aligned}$$

Roughly speaking, Proposition A.2 allows us to count preimages under F_k. That is, consider $z_0 \in \overline{\mathbb{C}}_n \subset \mathcal{S}_\ell$ and $w_0 \in \overline{\mathbb{C}}_{n+1} \subset \mathcal{S}_{\ell+1}$. Then, for sufficiently large k and for any $w \in \mathcal{A}_{J_{\ell+1},k}(\overline{\mathbb{C}})$ close to $\rho^{-1}_{\ell+1,n+1}(w_0)$, there exists δ preimages of w, under F_k, close to $\rho^{-1}_{\ell,n}(z_0)$ where

$$\delta = \begin{cases} d_{z_0}(\varphi_{n,n+1}), & \text{if } \tilde{\varphi}_{n,n+1}(z_0) \neq w_0, \\ d_{z_0}(\varphi_{n,n+1}) + \deg_{z_0} \tilde{\varphi}_{n,n+1}(z_0), & \text{if } \tilde{\varphi}_{n,n+1}(z_0) = w_0. \end{cases}$$

See Proposition A.2 for a precise statement.

Remark 3.2. In Berkovich space language, consider a type II point $x := x_n$ that maps onto $y := x_{n+1}$ by a rational function F, and let $D_x(z_0), D_y(w_0)$ be directions at x and y, respectively. Then the number of preimages in $D_x(z_0)$ of a given point $w \in D_y(w_0)$ is prescribed by a well known formula that depends on the *surplus multiplicity* and *multiplicity* of the direction $D_x(z_0)$ (see [7, Section 3]). The formula above is its Archimedean analogue. In this analogy, x_n, x_{n+1} correspond to $\overline{\mathbb{C}}_n, \overline{\mathbb{C}}_{n+1}$.

Remark 3.3. A basic example to consider is

$$f_k(z) = c_k z^2 + c_k \in \mathrm{Rat}_2,$$

where $c_k \to \infty$. For $n \ge 1$, let

$$A_{n,k}(z) := c_k^{-2^{n-1}} z.$$

Note that $A_{1,k} \circ f_k(z)$ converges to $z^2 + 1$, and for $n \ge 2$,

$$A_{n,k} \circ f_k^{\circ n}(z) \to z^{2^{n-1}},$$

uniformly in compact subsets of $\mathbb{C}$ as $k \to \infty$. Consequently, all times are fully ramified for $\{f_k\}$. It is not difficult to construct similar examples of polynomials in any degree, not necessarily unicritical. In a certain sense, the Structure Theorem says that a degenerating sequence of rational maps $\{f_k\}$ with at least three succesive fully ramified resembles a degeneration along polynomial maps (c.f. Theorem C).

Remark 3.4. If we have just two consecutive ramified times $m = 1, 2$, the spheres $\overline{\mathbb{C}}_1, \overline{\mathbb{C}}_2$ and $\overline{\mathbb{C}}_3$ are not necessarily organized in a linear concatenation. Indeed, consider the family of quadratic rational maps, parametrized by $0 < t \ll 1$:

$$f_t(z) := \frac{1 + t^3 z^2}{t z^2} + t^{-1/2}.$$

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Let $A_{1,t}(z) := tz$, $A_{2,t}(z) := -t^{-3/2} + zt^{-1}$, and $A_{3,t}(z) = -2^{-1}t^{-3/2}(z - t^{-1/2} - 1)$. As $t \to 0$, for a generic z, we have $A_{1,t} \circ f_t(z) \to z^{-2}$, $A_{2,t} \circ f_t \circ A_{1,t}^{-1}(z) \to z^{-2}$, and $A_{3,t} \circ f_t \circ A_{2,t}^{-1}(z) \to z$. Thus, $m = 0, 1$ are fully ramified times for any sequence $\{f_{t_k}\}$ such that $t_k \searrow 0$. However, the space associated to the scalings $\{A_{1,t_k}, A_{2,t_k}, A_{3,t_k}\}$ is a bouquet of three spheres with a common point. Note that, for k large, f_k has an attracting fixed point $z_k = t_k^{-1/2} + 1 + o(1)$, and therefore, a quadratic-like restriction.

3.2. Proof of the Structure Theorem

We work under the hypothesis of the Structure Theorem and consider $n_0 \geq m + 2$ such that n is a fully ramified time if $m \leq n \leq n_0$. In §3.2.1 we establish that $\overline{\mathbb{C}}_m, \ldots, \overline{\mathbb{C}}_{n_0}$ are organized in a linear concatenation respecting the order of the integers, and deduce Theorem C. The proof of the Structure Theorem will be by induction on $\ell \geq n_0$. In §3.2.2, we reformulate the assertions of the theorem in a manner suitable for induction and establish the base case. In §3.2.3, the inductive step for $\ell \geq n_0$ is proven. We rely on a series of lemmas, stated and established, under the assumptions and notation of Theorem 3.1.

3.2.1. Fully ramified times

Choose $n_0 \geq m + 2$ and assume that n is a fully ramified time for all n such that $m + 2 \leq n \leq n_0$.

The spheres $\overline{\mathbb{C}}_m, \ldots, \overline{\mathbb{C}}_{n_0+1}$ are pairwise distinct:

Lemma 3.5. *If $m \leq n < n' \leq n_0 + 1$, then $A_{(n',n)} \in \partial\mathrm{Rat}_1$.*

Proof. By contradiction, suppose that $A_{(n',n)}$ is nondegenerate. Then

$$[f_k^{\circ n'-n}] = [(A_{n,k} \circ A_{n',k}^{-1}) \circ (A_{n',k} \circ f_k^{\circ n'-n} \circ A_{n,k})] \to [A_{(n',n)} \circ \varphi_{n'-1,n'} \circ \cdots \circ \varphi_{n,n+1}] \in \mathrm{rat}_{d^{n'-n}},$$

which contradicts that $[f_k] \to [g] \in \partial\mathrm{rat}_d$, since the $(n'-n)$-th iterate map is proper (see [3, Proposition 4.1]). □

Let $\mathcal{S}'$ be the space associated to $\{A_{n,k} : m + 1 \leq n \leq n_0 + 1\}$. According to Proposition A.3,

$$\mathcal{F}_{n_0} : \mathcal{S}_{n_0} \to \mathcal{S}',$$

given by

$$\mathcal{F}_{n_0}(z) := \varphi_{n,n+1}(z) \in \overline{\mathbb{C}}_{n+1},$$

for $z \in \overline{\mathbb{C}}_n$ and $n \in J_{n_0}$, is continuous. Moreover, there exists a finite set $\mathrm{Crit}(\mathcal{F}_{n_0}) \subset \mathcal{S}_{n_0}$ of *critical points*, each with an assigned multiplicity, such that the total number of critical points is $2d - 2$, counted with multiplicities. Furthermore, $\mathrm{Crit}(\mathcal{F}_{n_0})$ has the following property: if $n \in J_{n_0}$ and $z \in \overline{\mathbb{C}}_n$, then the multiplicity of z as a critical point of $\tilde{\varphi}_{n,n+1}$ is the number of elements of $\mathrm{Crit}(\mathcal{F}_{n_0})$ in $\rho_{n_0,n}^{-1}(z)$, counted with multiplicities. A priori, $\mathrm{Crit}(\mathcal{F}_{n_0})$ might contain intersection points of distinct spheres.

Lemma 3.6. *There exists a total order $\prec$ in J_{n_0} such that*

$$\begin{array}{ccc} J_{n_0-1} & \to & J_{n_0} \\ n & \mapsto & n+1 \end{array}$$

is a monotone map. Moreover, given $n \neq n' \in J_{n_0}$, for $\overline{\mathbb{C}}_n, \overline{\mathbb{C}}_{n'} \subset \mathcal{S}_{n_0}$, the following hold:

1. *$\overline{\mathbb{C}}_n \cap \overline{\mathbb{C}}_{n'} \neq \emptyset$ if and only if n and n' are $\prec$-consecutive.*
2. *If $x \in \overline{\mathbb{C}}_n \cap \overline{\mathbb{C}}_{n'}$, then $\deg_x \varphi_{n,n+1} = d$.*
3. *If n, n' are $\prec$-consecutive in J_{n_0-1}, then $n + 1, n' + 1$ are $\prec$-consecutive in J_{n_0}.*

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Proof. To lighten notation, when possible, we omit sub-indices and write $\mathcal{S} := \mathcal{S}_{n_0}$, $J = J_{n_0}$ and $\mathcal{F} := \mathcal{F}_{n_0}$. Let us first show that given $n \in J$, there are at most two spheres, distinct from $\overline{\mathbb{C}}_n$, that intersect $\overline{\mathbb{C}}_n$. Suppose on the contrary that there are three distinct numbers $n_1, n_2, n_3 \neq n$ in J such that the corresponding spheres have nonempty intersection with $\overline{\mathbb{C}}_n$ at p_1, p_2, p_3, respectively. Simple connectivity of $\mathcal{S}$ yields that $X_j := \overline{\mathbb{C}}_{n_j} \setminus \{p_j\}$ are pairwise disjoint for $j = 1, 2, 3$, and hence $\mathcal{T}_j := \rho^{-1}_{n_0,n_j}(X_j)$ are also pairwise disjoint for $j = 1, 2, 3$. Since φ_{n_j,n_j+1} has degree d, its local degree at p_j is at most d. Thus X_j contains at least $d-1$ critical points of φ_{n_j,n_j+1}, counted with multiplicities. Therefore, $\mathcal{T}_j$ contains at least $d-1$ points of $\mathrm{Crit}(\mathcal{F})$, counted with multiplicity, by Proposition A.3. Then $\mathrm{Crit}(\mathcal{F})$ has at least $3d-3 > 2d-2$ points, counted with multiplicities, which contradicts Proposition A.3. A similar argument shows that three distinct spheres do not share a common point.

To define the order in J. Declare $m \prec m+1$. Consider the graph $\mathcal{T}$ with vertex set J and edges given by all pairs $\{n, n'\}$ such that $\overline{\mathbb{C}}_n \cap \overline{\mathbb{C}}_{n'} \neq \emptyset$ in $\mathcal{S}$. Since there are no triple intersection points and $\mathcal{S}$ is simply connected, by Proposition A.1, we conclude that $\mathcal{T}$ is a connected tree. Moreover, the previous paragraph also implies that each vertex of $\mathcal{T}$ has valence at most 2. Thus $\mathcal{T}$ is isomorphic to an interval graph: it has exactly two valence 1 vertices, say a and b. Without loss of generality, $m+1$ belongs to the subtree connecting m with b. Declare $n \prec n'$ if and only if n belongs to the connected component of $\mathcal{T} \setminus \{n'\}$ containing a; equivalently, generic elements of $\overline{\mathbb{C}}_n$ and $\overline{\mathbb{C}}_a$ lie in the same connected component of $\mathcal{S} \setminus \overline{\mathbb{C}}_{n'}$.

Suppose that $n \prec n'$ are two consecutive elements of J_{n_0}. Denote by $x \in \mathcal{S}$ the intersection point of $\overline{\mathbb{C}}_n$ and $\overline{\mathbb{C}}_{n'}$. We claim that the local degree of $\varphi_{n,n+1}$ at x is d. Indeed, the critical multiplicity of $\varphi_{n,n+1}$ at x is the number of elements of $\mathrm{Crit}(\mathcal{F})$, counted with multiplicity, in $\rho^{-1}_{n_0,n}(x) \supset \rho^{-1}_{n_0,n}(\overline{\mathbb{C}}_{n'} \setminus \{x\})$, which is at least $d-1$ since $\deg \varphi_{n',n'+1} = d$. Therefore, $\deg_x \varphi_{n,n+1} = d$. A similar argument shows that $\deg_x \varphi_{n',n'+1} = d$.

Finally, if $n, n' \in J_{n_0-1}$ are $\prec$-consecutive, then $n+1$ and $n'+1$ are consecutive in $J \setminus \{m\}$, since $\mathcal{F}(x)$ lies in the intersection of the corresponding spheres. Moreover, the translation map $n \mapsto n+1$ is injective and, being consecutive preserving, must also be monotone. □

We conclude that if the translation by 1 is $\prec$-monotone increasing, then

$$m \prec m+1 \prec \cdots \prec n_0 - 1 \prec n_0.$$

That is, the order $\prec$ coincides with the standard order of the integers. Otherwise, if the translation by 1 is $\prec$- monotone decreasing, then either

$$m \prec m+2 \prec \cdots \prec m+3 \prec m+1,$$

or

$$\cdots \prec m+2 \prec m \prec m+1 \prec m+3 \prec \cdots.$$

However, studying the location of the Julia set $\mathcal{J}(f_k)$ of f_k for sufficiently large k, we will obtain an obstruction for translation to be decreasing. The location of the Julia set is controlled by the following general result whose statement is illustrated in Figure 2.

Proposition 3.7. *Consider $\{h_k\} \subset \mathrm{Rat}_d$. Assume that $\{B_{n,k} : n = 0, 1\}$ are a pair of independent scalings such that the associated space is obtained from the identification of $a \in \overline{\mathbb{C}}_0$ with $b \in \overline{\mathbb{C}}_1$, and fix a point $w \in \overline{\mathbb{C}}_0$. Suppose that the following hold:*

1. *$B_{1,k} \circ h_k \circ B^{-1}_{0,k}$ converges uniformly to $\varphi \in \mathrm{Rat}_d$.*
2. *$\deg_w \varphi = d$.*
3. *$\varphi(w) \begin{cases} \neq b \text{ if } w = a, \\ = b \text{ if } w \neq a. \end{cases}$*

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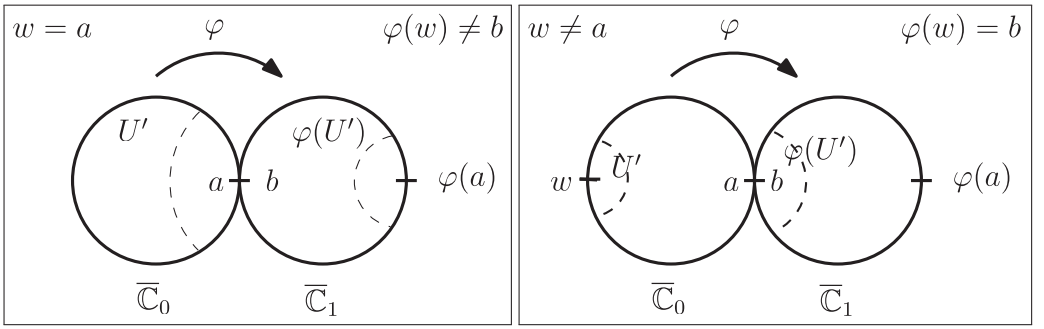


Figure 2. *Illustration of Proposition 3.7.*

Given a neighborhood V of b, for any k sufficiently large, the following statements hold:

(i) *$\mathcal{J}(h_k) \subset B_{1,k}^{-1}(V)$ and $\overline{\mathbb{C}} \setminus B_{1,k}^{-1}(V)$ is contained in a completely invariant attracting Fatou component of h_k.*

(ii) *There exist Jordan domains U_k' and $U_k := h_k(U_k')$ such that (U_k', U_k, h_k) is a degree d polynomial-like map with the modulus of $U_k \setminus \overline{U_k'}$ tending to ∞, as $k \to \infty$. Moreover,*

  (a) *$B_{0,k}(U_k')$ converges to a Jordan domain U' in $\overline{\mathbb{C}}_0$,*

  (b) *if $w = a$, then $\overline{\mathbb{C}}_0 \setminus \overline{U'}$ is a small neighborhood of a;*

  (c) *if $w \neq a$, then U' is a small neighborhood of w.*

As illustrated in Figure 2, we will choose a domain U' in $\overline{\mathbb{C}}_0$ that is mapped onto a domain $\varphi(U')$ in $\overline{\mathbb{C}}_1$ with degree d by φ. One should think that U' maps under φ onto a domain that "properly contains" U'; although $\varphi(U')$ is in a different sphere. The idea is that for k large, in the sphere $\overline{\mathbb{C}}$ where h_k acts, the domain $U_k' := B_{0,k}^{-1}(U')$ is, in fact, compactly contained in $U_k := h_k(U_k')$ since $B_{1,k}(U_k)$ converges to $\varphi(U')$.

Proof. Changing $B_{1,k}$ by $B \circ B_{1,k}$ for a suitable Möbius transformation B, we may assume that $b = \infty$ and $\varphi(a) = 0$. Similarly, changing $B_{0,k}$, we may assume that $a = 0$ and, in the case $w \neq a$, we may also assume $w = \infty$

Let $C_k := B_{0,k} \circ B_{1,k}^{-1}$. Since $b = \infty \in \overline{\mathbb{C}}_1$ is identified with $a = 0 \in \overline{\mathbb{C}}_0$, it follows that C_k converges to $C \in \partial \mathrm{Rat}_1$ such that $\tilde{C} = 0$ and the hole of C is at ∞. Write $T_k := B_{1,k} \circ h_k \circ B_{0,k}^{-1}$ and $H_k := B_{0,k} \circ h_k \circ B_{0,k}^{-1}$.

In view of (2) and (3), let D be a Jordan neighborhood of $z = 0$ $(= \varphi(a))$ so that $D' := \varphi^{-1}(D)$ is a Jordan neighborhood of $z = 0$ $(= a)$ which maps d-to-1 onto D, under φ. Note that $\overline{\mathbb{C}}_1 \setminus \overline{D}$ is a small neighborhood of $z = \infty (= b)$ if $w \neq a$. The uniform convergence of T_k to φ implies that $D_k' := T_k^{-1}(D)$ is a Jordan neighborhood of $z = 0$ converging to $\varphi^{-1}(D)$ in the Hausdorff topology. We will show that $U' = \overline{\mathbb{C}}_0 \setminus \varphi^{-1}(\overline{D})$ is the Jordan domain claimed in (ii-a). Note that $H_k = C_k \circ T_k$ maps D_k' onto $D_k := C_k(D)$ with degree d. The uniform convergence of C_k to 0, in D, yields that $D_k \Subset D_k'$ for k large. Then $H_k : \overline{\mathbb{C}}_0 \setminus \overline{D_k'} \to \overline{\mathbb{C}}_0 \setminus \overline{D}_k$ is a degree d polynomial-like map. Moreover, the modulus $\mathrm{Mod}(D_k' \setminus \overline{D_k})$ tends to infinity, as $k \to \infty$, since $C_k^{-1}(\overline{D_k}) = \overline{D}$ and $\overline{\mathbb{C}}_1 \setminus C_k^{-1}(D_k')$ converges, in the Hausdorff topology, to $\{\infty\}$. After conjugacy by $B_{0,k}^{-1}$, the polynomial-like map $(\overline{\mathbb{C}}_0 \setminus \overline{D_k'}, \overline{\mathbb{C}}_0 \setminus \overline{D}_k, H_k)$ becomes the desired polynomial-like restriction of h_k. Thus, statement (ii) holds. Note that D_k' must be contained in a completely invariant attracting Fatou component of H_k. Since C_k^{-1} converges uniformly to ∞ outside D_k', it follows that $C_k^{-1}(\mathcal{J}(H_k)) = B_{1,k}(\mathcal{J}(h_k))$ converges, in the Hausdorff topology, to $\{\infty\}$. In particular, for k large, $B_{1,k}(\mathcal{J}(h_k)) \subset V$, for any given neighborhood of $b = \infty$. Similarly, $C_k^{-1}(D_k')$ contains $\overline{\mathbb{C}} \setminus V$ for k large; therefore, $\overline{\mathbb{C}} \setminus B_{1,k}^{-1}(V)$ is contained in a completely invariant attracting Fatou component of h_k. □

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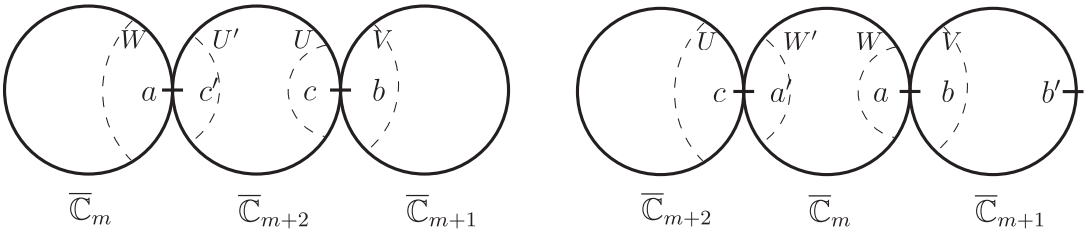


Figure 3. *Illustration of proof of Lemma 3.8: case (i) on the left and case (ii) on the right.*

We remark that the above polynomial-like map is extracted from a sequence $\{h_k\}$ with just 1 fully ramified time. However, this extraction requires that assumptions (2) and (3) are satisfied. To prove Theorem C, we will show that assumptions (2) and (3) always hold for sequences with 3 consecutive fully ramified times.

Now we are ready to rule out monotone decreasing translations. Note that translation by 2 is always monotone increasing from J_{n_0-2} into J_{n_0}.

Lemma 3.8. *The map $n \mapsto n+1$ is $\prec$-monotone increasing in J_{n_0-1}.*

Proof. We proceed by contradiction and suppose that $n \mapsto n+1$ is $\prec$-orientation reversing. Then one of following concatenations of spheres occurs:

(i) $\overline{\mathbb{C}}_m \prec \overline{\mathbb{C}}_{m+2} \prec \overline{\mathbb{C}}_{m+1}$.
(ii) $\overline{\mathbb{C}}_{m+2} \prec \overline{\mathbb{C}}_m \prec \overline{\mathbb{C}}_{m+1}$.

Set $a := a_{m+1,m}, b := a_{m,m+1}, c := a_{m+1,m+2}$, and consider small neighborhoods $W \subset \overline{\mathbb{C}}_m$ of a, $V \subset \overline{\mathbb{C}}_{m+1}$ of b and $U \subset \overline{\mathbb{C}}_{m+2}$ of c. See Figure 3 for an illustration of these and other neighborhoods that we introduce below.

Let $W_k := A_{m,k}^{-1}(W)$, $V_k := A_{m+1,k}^{-1}(V)$ and $U_k := A_{m+2,k}^{-1}(U)$. We will freely assume that k is sufficiently large for the assertions below to hold.

We first show that, in both cases,

$$\tilde{\varphi}_{m,m+1}(a) = b \quad \text{and} \quad \tilde{\varphi}_{m+1,m+2}(b) = c,$$

by counting preimages of generic points in $\overline{\mathbb{C}}_{m+2}$ and in $\overline{\mathbb{C}}_{m+1}$.

Indeed, if $\tilde{\varphi}_{m,m+1}(a) \neq b$, then for a generic point $z \in \overline{\mathbb{C}}_{m+2}$, under f_k, the point $A_{m+2,k}^{-1}(z)$ has at least 1 preimage outside W_k. However, $\deg \tilde{\varphi}_{m+1,m+2} = d$, so all d preimages of $A_{m+2,k}^{-1}(z)$ lie in W_k. It is a contradiction.

Similarly, if $\tilde{\varphi}_{m+1,m+2}(b) \neq c$, then for a generic point $w \in \overline{\mathbb{C}}_{m+1}$, under f_k, the point $A_{m+1,k}^{-1}(w)$ has at least 1 preimage outside V_k. However, $\deg \tilde{\varphi}_{m,m+1} = d$, so all d preimages of $A_{m+1,k}^{-1}(w)$ lie in V_k. This is again a contradiction.

With $\tilde{\varphi}_{m,m+1}(a) = b$ and $\tilde{\varphi}_{m+1,m+2}(b) = c$, we now consider cases (i) and (ii) separately.

Case (i): Let U' be a small neighborhood of $c' = a_{m,m+2}$ and $U_k' := A_{m+2,k}^{-1}(U')$. By Proposition 3.7, applied to $w = a$ and $\varphi = \varphi_{m+1,m+2} \circ \varphi_{m,m+1}$, we have that $\mathcal{J}(f_k) = \mathcal{J}(f_k^{\circ 2})$ converges to $c' \in \overline{\mathbb{C}}_{m+2}$. Viewed in $\overline{\mathbb{C}}_{m+1}$, the Julia sets converge to b. More precisely, $\mathcal{J}(f_k) \subset U_k' \subset V_k$. But, $\varphi_{m+1,m+2}(b) = c$, so $\mathcal{J}(f_k) = f_k(\mathcal{J}(f_k)) \subset U_k$. This is impossible because U_k and U_k' are disjoint. Hence (i) cannot occur.

Case (ii): To proceed with the argument, we first fix some notation. Let $a' := a_{m+2,m}$ and note that $b' := \tilde{\varphi}_{m,m+1}(a') \neq b$. Choose a small neighborhood W' of $a' \in \overline{\mathbb{C}}_m$ such that $\tilde{\varphi}_{m,m+1}(W')$, a neighborhood of b', is properly contained in $V^c := \overline{\mathbb{C}}_{m+1} \setminus \overline{V}$. Let $W_k' := A_{m,k}^{-1}(W')$, and $V_k^c := A_{m+1,k}^{-1}(V^c)$. Keep in mind that $f_k(W_k') \Subset V_k^c$.

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By Proposition 3.7, for a choice of a small neighborhood W of $a \in \overline{\mathbb{C}}_m$, the map $g_k := (W_k, N_k, f_k^{\circ 2})$ is a degree d^2 polynomial-like map where $N_k := f_k^{\circ 2}(W_k)$. Moreover, the modulus $\mathrm{Mod}(N_k \setminus \overline{W_k})$ diverges to ∞, as $k \to \infty$. By shrinking V if necessary, we may assume that V_k is properly contained in $f_k(W_k)$.

We now show that the filled Julia set $\mathcal{K}(g_k)$ of g_k is connected. Since $c \notin \varphi_{m+1,m+2}(V^c) \subset \overline{\mathbb{C}}_{m+2}$, we have that $f_k(V_k^c) \subset W_k'$. Therefore, $f_k^{\circ 2}(V_k^c) \Subset V_k^c$, which yields that V_k^c is in the basin of an attracting fixed point z_k of g_k. We claim that all the critical values of g_k lie in this basin. The set of critical values of g_k is the union of $X_k := g_k(\mathrm{Crit}(f_k) \cap W_k)$ and $Y_k := f_k(\mathrm{Crit}(f_k) \cap f_k(W_k))$. Note that, counting multiplicities, $d-1$ critical points of f_k are contained in $V_k^c \subset W_k$ and, $d-1$ in W_k'. Thus, $X_k \subset V_k^c$ and $Y_k \subset f_k(W_k') \subset V_k^c$. Therefore, all critical values of g_k are contained in the basin of z_k and g_k has connected filled Julia set.

Composing with appropriate coordinate changes and passing to a subsequence, if necessary, we claim that the sequence $f_k^{\circ 2}$ converges to a degree d^2 polynomial. Consider a scaling B_k such that $B_k(\mathcal{K}(g_k))$ has Euclidean diameter 1 and contains 0. Then

$$(B_k(W_k), B_k(N_k), B_k \circ f_k^{\circ 2} \circ B_k^{-1})$$

is a polynomial-like map whose filled Julia set is connected, has Euclidean diameter 1 and contains 0. Since the modulus of $B_k(N_k) \setminus \overline{B_k(W_k)}$ tends to ∞, as $k \to \infty$, by [14, Theorem 5.8] and a diagonal argument, the sequence $B_k \circ f_k^{\circ 2} \circ B_k^{-1} : B_k(W_k) \to B_k(N_k)$ has a convergent subsequence whose limit is a polynomial-like map with infinite modulus. Thus, this limit is a polynomial of degree d^2.

Therefore, passing to a subsequence if necessary, $[f_k^{\circ 2}]$ converges in rat_{d^2}. Since the iterate map $\mathrm{rat}_d \to \mathrm{rat}_{d^2}$ is proper, it follows that $[f_k]$ converges in rat_d. This contradicts the assumption that $[f_k]$ accumulates at a point of $\partial \mathrm{rat}_d$. Hence (ii) cannot occur, and the proof is complete. □

We are now ready to deduce Theorem C.

Proof of Theorem C. Assume that $\{f_k\} \subset \mathrm{Rat}_d$, $\{A_k\}$ and $\{B_k\}$ are as in the statement of the theorem. We proceed by contradiction. After passing to a subsequence, we suppose that f_k lacks of a degree d polynomial-like restriction for all k. Then $h_k := A_k \circ f_k \circ A_k^{-1}$ also has no degree d polynomial-like restriction. Passing to a further subsequence, $[\![A_k \circ f_k^{\circ n} \circ A_k^{-1}]\!]$ converges to some $[\![\varphi_n]\!]$ for all n. Moreover, $\varphi_3 \in \mathrm{Rat}_{d^3}$. From Lemma 2.4, the sequence $\{h_k\}$ has fully ramified times $n = 1, 2, 3$. Let $\{A_{n,k}\}$ be the collection of scalings such that $A_{n,k} \circ h_k^{\circ n} \to \varphi_n$. Then we are under the hypothesis of Theorem 3.1 for $\{h_k\}$ and $m = 1$. By Lemma 3.8, $\mathcal{S}_3$ is the concatenation of 3 spheres $\overline{\mathbb{C}}_1, \overline{\mathbb{C}}_2, \overline{\mathbb{C}}_3$. Moreover, $\varphi_{2,3}$ maps $\overline{\mathbb{C}}_2$ onto $\overline{\mathbb{C}}_3$ by a degree d map. By Lemma 3.6 (2), the intersection point of $\overline{\mathbb{C}}_2$ with $\overline{\mathbb{C}}_1$ maps, by $\varphi_{2,3}$, to the intersection point of $\overline{\mathbb{C}}_3$ with $\overline{\mathbb{C}}_2$ with local degree d. Thus, we are under the hypothesis of Proposition 3.7, for $\{A_{n,k} : n = 1, 2\}$, that yields the existence of a degree d polynomial-like restriction of h_k and, therefore, of f_k. □

3.2.2. Inductive hypothesis

Given $\ell \geq n_0$, consider the following assertions:

- $(0.\ell)$ For all $m \leq n, n' \leq \ell$, if $n \neq n'$, then $A_{(n,n')} \in \partial \mathrm{Rat}_1$.
- $(1.\ell)$ $\mathcal{S}_\ell$ is obtained by identifying $a_{n+1,n} \in \overline{\mathbb{C}}_n$ with $a_{n,n+1} \in \overline{\mathbb{C}}_{n+1}$ for $m \leq n < \ell$. Moreover, $a_{n-1,n} \neq a_{n+1,n}$ for $m < n < \ell$.
- $(2.\ell)$ $\tilde{\varphi}_{n,n+1}(a_{n-1,n}) = a_{n,n+1}$ if $m < n < \ell$, and $\tilde{\varphi}_{n,n+1}(a_{n+1,n}) = a_{n+2,n+1}$ if $m \leq n < \ell - 1$.
- $(3.\ell)$ $\deg \tilde{\varphi}_{n,n+1} = \deg_{a_{n,n-1}} \tilde{\varphi}_{n-1,n}$ for $m < n < \ell$.
- $(4.\ell)$ If $m < n < \ell$ and $n+1$ is a fully ramified time, then the local degree of $\varphi_{n,n+1}$ at $a_{n\pm 1,n}$ is d.
- $(5.\ell)$ For $m < n < \ell$,

$$d = d_{a_{n-1,n}}(\varphi_{n,n+1}) + \deg_{a_{n-1,n}} \tilde{\varphi}_{n,n+1}.$$

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In other words, $(0.\ell)$ states that, for $m \le n \le \ell$, the scalings $\{A_{k,n}\}$ are independent, so the corresponding spheres $\overline{\mathbb{C}}_m, \dots, \overline{\mathbb{C}}_\ell$ are distinct. Then $(1.\ell)$ says that these spheres are organized in a linear concatenation $\mathcal{S}_\ell$. Accordingly, $(2.\ell)$ claims that the induced map on this linear concatenation $\mathcal{S}_\ell$ is continuous. The inductive hypothesis $(4.\ell)$ asserts that the maps $\tilde{\varphi}_{n,n+1}$ are monomials in appropriate coordinates if n and $n+1$ are fully ramified times. Finally, $(3.\ell)$ and $(5.\ell)$ are formulas describing the degree of $\tilde{\varphi}_{n,n+1}$ and the depth of the holes of $\varphi_{n,n+1}$, which encode assertions (3) and (5) of the theorem, as explained below:

Proof of Theorem 3.1 assuming (0.ℓ)–(5.ℓ) for all ℓ ≥ n₀. Clearly $(0.\ell)$ for all ℓ yields the independence of $\{A_{n,k} : n \ge m\}$. Moreover, $(1.\ell)$ implies statement (1) of Theorem 3.1. For statement (2), we only need to check continuity at the intersection of consecutive spheres, which is the content of $(2.\ell)$. Provided we normalize $a_{n,n-1} = 0 \in \overline{\mathbb{C}}_{n-1}$ and $a_{n-1,n} = \infty \in \overline{\mathbb{C}}_n$, for all n, statement $(3.\ell)$ for all ℓ is the second equality in (3). Statement (4) is exactly $(4.\ell)$ for all ℓ. For statement (5) and the remaining part of statement (3), note that $d - \deg \tilde{\varphi}_{n,n+1}$ is the total number of holes of $\varphi_{n,n+1}$, counted with depths. Therefore,

$$d_{a_{n-1,n}}(\varphi_{n,n+1}) \le d - \deg \tilde{\varphi}_{n,n+1} \le d - \deg_{a_{n-1,n}} \tilde{\varphi}_{n,n+1}.$$

Thus, $(5.\ell)$ for all ℓ yields equality between all these quantities. Equivalently, $a_{n-1,n}$ is the unique hole of $\varphi_{n,n+1}$ and the local degree of $\tilde{\varphi}_{n,n+1}$ at this point is $\deg \tilde{\varphi}_{n,n+1}$. Statements (3) and (5) now follow. □

Lemma 3.9. *Statements $(0.n_0)$–$(5.n_0)$ hold.*

Proof. As observed above, Lemma 3.5 is $(0.n_0)$. By Lemmas 3.6 and 3.8, the space $\mathcal{S}_{n_0}$ is the concatenation of spheres indexed by J_{n_0} in the standard order of the integers. Therefore, statement $(1.n_0)$ holds. Statement $(2.n_0)$ follows from the continuity of $\mathcal{F}_{n_0}$, furnished by Proposition A.3. Statement $(3.n_0)$ is trivially true and statement $(4.n_0)$ is a direct consequence of Lemma 3.6 (2). Finally, statement $(5.n_0)$ follows since $\varphi_{n,n+1}$ has no holes and the local degree at $a_{n-1,n}$ is d. □

3.2.3. Proof of the Structure Theorem for not fully ramified times

We continue with the induction. Assume that $(0.\ell)$–$(5.\ell)$ from §3.2.2 hold for some $\ell \ge n_0$. The proofs of the corresponding statements for $\ell+1$ are organized in a series of lemmas below. Specifically, Lemma 3.10 shows that the scaling $\{A_{\ell+1,k}\}$ is independent of the ones corresponding to the spheres $\overline{\mathbb{C}}_m, \dots, \overline{\mathbb{C}}_\ell$. Thus, $\mathcal{S}_{\ell+1}$ consists of pairwise distinct spheres $\overline{\mathbb{C}}_m, \dots, \overline{\mathbb{C}}_{\ell+1}$, which are shown to be organized in a linear concatenation in Lemma 3.11. Then, Lemma 3.12 shows that the transition map $\tilde{\varphi}_{\ell,\ell+1} \colon \overline{\mathbb{C}}_\ell \to \overline{\mathbb{C}}_{\ell+1}$ is a polynomial with a unique hole at the intersection point with $\overline{\mathbb{C}}_{\ell-1}$. In Lemma 3.13, continuity of the induced map on $\mathcal{S}_\ell$ is established. Finally, Lemma 3.14 studies the degree of $\tilde{\varphi}_{\ell,\ell+1}$. We end this subsection by summarizing how all these lemmas yield the inductive step.

We employ the definitions and results from Appendix A. Let $J := \{n : m \le n \le \ell\}$ and consider the collection of independent scalings $\mathbf{A} := \{A_{n,k} : n \in J\}$ indexed by J. The space associated to $\mathbf{A}$ is $\mathcal{S}_\ell$, the concatenation of the spheres $\overline{\mathbb{C}}_m, \dots, \overline{\mathbb{C}}_\ell$. Denote by $\mathcal{A}_k$ the corresponding embedding of $\overline{\mathbb{C}}$ into $\overline{\mathbb{C}}^J$ and by $\rho_{\ell,j}$ the retraction of $\mathcal{S}_\ell$ onto $\overline{\mathbb{C}}_j$. For all $n \ne j$, note that $\rho_{\ell,j}(\overline{\mathbb{C}}_n) = \{a_{n,j}\} \subset \overline{\mathbb{C}}_j$.

Lemma 3.10. *$A_{(\ell+1,j)}$ is degenerate for all $j \in J$.*

Proof. By contradiction, suppose $A_{(\ell+1,j)}$ is nondegenerate. Without loss of generality, we may assume that $A_{\ell+1,k} = A_{j,k}$, for all k. In the language of Appendix A, the sequence $\{f_k\}$ maps $\mathbf{A}$ to $\mathbf{A}$ by $\tau : J \to J$ where $\tau(n) = n+1$ for $n < \ell$ and $\tau(\ell) = j$. Consider $G_k : \mathcal{A}_k(\overline{\mathbb{C}}) \to \mathcal{A}_k(\overline{\mathbb{C}})$ defined by

$$G_k := \mathcal{A}_k \circ f_k \circ \mathcal{A}_k^{-1}.$$

In view of the inductive hypothesis $(5.\ell)$ and Proposition A.2 applied to $j-1$, every point in $\mathcal{A}_k(\overline{\mathbb{C}})$, contained in a small neighborhood W_0 of $\rho_{\ell,j}^{-1}(a_{j-1,j}) = \overline{\mathbb{C}}_m \cup \cdots \cup \overline{\mathbb{C}}_{j-1}$, has d preimages under G_k,

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close to $\rho^{-1}_{\ell,j-1}(a_{j-2,j-1}) = \overline{\mathbb{C}}_m \cup \cdots \cup \overline{\mathbb{C}}_{j-2}$, which is bounded away from $\overline{\mathbb{C}}_\ell$. But W_0 contains points w_0 in $\overline{\mathbb{C}}_j$ close to $a_{j-1,j}$ with a preimage z_0 in $\overline{\mathbb{C}}_\ell$, under $\tilde{\varphi}_{\ell,j} = \tilde{\varphi}_{\ell,\ell+1}$. It follows that points in $\mathcal{A}_k(\overline{\mathbb{C}})$, close to w_0, simultaneously have a G_k-preimage close to $z_0 \in \overline{\mathbb{C}}_\ell$ and d preimages close to $\rho^{-1}_{\ell,j-1}(a_{j-2,j-1})$, which is impossible. □

Let $J' := \{j : m \le n \le \ell + 1\}$. From the previous lemma, $\mathbf{B} := \{A_{n,k} : n \in J'\}$ is a collection of independent scalings indexed by J'. It follows that $\{f_k\}$ maps $\mathbf{A}$ to $\mathbf{B}$ by $\tau(n) = n+1$. For all $j \in J'$, recall that $\rho_{\ell+1,j}$ is the retraction from $\mathcal{S}_{\ell+1}$, the space associated to $\mathbf{B}$, onto $\overline{\mathbb{C}}_j$. Let $\mathcal{B}_k$ be the corresponding embedding of $\overline{\mathbb{C}}$ into $\overline{\mathbb{C}}^{J'}$ and

$$F_k := \mathcal{B}_k \circ f_k \circ \mathcal{A}_k^{-1}.$$

Lemma 3.11. *$a_{\ell+1,\ell} \neq a_{\ell-1,\ell} \in \overline{\mathbb{C}}_\ell$. Moreover, $\mathcal{S}_{\ell+1}$ is obtained from the union of $\overline{\mathbb{C}}_m, \ldots, \overline{\mathbb{C}}_{\ell+1}$ by identifying $a_{n+1,n}$ with $a_{n,n+1}$ for $m \le n < \ell + 1$.*

Proof. By contradiction, suppose that $a_{\ell+1,\ell} = a_{\ell-1,\ell} \in \overline{\mathbb{C}}_\ell$. Then $\overline{\mathbb{C}}_{\ell-1}, \overline{\mathbb{C}}_\ell, \overline{\mathbb{C}}_{\ell+1}$ intersect at $a_{\ell,\ell-1} \in \overline{\mathbb{C}}_{\ell-1}$, which is identified with $a_{\ell-1,\ell} \in \overline{\mathbb{C}}_\ell$. Thus $\rho^{-1}_{\ell+1,\ell}(a_{\ell-1,\ell})$ contains $\overline{\mathbb{C}}_{\ell+1}$. Any point in $\mathcal{B}_k(\overline{\mathbb{C}})$ close to a generic point w_0 of $\overline{\mathbb{C}}_{\ell+1}$ has a F_k-preimage in $\mathcal{A}_k(\overline{\mathbb{C}})$ close to $\overline{\mathbb{C}}_\ell \subset \mathcal{S}_\ell$. Proposition A.2 and $(5.\ell)$ yield a contradiction, since the d preimages of w_0 are close to $\rho^{-1}_{\ell,\ell-1}(a_{\ell-2,\ell-1}) = \overline{\mathbb{C}}_m \cup \cdots \cup \overline{\mathbb{C}}_{\ell-2}$, which is bounded away from $\overline{\mathbb{C}}_\ell$ by $(1.\ell)$. Thus, $a_{\ell+1,\ell} \neq a_{\ell-1,\ell} \in \overline{\mathbb{C}}_\ell$

We conclude that in $\mathcal{S}_{\ell+1}$ the sphere $\overline{\mathbb{C}}_{\ell+1}$ has nonempty intersection only with $\overline{\mathbb{C}}_\ell$; for otherwise, if $\overline{\mathbb{C}}_{\ell+1} \cap \overline{\mathbb{C}}_n \neq \emptyset$ for some $n \le \ell - 1$, then $a_{\ell+1,\ell} = a_{\ell-1,\ell}$. Therefore, $\mathcal{S}_{\ell+1}$ is the claimed concatenation of spheres. □

According to the next lemma, in coordinates for $\overline{\mathbb{C}}_\ell$ and $\overline{\mathbb{C}}_{\ell+1}$ where $a_{\ell-1,\ell}$ and $a_{\ell,\ell+1}$ are at ∞, the map $\tilde{\varphi}_{\ell,\ell+1}$ becomes a polynomial. Moreover, $a_{\ell-1,\ell}$ is the unique hole of $\varphi_{\ell,\ell+1}$, if $\varphi_{\ell,\ell+1} \in \partial\mathrm{Rat}_d$. In particular, $(5.\ell + 1)$ and the first assertion of $(2.\ell + 1)$ hold.

Lemma 3.12. *Let $a := a_{\ell-1,\ell}$, $a' := a_{\ell,\ell+1}$ and $\varphi := \varphi_{\ell,\ell+1}$. Then $\tilde{\varphi}^{-1}(a') = a$ and*

$$d = \deg_a \tilde{\varphi} + d_a(\varphi).$$

Proof. Let z be such that $\tilde{\varphi}(z) = a'$. Since $\rho^{-1}_{\ell+1,\ell+1}(a')$ contains $\overline{\mathbb{C}}_n$ for all $n \le \ell$, in particular, it contains $\overline{\mathbb{C}}_{\ell-1}$. If $w_k \in \mathcal{B}_k(\overline{\mathbb{C}})$ is sufficiently close to a generic point of $\overline{\mathbb{C}}_{\ell-1}$, then it has a preimage close to $\rho^{-1}_{\ell,\ell}(z)$. However, all the F_k-preimages of w_k are close to of $\overline{\mathbb{C}}_m \cup \cdots \cup \overline{\mathbb{C}}_{\ell-2}$, by $(5.\ell)$ and Proposition A.2. It follows that $\rho^{-1}_{\ell,\ell}(z)$ is contained in $\overline{\mathbb{C}}_m \cup \cdots \cup \overline{\mathbb{C}}_{\ell-2}$, whose $\rho_{\ell,\ell}$-image is $\{a\}$. Thus $z = a$ is the unique $\tilde{\varphi}_{\ell,\ell+1}$-preimage of a'. Moreover, since all the F_k-preimages of w_k are close to $\rho^{-1}_{\ell,\ell}(a)$, by Proposition A.2, the second assertion holds. □

Now we prove the second assertion of $(2.\ell + 1)$:

Lemma 3.13. *Let $b := a_{\ell,\ell-1}$, $b' := a_{\ell+1,\ell}$ and $\varphi := \varphi_{\ell-1,\ell}$. Then $\tilde{\varphi}(b) = b'$.*

Proof. By contradiction, suppose that $\tilde{\varphi}(b) \neq b'$. Then $\tilde{\varphi}^{-1}(b')$ consists of finitely many points in $\overline{\mathbb{C}}_{\ell-1}$ bounded away from the rest of the spheres, by $(1.\ell)$ and $(2.\ell)$. Thus, $\rho^{-1}_{\ell,\ell-1}(z) = \{z\}$ for all $z \in \tilde{\varphi}^{-1}(b')$. Therefore, the F_k-preimages of points in $\mathcal{B}_k(\overline{\mathbb{C}})$ close to a generic point of $\overline{\mathbb{C}}_{\ell+1} = \rho^{-1}_{\ell+1,\ell}(b')$ are close to the finite set $\tilde{\varphi}^{-1}(b') \subset \overline{\mathbb{C}}_{\ell-1}$ far from $\overline{\mathbb{C}}_\ell$. This is impossible since $\tilde{\varphi}$ maps $\overline{\mathbb{C}}_\ell$ onto $\overline{\mathbb{C}}_{\ell+1}$. □

We also have the following result on degrees, which gives $(3.\ell + 1)$:

Lemma 3.14. *Let $b := a_{\ell,\ell-1}$. Then $\deg \tilde{\varphi}_{\ell-1,\ell} \ge \deg_b \tilde{\varphi}_{\ell-1,\ell} = \deg \tilde{\varphi}_{\ell,\ell+1}$.*

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Proof. Let $b' := a_{\ell+1,\ell}$. Note that b is not a hole of $\tilde{\varphi}_{\ell-1,\ell}$ by $(5.\ell)$. Let w_0 be a generic point of $\overline{\mathbb{C}}_{\ell+1} = \rho^{-1}_{\ell+1,\ell}(b')$. There are exactly $\deg_b \tilde{\varphi}_{\ell-1,\ell}$ preimages, under F_k, of every point in $\mathcal{B}_k(\overline{\mathbb{C}})$ close to w_0, which are close to $\overline{\mathbb{C}}_\ell = \rho^{-1}_{\ell,\ell-1}(b)$. But there are $\deg \tilde{\varphi}_{\ell,\ell+1}$ preimages of w_0 in $\overline{\mathbb{C}}_\ell$, so $\deg \tilde{\varphi}_{\ell,\ell+1} = \deg_b \tilde{\varphi}_{\ell-1,\ell} \le \deg \varphi_{\ell-1,\ell}$. □

We now conclude the induction by certifying that $(0.\ell+1)$–$(5.\ell+1)$ follow from the above lemmas. Indeed, $(0.\ell+1)$ is a consequence of Lemma 3.10, $(1.\ell+1)$ is Lemma 3.11. Lemma 3.12 implies the first assertion of $(2.\ell+1)$ together with $(5.\ell+1)$. The second assertion of $(2.\ell+1)$ is obtained from Lemma 3.13. Statement $(3.\ell+1)$ is contained in Lemma 3.14. Suppose that $n_0+1=\ell$ is not a fully ramified time. Then $(4.\ell+1)$ reduces to $(4.n_0)$, which we have already proven (Lemma 3.6 (2)). This finishes the proof of Theorem 3.1.

3.3. Measures and depths

Consider a sequence $\{f_k\}$ with a collection of scaling $\{A_{n,k}\}$ for which the Structure Theorem 3.1 applies. Proposition 3.7 says that the Julia set $\mathcal{J}(f_k)$, viewed in $\overline{\mathbb{C}}_n$ for $n > m$, converges to $a_{n-1,n}$. Measure-theoretically we have the following consequence:

Corollary 3.15. *Under the assumption and notation of Theorem 3.1, for any $n > m$, let $A_n := \lim A_{n,k}$ and $a_n := \tilde{A}_n$. Then the following hold:*

1. $a_n = a_{n-1,n}$.
2. $A_{n,k}(\mathcal{J}(f_k))$ *converges in the Hausdorff topology on compact sets to* $\{a_n\}$. *In particular,*

$$(A_{n,k})_*\mu_{f_k} \to \delta_{a_n}.$$

Proof. Consider $n > m$. By Theorem 3.1,

$$T_k := A_{n+m,k} \circ f_k^{\circ m} \circ A_{n,k}^{-1}$$

converges to a map $\varphi \in \mathrm{Rat}^*_{d^m}$ with a unique hole at $a_{n-1,n} = \infty$. If $a_n \neq a_{n-1,n}$, then $A_{n+m,k} \circ f_k^{\circ m} = T_k \circ A_{n,k}$ converges to $\tilde{\varphi}(a_n)$ for generic $z \in \overline{\mathbb{C}}$. By Theorem 3.1 (4), the unique preimage of $a_{m,n+m} = \infty$ under the polynomial $\tilde{\varphi}$ is $a_{n-1,n} = \infty$. Thus, $\tilde{\varphi}(a_n) \neq \infty$. However, $A_{m,k} \circ f_k^{\circ m}$ converges to map with nonconstant reduction $\tilde{\varphi}_m$. We conclude that, generically,

$$(A_{m+n,k} \circ A_{m,k}^{-1}) \circ (A_{m,k} \circ f_k^{\circ m})(z) \to a_{m,m+n} = \infty.$$

This is a contradiction. Hence statement (1) holds.

For statement (2), consider a small neighborhood W of a_n. By statement (1), we have that $a_n = a_{n-1,n}$ and, by Proposition 3.7 and Theorem 3.1, we conclude that W contains $A_{n,k}(\mathcal{J}(f_k))$ for all sufficiently large k. The conclusion follows since $A_{n,k}(\mathcal{J}(f_k))$ is the support of $(A_{n,k})_*\mu_{f_k}$. □

Sequences with 3 consecutive fully ramified times have well-behaved limits in $\widehat{\mathrm{Rat}}_d$:

Corollary 3.16. *Under the assumption and notation in Theorem 3.1, if $f_k^{\circ n}$ converges to $g_n \in \overline{\mathrm{Rat}}_{d^n}$ for all n (i.e., $\{f_k\}$ converges to (g_n) in $\widehat{\mathrm{Rat}}_d$), then for any given $n > m$ the following hold:*

1. $\eta_{g_n} = \eta_{g_{m+1}}$;
2. $\tilde{g}_n = \tilde{g}_{m+1}$ *is a constant in* $\overline{\mathbb{C}}$; *and*
3. *for any* $z \in \overline{\mathbb{C}}$,

$$\frac{d_z(g_n)}{d^n} = \frac{d_z(g_{m+1})}{d^{m+1}}.$$

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Proof. Consider $n > m$. Recall that, in convenient coordinates, we put $a_{n-1,n}$ at ∞. Let us first prove statement (2). The hole h of $A_n = \lim A_{n,k}$ is $\tilde{g}_n$. By Corollary 3.15 (1), generically, $A_{n,k}$ converges to $a_n = a_{n-1,n} = \infty \in \overline{\mathbb{C}}_n$. Off a_n, we have that $A_{n,k}^{-1}$ converges to h. Since $A_n \circ A_{n+1}^{-1}$ converges generically to $a_{n+1,n} \neq a_n$, it follows that $A_{n+1,k}^{-1} = A_{n,k}^{-1} \circ (A_n \circ A_{n+1}^{-1})$ also converges to h. Thus the hole of A_{n+1} is also h and $\tilde{g}_{n+1} = h$.

We now show statement (3), which immediately implies statement (1). It is sufficient to show that $d_z(g_{n+1}) = d_z(g_n) \cdot d$ for all $z \in \overline{\mathbb{C}}$. By Lemma 2.5, if $\tilde{\varphi}_{m+1}(z) \neq a_{m,m+1}$, then $d_z(g_n) = d_z(\varphi_n)$; and if otherwise, $d_z(g_n) = d_z(\varphi_n) + \deg_z \tilde{\varphi}_n$. To compute $d_z(g_{n+1})$, we consider two cases according to whether $\tilde{\varphi}_{m+1}(z) = a_{m,m+1}$ or not, equivalently, $\tilde{\varphi}_n(z) = a_{n-1,n} = \infty \in \overline{\mathbb{C}}_n$ or not since all transition maps are polynomials (Theorem 3.1 (4)).

Suppose that $\tilde{\varphi}_{m+1}(z) \neq a_{m,m+1}$. That is, $\tilde{\varphi}_n(z) \neq a_{n-1,n} = a_n$. Then $\tilde{\varphi}_n(z)$ is not a hole of $\varphi_{n,n+1}$ and $\tilde{\varphi}_{n+1}(z) \neq a_{n+1}$. Therefore, using the formula (2.2), we obtain that

$$d_z(g_{n+1}) = d_z(\varphi_{n+1}) = d_z(\varphi_n) \cdot d = d_z(g_n) \cdot d.$$

Suppose that $\tilde{\varphi}_{m+1}(z) = a_{m,m+1}$. That is, $\varphi_n(z) = a_{n-1,n} = a_n$. Then $\varphi_n(z)$ is the hole of $\varphi_{n,n+1}$, which has depth $d - \deg \tilde{\varphi}_{n,n+1}$ and the local degree at a_n is $\deg \tilde{\varphi}_{n,n+1}$. It follows that

$$\begin{aligned} d_z(g_{n+1}) &= d_z(\varphi_{n+1}) + \deg_z \tilde{\varphi}_{n+1} \\ &= d_z(\varphi_n) \cdot d + \deg_z \tilde{\varphi}_n \cdot d_{a_n}(\varphi_{n,n+1}) + \deg_z \tilde{\varphi}_n \cdot \deg_{a_n} \tilde{\varphi}_{n,n+1} \\ &= d_z(\varphi_n) \cdot d + \deg_z \tilde{\varphi}_n \cdot d \\ &= d_z(g_n) \cdot d. \end{aligned}$$

□

4. Iterations and measures for degenerate maps

In this section, we prove Theorem A. Given $\mathbf{g} = (g_n) \in \widehat{\mathrm{Rat}}_d$, we aim at proving the existence of a probability measure $\mu_{\mathbf{g}}$, continuously depending on $\mathbf{g}$, such that

$$\frac{1}{d^n}(g_n)^* \mu \to \mu_{\mathbf{g}},$$

for a generic probability measure μ. To this end, we consider a sequence $\{f_k\} \subset \mathrm{Rat}_d$ converging to $\mathbf{g}$. First we make some assumptions on the sequence $\{f_k\}$ and prove Theorem A under these assumptions. Later, using compactness of $\widehat{\mathrm{Rat}}_d$ and $M^1(\overline{\mathbb{C}})$, we deduce Theorem A in full generality.

We may simultaneously consider the limits of $f_k^{\circ n}$ and of $[\![f_k^{\circ n}]\!]$; to record these limits, let us introduce the following definition:

Definition 4.1. A sequence $\{f_k\} \subset \mathrm{Rat}_d$ is *totally convergent* if for any $n \geq 1$, there exist $g_n \in \overline{\mathrm{Rat}}_{d^n}$ and $\varphi_n \in \mathrm{Rat}^*_{d^n}$ such that the following hold:

1. $f_k^{\circ n}$ converges to $g_n \in \overline{\mathrm{Rat}}_{d^n}$ (i.e., $\{f_k\}$ converges to (g_n) in $\widehat{\mathrm{Rat}}_d$), and
2. $[\![f_k^{\circ n}]\!]$ converges to $[\![\varphi_n]\!] \in [\![\mathrm{Rat}^*_{d^n}]\!]$.

We say that $\{f_k\}$ *totally converges to* (g_n, φ_n).

To prove Theorem A, we assume that $\{f_k\} \subset \mathrm{Rat}_d$ totally converges to (g_n, φ_n), as $k \to \infty$ (see Definition 4.1). We also assume that the conjugacy classes $\{[f_k]\} \subset \overline{\mathrm{rat}}_d$ converge in moduli space to some $[g] \in \overline{\mathrm{rat}}_d$. Then we consider the following cases:

1. $\deg \tilde{\varphi}_n = o(d^n)$.
2. $\deg \tilde{\varphi}_n \neq o(d^n)$ and
   (a) $[g] \in \mathrm{rat}_d$ or,
   (b) $[g] \in \partial \mathrm{rat}_d$.

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In each case, we show that a measure $\mu_{\mathbf{g}}$ with the desired properties exists and that μ_{f_k} converges to $\mu_{\mathbf{g}}$. After discussing some preliminary facts in §4.1, case (1) is considered in §4.2, case (2a) in §4.3, and case (2b) in §4.4. Finally, in §4.5, we assemble these results to produce a proof of Theorem A.

4.1. Convergence of depth measures

Recall from Definition 2.7 that the depth measure of a map $g \in \partial\mathrm{Rat}_d$ is

$$\eta_g := \sum_{h \in \mathrm{Hole}(g)} d_h(g)\delta_h.$$

Lemma 4.2. *Let $\{f_k\} \subset \mathrm{Rat}_d$ be a sequence such that $[\![f_k^{\circ n}]\!] \to [\![\varphi_n]\!]$ in $[\![\mathrm{Rat}_{d^n}^*]\!]$ for each $n \geq 1$, as $k \to \infty$. Then there exists a purely atomic measure η such that, as $n \to \infty$,*

$$\frac{1}{d^n}\eta_{\varphi_n} \to \eta.$$

Proof. Apply Lemma 2.4 to write $\varphi_{n+1} = \varphi_{n,n+1} \circ \varphi_n$ where $\deg \tilde{\varphi}_{n,n+1} \geq 1$. For all $h \in \overline{\mathbb{C}}$, in view of (2.2), we have

$$d_h(\varphi_{n+1}) = d_h(\varphi_n) \cdot d + \deg_h \tilde{\varphi}_n \cdot d_{\tilde{\varphi}_n(h)}(\varphi_{n,n+1}).$$

Division by d^{n+1} yields that

$$\frac{\eta_{\varphi_n}(\{h\})}{d^n} = \frac{d_h(\varphi_n)}{d^n}$$

is nondecreasing. Let η be the purely atomic measure defined by

$$\eta(\{h\}) := \lim_n \frac{\eta_{\varphi_n}(\{h\})}{d^n}.$$

It follows that $\eta_{\varphi_n}/d^n \to \eta$. □

4.2. Small degree growth

Let $\{f_k\} \subset \mathrm{Rat}_d$ be a sequence totally converging to (g_n, φ_n). The growth of $\deg \tilde{\varphi}_n$ plays a crucial role in our argument. In the case of small degree growth, that is, $\deg \tilde{\varphi}_n = o(d^n)$, we have the following convergence of measures.

Proposition 4.3. *Consider a sequence $\{f_k\} \subset \mathrm{Rat}_d$ totally converging to (g_n, φ_n). Suppose μ is a nonexceptional probability measure for (g_n). Let $\eta := \lim(\eta_{\varphi_n}/d^n)$. Assume that*

$$\deg \tilde{\varphi}_n = o(d^n).$$

Then

$$\eta = \lim_{n\to\infty} \frac{1}{d^n}(g_n)^*\mu = \lim_{k\to\infty} \mu_{f_k}.$$

Proof. Let ν be an accumulation point of μ_{f_k}. Consider $A_{n,k} \in \mathrm{Rat}_1$ such that $A_{n,k} \circ f_k^{\circ n} \to \varphi_n$, as $k \to \infty$, for all n. Passing to a subsequence of $\{f_k\}$, we may assume that μ_{f_k} converges to ν and $(A_{n,k})_*\mu_{f_k}$ converges to some measure ν_n for all n. By Proposition 2.10,

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$$\nu = \frac{1}{d^n}(\varphi_n)^*\nu_n = \frac{1}{d^n}\eta_{\varphi_n} + \frac{1}{d^n}(\tilde{\varphi}_n)^*\nu_n \to \eta,$$

since $(\tilde{\varphi}_n)^*\nu_n(\overline{\mathbb{C}}) = \deg\tilde{\varphi}_n = o(d^n)$. Therefore, the measures μ_{f_k} converge to η.

To show the measure $(g_n)^*\mu/d^n$ converges to η, we now prove that the measures $(g_n)^*\mu$ and η_{φ_n} differ in at most $o(d^n)$. Although the conclusion is the same, there are two cases, according to whether $\tilde{g}_n$ is constant or not.

For all n such that $\tilde{g}_n$ is constant, from Lemma 2.5, the sequence $\{A_{n,k}\}$ converges to some $A_n \in \partial\mathrm{Rat}_1$, as $k \to \infty$, and

$$(g_n)^*\mu = \eta_{g_n} = \eta_{\varphi_n} + \sum_{z\in\tilde{\varphi}_n^{-1}(\tilde{A}_n)} \deg_z\tilde{\varphi}_n\cdot\delta_z = \eta_{\varphi_n} + \varepsilon_n,$$

where $\varepsilon_n(\overline{\mathbb{C}}) = \deg\tilde{\varphi}_n$.

If $\tilde{g}_n$ is not constant, then

$$(g_n)^*\mu = \eta_{g_n} + (\tilde{g}_n)^*\mu = \eta_{\varphi_n} + (\tilde{g}_n)^*\mu = \eta_{\varphi_n} + \varepsilon_n',$$

where $\varepsilon_n'(\overline{\mathbb{C}}) = \deg\tilde{g}_n$.

For all $n \geq 1$, we have that $\deg\tilde{g}_n \leq \deg\tilde{\varphi}_n = o(d^n)$. Therefore, both $\varepsilon_n(\overline{\mathbb{C}})$ and $\varepsilon_n'(\overline{\mathbb{C}})$ are $o(d^n)$, and hence

$$\lim_{n\to\infty}\frac{1}{d^n}(g_n)^*\mu = \lim_{n\to\infty}\frac{1}{d^n}\eta_{\varphi_n} = \eta.$$

This completes the proof. □

We will deal with sequences such that $\deg\tilde{\varphi}_n \neq o(d^n)$ in the next two subsections, but before, let us record the following observation:

Lemma 4.4. *Let $\{f_k\}$ be a sequence totally converging to (g_n, φ_n) and $g_1 \in \partial\mathrm{Rat}_d$. Suppose that $\deg\tilde{\varphi}_n \neq o(d^n)$. Then*

$$0 < \frac{1}{d^n}\deg\tilde{\varphi}_n \leq 1$$

is eventually constant (independent of n). Moreover, for all n

$$\deg\tilde{g}_n = 0.$$

Proof. Let $\varphi_{n,n+1} \in \mathrm{Rat}_d^*$ be such that $\varphi_{n+1} = \varphi_{n,n+1}\circ\varphi_n$, as in Lemma 2.4. Since $\deg\tilde{\varphi}_{n+1} = \deg\tilde{\varphi}_{n,n+1}\cdot\deg\tilde{\varphi}_n$ and $\deg\tilde{\varphi}_n \neq o(d^n)$, it follows that $\deg\tilde{\varphi}_{n,n+1} = d$ for all n sufficiently large. Therefore, $\deg\tilde{\varphi}_n/d^n$ is eventually constant in $(0, 1]$.

We now prove that $\deg\tilde{g}_n = 0$, by contradiction. First observe that $g_n \in \partial\mathrm{Rat}_{d^n}$ because $g_1 \in \partial\mathrm{Rat}_d$ and iteration is a proper map. Suppose that, for some n_0, we have $\deg\tilde{g}_{n_0} \geq 1$. Then $\deg\tilde{\varphi}_{\ell n_0} = \deg\tilde{g}_{\ell n_0} = \deg\tilde{g}_{n_0}^{\circ\ell} = o(d^{n_0\ell})$, as $\ell \to \infty$, since $\deg\tilde{g}_{n_0} < d^{n_0}$. Therefore, $\deg\tilde{\varphi}_{\ell n}/d^{n\ell}$ is not eventually constant, as $\ell \to \infty$, which is a contradiction. □

4.3. Potential good reduction

Here we consider the case in which $\{f_k\}$ totally converges to (g_n, φ_n) with $\deg\tilde{\varphi}_n \neq o(d^n)$ and $\{[f_k]\} \subset \mathrm{rat}_d$ converges in rat_d. In order to state the hypothesis that are really used, we consider a slightly more general case:

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Proposition 4.5. *Let* $\{f_k\} \subset \mathrm{Rat}_d$ *be a sequence converging to* $g \in \partial\mathrm{Rat}_d$. *Assume that* $\{[f_k]\}$ *converges in moduli space* rat_d. *Then* g *has a unique hole, say* h. *Moreover, if* $f_k^{\circ n} \to g_n$, *as* $k \to \infty$ *and* $\tilde{g}_n$ *is constant, for all* n, *then*

$$\lim_{k\to\infty} \mu_{f_k} = \delta_h = \lim_{n\to\infty} \frac{1}{d^n}(g_n)^*\mu$$

for all probability measures μ *that are nonexceptional for* (g_n).

Proof. Let $\{L_k\} \subset \mathrm{Rat}_1$ be such that $\psi_k := L_k \circ f_k \circ L_k^{-1}$ converges (uniformly) to some $\psi \in \mathrm{Rat}_d$, as $k \to \infty$. We may assume that $\{L_k\}$ converges to a degenerate Möbius transformation with a hole at h and with constant reduction a. By contradiction we prove that h is the unique hole of g. Suppose that $h' \neq h$ is a hole of g. Consider a small neighborhood U of h'. Pick a generic point $z_0 \in \overline{\mathbb{C}}$ such that $z_0 \neq \psi(a)$. For sufficiently large k, from Lemma 2.6 (1), we deduce that $z_0 \in L_k \circ f_k(U)$; therefore, $z_0 \in \psi_k \circ L_k(U)$. The uniform convergence, in U, of $\{L_k\}$ to the constant a implies that $z_0 = \psi(a)$, which contradicts the choice of z_0. Thus the first assertion holds.

We now prove the second assertion. Since the delta measure μ_ψ is nonexceptional for $L = \lim L_k$ and $\mu_{\psi_k} \to \mu_\psi$, we conclude that $\mu_{f_k} = (L_k)^*\mu_{\psi_k} \to \delta_h$, as $k \to \infty$, by Proposition 2.9. To establish $\delta_h = (g_n)^*\mu/d^n$, note that $L_k \circ f_k^{\circ n} \circ L_k^{-1}$ converges to $\psi^{\circ n} \in \mathrm{Rat}_{d^n}$. From the previous paragraph, we conclude that h is the unique hole of g_n, for all $n \geq 1$. Moreover, $d_h(g_n) = d^n$ because $\deg \tilde{g}_n = 0$. Hence $(g_n)^*\mu/d^n = \delta_h$, for all probability measures μ that are nonexceptional for (g_n). □

4.4. Degeneration with large degree growth

The aim of this section is to apply the Structure Theorem 3.1 and its corollaries, to prove the following:

Proposition 4.6. *Let* $\{f_k\}$ *be a sequence in* Rat_d *totally converging to* (g_n, φ_n) *such that* $[f_k]$ *converges to* $[g] \in \partial\mathrm{rat}_d$. *Assume that* $\deg \tilde{\varphi}_n \neq o(d^n)$. *Then there exists a probability measure* ν *such that*

$$\nu = \lim_{k\to\infty} \mu_{f_k} = \lim_{n\to\infty} \frac{1}{d^n}(g_n)^*\mu,$$

for any probability measure μ *that is nonexceptional for* (g_n).

Proof. Consider an accumulation point ν of $\{\mu_{f_k}\}$. After passing to a subsequence, we may assume $\mu_{f_k} \to \nu$. We may also assume that there exists a collection of scalings $\{A_{n,k} : n, k \geq 1\}$ for $\{f_k\}$ with convergent changes of coordinates. Since $\deg \tilde{\varphi}_n \neq o(d^n)$, any sufficiently large n is a fully ramified time. Thus, we may apply Corollary 3.15 (2) to conclude that $(A_{n,k})_*\mu_{f_k} \to \delta_{a_n}$, for all sufficiently large n. By Proposition 2.10, we have that

$$\nu = \frac{1}{d^n}(\varphi_n)^*\delta_{a_n},$$

and, therefore,

$$\begin{aligned}\nu &= \frac{1}{d^n}\left(\sum_{h\in\mathrm{Hole}(\varphi_n)} d_h(\varphi_n)\cdot\delta_h + (\tilde{\varphi}_n)^*\delta_{a_n}\right)\\ &= \frac{1}{d^n}\left(\sum_{h\in\mathrm{Hole}(\varphi_n)} d_h(\varphi_n)\cdot\delta_h + \sum_{\tilde{\varphi}_n(h)=a_n} \deg_h \tilde{\varphi}_n\cdot\delta_h\right).\end{aligned}$$

By Lemma 4.4, we may assume that g_n has constant reduction and apply Lemma 2.5 to conclude that $\mathrm{Hole}(g_n) = \mathrm{Hole}(\varphi_n) \cup \tilde{\varphi}_n^{-1}(a_n)$. Then

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$$\nu = \frac{1}{d^n} \sum_{h \in \mathrm{Hole}(g_n)} d_h(g_n) \cdot \delta_h = \frac{1}{d^n} g_n^* \mu,$$

for any probability measure μ that is nonexceptional for g_n and, as a consequence, μ_{f_k} converges to ν. □

4.5. *Proof of Theorem A*

Given any sequence $\{f_k\} \subset \mathrm{Rat}_d$ converging to $\mathbf{g} = (g_n) \in \widehat{\mathrm{Rat}}_d$, we simultaneously show that there exists a probability measure $\mu_{\mathbf{g}}$ such that

$$\frac{1}{d^n} (g_n)^* \mu \to \mu_{\mathbf{g}},$$

and that if along a subsequence of $\{f_k\}$ we have convergence of the measures μ_{f_k} to a measure ν, then $\nu = \mu_{\mathbf{g}}$.

So given a subsequence such that the measures of maximal entropy converge, there exists a further subsequence $\{f_{k_i}\}$ such that $\{f_{k_i}\}$ totally converges to (g_n, φ_n) and $[f_{k_i}] \to [g] \in \overline{\mathrm{rat}}_d$. If $\deg \tilde{\varphi}_n = o(d^n)$ or $[g] \in \mathrm{rat}_d$, by Propositions 4.3 and 4.5, there exists a probability measure $\mu_{\mathbf{g}}$ such that

$$\frac{1}{d^n} (g_n)^* \mu \to \mu_{\mathbf{g}} = \nu.$$

If $[g] \in \partial \mathrm{rat}_d$ and $\deg \tilde{\varphi}_n \neq o(d^n)$, then we apply Proposition 4.6 to obtain the same conclusion.

5. Continuous extension of barycentered measures

The moduli space rat_d is a complex orbifold of dimension $2d-2$. Its compactification $\overline{\mathrm{rat}}_d$ is the categorical quotient of semistable points in $\overline{\mathrm{Rat}}_d$ by the $\mathrm{SL}(2,\mathbb{C})$ conjugation action, see [17]. According to [3, Section 3], the (semi)stable points can be characterized by the depths; more precisely,

- a point $\varphi \in \overline{\mathrm{Rat}}_d$ is *semistable* if $d_z(\varphi) \leq (d+1)/2$ for any $z \in \overline{\mathbb{C}}$ and $\tilde{\varphi}(h) \neq h$ for any $h \in \overline{\mathbb{C}}$ with $d_h(\varphi) \geq d/2$;
- a point $\varphi \in \overline{\mathrm{Rat}}_d$ is *stable* if $d_z(\varphi) \leq d/2$ for any $z \in \overline{\mathbb{C}}$ and $\tilde{\varphi}(h) \neq h$ for any $h \in \overline{\mathbb{C}}$ with $d_h(\varphi) \geq (d-1)/2$.

In §5.1, we review basic facts from [3]. In particular, to each element in rat_d one may continuously assign a well-defined barycentered measure of maximal entropy, modulo push-forward by a rotation in $\mathrm{SO}(3)$. In §5.2, we establish Theorem B, which asserts continuous extension of this assignment to the resolution space $\widehat{\mathrm{rat}}_d \subset \Pi_{n\geq 1} \overline{\mathrm{rat}}_{d^n}$ of the iterate map $\overline{\mathrm{rat}}_d \dashrightarrow \Pi_{n\geq 1} \overline{\mathrm{rat}}_{d^n}$ discussed in the introduction §1.

5.1. *Barycentered measures*

Recall that $M^1(\overline{\mathbb{C}})$ is the space of probability measures endowed with the weak*-topology. Let $M_o^1(\overline{\mathbb{C}}) \subset M^1(\overline{\mathbb{C}})$ be the subset of probability measures with no atom of weight $\geq 1/2$. Then, under the identifications of $\overline{\mathbb{C}}$ with $S^2 \subset \mathbb{R}^3$ by the stereographic projection and of hyperbolic space $\mathbb{H}^3$ with the unit ball in $\mathbb{R}^3$, the *conformal barycenter* $C(\mu) \in \mathbb{H}^3$ of $\mu \in M_o^1(\overline{\mathbb{C}})$ is well-defined and continuous on $M_o^1(\overline{\mathbb{C}})$. More precisely, given $\mu \in M_o^1(\overline{\mathbb{C}})$, the barycenter $C(\mu)$ is uniquely determined by the following two properties:

1. $C(\mu) = 0 \in \mathbb{H}^3$ if and only if $\int_{S^2} x d\mu(x) = 0$; and
2. $C(A_* \mu) = A(C(\mu))$ for any $A \in \mathrm{Rat}_1 \cong \mathrm{Isom}^+ \mathbb{H}^3$.

The conformal barycenter $C(\mu)$ of any $\mu \in M^1(\overline{\mathbb{C}}) \setminus M_o^1(\overline{\mathbb{C}})$ is undefined. For more details, see [3, Section 8] and [5].

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A measure $\mu \in M^1(\overline{\mathbb{C}})$ is *barycentered* if $C(\mu)$ is well-defined and equals 0. Denote by $M^1_{bc}(\overline{\mathbb{C}}) \subset M^1(\overline{\mathbb{C}})$ the subset of barycentered measures, and let $\overline{M}^1_{bc}(\overline{\mathbb{C}})$ be the closure of $M^1_{bc}(\overline{\mathbb{C}})$ in $M^1(\overline{\mathbb{C}})$. Each element in $\overline{M}^1_{bc}(\overline{\mathbb{C}})$ is either barycentered or of the form $(\delta_a + \delta_{-1/\bar{a}})/2$. for some $a \in \overline{\mathbb{C}}$, see [3, Lemma 8.3]. Then both $M^1_{bc}(\overline{\mathbb{C}})$ and $\overline{M}^1_{bc}(\overline{\mathbb{C}})$ are invariant under the push-forward action of the compact group of rotations $\mathrm{SO}(3) \subset \mathrm{PSL}(2,\mathbb{C})$. Consider the quotient topological spaces

$$M^1_{dm}(\overline{\mathbb{C}}) := M^1_{bc}(\overline{\mathbb{C}})/\mathrm{SO}(3) \quad \text{and} \quad \overline{M}^1_{dm}(\overline{\mathbb{C}}) := \overline{M}^1_{bc}(\overline{\mathbb{C}})/\mathrm{SO}(3).$$

The space $M^1_{dm}(\overline{\mathbb{C}})$ is locally compact, Hausdorff and its one-point compactification is $\overline{M}^1_{dm}(\overline{\mathbb{C}})$, see [3, Theorem 8.1]. Denote by $[\infty]$ the unique point in $\overline{M}^1_{dm}(\overline{\mathbb{C}}) \setminus M^1_{dm}(\overline{\mathbb{C}})$.

For each $\mu \in M^1_o(\overline{\mathbb{C}})$, there exists $A \in \mathrm{Rat}_1$ such that $A_*\mu \in M^1_{bc}(\overline{\mathbb{C}})$. Writing $[\mu] := [A_*\mu] \in M^1_{dm}(\overline{\mathbb{C}})$, one obtains a continuous map

$$\begin{array}{ccc} M^1_o(\overline{\mathbb{C}}) & \to & M^1_{dm}(\overline{\mathbb{C}}) \\ \mu & \mapsto & [\mu], \end{array}$$

which induces a continuous map $\mathrm{rat}_d \to M^1_{dm}(\overline{\mathbb{C}})$, sending $[f]$ to $[\mu_f]$. We remark here that the above continuous map can not extend continuously to $M^1(\overline{\mathbb{C}})$; for instance, considering $\mu_n \in M^1(\overline{\mathbb{C}})$ such that $\mu_n(\{0\}) = \mu_n(\{1/n\}) = \mu_n(\{1\}) = 1/3$, we have that $[\lim_{n\to\infty} \mu_n] = [\infty] \neq \lim_{n\to\infty}[\mu_n]$.

5.2. *Proof of Theorem B*

Consider a sequence $\{[f_k]\} \subset \mathrm{Rat}_d$ converging to $([g_n])$ in $\widehat{\mathrm{rat}}_d$. Let $\mu \in M^1(\overline{\mathbb{C}})$ be a nonexceptional measure for (g_n). For notational simplicity,

$$\mu_k := \mu_{f_k}$$
$$\nu_n := \frac{(g_n)^*\mu}{d^n}.$$

Given accumulation points $[\mu_\infty]$ of $\{[\mu_k]\}$ and $[\nu]$ of $\{[\nu_n]\}$, after some work, we will show that $[\nu] = [\mu_\infty]$, that is, Theorem B holds. By continuity of the measure of maximal entropy for nondegenerate rational maps [13], if $[g_1] \in \mathrm{rat}_d$, equivalently, $[f_k]$ converges in rat_d, then $[\nu] = [\mu_\infty]$. Thus we work under the assumption that

$$[g_1] \in \partial \mathrm{rat}_d.$$

Passing to a subsequence in k, we assume that $[\mu_k]$ converges to $[\mu_\infty]$. Passing to a subsequence n_i, we assume that $[\nu_{n_i}]$ converges to $[\nu]$. It is sufficient to establish that $[\mu_\infty]$ and $[\nu]$ coincide when $[\mu_\infty] \neq [\infty]$ or $[\nu] \neq [\infty]$.

Lemma 5.1. *Assume that* $[\mu_\infty] \neq [\infty]$. *Then*

$$[\nu] = \lim_{n\to\infty} [\nu_n] = [\mu_\infty].$$

Proof. After changing the representative of $[f_k]$, we may assume that μ_{f_k} is barycentered, for all k. Passing to a subsequence, $\{f_k\}$ converges in $\widehat{\mathrm{Rat}}_d$, say to $\mathbf{h} = (h_n)$. By Theorem A, we have that μ_{f_k} converges to some measure $\mu_{\mathbf{h}}$. Since $[\mu_\infty] \neq [\infty]$, we have $[\mu_{\mathbf{h}}] = [\mu_\infty] \neq [\infty]$. Therefore, $\mu_{\mathbf{h}}(z) < 1/2$ for all $z \in \overline{\mathbb{C}}$. Hence, for n sufficiently large, h_n is GIT-stable. Therefore, $[f_k^{\circ n}]$ converges to $[h_n]$ and $[h_n] = [g_n]$. In particular, $[(h_n)^*\mu/d^n] = [\nu_n]$ provided that μ is not exceptional for h_n, for all n. The sequence (h_n) is uniquely determined up to $\mathrm{SO}(3)$ conjugacy (the same for all n). For any

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given n, there are at most countably many elements of $\mathrm{SO}(3)$ so that μ is exceptional for the conjugacy of h_n by that element. Hence we may find a conjugate so that μ is not exceptional for all h_n. Thus the conclusion holds. □

Now we assume that $[\nu] \neq [\infty]$. Passing to a further subsequences of $\{f_k\}$ and $\{g_{n_i}\}$ the following hold:

1. There exists $\varepsilon > 0$ such that, for all n_i and all $z \in \overline{\mathbb{C}}$,
$$d_z(g_{n_i}) < \left(\frac{1}{2} - 2\varepsilon\right) \cdot d^{n_i}.$$
2. For each n_i, there exists $B_{n_i,k} \in \mathrm{Rat}_1$ such that, as $k \to \infty$,
$$B_{n_i,k} \circ f_k^{\circ n_i} \circ B_{n_i,k}^{-1} \to g_{n_i}.$$
3. Let
$$f_{n_i,k} := B_{n_i,k} \circ f_k \circ B_{n_i,k}^{-1}.$$
Then, for all n_i and m there exists $g_{n_i,m} \in \overline{\mathrm{Rat}}_{d^m}$ such that
$$f_{n_i,k}^{\circ m} \to g_{n_i,m}.$$
4. For each n_i and each $m \geq 1$, there exist a collection of scalings $\{A_{n_i,m,k}\}$ and maps $\psi_{n_i,m} \in \mathrm{Rat}_{d^m}^*$ such that, as $k \to \infty$,
$$A_{n_i,m,k} \circ f_{n_i,k}^{\circ m} \to \psi_{n_i,m}.$$
5. There exists $c \in [0,1]$ such that, as $n_i \to \infty$,
$$\frac{\deg \tilde{\psi}_{n_i,n_i}}{d^{n_i}} \to c \in [0,1].$$

Note that g_{n_i} is GIT-stable for each n_i by statement (1). Thus we may choose $B_{n_i,k}$ as in statement (2). Statements (3) and (4) say that $\{f_{n_i,k}\}$ totally converges to $(g_{n_i,m}, \psi_{n_i,m})$ with associated scalings $\{A_{n_i,m,k}\}$.

Observe that $g_{n_i,n_i} = g_{n_i}$. If $\tilde{g}_{n_i,m}$ is nonconstant, then in (4) we choose the scaling $A_{n_i,m,k}$ to be the identity and, therefore, $\psi_{n_i,m} = g_{n_i,m}$.

For brevity, write $\psi_{n_i} := \psi_{n_i,n_i}$. We establish the following two lemmas according to the growth of $\deg \tilde{\psi}_{n_i}$; more precisely, according to whether $c > 0$ or $c = 0$.

Lemma 5.2. *Suppose that $c > 0$, that is, $\deg \tilde{\psi}_{n_i} \neq o(d^{n_i})$. Given a sufficiently large n_i, for any $\ell \geq n_i$ and any $z \in \overline{\mathbb{C}}$,*
$$\frac{d_z(g_{n_i,\ell})}{d^\ell} = \frac{d_z(g_{n_i})}{d^{n_i}}.$$
Moreover,
$$[\nu] = [\nu_{n_i}] = [\mu_\infty].$$

Proof. Since $c > 0$, for sufficiently large n_i, the number of times $m < n_i$ such that $\deg \tilde{\psi}_{n_i,m+1} < d \cdot \deg \tilde{\psi}_{n_i,m}$ is bounded independently of n_i. Thus, we can pick N sufficiently large so that for any $n_i \geq N$, there is $1 \leq m_i \leq n_i - 3$ satisfying that $A_{n_i,m_i+3,k} \circ f_{n_i,k}^{\circ 3} \circ A_{n_i,m_i,k}^{-1}$ converges in Rat_{d^3}. Since

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we are working under the assumption that $[g_1] \in \partial \mathrm{rat}_d$, by Corollary 3.16, we conclude that for any $\ell \geq n_i$ and any $z \in \overline{\mathbb{C}}$, the reduction $\tilde{g}_{n_i,\ell}$ is a constant independent of ℓ and

$$\frac{d_z(g_{n_i,\ell})}{d^\ell} = \frac{d_z(g_{n_i})}{d^{n_i}}.$$

Hence

$$\frac{(g_{n_i,\ell})^*\mu}{d^\ell} = \frac{(g_{n_i})^*\mu}{d^{n_i}} = \nu_{n_i}.$$

Since by Theorem A

$$\lim_{\ell\to\infty} \frac{(g_{n_i,\ell})^*\mu}{d^\ell} = \lim_{k\to\infty} \mu_{f_{n_i,k}},$$

we have that

$$\nu_{n_i} = \lim_{k\to\infty} \mu_{f_{n_i,k}}.$$

Observing that $\nu_{n_i} \in M_o^1(\overline{\mathbb{C}})$ since g_{n_i} is stable, by the continuity of the map $M_o^1(\overline{\mathbb{C}}) \to M_{dm}^1(\overline{\mathbb{C}})$, we obtain that

$$[\nu_{n_i}] = \lim_{k\to\infty} [\mu_{f_{n_i,k}}] = \lim_{n\to\infty} [\mu_{f_k}] = [\mu_\infty].$$

Thus the conclusion follows. □

Lemma 5.3. *Suppose that $c = 0$, that is, $\deg \tilde{\psi}_{n_i} = o(d^{n_i})$. Then*

$$[\nu] = [\mu_\infty].$$

Proof. By Lemma 5.1, we may assume that $[\mu_\infty] = [\infty]$. Passing to a subsequence in k we can assume that $B_{n_i,k} \circ B_{n_j,k}^{-1}$ converges to some $B_{n_j,n_i} \in \overline{\mathrm{Rat}_1}$ for any $n_i \neq n_j$. Then passing to a further subsequence in n_i if necessary, we can assume that $B_{n_j,n_i} \in \mathrm{Rat}_1$ for all n_i, n_j or $B_{n_j,n_i} \in \partial\mathrm{Rat}_1$ for all n_i, n_j.

Case 1: $B_{n_j,n_i} \in \mathrm{Rat}_1$ for all $n_i \neq n_j$. Since $\{f_{n_1,k}\}$ totally converges to $(g_{n_1,m}, \psi_{n_1,m})$, passing to a subsequence in k, we may choose $\{A_{n_1,n_i,k}\}$ so that $\psi_{n_i} = \psi_{n_1,n_i} \circ C_{n_i}$ for some $C_{n_i} \in \mathrm{Rat}_1$.

By Lemma 4.2, let

$$\nu' := \lim_{i\to\infty} \eta_{\psi_{n_1,n_i}} \in M^1(\overline{\mathbb{C}}),$$

where $\eta_{\psi_{n_1,n_i}}$ is the corresponding depth measure from Definition 2.7. We claim that $\nu' \in M_o^1(\overline{\mathbb{C}})$ and hence $[\nu'] \neq [\infty]$. Indeed, given $z \in \overline{\mathbb{C}}$, let $w_i = C_{n_i}^{-1}(z)$, then

$$\nu'(\{z\}) = \lim \frac{d_z(\psi_{n_1,n_i})}{d^{n_i}} = \lim \frac{d_{w_i}(\psi_{n_i})}{d^{n_i}} \leq \lim \frac{d_{w_i}(g_{n_i})}{d^{n_i}} \leq \frac{1}{2} - 2\varepsilon.$$

The first equality follows from the proof of Lemma 4.2 since $\nu'(\{z\})$ is the limit of the nondecreasing sequence $\{\eta_{\psi_{n_1,n_i}}(\{z\}) = d^{-n_i} \cdot d_z(\psi_{n_1,n_i})\}$. The second equality follows from $\psi_{n_i} = \psi_{n_1,n_i} \circ C_{n_i}$. The first inequality is Lemma 2.5 applied to $z = w_i$ and $\{f_{n_i,k}^{\circ n_i}\}$. The second inequality is statement (1).

Moreover, since $\deg \tilde{\psi}_{n_i} = o(d^{n_i})$, we have that $\deg \tilde{\psi}_{n_1,n_i} = o(d^{n_i})$, as $i \to \infty$, and hence $\deg \tilde{\psi}_{n_1,n} = o(d^n)$, as $n \to \infty$. Thus, applying Proposition 4.3 to $\{f_{n_1,k}\}$, we conclude that

$$\lim_{k\to\infty} \mu_{f_{n_1,k}} = \lim_{i\to\infty} \frac{(g_{n_1,n_i})^*\mu'}{d^{n_i}} = \nu',$$

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whenever $\mu' \in M^1(\overline{\mathbb{C}})$ is any nonexceptional measure for $(g_{n_1,n})$. Continuity of the map $M_o^1(\overline{\mathbb{C}}) \to M_{dm}^1(\overline{\mathbb{C}})$ yields

$$[\mu_\infty] = \lim_{n\to\infty} [\mu_{f_k}] = \lim_{k\to\infty} [\mu_{f_{n_1,k}}] = [\nu'].$$

Hence $[\mu_\infty] \neq [\infty]$, which is a contradiction.

Case 2: $B_{n_j,n_i} \in \partial \mathrm{Rat}_1$ for any $n_i \neq n_j$. We show that this case cannot occur under our assumptions; more precisely, by counting preimages, we will obtain a contradiction with statement (1).

Since $\deg \tilde{\psi}_{n_i} = o(d^{n_i})$, we consider n_{i_0} such that for $n_i \geq n_{i_0}$,

$$\deg \tilde{\psi}_{n_i} < \varepsilon \cdot d^{n_i}.$$

Let $\ell := n_{i_0}$ and $s := n_{i_0+1}$. For brevity write

$$\gamma_k := B_{\ell,k} \circ B_{s,k}^{-1}$$

with $\gamma_k \to B_{(s,\ell)}$ and $\gamma_k^{-1} \to B_{(\ell,s)}$ in $\overline{\mathrm{Rat}_1}$. Let $b_\ell := \tilde{B}_{s,\ell}$ and $b_s := \tilde{B}_{\ell,s}$.

Intuitively, our argument is as follows. We restrict our attention to the collection of scalings $\{B_{\ell,k}, B_{s,k}\}$ with associated space formed by two spheres $\overline{\mathbb{C}}_\ell$ and $\overline{\mathbb{C}}_s$. We consider the set of preimages S of a generic point in $\overline{\mathbb{C}}_s$, under an appropriate iterate $f_{s,k}^{\circ s}$. Then statement (1) implies that less than half of the elements of S map, under $f_{s,k}^{\circ s} \circ \gamma_k^{-1}$, close to b_ℓ, the hole of $B_{(\ell,s)}$. So more than half of the elements are bounded away from b_ℓ. Applying the change of coordinates γ_k^{-1}, one concludes that more than half of the elements of S are close to b_s, which is a contradiction with statement (1).

More precisely, consider

$$\alpha_k := B_{s,k} \circ f_k^{\circ s-\ell} \circ B_{\ell,k}^{-1},$$
$$\beta_k := B_{\ell,k} \circ f_k^{\circ \ell} \circ B_{\ell,k}^{-1}.$$

Note that

$$f_{s,k}^{\circ s} = \alpha_k \circ \beta_k \circ \gamma_k \to g_s.$$

Let V be a small neighborhood of b_s. For $w \in \overline{\mathbb{C}}$, we now estimate $\#(f_{s,k}^{-s}(w) \cap V)$. First observe that $\#(\alpha_k)^{-1}(w) = d^{s-\ell}$. Given a small neighborhood D of b_ℓ, since $d_{b_\ell}(g_\ell) < (1/2 - 2\varepsilon) \cdot d^\ell$ from the statement (1) in the assumptions on $\{f_k\}$ and $\{g_{n_i}\}$, by Lemma 2.6 (2), we have that

$$\#((\alpha_k \circ \beta_k)^{-1}(w) \cap D) < (1/2 - 2\varepsilon) \cdot d^s + \varepsilon \cdot d^s = (1/2 - \varepsilon) \cdot d^s.$$

Therefore,

$$\#((\alpha_k \circ \beta_k)^{-1}(w) \cap (\overline{\mathbb{C}} \setminus D)) \geq (1/2 + \epsilon) \cdot d^s.$$

In $\overline{\mathbb{C}} \setminus D$, we have uniform convergence of γ_k^{-1} to the constant b_s. We conclude that

$$\#(f_{s,k}^{\circ -s}(w) \cap V) \geq (1/2 + \varepsilon) \cdot d^s.$$

By Lemma 2.6 (2), we have that $d_{b_s}(g_s) \geq d^s/2$, which contradicts statement (1). □

This completes the proof of Theorem B.

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A. Trees of bouquets of spheres

A.1. Hausdorff limits of spheres

Let J be a finite set of indices. Consider a collection of scalings

$$\mathbf{A} := \{A_{n,k} : n \in J, k \geq 1\} \subset \mathrm{Rat}_1,$$

indexed by J. Assume that for all $n, n' \in J$ with $n \neq n'$, there exists $A_{(n,n')} \in \partial\mathrm{Rat}_1$ such that

$$A_{n',k} \circ A_{n,k}^{-1} \to A_{(n,n')} \text{ as } k \to \infty.$$

We say that $\mathbf{A}$ is *a collection of independent scalings indexed by J.*

For $n \neq n'$, let

$$a_{n,n'} := \widetilde{A}_{(n,n')}.$$

Note that

$$a_{n',n} = \mathrm{Hole}(A_{(n,n')}).$$

For each $n \in J$, consider a copy of $\overline{\mathbb{C}}$ inside $\overline{\mathbb{C}}^J$:

$$\overline{\mathbb{C}}_n := \left\{(w_j) \in \overline{\mathbb{C}}^J : w_j = a_{n,j} \text{ for all } j \neq n\right\}.$$

Denote by $\rho_n : \overline{\mathbb{C}}^J \to \overline{\mathbb{C}}$ the projection to the n-th coordinate. This projection provides a *standard coordinate for* $\overline{\mathbb{C}}_n$. That is, $\rho_n : \overline{\mathbb{C}}_n \to \overline{\mathbb{C}}$ is a conformal isomorphism.

Consider the union of Riemann spheres

$$\mathcal{S} := \bigcup_{n \in J} \overline{\mathbb{C}}_n \subset \overline{\mathbb{C}}^J.$$

We will show that $\mathcal{S}$ is a tree of bouquets of spheres that arises as the Hausdorff limit of Riemann spheres. We say that $\mathcal{S}$*is the space associated to* $\mathbf{A}$.

For all $n \neq n' \in J$, note that $\rho_n(\overline{\mathbb{C}}_{n'}) = \{a_{n',n}\}$. Moreover, the intersection $\overline{\mathbb{C}}_{n'} \cap \overline{\mathbb{C}}_n$ is not empty if and only if

$$a_{n',n''} = a_{n,n''}$$

for all $J \ni n'' \neq n, n'$. In this case, the intersection point is $a_{n',n} \in \overline{\mathbb{C}}_n$ and $a_{n,n'} \in \overline{\mathbb{C}}_{n'}$ in the respective standard coordinates.

An alternative description of $\mathcal{S}$ can be given as the quotient of the disjoint union of abstract Riemann spheres $\sqcup_{n \in J} \overline{\mathbb{C}}_n$ by an equivalence relation $\sim$. In fact, let $\sim$ be the equivalence relation such that $a_{n',n} \in \overline{\mathbb{C}}_n$ is identified with $a_{n,n'} \in \overline{\mathbb{C}}_{n'}$ if and only if

$$a_{n,n''} = a_{n',n''}$$

for all $n'' \neq n, n'$. All other $\sim$-classes are trivial. Then

$$\mathcal{S} \equiv \sqcup_{n \in J} \overline{\mathbb{C}}_n / \sim .$$

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Associated to $\mathbf{A}$ we also have a sequence of embedding of $\overline{\mathbb{C}}$ in $\overline{\mathbb{C}}^J$:

$$\begin{aligned} \mathcal{A}_k : \overline{\mathbb{C}} &\to \quad \overline{\mathbb{C}}^J \\ z &\mapsto (A_{n,k}(z))_{n\in J}. \end{aligned}$$

Proposition A.1. *Consider a collection* $\mathbf{A}$ *of independent scalings indexed by* J*. Let* $\mathcal{S} \subset \overline{\mathbb{C}}$ *be the space associated to* $\mathbf{A}$*. If* $\mathcal{A}_k : \overline{\mathbb{C}} \to \overline{\mathbb{C}}^J$ *is the associated sequence of embeddings, then*

$$\mathcal{S} = \cap_{k_0\geq 1}\overline{\cup_{k\geq k_0}\mathcal{A}_k(\overline{\mathbb{C}})}.$$

Moreover, $\mathcal{S}$ *is simply connected.*

Proof. Consider points $(w_j) \in \mathcal{S}$ such that $w_\ell \neq a_{j,\ell}$ and $w_j = a_{\ell,j}$ for all $j \neq \ell$. Such elements are dense in $\overline{\mathbb{C}}_\ell$. Denoting by $z_k := A^{-1}_{\ell,k}(w_\ell)$, we have that $\mathcal{A}_k(z_k) \to (w_j)$. So $\mathcal{S}$ is contained in the Hausdorff limit of $\{\mathcal{A}_k(\overline{\mathbb{C}})\}$.

We prove that the Hausdorff limit of $\{\mathcal{A}_k(\overline{\mathbb{C}})\}$ is contained in $\mathcal{S}$. Suppose on the contrary that $\{z_k\}$ is a sequence such that $\mathcal{A}_k(z_k)$ converges to some $(u_j) \notin \mathcal{S}$. Then for all $j \in J$, we may choose $J \ni s(j) \neq j$ such that $u_{s(j)} \neq a_{j,s(j)}$. Therefore, $u_j = a_{s(j),j}$ since the change of coordinates $A_{j,k} \circ A^{-1}_{s(j),k}$ converges uniformly in compact subsets of $\overline{\mathbb{C}} \setminus \{a_{j,s(j)}\}$ to $a_{s(j),j}$. Now consider $j_0 \in J$ and let $j_i = s^{\circ i}(j_0)$ for $i \geq 1$. We may assume that $j_p = j_0$ for a minimal $p \geq 2$. Since $a_{j_0,j_1} \neq u_{j_1} = a_{j_2,j_1}$, it follows that $a_{j_2,j_0} = a_{j_1,j_0}$. Recursively, we obtain that $a_{j_{p-1},j_0} = a_{j_1,j_0} = u_{j_0}$. However, $u_{j_0} = u_{s(j_{p-1})} \neq a_{j_{p-1},j_0}$. It is a contradiction.

Any Hausdorff limit of connected sets is connected, so $\mathcal{S}$ is connected. We now show that $\mathcal{S}$ is simply connected. By contradiction, suppose that for some $q \geq 2$, there exists a cycle of spheres of length q. That is, there exist a chain of pairwise distinct spheres $\overline{\mathbb{C}}_{\ell_0}, \ldots, \overline{\mathbb{C}}_{\ell_{q-1}}$ that intersect at pairwise distinct points $x_0, \cdots, x_{q-1}$; that is, $\{x_i\} = \overline{\mathbb{C}}_{\ell_i} \cap \overline{\mathbb{C}}_{\ell_{i+1}}$ for all $0 \leq i < q$, subscripts mod q. Then $x_{j+1} - x_j$ is a vector in $\mathbb{C}^J$ with unique nonvanishing component in its ℓ_j-coordinate for all j. But $\sum x_{j+1} - x_j = x_0 - x_0 = 0$, which is a contradiction. □

A.2. Maps between trees of bouquets of spheres

Let J, J' be finite sets. Consider collections

$$\mathbf{A} := \{A_{n,k} : n \in J, k \geq 1\} \subset \mathrm{Rat}_1,$$

$$\mathbf{B} := \{B_{n,k} : n \in J', k \geq 1\} \subset \mathrm{Rat}_1.$$

Assume that $\mathbf{A}$ and $\mathbf{B}$ are collections of independent scalings.

Consider a sequence $\{f_k\} \subset \mathrm{Rat}_d$. We say that $\{f_k\}$ *maps* $\mathbf{A}$ *into* $\mathbf{B}$ *by* $\tau : J \to J'$ if for all $n \in J$,

$$B_{\tau(n),k} \circ f_k \circ A^{-1}_{n,k}$$

converges to some element $\varphi_{n,\tau(n)} \in \mathrm{Rat}^*_d$. When $\varphi_{n,\tau(n)} \in \mathrm{Rat}_d$, for all $n \in J$, we say that $\{f_k\}$ is *fully ramified* at $\mathbf{A}$.

In coordinates, given by the associated embeddings $\mathcal{A}_k$ and $\mathcal{B}_k$ of $\overline{\mathbb{C}}$ into $\overline{\mathbb{C}}^J$ and $\overline{\mathbb{C}}^{J'}$, the maps f_k become

$$\begin{aligned} F_k : \mathcal{A}_k(\overline{\mathbb{C}}) &\to \quad \mathcal{B}_k(\overline{\mathbb{C}}) \\ w &\mapsto \mathcal{B}_k \circ f_k \circ \mathcal{A}_k^{-1}(w). \end{aligned}$$

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In the limit, F_k is, in a certain sense, semiconjugate to the action of $\varphi_{n,\tau(n)}$ on $\overline{\mathbb{C}}_n$. More precisely, provided that z is not a hole of $\varphi_{n,\tau(n)}$,

$$\rho_{\tau(n)} \circ F_k(z) \to \tilde{\varphi}_{n,\tau(n)}(\rho_n(z)).$$

In general, we have the following:

Proposition A.2. *Consider a sequence $\{f_k\} \subset \mathrm{Rat}_d$ and collections of independent scalings* **A** *and* **B** *indexed by J and J' with associated spaces $\mathcal{S}$ and $\mathcal{S}'$, respectively, such that $\{f_k\}$ maps* **A** *to* **B** *by $\tau : J \to J'$. Let $\mathcal{A}_k$ and $\mathcal{B}_k$ be the associated embeddings of $\overline{\mathbb{C}}$ into $\overline{\mathbb{C}}^J$ and $\overline{\mathbb{C}}^{J'}$, respectively. Let*

$$\begin{aligned} F_k : \mathcal{A}_k(\overline{\mathbb{C}}) &\to \mathcal{B}_k(\overline{\mathbb{C}}) \\ w &\mapsto \mathcal{B}_k \circ f_k \circ \mathcal{A}_k^{-1}(w). \end{aligned}$$

Consider $z_0 \in \overline{\mathbb{C}}_j \subset \mathcal{S}$ and $w_0 \in \overline{\mathbb{C}}_{\tau(j)} \subset \mathcal{S}'$. Then, for every small neighborhood $U_0 \subset \overline{\mathbb{C}}^J$ of z_0, there exists a neighborhood $W_0 \subset \overline{\mathbb{C}}^{J'}$ of w_0 such that for all $w \in \mathcal{B}_k(\overline{\mathbb{C}}) \cap W_0$,

$$\#F_k^{-1}(w) \cap U_0 = \begin{cases} d_{z_0}(\varphi_{j,\tau(j)}), & \text{if } \tilde{\varphi}_{j,\tau(j)}(z_0) \neq w_0, \\ d_{z_0}(\varphi_{j,\tau(j)}) + \deg_{z_0} \tilde{\varphi}_{j,\tau(j)}, & \text{if } \tilde{\varphi}_{j,\tau(j)}(z_0) = w_0. \end{cases}$$

Proof. For simplicity, let $S_k := \mathcal{A}_k(\overline{\mathbb{C}})$ and $S'_k := \mathcal{B}_k(\overline{\mathbb{C}})$. Note that $S_k \cap \rho_j^{-1}(U) = \mathcal{A}_k \circ A_{j,k}^{-1}(U)$ for any $U \subset \overline{\mathbb{C}}$. A similar statement holds of S'_k.

Given a small neighborhood U_0 of z_0, by compactness, it contains $\rho_j^{-1}(U'_0)$ for some neighborhood U'_0 of $\rho_j(z_0)$ in $\overline{\mathbb{C}}$. Let $h_k := A_{\tau(j),k} \circ f_k \circ A_{j,k}^{-1}$. Then $h_k \to \varphi_{j,\tau(j)}$. By Lemma 2.6 (1), there exists a neighborhood W'_0 of $w'_0 := \rho_j(w_0)$ in $\overline{\mathbb{C}}$ such that

$$\#h_k^{-1}(w') \cap U'_0 = d_{z_0}(\varphi_{j,\tau(j)}) + \deg_{z_0} \varphi_{j,\tau(j)},$$

for all $w' \in W'_0$. Let $W_0 := \rho_{\tau(j)}^{-1}(W'_0)$. Given $w \in S_k \cap W_0$, consider $w' \in W'_0$ such that $\mathcal{B}_k \circ A_{\tau(j),k}^{-1}(w') = w$. Now, $z' \in U'_0$ is such that $h_k(z') = w'$ if and only if $z = \mathcal{A}_k \circ A_{j,k}^{-1}(z')$ is such that $F_k(z) = w$. That is, $\mathcal{A}_k \circ A_{j,k}^{-1}$ is a bijection between F_k-preimages of w in U_0 and h_k-preimages of w' in U'_0. □

A.3. Fully ramified maps

Let us now consider the special case in which the limiting degree at each sphere is maximal.

Proposition A.3. *Consider a sequence $\{f_k\} \subset \mathrm{Rat}_d$ that maps* **A** *to* **B** *by a bijection $\tau : J \to J'$, where* **A** *and* **B** *are independent collections of scalings with associated spaces $\mathcal{S}$ and $\mathcal{S}'$. If $\{f_k\}$ is fully ramified at* **A***, then there is a well defined and continuous map $\mathcal{F} : \mathcal{S} \to \mathcal{S}'$ given by:*

$$\mathcal{F}(z) := \varphi_{n,\tau(n)}(z) \in \overline{\mathbb{C}}_{\tau(n)},$$

for $z \in \overline{\mathbb{C}}_n$ and $n \in J$. Moreover, there exist a finite set $\mathrm{Crit}(\mathcal{F}) \subset \mathcal{S}$ and a (multiplicity) function $m : \mathrm{Crit}(\mathcal{F}) \to \mathbb{N}$ such that the following hold:

1.

$$\sum_{\omega \in \mathrm{Crit}(\mathcal{F})} m(\omega) = 2d - 2.$$

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2. *The multiplicity of a critical point x of $\varphi_{n,\tau(n)}$ is*

$$\sum_{\omega\in\rho_n^{-1}(x)\cap \mathrm{Crit}(\mathcal{F})} m(\omega).$$

We say that $\omega \in \mathrm{Crit}\,\mathcal{F}$ is a *critical point of multiplicity* $m(\omega)$ of $\mathcal{F}$.

Proof. Recall that to obtain $\mathcal{S}$, we identify $a_{n,n'}$ with $a_{n',n}$ if and only if

$$a_{n',n''} = a_{n,n''}$$

for all $J \ni n'' \neq n, n'$. Similarly, we identify points of different spheres in $\mathcal{S}'$. Since $\tau : J \to J'$ is a bijection, for the continuity of $\mathcal{F}$, it suffices to check that $\varphi_{n,\tau(n)}(a_{n',n}) = a_{\tau(n'),\tau(n)}$ for all $n \neq n' \in J$. Choose $z \in \overline{\mathbb{C}}_{n'}$ such that $z \neq a_{n,n'}$ and $\varphi_{n',\tau(n')}(z) \neq a_{\tau(n),\tau(n')}$. Then, uniformly in a neighborhood of z, the change of coordinates $A_{n,k} \circ A_{n',k}^{-1}$ converges to $a_{n',n}$. Thus

$$(A_{\tau(n),k} \circ f_k \circ A_{n,k}^{-1}) \circ (A_{n,k} \circ A_{n',k}^{-1})(z) \to \varphi_{n,\tau(n)}(a_{n',n}).$$

Reorganizing the above expression yields

$$(A_{\tau(n),k} \circ A_{\tau(n'),k}^{-1}) \circ (A_{\tau(n'),k} \circ f_k \circ A_{n',k}^{-1})(z) \to a_{\tau(n'),\tau(n)}.$$

Hence $\varphi_{n,\tau(n)}(a_{n',n}) = a_{\tau(n'),\tau(n)}$, as claimed.

Passing to a subsequence, we may label the critical points of f_k by $c_k(1), \ldots, c_k(2d-2)$ with repetitions according to multiplicity. We may also assume that $\mathcal{A}_k(c_k(i))$ converges to $c(i) \in \mathcal{S}_{N_0,N}$, as $k \to \infty$, for all i. We say that $\omega \in \mathcal{S}_N$ is a *critical point of* $\mathcal{F}$ *of multiplicity* m if there exist exactly m indices i such that $c(i) = \omega$. Let $\mathrm{Crit}(\mathcal{F})$ be the set formed by the critical points. Note that, a priori, $\mathrm{Crit}(\mathcal{F})$ depends on the choice of the subsequence. It can be shown that it is independent, but we will not use nor prove this fact.

Given $x \in \overline{\mathbb{C}}$, since $A_{n+1,k} \circ f_k \circ A_{n,k}^{-1}$ converges locally uniformly to $\tilde{\varphi}_{n,n+1}$ away from the holes of $\varphi_{n,n+1}$, the number of critical points of $A_{n+1,k} \circ f_k \circ A_{n,k}^{-1}$ in a neighborhood W of x, counted with multiplicity, is $m := \deg_x \tilde{\varphi}_{n,n+1} - 1$. That is, the number of critical point of f_k in $A_{n,k}^{-1}(W)$ is m. Therefore, $\mathcal{A}_k(A_{n,k}^{-1}(W))$ contains m critical points of $\mathcal{F}$ for all k sufficiently large. That is, $\rho_n^{-1}(x)$ contains m critical points $\omega \in \mathrm{Crit}\,\mathcal{F}$, counted with multiplicities. □

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