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AFFINE REFLECTION SUBGROUPS OF COXETER GROUPS

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Abstract

In this paper, we study affine reflection subgroups and their root subsystems in arbitrary infinite Coxeter groups of finite rank. We give a characterization of limit roots (the accumulation points of the projections of roots onto suitably chosen hyperplanes) associated with affine reflection subgroups. We describe the relationship between these limit roots and all other limit roots. In particular, we show that each limit root associated with an affine reflection subgroup may only produce positive values with finitely many positive roots under the bilinear form afforded by the Tits representation (thus satisfying a so-called *Pos-finite* condition); and we show that an arbitrary limit root producing positive bilinear form values with only finitely many positive roots must be a convex combination of limit roots associated with affine reflection subgroups. As an application of this Pos-finite condition, we give an independent characterization of when a finitely generated infinite Coxeter group may possess affine reflection subgroups. We also use this Pos-finite analysis to partially resolve the question of whether the projections of an infinite injective sequence of roots totally ordered by the dominance partial ordering possesses or not a unique point of accumulation.

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1. Introduction

A *Coxeter system* (W, R) consists of an abstract group W and a generating set R comprising involutions. The group W, called a *Coxeter group*, is generated by elements of R subject only to braid relations on pairs of generators. Such a group W can be realized, via the classical *Tits representation*, as a reflection group acting on a real vector space V, where the set of *reflections* is taken to mean the set $T := \bigcup_{w \in W} wRw^{-1}$. A *reflection subgroup* of W is a subgroup of W generated by a subset of T. It is well known that a reflection subgroup of a Coxeter group is itself

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a Coxeter group. The classification of the reflection subgroups of Coxeter groups is of considerable importance and it arises in a multitude of mathematical contexts. Complete classifications of the reflection subgroups of finite Coxeter groups and affine Coxeter groups can be found in [12]. However, for general Coxeter groups, the classification of their reflection subgroups is yet to be completed. Among all reflection subgroups, those isomorphic to affine Coxeter groups are called *affine reflection subgroups*. In this paper, we study affine reflection subgroups in Coxeter groups with finitely many generators.

The *root system* of a Coxeter group W is the collection of all *roots* that are representative -1-eigenvectors of the reflections in W. It is well known that the root system of a Coxeter group W is a discrete set in V. However, through an approach first systematically developed in [21] (whose underlying philosophy may trace back to the work in [24]), it has been observed that, when dealing with finitely generated infinite Coxeter groups, the projections of their roots onto suitably chosen hyperplanes are always contained in certain compact sets. These projections of roots are called *normalized roots* and they may exhibit intricate asymptotic behaviours as observed in [11, 21]. The set of accumulation points of normalized roots for a Coxeter group W is denoted by $E(W)$ and its elements are called the *limit roots* of W. It was observed in [21] that $E(W)$ is contained in the isotropic cone of V with respect to the bilinear form arising from the Tits representation, and there is a natural W-action on $E(W)$ and it was later shown in [11] that this action was minimally faithful. In [10, 11], it has been further proven that the convex hull of $E(W)$ is the topological closure of the projection of the so-called *imaginary cone* onto the chosen hyperplane in which the normalized roots lie. The notion of an imaginary cone was first introduced in the context of Kac–Moody algebra as the cone generated by positive imaginary roots (for an accessible reference, please refer to [24]) and has been later adapted to the setting of Coxeter groups as a (potentially not strict) subset of the so-called *dual of the Tits cone*. An authoritative work on the imaginary cones of Coxeter groups can be found in [10]. Also, in [18], one may uncover a treasure trove of related materials.

In this paper, we study the limit roots arising from affine reflection subgroups of a Coxeter group W. In the case when W is infinite but finitely generated, we give a characterization of the set of limit roots arising from affine reflection subgroups. If η is one such limit root, we show that η satisfies the following.

(i) The cardinality of the set $\{x \in \Phi^+ \mid (x, \eta) > 0\}$ is finite, where Φ^+ denotes the set of positive roots and $(\ ,\ )$ denotes the bilinear form arising from the Tits representation, henceforth known as the *Pos-finite* condition (Theorem 5.5).

(ii) The support of η is a connected set and remains connected (in the sense of Coxeter graphs) under the action of W (Proposition 5.22).

(iii) η is in the intersection of the imaginary cone and the normalized isotropic cone (Proposition 5.11).

We prove that if a limit root η satisfies Conditions (i) and (ii), then η must be a limit root arising from an affine reflection subgroup (Theorem 5.23), and we show that a point in

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the normalized isotropic cone is a limit root arising from an affine reflection subgroup if and only if it satisfies Condition (ii) and can be acted on by an element of W into the fundamental domain of the imaginary cone (Theorem 5.25). We also show that, if a limit root η satisfies Condition (i) but not Condition (ii), then η must be a nontrivial linear combination of limit roots arising from affine reflection subgroups. In fact, we give a characterization of when every point in the convex hull of certain limit roots arising from affine reflection subgroups are themselves limit roots (Proposition 5.30).

Moreover, by studying the properties of limit roots arising from affine reflection subgroups of finitely generated infinite Coxeter groups, we obtain an independent characterization of when (irreducible) finitely generated infinite Coxeter groups may possess affine reflection subgroups; indeed, in Corollary 5.27, we observe that irreducible affine reflection subgroups of finitely generated infinite Coxeter groups are precisely the infinite reflection subgroups of conjugates of standard affine parabolic subgroups. Similar observations concerning affine reflection subgroups with three or more generators can be found in [5, Theorem 3.3].

Furthermore, to the best of our knowledge, the following two natural questions concerning limit roots and the imaginary cone are not explicitly answered in the literature: which points in the isotropic set are necessarily limit roots; and is the containment of the imaginary cone in the dual of the Tits cone proper? It turns out that by studying those limit roots satisfying the Pos-finite condition and those limit roots not satisfying the Pos-finite condition, we prove that the containment of the imaginary cone in the dual of the Tits cone is strict in the case of finitely generated nonaffine infinite Coxeter groups (Corollary 5.12), with the imaginary cone containing precisely those limit roots satisfying the Pos-finite condition and the complement of the imaginary cone in the dual of the Tits cone containing those limit roots that do not satisfy the Pos-finite condition. In this process, we actually obtain a description of which points in the isotropic cone must be limit roots (Proposition 5.19).

Finally, we give an application of limit roots corresponding to affine reflection subgroups to partially resolve an open problem of whether or not an infinite sequence of normalized roots whose corresponding positive roots are totally ordered by an important partial ordering, called *dominance*, possesses a unique point of accumulation (conjectured in [11]). The dominance partial ordering, first introduced in [4], was defined on the root systems of infinite Coxeter groups, and has a natural interpretation and application in geometric group theory, known as *parallel walls*. Refinement of the existing literature on this partial ordering may supply experts in both representation theory and geometric group theory with further tools to study infinite Coxeter groups. In this paper, we show how the study of limit roots and dominance of finitely generated infinite Coxeter groups may intertwine. Specifically, we prove in Proposition 6.5 the following result: if any accumulation point of an infinite sequence of normalized roots whose corresponding positive roots are totally ordered by dominance is a limit root arising from affine reflection subgroups, then that sequence of normalized roots is, in fact, convergent, with the affine limit root being the sole point of accumulation.

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Moreover, this enables us to construct concrete prototypes of convergent sequences of normalized roots other than the well-known dihedral cases.

The organization of this paper beyond this introduction section is as follows: in Section 2, we recall basic definitions and properties of Coxeter groups. It contains a slight generalization to the classical Tits representation of Coxeter groups in a construction known as a *Coxeter datum* (following the works of [14, 17, 25]). We recall properties of reflection subgroups in Coxeter groups and, specifically, we recall the beautiful characterization of *canonical generators* of reflection subgroups, as obtained in [8, 9]. In Section 3, we recall a characterization of the root systems of affine Coxeter groups taken from [24]. Also, we recall an elegant result from [25, 26] establishing a Coxeter group to be affine if the bilinear form associated with its Tits representation admits a nonzero radical. We also collect a few basic properties of the dominance partial ordering defined in the root systems of infinite Coxeter groups (as contained in [4, 13, 14]). In Section 4, we recall the construction of normalized roots and limit roots as pioneered in [21]. In Section 5, we present the new results of this paper as described in the earlier part of this introduction. For readers familiar with foundational results in Coxeter groups [2, 23], reflection subgroups [8, 9], the dominance partial ordering [4], and limit roots [21], they may commence directly from Section 5. In Section 6, we give an application of limit roots relating to affine reflection subgroups to the study of infinite injective sequences of roots totally ordered by dominance. In Proposition 6.5, we partially resolve a conjecture raised in [11]. We also give some characterizations of when a trio of roots ordered by dominance generate an affine reflection subgroup.

2. Definitions and notation

For a set A, define $\mathrm{PLC}(A)$, the *positive linear combination of* A, to be the set given by

$$\left\{ \sum_{a \in A'} \lambda_a a \mid A' \subseteq A, A' \text{ is finite and nonempty, } \lambda_a > 0 \text{ for all } a \in A' \right\};$$

and we define $\mathrm{cone}(A)$, the *positive cone spanned by* A, to be the set

$$\mathrm{cone}(A) = \mathrm{PLC}(A) \cup \{0\};$$

and we use the notation of $\mathrm{conv}(A)$ to denote the *convex hull* of A.

Suppose that (W, R) is a Coxeter system in the sense of [20] or [23] with R being a set of involutions generating W subject only to the condition that $(st)^{m_{st}} = 1$ for all $s \neq t \in R$ for some $m_{st} \in \mathbb{N}_{\geq 2} \cup \{\infty\}$. We stress that the same abstract Coxeter group W may be realized in a number of different ways, that is, different Coxeter systems may give rise to the same abstract group with distinct generating sets and, possibly more importantly, different sets of reflections. For an elementary example, one may consider the dihedral group of order 20 and the direct product of $\mathbb{Z}_2$ with the dihedral group of order 10, in which the same abstract group can be realized differently as Coxeter groups with incompatible sets of reflections. Having this in mind, in this paper, in most cases,

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we take the preference of referring to a Coxeter system (W, R) instead of just referring to a Coxeter group W; and when we simply use the expression 'let W be a Coxeter group', the reader should interpret it as a statement concerning not only the group W, but more importantly, the particular presentation specified by a Coxeter system (W, R).

Following the tradition of [25, 26], we make the following definition of a root basis.

DEFINITION 2.1. Given a Coxeter system (W, R), let V be a vector space over $\mathbb{R}$ and let $(\ ,\ )$ be a symmetric bilinear form on V, and let $\Pi = \{\alpha_s \mid s \in R\}$ be a set of vectors of V in bijective correspondence with R. Then, Π is called a *root basis* if the following hold:

(C1) $(\alpha_s, \alpha_s) = 1$ for all $s \in R$, and if s, t are distinct generators, then either $(\alpha_s, \alpha_t) = -\cos(\pi/m_{st})$ if m_{st} is a finite integer, or else $(\alpha_s, \alpha_t) \leq -1$ if $m_{st} = \infty$;
(C2) $0 \notin \mathrm{PLC}(\Pi)$.

When Π is a root basis, the triple $\mathscr{C} = (V, \Pi, (\ ,\ ))$ is called a *Coxeter datum*. For each nonisotropic $a \in V$ (that is, $(a, a) \neq 0$), let ρ_a be the *reflection* corresponding to a; that is, $\rho_a \in \mathrm{GL}(V)$ is defined by $\rho_a x = x - 2((x, a)/(a, a))a$ for all $x \in V$. Let $G_{\mathscr{C}}$ be the subgroup of $\mathrm{GL}(V)$ generated by $\{\rho_{\alpha_s} \mid s \in R\} = \{\rho_a \mid a \in \Pi\}$. Let $\phi_{\mathscr{C}} \colon W \to G_{\mathscr{C}}$ be the *Tits representation*. It is the unique isomorphism with

$$\phi_{\mathscr{C}} \colon W \to G_{\mathscr{C}}, \text{ satisfying } \phi_{\mathscr{C}}(r_a) = \rho_a \quad \text{for all } a \in \Pi.$$

Under the Tits representation, there is a $(\ ,\ )$-preserving W-action on V: for each $w \in W$ and $x \in V$, define $wx \in V$ by $wx = \phi_{\mathscr{C}}(w)x$. We call (W, R) the *abstract Coxeter system* associated to the Coxeter datum $\mathscr{C}$ and we call W a *Coxeter group* of rank $\#R$ (where $\#$ denotes cardinality). We use $\ell \colon W \to \mathbb{N}$ to denote the *length function* of W (with respect to R).

Let $\Phi \subset V$ be the *root system* of the Coxeter system (W, R) arising from the Coxeter datum $\mathscr{C}$. That is,

$$\Phi = W\Pi = \{wa \mid w \in W,\ a \in \Pi\}.$$

The set $\Phi^+ = \Phi \cap \mathrm{PLC}(\Pi)$ is called the set of *positive roots* and the set $\Phi^- = -\Phi^+$ is called the set of *negative roots*.

When Π is a basis for the space V, for each $x \in V$, there is a unique expression

$$x = \sum_{a \in \Pi} \mathrm{coeff}_a(x)a, \quad \mathrm{coeff}_a(x) \in \mathbb{R} \quad \text{for all } a \in \Pi,$$

and we define the *support* of x to be

$$\mathrm{supp}(x) = \{a \in \Pi \mid \mathrm{coeff}_a(x) \neq 0\}.$$

Define $T = \bigcup_{w \in W} wRw^{-1}$. We call T the *set of reflections* in W. For each $x \in \Phi$, we have $x = w\alpha_s$ for some $w \in W$ and $s \in R$ (or equivalently, $\alpha_s \in \Pi$). Define $r_x = wsw^{-1} \in T$. Then, $\phi_{\mathscr{C}}(r_x) = \rho_x$ and we call r_x the *reflection corresponding to* x. It is readily checked that $T = \{r_x \mid x \in \Phi\}$.

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Define functions $N\colon W \to \mathcal{P}(\Phi^+)$ and $\overline{N}\colon W \to \mathcal{P}(T)$ (where $\mathcal{P}$ denotes the power set) by requiring $N(w) = \{x \in \Phi^+ \mid wx \in \Phi^-\}$ and $\overline{N}(w) = \{t \in T \mid \ell(wt) < \ell(w)\}$ for all $w \in W$. Note that then, $\overline{N}(w) = \{r_x \mid x \in N(w)\}$.

DEFINITION 2.2. A subgroup W' of W is a *reflection subgroup* of W if $W' = \langle T' \rangle$ for some $T' \subseteq T$. If W' is a reflection subgroup of W, we write $W' \leq W$; furthermore, we let

$$R(W') = \{t \in T \mid \overline{N}(t) \cap W' = \{t\}\}$$

and

$$\Pi(W') = \{x \in \Phi^+ \mid r_x \in R(W')\}.$$

We call $R(W')$ the set of *canonical generators* of W' and we call $\Pi(W')$ the set of *canonical roots* of W'.

It was shown by Dyer [9, Theorems 3.3] and Deodhar [6, 7] that $(W', R(W'))$ is a Coxeter system whenever W' is a reflection subgroup of W, and furthermore, [9, Theorem 4.4] gave a characterization of $\Pi(W')$: for $\Delta \subseteq \Phi^+$, let $W' = \langle r_x \mid x \in \Delta \rangle$. Then, $\Delta = \Pi(W')$ if and only if

$$(a, b) \in \{-\cos(\pi/n) \mid n \in \mathbb{N} \text{ and } n \geq 2\} \cup (-\infty, -1],$$

whenever $a, b \in \Delta$ are distinct. We use this characterization quite often in the rest of this paper.

Let $(\ ,\ )'$ be the restriction of $(\ ,\ )$ to the subspace $\mathbb{R}(\Pi(W'))$ of V spanned by $\Pi(W')$. Then, $\mathscr{C}' = (\mathbb{R}(\Pi(W')), \Pi(W'), (\ ,\ )')$ is a Coxeter datum with $(W', R(W'))$ being the associated abstract Coxeter system. Thus, the notion of a root system applies to $\mathscr{C}'$. We let $\Phi(W')$, $\Phi^+(W')$, and $\Phi^-(W')$ be respectively the set of roots, positive roots, and negative roots for the datum $\mathscr{C}'$. Then, $\Phi(W') = W'\Pi(W')$ and [9, Theorem 3.3] yields that

$$\Phi(W') = \{x \in \Phi \mid r_x \in W'\},$$

and we call $\Phi(W')$ the *root subsystem* of the reflection subgroup W'. We denote $\Phi^+(W') = \Phi(W') \cap \mathrm{PLC}(\Pi(W'))$ and $\Phi^-(W') = -\Phi^+(W')$, the sets of positive and negative roots in this subsystem. In this paper, a reflection subgroup W' is called a *dihedral reflection subgroup* if $\#R(W') = 2$. Moreover, we distinguish two types of infinite dihedral reflection subgroups: *infinite affine reflection subgroups* and *infinite nonaffine dihedral reflection subgroups*. Specifically, if W' is an infinite dihedral reflection subgroup of W and $\Pi(W') = \{a, b\} \subset \Phi^+$, then

$$(a, b) \leq -1.$$

If $(a, b) = -1$, then W' is an infinite affine dihedral reflection subgroup; whereas if $(a, b) < -1$, then W' is an infinite nonaffine dihedral reflection subgroup. Note that the two types of infinite dihedral reflection groups are each isomorphic to the abstract infinite dihedral Coxeter group $\widetilde{A}_1$.

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A *standard parabolic subgroup* W_M of W is defined as follows: for $M \subseteq \Pi$, let $R_M := \{r_a \mid a \in M\}$ and $W_M := \langle R_M \rangle$. Observe that $\Phi(W_M) = \Phi \cap \mathbb{R}M$. We may also use the notation W_{R_M} in place of W_M. A *parabolic subgroup* of W is any conjugate of a standard parabolic subgroup of W.

3. Some background results

DEFINITION 3.1. Let (W, R) be a Coxeter system with $\mathscr{C} = (V, \Pi, (\ ,\ ))$ being an associated Coxeter datum. We define the following two sets related to $(\ ,\ )$:

(i) the *radical* of $(\ ,\ )$, denoted by Rad is given by

$$\text{Rad} := \{v \in V \mid (v, u) = 0 \text{ for all } u \in V\};$$

(ii) the *isotropic cone* of $(\ ,\ )$, denoted by Q, is given by

$$Q := \{v \in V \mid (v, v) = 0\}.$$

It is well known that in an irreducible affine Coxeter group W, the bilinear form $(\ ,\ )$ afforded by the Tits representation of W has a radical with the following properties.

PROPOSITION 3.2 [23, Proposition 2.6 and Theorem 2.7]. *Let (W, R) be a Coxeter system in which W is an irreducible affine Coxeter group and R is a finite set of generators, and let $\mathscr{C} = (V, \Pi, (\ ,\ ))$ be an associated Coxeter datum. The bilinear form $(\ ,\ )$ has a radical* Rad *such that:*

(i) $\dim(\text{Rad}) = 1$*;*
(ii) *the radical* Rad *coincides with the isotropic cone Q.*

LEMMA 3.3 [24, Proposition 6.3]. *Let (W, R) be a Coxeter system in which W is an irreducible affine Coxeter group (in the sense of [23]), with associated Coxeter datum $\mathscr{C} = (V, \Pi, (\ ,\ ))$ and corresponding root system Φ. Each element of Φ is congruent modulo* Rad *to an element of the root subsystem corresponding to some fixed finite standard parabolic subgroup of W.*

The following is a key result that affords an elegant method to ascertain whether a Coxeter group W is affine or not, based on the radical associated with its Coxeter datum $\mathscr{C} = (V, \Pi, (\ ,\ ))$.

LEMMA 3.4 [26, Lemma 6.1.1]. *Let W be an irreducible infinite Coxeter group of finite rank. Then, W is nonaffine implies that*

$$\text{Rad} \cap \text{cone}(\Pi) = \{0\}.$$

The next result is also well known.

PROPOSITION 3.5 [21, Lemma 4.9]. *Suppose that W is a Coxeter group of finite rank. Then, W is infinite if and only if W contains an infinite dihedral reflection subgroup.*

DEFINITION 3.6. Let W be a Coxeter group and let W' be a reflection subgroup of W with $x, y \in \Phi(W')$.

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(i) We say that x *dominates* y with respect to W' if

$$\{w \in W' \mid wx \in \Phi^-(W')\} \subseteq \{w \in W' \mid wy \in \Phi^-(W')\}.$$

If x dominates y with respect to W', then we write $x \operatorname{dom}_{W'} y$; furthermore, we write $x \overset{\neq}{\operatorname{dom}}_{W'} y$ if $x \operatorname{dom}_{W'} y$ with $x \neq y$ and we simply write $x \operatorname{dom} y$ in the case where $x \overset{\neq}{\operatorname{dom}}_{W} y$.

(ii) Define $D(x) = \{y \in \Phi^+ \mid x \overset{\neq}{\operatorname{dom}} y\}$. If $x \in \Phi^+$ and $D(x) = \emptyset$, then we call x *elementary*. For each nonnegative integer n, define $D_n = \{x \in \Phi^+ \mid \#D(x) = n\}$.

It is clear from the above definition that for each $x \in \Phi^+$, the set $D(x)$ is finite; indeed, if $x \operatorname{dom} y$ and $y \in \Phi^+$, then $r_x y \in \Phi^-$, whence $y \in N(r_x)$, a finite set of cardinality $\ell(r_x)$.

It was shown in [4] by Brink and Howlett that when a Coxeter group W is finitely generated, then the set of elementary roots is finite. This finiteness property then enabled Brink and Howlett to establish that all finitely generated Coxeter groups are automatic. Brink later gave a complete construction of D_0 for all such Coxeter groups in [3]. Subsequently, in [13], it was shown that in a finitely generated Coxeter group W, the sets D_n are finite for all $n \in \mathbb{N}$, and furthermore, each D_n is nonempty for all infinite Coxeter groups.

It is readily checked that dominance with respect to any reflection subgroup W' of a Coxeter group W is a partial ordering on $\Phi(W')$. The following lemma summarizes some basic properties of dominance.

LEMMA 3.7 [13, Lemma 2.2].

(i) *Let $x, y \in \Phi$ be arbitrary. There is dominance between x and y if and only if $(x, y) \geq 1$.*

(ii) *Dominance is W-invariant, that is, if $x \operatorname{dom} y$, then $wx \operatorname{dom} wy$ for all $w \in W$.*

(iii) *Let $x, y \in \Phi$ be such that $x \operatorname{dom} y$. Then, $-y \operatorname{dom} -x$.*

We close this section with the following easy but useful result.

LEMMA 3.8. *Suppose that $a, b \in \Phi^+$ with $(a, b) \leq -1$ and suppose that $x \in \Phi$ with $x \operatorname{dom} a$. Then, $(x, b) \leq -1$.*

PROOF. Since $(a, b) \leq -1$, it follows that $a \operatorname{dom} -b$. Thus, we have $x \operatorname{dom} a \operatorname{dom} -b$ and, hence, $(x, b) \leq -1$. □

4. Limit roots

Throughout this section, we fix a Coxeter system (W, R) and an associated Coxeter datum $\mathscr{C} = (V, \Pi, (\ ,\ ))$ with Φ being the corresponding root system. Furthermore, we impose that $\#\Pi = \#R < \infty$. It is well known that Φ is a discrete set (with respect to the standard topology), since W acts discretely on V. Nevertheless, following the approach of studying the so-called *normalized roots*, as pioneered in [21], we may study the asymptotic behaviors that these normalized roots exhibit if we consider the elements

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of Φ as representatives of the directions whose corresponding reflections generate the reflection group W. In [21], the following novel approach has been adopted to obtain such a set of representative directions: consider a projective version of the infinite root system Φ by cutting the cone $\mathrm{PLC}(\Pi)$, in which the positive roots are located, by a suitable affine hyperplane $\mathscr{H}$; and in doing so, we obtain the so-called *normalized root system* $\widehat{\Phi}$, consisting of the intersections of the rays spanned by the roots in Φ^+ with the hyperplane $\mathscr{H}$. The great advantage of this construction is that the properties of a root system then ensure that these normalized roots are contained in the convex hull of $\widehat{\Pi}$ (consisting of the intersections of the rays $\{\mathbb{R}x\}_{x\in\Pi}$ with the hyperplane $\mathscr{H}$). Under the assumption that $\#\Pi$ is finite, the convex hull of $\widehat{\Pi}$ forms a compact polytope and, thus, allows $\widehat{\Phi}$ to possess a nonempty set of accumulation points, and by studying these accumulation points, we may gain a glimpse of the distribution of roots in infinite root systems. For this construction to work properly, the hyperplane $\mathscr{H}$ should conform with the following definition.

DEFINITION 4.1 [11, 21]. An affine hyperplane V_1 of codimension 1 in V is called *transverse* to Φ^+ if for each $a \in \Pi$, the ray $\mathbb{R}_{>0}a$ intersects V_1 in exactly one point. Given a hyperplane V_1 transverse to Φ^+, let V_0 be the hyperplane that is parallel to V_1 and contains the origin.

REMARK 4.2. Recall the standard result that $\Phi = \Phi^+ \uplus \Phi^-$ (where $\uplus$ denotes disjoint union). This combined with the requirement that $0 \notin \mathrm{PLC}(\Pi)$ immediately imply that it is always possible to find a hyperplane containing the origin that separates Φ^+ and Φ^- (see [21, Section 5.2] for more details). By suitably translating this hyperplane, it is always possible to find a hyperplane transverse to Φ^+.

Let V_1 be a transverse hyperplane and let V_0 be as in the preceding definition. Let V_0^+ be the open half space induced by V_0 that contains V_1. Observe that V_0^+ contains $\mathrm{PLC}(\Pi)$. Since $\Phi^+ \subset \mathrm{PLC}(\Pi) \subset V_0^+$ and V_1 is parallel to the boundary of V_0^+, it follows that

$$\#(V_1 \cap \mathbb{R}_{>0}\beta) = 1$$

for each $\beta \in \Phi^+$.

DEFINITION 4.3. Let V_1 be a transverse hyperplane in V and let V_0 be obtained from V_1, as in Definition 4.1.

(1) For each $v \in V \setminus V_0$, the unique intersection point of $\mathbb{R}v$ and the transverse hyperplane V_1 is denoted $\widehat{v}$. The *normalization map* $\pi_{V_1} : V \setminus V_0 \to V_1$ is given by $\pi_{V_1}(v) := \widehat{v}$ for all $v \in V \setminus V_0$. Set $\widehat{\Phi} := \pi_{V_1}(\Phi)$ and call the elements $\widehat{x} \in \widehat{\Phi}$ *normalized roots*.
(2) Let $|\cdot|_1 : V \to \mathbb{R}$ be the unique linear map satisfying the requirement that $|v|_1 = 0$ for all $v \in V_0$, and $|v|_1 = 1$ for all $v \in V_1$.

Observe that $\pi_{V_1}(-x) = \pi_{V_1}(x)$ for all $x \in V$, and $\widehat{y} = (y/(|y|_1))$ for all $y \in V \setminus V_0$.

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Also observe that $\widehat{\Phi} \subseteq \mathrm{conv}(\widehat{\Pi})$ (recall that $\mathrm{conv}(X)$ denotes the convex hull of a set X) and $\widehat{\Pi} = \{\widehat{x} \mid x \in \Pi\}$. Since W is finitely generated, Π is a finite set and then we see that $\widehat{\Phi}$ is contained in the compact set $\mathrm{conv}(\widehat{\Pi})$. Consequently, if Π is finite, then the accumulation points of $\widehat{\Phi}$ are contained in $\mathrm{conv}(\widehat{\Pi})$.

DEFINITION 4.4. Keep all the notation of the previous definition.

(1) The set of *limit roots* $E(W)$ of the Coxeter group W (with respect to V_1) is the set of accumulation points of $\widehat{\Phi}$. For a reflection subgroup $W' \leq W$, the set of limit roots $E(W')$ arising from the reflection subgroup W' (with respect to V_1) is the set of accumulation points of $\widehat{\Phi(W')}$.
(2) The *normalized isotropic cone* $\widehat{Q}$ (associated to the bilinear form $(\,,\,)$ and with respect to V_1) is the set $\widehat{Q} = Q \cap V_1$.

EXAMPLE 4.5. Suppose that Π forms a basis for the space V. The hyperplane $V_1 := \{v \in V \mid \sum_{a\in\Pi} \mathrm{coeff}_a(v) = 1\}$ is a transverse hyperplane and the corresponding hyperplane $V_0 := \{v \in V \mid \sum_{a\in\Pi} \mathrm{coeff}_a(v) = 0\}$ is obtained from translating V_1 to contain the origin. Observe that under these conditions, the corresponding $|\cdot|_1$ has the property that $|v|_1 = \sum_{a\in\Pi} \mathrm{coeff}_a(v)$ for all $v \in V$.

REMARK 4.6. For the rest of this section and the whole of the next section, we adopt the set up as in the preceding example, that is, we assume that Π forms a basis for V and we take

$$V_1 = \left\{ v \in V \,\middle|\, \sum_{a\in\Pi} \mathrm{coeff}_a(v) = 1 \right\}.$$

Next, we have an elegant result from [21].

THEOREM 4.7 [21, Theorem 2.7 and Proposition 2.14]. *Let V_1 be a transverse hyperplane and let $\widehat{Q}$ be the associated normalized isotropic cone. Then:*

(1) *$E(W) \neq \emptyset$ if and only if the Coxeter group W is infinite;*
(2) *$E(W) \subseteq \widehat{Q}$.*

Following the convention set in [21], we define

$$D = \bigcap_{w\in W} w(V \setminus V_0) \cap V_1 = V_1 \setminus \bigcup_{w\in W} wV_0,$$

and we define the $\cdot$-action of W on D as follows: for any $w \in W$ and $x \in D$,

$$w \cdot x = \widehat{wx}.$$

Observe that the property that $WD = \bigcup_{w\in W} wD \subseteq V \setminus V_0$ guarantees that this $\cdot$-action of W on D is well defined. Furthermore, we observe that each $w \in W$ acts continuously on D. The next result is taken from [21], which summarizes a number of key facts.

PROPOSITION 4.8 [21, Proposition 3.1]. *Let V_1 be a transverse hyperplane, and let $\widehat{\Phi}$ and $E(W)$ be the corresponding normalized roots and limit roots.*

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(i) $\widehat{\Phi}$ *and* $E(W)$ *are contained in* D.
(ii) $\widehat{\Phi}$ *and* $E(W)$ *are stable under the* $\cdot\cdot$*-action of* W*; moreover,*

$$\widehat{\Phi} = W \cdot \Pi.$$

(iii) *The topological closure* $\widehat{\Phi} \uplus E(W)$ *is stable under the* $\cdot\cdot$*-action of* W.

Furthermore, it has been observed in [21] that the $\cdot\cdot$-action has the following nice geometric description.

PROPOSITION 4.9 [21, Proposition 3.5]. *Keep the previous notation.*

(i) *Let* $\alpha \in \Phi$ *and* $x \in D \cap Q$. *Denote by* $L(\widehat{\alpha}, x)$ *the line containing* $\widehat{\alpha}$ *and* x. *Then:*
   (a) *if* $(\alpha, x) = 0$, *then* $L(\widehat{\alpha}, x)$ *intersects* Q *only at* x *and* $r_\alpha \cdot x = x$;
   (b) *if* $(\alpha, x) \neq 0$, *then* $L(\widehat{\alpha}, x)$ *intersects* Q *in two distinct points, namely,* x *and* $r_\alpha \cdot x$.

(ii) *Let* α_1 *and* α_2 *be two distinct roots in* Φ, $x \in L(\widehat{\alpha_1}, \widehat{\alpha_2}) \cap Q$, *and* $w \in W$. *Then,* $w \cdot x \in L(w \cdot \alpha_1, w \cdot \alpha_2) \cap Q$.

Given a transverse hyperplane V_1, the limit roots coming from a given rank-2 reflection subgroup can be observed inside $E(\Phi)$. Take two distinct positive roots a and b, and denote by $W' = \langle r_a, r_b \rangle$ the dihedral reflection subgroup of W generated by the reflections r_a and r_b corresponding to the two positive roots. Let $\Pi(W') = \{a', b'\}$ be the set of canonical roots for the dihedral reflection subgroup W'. Then, [9, Theorem 4.4] yields that $(a', b') \in (-\infty, -1] \cup \{-\cos(\pi/n) \mid n \in \mathbb{N}_{\geq 2}\}$. It is clear that $0 \notin \mathrm{PLC}(\{a', b'\})$, since $a', b' \in \mathrm{PLC}(\Pi)$ and $0 \notin \mathrm{PLC}(\Pi)$ by the definition of Π. Now, let $V' = \mathbb{R}a' \oplus \mathbb{R}b'$, a two-dimensional subspace of V spanned by $\Pi' = \{a', b'\}$, and let $(\ ,\ )'$ be the restriction of $(\ ,\ )$ on V'. Then, $\mathscr{C}' = (V', \Pi', (\ ,\ )')$ is also a Coxeter datum with associated root system $\Phi' = W'\Pi'$. Observe that the hyperplane V_1 is also transverse with respect to Φ'. Let $E(W')$ denote the set of limit roots of W' in the root system Φ' with respect to V_1. The following was observed in [21, Section 2.3].

PROPOSITION 4.10 [21]. *Given the set up in the preceding paragraph:*

(i) $E(W') = Q \cap L(\widehat{a'}, \widehat{b'}) = E(W) \cap L(\widehat{a'}, \widehat{b'})$;
(ii) *the cardinality of* $E(W')$ *is* 0, 1, *or* 2 *precisely when respectively* $|(a', b')| < 1$, $|(a', b')| = 1$, *or* $|(a', b')| > 1$.

It turns out that by considering the cardinality of $E(W)$, we may easily ascertain whether W is affine or not.

PROPOSITION 4.11 [21, Corollary 2.16]. *Suppose that* W *is an irreducible Coxeter group. Then,* $E(W)$ *is a singleton set if and only if* W *is an affine Coxeter group.*

The last two propositions allow us to deduce that every infinite dihedral reflection subgroup of an irreducible affine Coxeter group must be affine. Conversely, as a corollary to [11, Theorem 6.2(a)], one can establish that an infinite nonaffine Coxeter

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group must possess an infinite dihedral reflection subgroup that is nonaffine. Thus, we have the following proposition.

PROPOSITION 4.12. *Let W be an irreducible Coxeter group of finite rank. Then, W is affine if and only if every infinite dihedral reflection subgroup of W is affine.*

To close this section on necessary introductory material on limit roots, we further recall the following important result that establishes that when W is a finitely generated irreducible and infinite Coxeter group, the limit set $E(W)$ itself is a W-invariant object (with respect to the $\cdot$-action of W), and this action is minimally faithful.

THEOREM 4.13 [21, Proposition 3.2] [11, Theorem 3.1]. *Suppose that (W, R) is an irreducible Coxeter system in which $|R| < \infty$ and $|W| = \infty$. Then:*

(i) $|wx|_1 > 0$ *for all* $w \in W$ *and* $x \in E(W)$;
(ii) *if* $x \in E(W)$, *then* $E(W) = \overline{W \cdot x} = \overline{\{w \cdot x \mid w \in W\}}$, *that is, the $\cdot$-action of W on $E(W)$ is minimal.*

With a more technical but otherwise predictable generalization, it can be observed that the irreducibility requirement in part (i) of Theorem 4.13 can be removed.

5. Limit roots relating to affine reflection subgroups

Throughout this section, each Coxeter system (W, R) is understood to have an associated Coxeter datum $\mathscr{C} = (V, \Pi, (\ ,\ ))$ with Π being a finite set and Φ being the corresponding root system. Further, unless otherwise stated, the Coxeter group W is assumed to be infinite. Keeping the set up as in Remark 4.6, specifically, we assume that Π forms a basis for V and we let V_1 be the transverse hyperplane as in Remark 4.6, and for any reflection subgroup $W' \leq W$, let $E(W')$ be the set of limit roots of W' with respect to the transverse hyperplane V_1.

A number of results in this section, including Theorem 5.5, and Propositions 5.14 and 5.25, can be seen as consequences of results contained in [10]. More specifically, in [10], a novel approach of realizing the Tits cone of a Coxeter group W in the same vector space V bearing the Tits representation of W (rather than in the algebraic dual of V, as in the more conventional treatment) has been adopted; and a certain functoriality condition was established on the imaginary cones of a Coxeter group W and the reflection subgroups of W, namely, the imaginary cone of a reflection subgroup is contained in the imaginary cone of the ambient group W. These results are significant and they often require working in set-ups different from the more conventional ones, and therefore, for the purposes of self-containedness and ease of reading, we have included independent proofs using more standard methodologies. We sincerely thank an anonymous referee who pointed out these alternative possibilities to us.

Suppose that W' is an irreducible affine reflection subgroup of W and let $\Pi(W') = \{a_1, a_2, \ldots, a_s\}$. Let η be the unique limit root in $E(W')$. By Lemma 3.3, the limit root η is in the radical of the bilinear form $(\ ,\ )$ restricted to the subspace $V_{W'}$

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spanned by $a_1, a_2, \ldots, a_s$, and it then follows that $(\eta, x) = 0$ for all $x \in \Phi(W') \subset V_{W'}$. However, it is possible that $(\eta, x) > 0$ for roots $x \in \Phi^+ \setminus \Phi(W')$ and such positive roots may potentially form an infinite set. In this section, among other things, we prove that $(\eta, x) > 0$ for only finitely many positive roots $x \in \Phi^+$. Note that Propositions 3.5, 4.11, and 4.12 together imply that if W' is an irreducible affine reflection subgroup of W, then W' has an affine dihedral reflection subgroup $W'' \subseteq W'$ with $E(W'') = E(W')$. Hence, many discussions on general affine reflection subgroups, especially their limit roots, can be simplified to discussions on affine dihedral reflection subgroups.

Before we give a characterization of those limit roots in $E(W)$ arising from affine reflection subgroups, let us first look at some limit roots that possibly do not arise from affine reflection subgroups. The first candidate for such possibly nonaffine limit roots might come from infinite nonaffine dihedral reflection subgroups.

Choose $a, b \in \Phi^+$ such that $(a, b) = -\cosh\theta < -1$ for some $\theta > 0$, and let $W' = \langle r_a, r_b \rangle$ be the dihedral reflection subgroup generated by the reflections corresponding to a and b. Note first that W' is infinite and nonaffine. Proposition 4.10 then yields that $E(W')$ consists of two distinct limit roots. Direct calculations then show that the intersection of the isotropic cone Q with the subspace $\mathbb{R}a \oplus \mathbb{R}b$ consists of two lines:

$$\mathbb{R}((\cosh\theta + \sinh\theta)a + b)$$

and

$$\mathbb{R}((\cosh\theta - \sinh\theta)a + b) = \mathbb{R}(a + (\cosh\theta + \sinh\theta)b),$$

and if we let η_1 and η_2 be as follows, then $E(W') = \{\eta_1, \eta_2\}$:

$$\eta_1 := \frac{(\cosh\theta + \sinh\theta)|a|_1}{(\cosh\theta + \sinh\theta)|a|_1 + |b|_1}\widehat{a} + \frac{|b|_1}{(\cosh\theta + \sinh\theta)|a|_1 + |b|_1}\widehat{b}$$

and

$$\eta_2 := \frac{(\cosh\theta - \sinh\theta)|a|_1}{(\cosh\theta - \sinh\theta)|a|_1 + |b|_1}\widehat{a} + \frac{|b|_1}{(\cosh\theta - \sinh\theta)|a|_1 + |b|_1}\widehat{b}.$$

Furthermore,

$$\eta_1 = \lim_{i\to\infty} (\widehat{r_a r_b})^i a \in \mathbb{R}((\cosh\theta + \sinh\theta)a + b)$$

and

$$\eta_2 = \lim_{i\to\infty} (\widehat{r_b r_a})^i b \in \mathbb{R}((\cosh\theta - \sinh\theta)a + b).$$

It follows readily that

$$(\eta_1, a) = \frac{\sinh\theta}{(\cosh\theta + \sinh\theta)|a|_1 + |b|_1} > 0$$

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and

$$(\eta_2, b) = \frac{\sinh\theta(\cosh\theta - \sinh\theta)}{(\cosh\theta - \sinh\theta)|a|_1 + |b|_1} > 0.$$

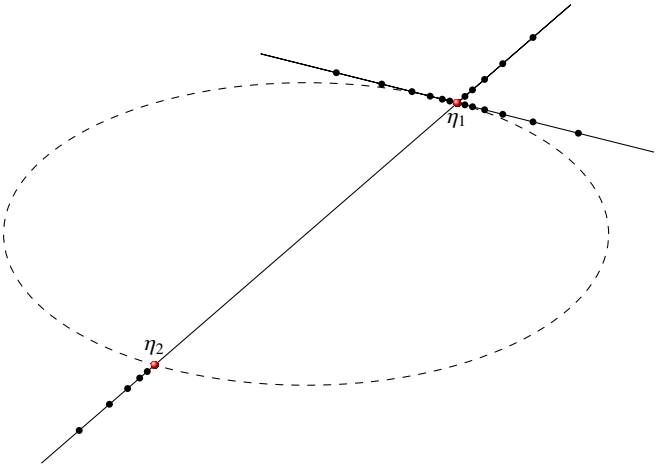
Note that $(r_b r_a)^i \cdot \eta_1 = \eta_1$ and $(r_a r_b)^i \cdot \eta_2 = \eta_2$ for all $i \in \mathbb{N}$. Consequently, for all $i \in \mathbb{N}$,

$$(\eta_1,\ (r_a r_b)^i a) > 0 \quad \text{and} \quad (\eta_2,\ (r_b r_a)^i b) > 0.$$

In particular,

$$(\eta_1, x) > 0 \text{ and } (\eta_2, y) > 0 \text{ for infinitely many } x, y \in \Phi^+.$$

However, the above discussion does not immediately rule out the possibility that an affine dihedral reflection subgroup and an infinite nonaffine dihedral reflection subgroup share the same limit root, a situation illustrated in the following schematic diagram.



In this schematic diagram, the dotted circle represents the normalized isotropic cone, and the normalized root subsystems of two infinite dihedral reflection subgroups, one affine and the other nonaffine, are contained in the two straight lines, with the black dots schematically representing normalized roots and the two red dots representing possible limit roots.

It turns out that this situation does not arise. Indeed, in this section, we give a characterization of the set of limit roots arising from affine reflection subgroups. For each η arising from an affine reflection subgroup of an infinite Coxeter group of finite rank, we establish where in $\widehat{Q}$ such an η must have been located; and we establish the finiteness of the cardinality of the set $\{x \in \Phi^+ \mid (x, \eta) > 0\}$. We also establish that the support of η is connected and remains connected under the $\cdot$-action of W (recall that a set $M \subseteq \Pi$ is *disconnected* if there are nonempty proper subsets M_1 and M_2 of M with $M = M_1 \uplus M_2$ such that $(a, b) = 0$ for each $a \in M_1$ and $b \in M_2$; otherwise M is said to be *connected* and in that case, we may say that M forms a connected component of Π).

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Moreover, we establish that a limit root simultaneously satisfying these finiteness and connectedness properties must be arising from an affine reflection subgroup and no other limit roots may satisfy simultaneously these two properties.

DEFINITION 5.1. Let (W, R) be a Coxeter system in which W is an infinite Coxeter group. A limit root $\eta \in E(W)$ is called an *affine limit root* (or is simply called *affine*) if there exists an irreducible affine reflection subgroup $W' \leq W$ with $E(W') = \{\eta\}$ and we use the set

$$E_{\text{aff}} := \{\eta \in E(W') \mid W' \leq W \text{ is affine}\}$$

to denote the set of all affine limit roots in W. Also, we use the following set to denote the set of all nonaffine dihedral limit roots in W:

$$E^2_{\text{nonaff}} := \{\eta \in E(W') \mid W' \leq W \text{ is nonaffine dihedral}\}.$$

Note that the calculations immediately before the schematic diagram show that if $\eta \in E^2_{\text{nonaff}}$, then $(\eta, x) > 0$ for infinitely many $x \in \Phi^+$ and, consequently, if we could show that $(\eta, x) > 0$ only for finitely many $x \in \Phi^+$ whenever $\eta \in E_{\text{aff}}$, then we would have established that $E_{\text{aff}} \cap E^2_{\text{nonaff}} = \emptyset$; thus ruling out the possibility raised in the diagram. To achieve this, we need some preparatory work first.

DEFINITION 5.2. For each $v \in V$, define $\text{Pos}(v) \subseteq \Phi^+$ by

$$\text{Pos}(v) = \{x \in \Phi^+ \mid (v, x) > 0\}.$$

Further, define

$$\begin{aligned}\mathscr{K} &:= \{v \in \text{cone}(\Pi) \mid \text{Pos}(v) = \emptyset\} \\ &= \{v \in \text{cone}(\Pi) \mid (v, a) \leq 0 \text{ for all } a \in \Phi^+\}.\end{aligned}$$

Next, we recall the following fundamental result for infinite Coxeter groups of finite rank that is going to be a key ingredient in our proof that $E_{\text{aff}} \cap E^2_{\text{nonaff}} = \emptyset$.

LEMMA 5.3 [1, Proposition 4.5.5]. *The set*

$$\{(\alpha, \beta) \mid \alpha, \beta \in \Phi,\ |(\alpha, \beta)| < 1\}$$

is finite. In particular, there exists a fixed $\epsilon > 0$ such that $|(\alpha, \beta)| > \epsilon$ whenever $\alpha, \beta \in \Phi$ satisfy $(\alpha, \beta) \neq 0$.

REMARK 5.4. Note that by Propositions 3.5, 4.11, and 4.12, $\eta \in E_{\text{aff}}$ if and only if there exists an irreducible affine dihedral reflection subgroup $W' \leq W$ with $E(W') = \{\eta\}$.

Now, we are ready to prove that $(\eta, x) > 0$ only for finitely many $x \in \Phi^+$ whenever $\eta \in E_{\text{aff}}$.

THEOREM 5.5. *If $\eta \in E_{\text{aff}}$, then $\text{Pos}(\eta)$ is a finite set.*

PROOF. Let $D \leq W$ be an affine dihedral reflection subgroup with $E(D) = \{\eta\}$ and let $a, b \in \Phi^+$ be such that $\Pi(D) = \{a, b\}$. Then, $(a, b) = -1$ and $(\eta, a) = (\eta, b) = 0$. Letting

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$a_k = k(a+b)+a$ and $b_k = k(a+b)+b$ (for $k \in \mathbb{N}_{\geq 0}$), we have that $\widehat{a_k} \to \eta$ and $\widehat{b_k} \to \eta$, and the root subsystem of D satisfies $\Phi^+(D) = \{a_k, b_k \mid k \geq 0\}$. Note also that $a_k \operatorname{dom} a$ and $b_k \operatorname{dom} b$ for all $k \geq 0$.

Let $x \in \mathrm{Pos}(\eta)$. Since $\widehat{a_k} \to \eta$ and $\widehat{b_k} \to \eta$, for sufficiently large k, we have $(x, a_k) > 0$ and $(x, b_k) > 0$. Fixing such a k,

$$0 < (x, a_k) + (x, b_k) = (x, (2k+1)(a+b)) = (2k+1)(x, a+b).$$

Therefore, $(x, a+b) > 0$. Observe that $(x, a) > -1$, since otherwise, by Lemma 3.7, $x \operatorname{dom} -a \operatorname{dom} -a_k$ for all $k \in \mathbb{N}$, which implies that $(x, a_k) \leq -1$ for all $k \in \mathbb{N}$, but this means that $(x, \eta) \leq 0$, which contradicts $x \in \mathrm{Pos}(\eta)$. Similarly, $(x, b) > -1$.

We now observe that there exists $\epsilon > 0$ such that $(x, a+b) > \epsilon$ for all $x \in \mathrm{Pos}(\eta)$. To see this, let F denote the finite set

$$F = \{(\alpha, \beta) \mid \alpha, \beta \in \Phi,\ |(\alpha, \beta)| < 1\},$$

and define $\epsilon' = \min\{r + s \mid r, s \in F, r + s > 0\}$ and $\epsilon'' = 1 + \min F$. Setting $\epsilon = \min(\epsilon', \epsilon'') > 0$,

$$(x, a+b) = (x, a) + (x, b) \geq \epsilon > 0.$$

Therefore, since

$$(x, a_k) = k(x, a+b) + (x, a)$$

and

$$(x, b_k) = k(x, a+b) + (x, b),$$

we may deduce that there is a fixed $M \in \mathbb{N}$ such that

$$(x, a_k) \geq 1 \quad \text{and} \quad (x, b_k) \geq 1 \quad \text{for all } x \in \mathrm{Pos}(\eta) \text{ and for all } k \geq M. \tag{5-1}$$

Now, for a contradiction, suppose that $\mathrm{Pos}(\eta)$ is an infinite set. With M as above and for some fixed $k \geq M$, (5-1) then implies that there is dominance between the infinitely many $x \in \mathrm{Pos}(\eta)$ and each of a_k and b_k. Since a given root can dominate only finitely many positive roots (see the remark immediately after Definition 3.6), it follows that there are infinitely many positive roots $x \in \mathrm{Pos}(\eta)$ satisfying $x \operatorname{dom} a_k$ and $x \operatorname{dom} b_k$, but then each of such x would simultaneously dominate both a and b, and we have a contradiction to Lemma 3.8. □

It turns out that for a limit root $\eta \in E(W)$, the finiteness of $\mathrm{Pos}(\eta)$ is not a sufficient condition to ensure that $\eta \in E_{\mathrm{aff}}$. To ensure that η is the limit root of an affine reflection subgroup of W, more conditions on η are required. The following is an example demonstrating this general fact. We gratefully acknowledge the comments made by Professor M. Dyer on an earlier version of this paper and, in particular, we thank him for directing our attention to this example, which first appeared in [21, Example 5.8].

EXAMPLE 5.6. Let (W, R) be a rank-5 Coxeter system with an associated Coxeter datum $\mathscr{C} = (V, \Pi, (\ ,\ ))$ such that $\Pi := \{a, b, c, d, e\}$ is a basis for V, and $(\ ,\ )$ satisfies

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the requirement that the only nonzero values between a pair of simple roots are the following:

$$(a,b) = (d,e) = -1 \quad \textit{and} \quad (b,c) = (c,d) = -\frac{1}{2}.$$

A direct calculation shows that for each $n \in \mathbb{N}$,

$$(r_a r_b r_e r_d)^n c = c + n^2 b + n^2 d + (n^2+n)a + (n^2+n)e.$$

Thus,

$$\eta_0 := \lim_{n\to\infty} (r_a \widehat{r_b r_e r_d})^n c = \frac{a+b+d+e}{4} \in E(W).$$

Further, note that $\mathrm{Pos}(\eta_0) = \emptyset$, but η_0 is not a limit arising from an affine reflection subgroup of W. Instead, η_0 can be seen as the *normalized sum* of the limit roots from two mutually orthogonal affine reflection subgroups, namely, $W_1 := \langle r_a, r_b\rangle$ and $W_2 := \langle r_d, r_e\rangle$. Moreover, it can be checked that every point on the line segment in the interval between the two affine limit roots $(a+b)/2$ and $(d+e)/2$ is in fact a limit root. For more details on this interesting behavior, please see [10, Example 7.12]. It should also be pointed out that even though $\mathrm{supp}(\eta_0) = \{a,b,d,e\}$, it happens that $\eta_0 \notin E(\langle r_a, r_b, r_d, r_e\rangle)$.

In addition to the finiteness condition of the respective Pos, a complete characterization of affine limit roots also rests on the number of connected components of their supports. To prepare us for such a characterization, we still need a few observations. We begin by recalling a few results on the imaginary cones associated to Coxeter groups.

DEFINITION 5.7. Let (W,R) be an arbitrary Coxeter system and let $\mathscr{C} = (V, \Pi, (\ ,\ ))$ be an associated Coxeter datum with Φ being the corresponding root system. We define the *imaginary cone*, denoted by $\mathscr{Y}$, to be

$$\mathscr{Y} := \bigcup_{w\in W} w\mathscr{K},$$

where we recall $\mathscr{K} = \{v \in \mathrm{cone}(\Pi) \mid \mathrm{Pos}(v) = \emptyset\}$, as introduced in Definition 5.2.

The concept of the imaginary cone was first introduced in [24] in the context of Kac–Moody Lie algebras as the pointed cone spanned by the positive imaginary roots. This concept is later generalized to Coxeter groups. A definitive reference on the imaginary cones of a Coxeter group can be found in [10]. We show, among other things, that in a finitely generated Coxeter group W, the only limit roots in $\mathscr{Y}$ are precisely those limit roots relating to affine reflection subgroups of W (more precisely, linear combinations of affine limit roots). Furthermore, we show that every point in the intersection of the imaginary cone and the normalized isotropic cone is, in fact, a limit root (expressible as a linear combination of affine limit roots).

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The following characterization of the imaginary cone of a finitely generated Coxeter group is taken from [14].

PROPOSITION 5.8 [14, Lemma 4.4 and Proposition 4.22]. *Let $\mathscr{Z}$ be the imaginary cone of a finitely generated Coxeter group W. Then,*

$$\mathscr{Z} = \{v \in U^* \mid (v, a) \leq 0 \text{ for all but finitely many } a \in \Phi^+\},$$

where U^* is the dual of the Tits cone given by

$$U^* = \bigcap_{w \in W} w(\mathrm{cone}(\Pi)).$$

REMARK 5.9. Theorem 5.5 here may be alternatively established by considering the imaginary cones associated with the affine reflection subgroups of a finitely generated Coxeter group. Suppose that W is a finitely generated infinite Coxeter group possessing an affine reflection subgroup W'. Propositions 3.5, 4.11, and 4.12 yield that there exists some affine dihedral reflection subgroup W'' of W' (and, hence, also a reflection subgroup of W) with $E(W'') = E(W') = \{\eta\}$. Let V'' be the two-dimensional subspace spanned by the canonical roots $\Pi(W'')$ of the affine dihedral reflection subgroup W'' and let $(\ ,\ )''$ be the restriction of the bilinear form $(\ ,\ )$ to V''. Let $\Phi(W'')$ be the root subsystem of the affine dihedral reflection subgroup W''. Let

$$\mathscr{K}_{W''} = \{v \in \mathrm{cone}(\Pi(W'')) \mid (v, a) \leq 0 \text{ for all } a \in \Phi^+(W'')\},$$

and let $\mathscr{Z}_{W''} = \bigcup_{w \in W''} w\mathscr{K}_{W''}$ be the imaginary cone of W'' with respect to the Coxeter datum $(V'', \Pi(W''), (\ )'')$. Then,

$$\{\eta\} = \mathscr{K}_{W''} \subset \mathscr{Z}_{W''}.$$

It is a remarkable result of [10, Theorem 6.3] that

$$\mathscr{Z}_{W''} \subseteq \mathscr{Z}.$$

Consequently, $\eta \in \mathscr{Z}$ and it follows from Proposition 5.8 immediately that $\#\mathrm{Pos}(\eta) < \infty$.

To establish the connection between limit roots of finitely generated infinite Coxeter groups and the associated imaginary cones, we recall the following key result from [11].

THEOREM 5.10 [11, Theorem 2.3]. *Setting $Z := \mathscr{Z} \cap V_1$, the convex hull of $E(W)$ is the topological closure of Z.*

From Theorem 5.10, it is immediately clear that $E(W) \subseteq \overline{Z}$, where $\overline{\cdot}$ denotes the topological closure of $\cdot$.

It has been shown in [11, Lemma 2.4] that if (W, R) is an irreducible Coxeter system in which W is a nonaffine infinite Coxeter group, then $\mathrm{Int}(\mathscr{K}) \neq \emptyset$ (where $\mathrm{Int}(\cdot)$ denotes the topological interior of $\cdot$). Note that if $x \in \mathrm{Int}(\mathscr{K})$, then it is clear from the definition of $\mathscr{K}$ that $(x, a) < 0$ for all $a \in \Pi$ and, in particular, $(x, x) < 0$. Thus, no

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limit root can be in the interior of $\mathscr{K}$ and indeed, no limit root can be in the interior of $\mathscr{Z}$.

By combining Proposition 5.8, and Theorems 4.13 and 5.5, we can establish that affine limit roots are in the imaginary cone (in fact, every limit root η with $\#\mathrm{Pos}(\eta) < \infty$ is in the imaginary cone).

PROPOSITION 5.11. *If (W, R) is an irreducible Coxeter system, then $E(W) \subseteq U^*$. Moreover, if $\eta \in E(W)$ is such that $\#\mathrm{Pos}(\eta) < \infty$, then $\eta \in \mathscr{Z}$ and, in particular, $E_{\mathrm{aff}} \subseteq Z$.*

We are now able to establish that the containment of the imaginary cone in the dual of the Tits cone is strict in the case of nonaffine infinite Coxeter groups of finite rank.

COROLLARY 5.12. *Let (W, R) be a Coxeter system in which W is an irreducible infinite nonaffine Coxeter group and R is a finite generating set. The imaginary cone $\mathscr{Z}$ is a proper subset of the dual of the Tits cone U^*. In particular,*

$$E^2_{\mathrm{nonaff}} \subseteq U^* \setminus \mathscr{Z} \neq \emptyset.$$

PROOF. First, we recall Proposition 3.5, which stated that W is infinite if and only if it contains an infinite dihedral reflection subgroup. Since W is nonaffine, it follows from Proposition 4.12 that there exists an infinite nonaffine dihedral reflection subgroup $W' \leq W$ and so $\emptyset \neq E(W') \subset E^2_{\mathrm{nonaff}}$. Now, let $\eta \in E^2_{\mathrm{nonaff}}$ be arbitrary. Then, the calculations at the beginning of this section yield that $\#\mathrm{Pos}(\eta) = \infty$. Consequently, it follows from Proposition 5.8 that $\eta \notin \mathscr{Z}$ and so $E^2_{\mathrm{nonaff}} \subseteq U^* \setminus \mathscr{Z} \neq \emptyset$. □

For any finitely generated infinite Coxeter group W, it was proven in [16] that there is a W-equivariant map from the Davis complex of W into the imaginary cone of W, and this map is a homeomorphism onto its image. The Davis complex has many known topological properties and the existence of such a map may assist in the discovery of comparable properties of the imaginary cone, whose general topological features are yet to be explored. The construction of this map is built on earlier works in [25, 26], where a similar map from the Davis complex of W to the dual of the Tits cone of W was constructed. It was not entirely clear that the map obtained in [16] was in fact an improvement to the map constructed in [25, 26]. Indeed, it was not clear that the containment of the imaginary cone in the dual of the Tits cone was strict. This issue is now resolved by Corollary 5.12 in the case of infinite nonaffine Coxeter groups.

Next, we strive to complete a characterization of the limit roots in E_{aff}. So far, we have affirmed that

$$E_{\mathrm{aff}} \subseteq \{\eta \in E(W) \mid \#\mathrm{Pos}(\eta) < \infty\}.$$

We show that if the group W is not an affine Coxeter group, then this containment is strict. Further, we complete a characterization of the set $\{\eta \in E(W) \mid \#\mathrm{Pos}(\eta) < \infty\}$ itself and show that this set is actually closely related to affine limit roots. Furthermore, we establish the connection of this set of limit roots with the imaginary cone $\mathscr{Z}$.

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In particular, we are able to ascertain that each point in $\mathscr{Z} \cap \widehat{Q}$ is in fact a limit root. To complete these tasks, we still need a few observations on limit roots.

LEMMA 5.13. *Let $\eta \in E(W)$ be such that $\#\mathrm{Pos}(\eta) < \infty$. There exists some $w \in W$ with $w\eta \in \mathscr{K}$.*

PROOF. By Propositions 5.8 and 5.11, the fact that $\#\mathrm{Pos}(\eta) < \infty$ implies that $\eta \in Z = \mathscr{Z} \cap V_1$. Then, the definition of the imaginary cone gives that $\eta \in \bigcup_{w \in W} w\mathscr{K}$. Consequently, there exists some $w \in W$ with $w\eta \in \mathscr{K}$. □

It turns out that in a Coxeter group W of finite rank, if a limit root $\eta \in E(W)$ satisfies the condition that $\#\mathrm{Pos}(\eta) < \infty$, then the W-orbit of η contains a unique element in $\mathscr{K}$. This is a consequence of a more general fact that $\mathscr{K}$ is a *fundamental domain* for the action of W on $\mathscr{Z}$. That is, the W-orbit of every element of $\mathscr{Z}$ contains one and only one element of $\mathscr{K}$.

PROPOSITION 5.14. *Keep the notation of Definition 5.7. Let $\eta \in \mathscr{K}$ and let M be the set of all $a \in \Pi$ such that $(a, \eta) = 0$. Then,*

$$\{w \in W \mid w\eta \in \mathscr{K}\} = W_M = \{w \in W \mid w\eta = \eta\}. \tag{5-2}$$

That is, $\mathscr{K}$ is a fundamental domain for the action of W on $\mathscr{Z}$.

In particular, in a Coxeter group W of finite rank, if a limit root $\eta \in E(W)$ satisfies the condition that $\#\mathrm{Pos}(\eta) < \infty$, then the W-orbit of η contains a unique element of $\mathscr{K}$.

PROOF. Note that by the definition of $\mathscr{Z}$, it is clear that the W-orbit of every element of $\mathscr{Z}$ contains some element(s) of $\mathscr{K}$ and we only need to prove (5-2) to establish the uniqueness.

Clearly, $W_M \subseteq \{w \in W \mid w\eta = \eta\} \subseteq \{w \in W \mid w\eta \in \mathscr{K}\}$. Hence, it is enough to show that $\{w \in W \mid w\eta \in \mathscr{K}\} \subseteq W_M$.

Let $w \in W$ such that $w\eta = \eta' \in \mathscr{K}$. If $w \neq 1$, then we may choose $a \in \Pi$ with $w = w'r_a$ and $\ell(w) = \ell(w') + 1$. Note that $w'a \in \Phi^+$ and, hence,

$$0 \geq (w'a, \eta') = (w'a, w'r_a\eta) = (a, r_a\eta) = (r_a a, \eta) = -(a, \eta) \geq 0,$$

since η, $\eta' \in \mathscr{K}$ and $(\ ,\ )$ is W-invariant. Consequently, $a \in M$. Setting $w_0 = w$ and $w_1 = w'$, we have $\eta' = w\eta = w_0\eta = w'r_a\eta = w_1\eta$ and $w_0W_M = w_1W_M$. Now, if $w_1 \neq 1$, then we may repeat the above argument and find some $w_2 \in W$ satisfying the following three conditions: $\ell(w_2) = \ell(w_1) - 1 = \ell(w_0) - 2$; $w_2\eta = w_1\eta = w_0\eta = \eta'$; and $w_2W_M = w_1W_M = w_0W_M$. If $w_2 \neq 1$, we can repeat the argument again and so construct a sequence $w_0, w_1, w_2, \cdots \in W$. Since the length decreases at each step, this sequence must terminate; however, this process can be continued so long as $w_i \neq 1$. Thus, $w_i = 1$ for some finite i and

$$wW_M = w_0W_M = w_1W_M = w_2W_M = \cdots = W_M,$$

proving $w \in W_M$.

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Finally, note that under the given conditions, $\#\mathrm{Pos}(\eta) < \infty$ implies that $\eta \in \mathscr{Z}$ and the rest of the proposition follows from the above proof. □

REMARK 5.15. For those readers who are familiar with the treatment of the Tits cone being contained in the same vector space V bearing the Tits representation as pioneered in [10], Proposition 5.14 above translates into the statement that $\mathscr{K}$ is contained in the negative of the fundamental chamber on V in the set-up of [10].

LEMMA 5.16. *Let $x \in \mathscr{K} \cap Q$ and $x \neq 0$. Let $M := \mathrm{supp}(x) \subseteq \Pi$. The restriction of the bilinear form $(\ ,\ )$ to the subspace spanned by M has a nonzero radical* $\mathrm{Rad}(M)$ *and* $x \in \mathrm{Rad}(M)$.

PROOF. Write $x = \sum_{a \in M} \mathrm{coeff}_a(x)a$. Since $0 \neq x \in \mathscr{K} \subseteq \mathrm{PLC}(\Pi)$, it follows that $\mathrm{coeff}_a(x) > 0$ for all $a \in M$. Now, $x \in Q$ implies that

$$0 = (x, x) = \sum_{a \in M} \mathrm{coeff}_a(x)(x, a). \tag{5-3}$$

Since $x \in \mathscr{K}$, it follows that $(x, a) \leq 0$ for all $a \in \Pi$ and, hence, (5-3) yields that $(x, a) = 0$ for all $a \in M$. Consequently, the restriction of the bilinear form $(\ ,\ )$ on the subspace spanned by M has a nonzero radical, $\mathrm{Rad}(M)$, and moreover, $x \in \mathrm{Rad}(M)$. □

We can deduce that the converse of Lemma 5.16 is also true.

LEMMA 5.17. *Let $M \subseteq \Pi$ be such that the restriction of the bilinear form $(\ ,\ )$ on the subspace spanned by M has a nonzero radical* $\mathrm{Rad}(M)$ *and suppose that, moreover,* $\mathrm{Rad}(M) \cap \mathrm{PLC}(M) \neq \emptyset$. *Then,* $\mathrm{Rad}(M) \cap \mathrm{PLC}(M) \subseteq \mathscr{K} \cap Q$.

PROOF. Let $x \in \mathrm{Rad}(M) \cap \mathrm{PLC}(M)$ be arbitrary. Then, $(x, a) = 0$ for all $a \in M$, and moreover, $(x, a') \leq 0$ for all $a' \in \Pi \setminus M$ by the definition of Π and the fact that $x \in \mathrm{PLC}(M)$. Therefore, $x \in \mathscr{K}$.

Next, to establish that $x \in Q$, it is enough to observe that the fact $x \in \mathrm{Rad}(M)$ implies that $(x, x) = \sum_{a \in M} \mathrm{coeff}_a(x)(x, a) = 0$. □

PROPOSITION 5.18. *Suppose that $\eta \in E(W)$ such that* $\mathrm{Pos}(\eta)$ *is a finite set. There exists $M \subseteq \Pi$ such that the restriction of the bilinear form $(\ ,\ )$ on the subspace spanned by M has a nonzero radical* $\mathrm{Rad}(M)$ *satisfying the condition that* $\mathrm{Rad}(M) \cap \mathrm{PLC}(M) \neq \emptyset$ *and there exists some $w \in W$ such that*

$$w \cdot \eta \in \mathrm{Rad}(M).$$

PROOF. By Lemma 5.13, there exists some $w \in W$ such that $w \cdot \eta \in \mathscr{K}$. Note that then, $0 \neq w \cdot \eta \in \mathscr{K} \cap Q$. Consequently, it follows from Lemma 5.16 that there exists some $M \subseteq \Pi$ such that the restriction of the bilinear form $(\ ,\ )$ to the subspace spanned by M has a nonzero radical $\mathrm{Rad}(M)$, with $\mathrm{Rad}(M) \cap \mathrm{PLC}(M) \neq \emptyset$ and $w \cdot \eta \in \mathrm{Rad}(M)$. □

The next technical result follows from [10, Section 8.4], but for the sake of completeness, we include a proof here.

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PROPOSITION 5.19. *Let $\eta \in \mathscr{K} \cap \widehat{Q}$ be such that $M := \operatorname{supp}(\eta)$ is connected. Then, $\eta \in E_{\mathrm{aff}}$ and the standard parabolic subgroup W_M is an irreducible affine reflection subgroup of W with $E(W_M) = \{\eta\}$.*

PROOF. Lemma 5.16 yields that the restriction of the bilinear form $(\,,\,)$ on the subspace spanned by M has a nonzero radical $\operatorname{Rad}(M)$ with $\eta \in \operatorname{Rad}(M) \cap \operatorname{PLC}(M)$. Now, by Lemma 3.4, the conditions M being connected (and obviously finite) and $\operatorname{Rad}(M) \cap \operatorname{PLC}(M) \neq \emptyset$ together imply that the reflection subgroup $W_M := \langle r_a \mid a \in M\rangle$ is an irreducible affine Coxeter group. Proposition 3.2 yields that $\operatorname{Rad}(M)$ is one dimensional and it coincides with the isotropic cone restricted to the subspace spanned by M. Consequently, the normalized isotropic cone restricted to the subspace spanned by M consists of a single point and, hence, $E(W_M) = \{\eta\}$. □

It turns out that $\mathscr{K} \cap \widehat{Q} \subset E(W)$, and indeed, $\mathscr{Z} \cap \widehat{Q} \subset E(W)$. We show these once we have a better understanding of those limit roots whose supports are not connected.

LEMMA 5.20. *Let M be a connected subset of Π such that the restriction of the bilinear form $(\,,\,)$ on the subspace spanned by M has a nonzero radical $\operatorname{Rad}(M)$ satisfying $\operatorname{Rad}(M) \cap \operatorname{PLC}(M) \neq \emptyset$. Then, $\dim(\operatorname{Rad}(M)) = 1$.*

PROOF. Note that $\#M < \infty$, since $\#\Pi < \infty$. By Lemma 3.4, the conditions M being connected and $\operatorname{Rad}(M) \cap \operatorname{PLC}(M) \neq \emptyset$ establish that the standard parabolic subgroup $W_M := \langle r_a \mid a \in M\rangle$ is an irreducible affine Coxeter group. Proposition 3.2 yields that $\operatorname{Rad}(M)$ is one dimensional. □

Lemma 5.20 has the following easily seen consequence.

COROLLARY 5.21. *Let M be a finite subset of Π such that the restriction of the bilinear form $(\,,\,)$ on the subspace spanned by M has a nonzero radical $\operatorname{Rad}(M)$ and there exists some $x \in \operatorname{Rad}(M) \cap \operatorname{PLC}(M)$ with $\operatorname{supp}(x) = M$. Let $M = M_1 \uplus \cdots \uplus M_m$ be a decomposition of M into connected components. For each $i = 1, \ldots, m$, the restriction of the bilinear form $(\,,\,)$ on the subspace spanned by M_i has a one-dimensional radical $\operatorname{Rad}(M_i)$ with $\operatorname{Rad}(M_i) \cap \operatorname{PLC}(M_i) \neq \emptyset$. Furthermore, $\operatorname{Rad}(M) = \operatorname{Rad}(M_1) \oplus \cdots \oplus \operatorname{Rad}(M_m)$.*

PROOF. Under the given conditions, we may write $x = x_1 + \cdots + x_m$, where $x_i \in \operatorname{PLC}(M_i)$ and $\operatorname{supp}(x_i) = M_i$ for $i = 1, \ldots, m$. Since the M_i are pairwise disconnected, it follows from the properties of a root basis that $(a, b) = 0$ whenever $a \in \mathbb{R}M_i$ and $b \in \mathbb{R}M_j$ with $i \neq j$. Similarly, if $a \in \Pi \setminus M_i$, then $(x_i, a) \leq 0$; however, Lemma 5.17 ensures that $x \in \mathscr{K}$ and so if $a \in M_i$, then $(x_i, a) = (x, a) \leq 0$. Thus, $x_i \in \mathscr{K}$ for each i. Now, since $0 = (x, x) = \sum_{i=1}^{m} (x_i, x_i)$ and $(x_i, x_i) = \sum_{a \in M_i} |\operatorname{coeff}_a(x_i)|(x_i, a) \leq 0$, it follows that $x_i \in Q$ for each i. Lemma 5.16 yields that for each i, the restriction of the bilinear form $(\,,\,)$ to the subspace spanned by M_i has a nonzero radical $\operatorname{Rad}(M_i)$ such that $0 \neq x_i \in \operatorname{Rad}(M_i) \cap \operatorname{PLC}(M_i)$. Because each M_i is connected, Lemma 5.20 yields that $\dim(\operatorname{Rad}(M_i)) = 1$. Finally, it is readily checked that $\operatorname{Rad}(M) = \operatorname{Rad}(M_1) \oplus \cdots \oplus \operatorname{Rad}(M_m)$. □

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PROPOSITION 5.22. *Suppose that $\eta \in E(W)$ is a limit root arising from an irreducible affine reflection subgroup $W' \leq W$. Then,* $\mathrm{supp}(\eta)$ *is a connected subset of* Π.

PROOF. Note that Propositions 4.11 and 4.12 yield that we may choose an affine dihedral reflection subgroup $W'' \leq W'$ such that $\{\eta\} = E(W'')$. Set $\Pi(W'') = \{a, b\}$. Then, $a, b \in \Phi^+$ and $(a, b) = -1$. Then, $\mathrm{supp}(\eta) = \mathrm{supp}(a) \cup \mathrm{supp}(b)$. Since a and b are roots, it can be easily checked that $\mathrm{supp}(a)$ and $\mathrm{supp}(b)$ are both connected. Observe then that the fact $(a, b) = -1$ implies that $\mathrm{supp}(a) \cup \mathrm{supp}(b)$ is also connected. For otherwise,

$$a = \sum_{\alpha \in \mathrm{supp}(a)} \mathrm{coeff}_\alpha(a)\alpha \quad \text{where all the } \mathrm{coeff}_\alpha(a) > 0,$$

and

$$b = \sum_{\beta \in \mathrm{supp}(b)} \mathrm{coeff}_\beta(b)\beta \quad \text{where all the } \mathrm{coeff}_\beta(b) > 0,$$

and if $\mathrm{supp}(a) \cup \mathrm{supp}(b)$ is not connected, then note that $(\alpha, \beta) = 0$ for all $\alpha \in \mathrm{supp}(a)$ and for all $\beta \in \mathrm{supp}(b)$. However, then

$$(a, b) = \sum_{\alpha \in \mathrm{supp}(a)} \sum_{\beta \in \mathrm{supp}(b)} \mathrm{coeff}_\alpha(a)\, \mathrm{coeff}_\beta(b)(\alpha, \beta) = 0,$$

which contradicts that $(a, b) = -1$. □

THEOREM 5.23. *Let $\eta \in E(W)$ be such that* $\#\mathrm{Pos}(\eta) < \infty$ *and* $\mathrm{supp}(w \cdot \eta)$ *is connected for each $w \in W$. Then, there exists an irreducible affine reflection subgroup W' of W satisfying $E(W') = \{\eta\}$.*

PROOF. By Lemma 5.13, the fact $\#\mathrm{Pos}(\eta) < \infty$ implies that there exists $w \in W$ such that

$$0 \neq w \cdot \eta \in \mathscr{K} \cap Q.$$

Let $M := \mathrm{supp}(w \cdot \eta)$ and note that M is connected. Then, Proposition 5.19 yields that $W_M := \langle r_a \mid a \in M \rangle \leq W$ is an irreducible affine standard parabolic subgroup and $\{w \cdot \eta\} = E(W_M)$. It follows that $w^{-1} W_M w$, being a conjugate of an irreducible affine reflection subgroup of W, is itself an irreducible affine reflection subgroup of W and $\{\eta\} = E(w^{-1} W_M w)$. □

Combining Theorem 5.5, Proposition 5.22, and Theorem 5.23, we immediately have the following characterization of affine limit roots in an infinite Coxeter group of finite rank.

THEOREM 5.24. *Let $\eta \in E(W)$. Then, $\eta \in E(W')$ for some affine reflection subgroup W' of W if and only if* $\mathrm{Pos}(\eta)$ *is a finite set and* $\mathrm{supp}(w \cdot \eta)$ *is connected for each $w \in W$.*

In fact, the set of affine limit roots also admits the following alternative characterization.

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THEOREM 5.25. *If $\eta \in E(W)$, then $\eta \in E_{\text{aff}}$ if and only if there exists some $w \in W$ such that $w \cdot \eta \in \mathscr{K}$ with $\text{supp}(w \cdot \eta)$ being connected.*

PROOF. Note that under the given conditions, if $\eta \in E_{\text{aff}}$, then Theorem 5.5 yields that $\#\text{Pos}(\eta) < \infty$ and Proposition 5.13 then yields that there exists some $w \in W$ with $w \cdot \eta \in \mathscr{K}$, and moreover, Proposition 5.22 ensures that $\text{supp}(w \cdot \eta)$ is connected.

Conversely, let $\eta \in E(W)$ be such that there exists some $w \in W$ with $w \cdot \eta \in \mathscr{K} \cap Q$ and $M := \text{supp}(w \cdot \eta)$ being connected. Proposition 5.19 yields that W_M is an irreducible affine reflection subgroup of W and $\{w \cdot \eta\} = E(W_M)$. Consequently, $w^{-1}W_M w$ is an irreducible affine reflection subgroup of W with $\{\eta\} = E(w^{-1}W_M w) \subseteq E_{\text{aff}}$. □

REMARK 5.26. For those readers who are familiar with the treatment of realizing the Tits cone in the same vector space bearing the Tits representation, Theorem 5.25 above can be proved alternatively: if $\eta \in E$ such that $\text{Pos}(\eta)$ is finite, then there exists $w \in W$ such that $w\eta$ is in the negative of the Tits cone realized in V. Furthermore, since $w \cdot \eta \in E \subset \text{PLC}(\Pi)$, it follows that $w\eta \in \text{PLC}(\Pi) \cap \mathscr{K}$. Then, [10, Sections 8.4 and 8.8] readily establishes that $w\eta$ is a convex combination of limit roots from E_{aff} and, consequently, so is η.

COROLLARY 5.27. *An infinite Coxeter group W of finite rank contains an affine reflection subgroup if and only if W contains an affine standard parabolic subgroup. In particular, irreducible affine reflection subgroups in W are precisely the irreducible infinite reflection subgroups of affine parabolic subgroups of W.*

PROOF. Suppose that W has an affine standard parabolic subgroup. Then, there is nothing to prove, for a standard parabolic subgroup of W is, *a priori*, a reflection subgroup of W.

Conversely, suppose that W has an irreducible affine reflection subgroup W' and suppose that $\{\eta\} = E(W')$. Then, Theorem 5.25 yields that there exists some $w \in W$ such that $w \cdot \eta \in \mathscr{K}$ and $N := \text{supp}(w \cdot \eta)$ is connected. It follows from Proposition 5.19 that W_N is an irreducible affine reflection subgroup of W. Next, let

$$M := \{a \in \Pi \mid (a, w \cdot \eta) = 0\}.$$

Note that Proposition 5.14 then yields that $N \subseteq M$. Now, if $a \in M \setminus N$, then $(a, b) = 0$ for all $b \in N$, since otherwise, there exists $b \in N$ with $(a, b) < 0$, and since

$$w \cdot \eta = \text{coeff}_b(w \cdot \eta)b + \sum_{c \in N \setminus \{b\}} \text{coeff}_c(w \cdot \eta)c,$$

with all the coefficients being strictly positive, then it follows from the definition of a root basis that

$$(a, w \cdot \eta) = \left(a, \text{coeff}_b(w \cdot \eta)b + \sum_{c \in N \setminus \{b\}} \text{coeff}_c(w \cdot \eta)c\right)$$

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$$= \operatorname{coeff}_b(w \cdot \eta)(a, b) + \sum_{c \in N \setminus \{b\}} \operatorname{coeff}_c(w \cdot \eta)(a, c)$$
$$< 0,$$

which contradicts that $a \in M$. Thus, $M = N \uplus N'$, where N and N' are disconnected subsets of Π, and W_M is the direct product of standard parabolic subgroups W_N and $W_{N'}$. Now, for any $w' \in W'$,

$$(ww'w^{-1}) \cdot (w \cdot \eta) = (ww'w^{-1}w) \cdot \eta = (ww') \cdot \eta = w \cdot (w' \cdot \eta)$$
$$= w \cdot \eta.$$

That is, $wW'w^{-1}$ stabilizes $w \cdot \eta \in \mathscr{K}$. It follows from Proposition 5.14 that $wW'w^{-1} \subseteq W_M$ and, indeed, $wW'w^{-1} \subseteq W_N$. □

Observations comparable to Corollary 5.27 may be found in [10, Corollary 8.8] and [5, Theorem 3.3] (ultimately tracing back to [25, 26] and other unpublished works of Daan Krammer) for affine reflection subgroups of ranks greater than or equal to three. Corollary 5.27 provides a unified approach to deal with arbitrary affine reflection subgroups.

PROPOSITION 5.28. *Suppose that* $\eta \in E(W)$ *such that* $\#\mathrm{Pos}(\eta) < \infty$ *and* $\eta \notin E_{\mathrm{aff}}$. *Then,* η *is in the convex hull of limit roots from* E_{aff}.

PROOF. Since $\#\mathrm{Pos}(\eta) < \infty$, it follows from Lemma 5.13 that there exists some $w \in W$ such that $w \cdot \eta \in \mathscr{K} \cap Q$. Let $M := \mathrm{supp}(w \cdot \eta)$. Lemma 5.16 gives that the bilinear form $(\,,\,)$ restricted to the subspace spanned by M has a nonzero radical $\mathrm{Rad}(M)$ such that $w \cdot \eta \in \mathrm{Rad}(M) \cap \mathrm{PLC}(M)$. Since $\eta \notin E_{\mathrm{aff}}$, it follows that $w \cdot \eta \notin E_{\mathrm{aff}}$, and then Proposition 5.14 and Theorem 5.25 together yield that M is not connected. Now, suppose that $M = M_1 \uplus \cdots \uplus M_m$, where $M_1, \ldots, M_m$ form a complete list of connected components of M. Corollary 5.21 and Lemma 3.4 together imply that the standard parabolic subgroups $W_{M_1}, \ldots, W_{M_m}$ are all irreducible affine reflection subgroups. Let $\{\eta_i\} := E(W_{M_i}) \subset E_{\mathrm{aff}}$ for each $i = 1, \ldots, m$. It follows from Corollary 5.21 that $w \cdot \eta$ can be expressed as a positive linear combination of these η_i. That is, $w \cdot \eta = \sum_{i=1}^{m} \lambda_i \eta_i$ for some $\lambda_i \geq 0$ with $\sum_{i=1}^{m} \lambda_i = 1$. Consequently,

$$\eta = \sum_{i=1}^{m} \lambda_i' w^{-1} \cdot \eta_i \quad \text{where } \lambda_i' = |w\eta|_1 |w^{-1}\eta_i|_1 \lambda_i,$$

where $w^{-1} \cdot \eta_i$ is the limit root of the affine reflection subgroup $w^{-1}W_{M_i}w$ for each $1, \ldots, m$. □

The following is a well-known result and for completeness, we include a proof here.

LEMMA 5.29. *Let* W *be a finitely generated irreducible Coxeter group that may be either finite or infinite. There exists a positive root* x *such that* $\mathrm{supp}(x) = \Pi$.

PROOF. We give an inductive algorithm for computing one such positive root x.

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Since W is irreducible, its Coxeter graph is connected. We start with $x_1 = \alpha$, where α is an arbitrarily chosen simple root. Suppose that we already have $x_k \in \Phi^+$ such that $\#\mathrm{supp}(x_k) = k < \#\Pi$. Since W is irreducible, it follows that there exists some $\beta \in \Pi \setminus \mathrm{supp}(x_k)$ and $\gamma \in \mathrm{supp}(x_k)$ with $(\beta, \gamma) < 0$. Then, set $x_{k+1} := r_\beta x_k \in \Phi^+$ and $\#\mathrm{supp}(x_{k+1}) = k + 1$. We may continue this process until we have found an x_m with $\mathrm{supp}(x_m) = \Pi$. □

PROPOSITION 5.30. *Let (W, R) be a Coxeter system in which W is an irreducible infinite nonaffine Coxeter group and R is a finite generating set. Let $M_1, \ldots, M_m$ be connected but pairwise disconnected subsets of Π that further satisfy the requirement that their corresponding standard parabolic subgroups $W_{M_1}, \ldots, W_{M_m}$ are irreducible affine reflection subgroups. Let $\eta_1, \ldots, \eta_m$ be the limit roots of $W_{M_1}, \ldots, W_{M_m}$, respectively. Then,*

$$\left\{ \sum_{i=1}^{m} \lambda_i \eta_i \,\middle|\, \sum_{i=1}^{m} \lambda_i = 1 \text{ and } 0 \le \lambda_i \le 1 \text{ for all } i = 1, \ldots, m \right\} \subseteq E(W).$$

That is, each point in the convex hull of the affine limit roots $\eta_1, \ldots, \eta_m$ is a limit root.

PROOF. Fix a positive root x with $\mathrm{supp}(x) = \Pi$ whose existence is guaranteed by Lemma 5.29.

For each $i = 1, \ldots, m$, let $\Phi(W_{M_i})$ denote the root subsystem of the affine reflection subgroup W_{M_i}. For each M_i, fix a simple root $\alpha_i \in M_i$. By [22, Proposition 3.13], we can find some $\beta_i \in \Phi^+(W_{M_i})$ satisfying $\beta_i \overset{\neq}{\mathrm{dom}}\, \alpha_i$. Since W_{M_i} is affine, it follows that $(\alpha_i, \beta_i) = 1$. Let $\gamma_i := r_{\alpha_i}\beta_i \in \Phi^+(W_{M_i})$ and

$$(\alpha_i, \gamma_i) = (\alpha_i, r_{\alpha_i}\beta_i) = -(\alpha_i, \beta_i) = -1.$$

Then, by [9, Theorem 4.4], $\{\alpha_i, \gamma_i\}$ is the set of canonical generators for the dihedral reflection subgroup $\langle r_{\alpha_i}, r_{\gamma_i} \rangle$. Set $\delta_i = \alpha_i + \gamma_i \in \mathrm{PLC}(\Pi)$. Note that $(\delta_i, \alpha_i) = (\delta_i, \gamma_i) = 0$. Direct calculations show that

$$\Phi^+(\langle r_{\alpha_i}, r_{\gamma_i} \rangle) = \{ n\alpha_i + (n \pm 1)\gamma_i \mid n \in \mathbb{N} \},$$

and hence, $\alpha_i + n\delta_i \in \Phi^+(\langle r_{\alpha_i}, r_{\gamma_i} \rangle)$ for all $n \in \mathbb{N}$. Thus, $\widehat{\delta_i}$ is the sole limit root in $E(\langle r_{\alpha_i}, r_{\gamma_i} \rangle)$. Since W_{M_i} is an irreducible affine reflection subgroup and $\langle r_{\alpha_i}, r_{\gamma_i} \rangle \subseteq W_{M_i}$, it follows that $\eta_i = \widehat{\delta_i}$ and, in particular, $(\delta_i, a) = 0$ for all $a \in M_i$. Next, note that for each $\ell \in \mathbb{N}$,

$$\begin{aligned} r_{\alpha_i + \ell\delta_i}(x) &= x - 2(x, \alpha_i + \ell\delta_i)(\alpha_i + \ell\delta_i) \\ &= x - 2(x, \alpha_i)\alpha_i - 2\ell(x, \delta_i)\alpha_i - 2\ell(x, \alpha_i)\delta_i - 2\ell^2(x, \delta_i)\delta_i. \end{aligned}$$

Observe that this is a positive root (since $\mathrm{supp}(x) = \Pi$). Now, we let $p_i := -2(x, \alpha_i)$ and $q_i := -2(x, \delta_i)$. Note that $q_i > 0$ (since W is irreducible). Thus, the above can be written as

$$r_{\alpha_i + \ell\delta_i}(x) = x + p_i\alpha_i + \ell q_i\alpha_i + \ell p_i\delta_i + \ell^2 q_i\delta_i. \tag{5-4}$$

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Since $M_1, \ldots, M_m$ are pairwise disconnected subsets of Π, it follows that $(x_i, x_j) = 0$ for all $x_i \in \mathbb{R}M_i$ and $x_j \in \mathbb{R}M_j$, whenever $i \neq j$. Thus, from (5-4), we see that applying the pairwise commuting reflections corresponding to the positive roots $\alpha_1 + \ell_1\delta_1, \ldots, \alpha_m + \ell_m\delta_m$ to x (where $\ell_1, \ldots, \ell_m \in \mathbb{N}$) produces the following positive root:

$$r_{\alpha_m+\ell_m\delta_m} \cdots r_{\alpha_i+\ell_i\delta_i} \cdots r_{\alpha_1+\ell_1\delta_1}(x) = x + \sum_{i=1}^{m} p_i\alpha_i + \sum_{i=1}^{m} \ell_i q_i \alpha_i + \sum_{i=1}^{m} \ell_i p_i \delta_i + \sum_{i=1}^{m} \ell_i^2 q_i \delta_i.$$

It can be checked that for any $t_i \in (0, 1)$ with $\sum_{i=1}^{m} t_i = 1$, we may find integral sequences $(\ell_{1_n}), \ldots, (\ell_{m_n})$ with

$$\lim_{n\to\infty} \ell_{i_n} = \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{\ell_{i_n}^2}{\sum_{i=1}^{m} \ell_{i_n}^2} = t_i.$$

Consequently,

$$\left\{ \sum_{i=1}^{m} \lambda_i \eta_i \,\middle|\, \sum_{i=1}^{m} \lambda_i = 1 \text{ and } 0 \leq \lambda_i \leq 1 \text{ for all } i = 1, \ldots, m \right\} \subseteq E(W). \qquad \square$$

THEOREM 5.31. *Let W be an irreducible Coxeter group. Then,*

$$\mathscr{Z} \cap \widehat{Q} \subseteq E(W).$$

PROOF. Let $\eta \in \mathscr{Z} \cap \widehat{Q}$ be arbitrary. By the definition of the imaginary cone, there exists some $w \in W$ such that $w \cdot \eta \in \mathscr{K} \cap \widehat{Q}$. Let $M := \operatorname{supp}(w \cdot \eta)$. Lemma 5.16 gives that the bilinear form $(\ ,\ )$ restricted to the subspace spanned by M has a nonzero radical $\operatorname{Rad}(M)$ with $w \cdot \eta \in \operatorname{Rad}(M)$.

If M is connected, then Proposition 5.19 yields that $w \cdot \eta \in E_{\text{aff}}$ and this, in turn, means that $\eta \in E_{\text{aff}}$.

If M is disconnected, then let $M = M_1 \uplus \cdots \uplus M_m$ be a decomposition of M into connected components. Corollary 5.21 and Lemma 3.4 together establish that the standard parabolic subgroups $W_{M_1}, \ldots, W_{M_m}$ are all irreducible affine reflection subgroups of W. Denote the limit roots of $W_{M_1}, \ldots, W_{M_m}$ by $\eta_1, \ldots, \eta_m$, respectively. Corollary 5.21 further yields that $w \cdot \eta = \widehat{\sum_{i=1}^{m} \lambda_i \eta_i}$, where the coefficients λ_i are all nonnegative. That is, $w \cdot \eta$ is in the convex hull of the affine limit roots. It follows from Proposition 5.30 that $w \cdot \eta \in E(W)$ and, consequently, $\eta \in E(W)$. $\square$

DEFINITION 5.32. Define $E_{\text{aff-type}}$ to be

$$\{\eta \in E(W) \mid \#\operatorname{Pos}(\eta) < \infty \text{ and } \operatorname{supp}(w \cdot \eta) \text{ is disconnected for some } w \in W\},$$

the set of limit roots that are nontrivial sums of affine limit roots.

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In a Coxeter system (W, R) in which W is an infinite irreducible nonaffine Coxeter group and R is a finite generating set, it follows readily from Theorem 5.24, and Propositions 5.28 and 5.30 that $E^2_{\text{nonaff}} \cap (E_{\text{aff}} \uplus E_{\text{aff-type}}) = \emptyset$. Furthermore, by combining Proposition 5.8, Theorem 5.24, Proposition 5.28, and Theorem 5.31, we immediately arrive at the following conclusion.

COROLLARY 5.33. *We have*

$$\mathscr{Z} \cap \widehat{Q} = E(W) \cap \mathscr{Z} = E_{\text{aff}} \uplus E_{\text{aff-type}} = \{\eta \in E(W) \mid \#\text{Pos}(\eta) < \infty\}.$$

6. Connection with dominance chains

We close this paper with an application of the Pos-finite analysis developed in the previous section to the dominance partial ordering. In [11, Open Question 7.7], it was asked: if $\{x_i\}_{i\in\mathbb{N}} \subset \Phi^+$ is an infinite injective sequence of positive roots totally ordered by dominance, that is, $x_{i+1} \underset{\neq}{\text{dom}}\, x_i$ for all $i \in \mathbb{N}$, will the corresponding sequence of normalized roots $\{\widehat{x_i}\}_{i\in\mathbb{N}}$ possess a unique limit root. It has been observed in [11, Proposition 6.8] that if such a normalized sequence of an infinite injective sequence of positive roots totally ordered by dominance possess two distinct accumulation points, η_1 and η_2, then $(\eta_1, \eta_2) = 0$ and they span a two-dimensional totally isotropic subspace of V. In [19, Lemma 2.10], it has been further shown that in all Coxeter groups where the bilinear forms afforded by the Tits realization are of signature $(n-1, 1)$, such an infinite sequence possesses a unique accumulation point. Moreover, in a forthcoming paper, we show that in all infinite Coxeter groups of ranks 3 and 4, there is also unique accumulation. Based on these findings, we have some confidence that the uniqueness of accumulation can be generally expected for all infinite Coxeter groups of finite rank. Although at this present stage we are unable to resolve this question in its entirety, in this section, nevertheless, we prove that if such a normalized sequence possesses an accumulation point $\eta \in E_{\text{aff}}$, then this normalized sequence is convergent with η being the only accumulation point.

We believe that the significance of resolving this possible uniqueness of accumulation among infinite dominance chains is not to be underestimated. So far, in the exploration of limit roots, examples of a convergent infinite sequence of normalized roots are scarce and indeed, for the time being, we can only rely on an infinite sequence coming from infinite dihedral reflection subgroups or affine reflection subgroups. Needless to say that such examples are severely limited in their scopes, and the need to construct other possibly more nontrivial examples of convergent infinite sequences of normalized roots is somewhat pressing. If one could prove that in a finitely generated infinite Coxeter group, the normalized sequence corresponding to an infinite injective sequence of normalized roots is convergent, then we would have succeeded in some way to construct a nontrivial example of convergence. Furthermore, since dominance partial ordering is compatible with reflection subgroups (containing the reflections corresponding to those roots satisfying a dominance condition), such convergence behaviors work well with reflection subgroups.

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We first list a number of observations concerning dominance, establishing certain inequalities that may prove to be useful; we also prove a result establishing that when three distinct roots satisfy dominance, their corresponding reflections generate an affine reflection subgroup. We recall the following technical result taken from [14] (which is a refinement in one direction of [14, Proposition 4.10] that for two roots, x and y, we have $x \operatorname{dom} y$ if and only if $x - y \in \mathscr{Z} \subset U^*$).

PROPOSITION 6.1. *Suppose that $x, y \in \Phi$ such that $x \underset{\neq}{\operatorname{dom}} y$. There exists some $w \in W$ such that $wx \in \Phi^+$, $wy \in \Phi^-$, and $(w(x-y), z) \leq 0$ for all $z \in \Phi^+$.*

From the above-mentioned proposition, we may deduce a number of inequalities.

LEMMA 6.2. *Suppose that $a, b, c \in \Phi$ such that $a \operatorname{dom} b \operatorname{dom} c$. Then, $(a, c) \geq (b, c)$.*

PROOF. There is nothing to prove if either $a = b$ or $b = c$. Thus, we may assume that the dominance relations are strict. Since a strictly dominates b, Proposition 6.1 then yields that there exists some $w \in W$ with $wa \in \Phi^+$, $wb \in \Phi^-$, and $(w(a-b), z) \leq 0$ for all $z \in \Phi^+$. Since $b \operatorname{dom} c$ and $wb \in \Phi^-$, it follows that $wc \in \Phi^-$ and, consequently, $(w(a-b), wc) \geq 0$; that is, $(a-b, c) \geq 0$. □

An entirely similar proof to the previous one also establishes the following lemma.

LEMMA 6.3. *Suppose that $a, b, c \in \Phi$ such that $a \operatorname{dom} b \operatorname{dom} c$. Then, $(a, b) \leq (a, c)$.*

PROPOSITION 6.4. *Suppose that x, y, z are roots with $x \underset{\neq}{\operatorname{dom}} y \underset{\neq}{\operatorname{dom}} z$. If further $(x, z) = (x, y)$, then*

$$(x, z) = (x, y) = (y, z) = 1,$$

and the reflection subgroup $W' := \langle \rho_x, \rho_y, \rho_z \rangle$ is affine dihedral.

PROOF. First, since $x \underset{\neq}{\operatorname{dom}} y \underset{\neq}{\operatorname{dom}} z$, it follows from [14, Proposition 4.10] that $x - y \in U^*$ and $y - z \in U^*$. Then,

$$0 \geq (x-y, y-z) = \underbrace{(x, y-z)}_{=(x,y)-(x,z)=0} - \overbrace{(y, y-z)}^{=1-(y,z)\leq 0},$$

where $0 \geq (x-y, y-z)$ follows from [15, Theorem 5.8]. Consequently, $(y, y-z) = 0$ and so $(y, z) = 1$. Then,

$$(x, y-z) = 0, \quad (y, y-z) = 0, \quad \text{and} \quad (z, y-z) = 0.$$

That is, $y - z$ is in the radical of the bilinear form $(\ ,\ )$ restricted to the subspace spanned by x, y, and z. Furthermore,

$$w \cdot \widehat{(y-z)} = \widehat{y-z} \quad \text{for all } w \in W',$$

which in turn establishes that $E(W') = \{\widehat{y-z}\}$ is a singleton (following [11, Theorem 3.1]). Now, suppose for a contradiction that W' is nonaffine. It follows from

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Proposition 4.12 that there exists a nonaffine dihedral reflection subgroup $W'' \subset W'$ with $|E(W'')| = 2$, which contradicts $E(W'') \subseteq E(W')$. Consequently, W' is affine, whence

$$(x, y) = (y, z) = (x, z) = 1.$$

Then, $(x - y, x) = (x - y, y) = (x - y, z) = 0$, and $x - y$ is also in the radical of the bilinear form $(\ ,\ )$ restricted to the subspace spanned by x, y, and z. Now, if W' is of rank 3, then $\{x, y, z\}$ is linearly independent, and the radical of the bilinear form $(\ ,\ )$ restricted to the subspace spanned by x, y, and z is of dimension 2 spanned by $x - y$ and $y - z$. However, it is well known (for instance, see [23, Section 2.6]) that the radical of the restriction of the bilinear form $(\ ,\ )$ on the subspace spanned by x, y, and z is one-dimensional if W' is irreducible affine, and we have a contradiction, and whence W' is dihedral. □

PROPOSITION 6.5. *Let $\{x_i\}_{i\in\mathbb{N}} \subset \Phi^+$ be an infinite sequence of positive roots satisfying $x_{i+1} \underset{\neq}{\operatorname{dom}} x_i$ for all $i \in \mathbb{N}$. Suppose that there is an accumulation point η of the corresponding normalized sequence $\{\widehat{x_i}\}_{i\in N}$ and $\eta \in E_{\mathrm{aff}}$. The normalized sequence $\{\widehat{x_i}\}_{i\in\mathbb{N}}$ is convergent with η being the limit.*

PROOF. Suppose for a contradiction that there is yet another accumulation point for the normalized sequence $\{\widehat{x_i}\}_{i\in\mathbb{N}}$, say η'. Choose two subsequences $\{y_i\}_{i\in\mathbb{N}}$ and $\{z_i\}_{i\in\mathbb{N}}$ of $\{x_i\}_{i\in\mathbb{N}}$ such that $\lim_{i\to\infty} \widehat{y_i} = \eta$ and $\lim_{i\to\infty} \widehat{z_i} = \eta'$; furthermore, we require that

$$z_{i+1} \underset{\neq}{\operatorname{dom}} y_{i+1} \underset{\neq}{\operatorname{dom}} z_i \quad \text{for all } i \in \mathbb{N}.$$

It follows from Corollary 5.27 that there exists some $w \in W$ such that $\{w \cdot \eta\} = E(W_I)$ for some irreducible affine standard parabolic subgroup W_I. Note that $\{wx_i\}_{i\geq N}$ for some fixed $N \geq \ell(w)$ is still an infinite injective sequence of positive roots totally ordered by dominance; further, $w \cdot \eta$ and $w \cdot \eta'$ are now accumulation points for $\{\widehat{wy_i}\}_{i\geq N}$ and $\{\widehat{wz_i}\}_{i\geq N}$, respectively. Since now $\mathrm{Pos}(w \cdot \eta) = \emptyset$, it follows that

$$(w \cdot \eta, \widehat{wy_i}) \leq 0 \quad \text{and} \quad (w \cdot \eta, \widehat{wz_i}) \leq 0 \quad \text{for all } i \geq N.$$

However, the dominance in the two subsequences implies that, for all $i \geq N$,

$$(w \cdot \eta, \widehat{wy_i}) = \lim_{j\to\infty} (\widehat{wy_j}, \widehat{wy_i}) \geq 0,$$

and

$$(w \cdot \eta, \widehat{wz_i}) = \lim_{j\to\infty} (\widehat{wy_j}, \widehat{wz_i}) \geq 0.$$

Thus, for all $i \geq N$,

$$(w \cdot \eta, \widehat{wy_i}) = (w \cdot \eta, \widehat{wz_i}) = 0.$$

Note now that I, being a connected component of Π, is the set of simple roots for W_I, and furthermore, $I = \mathrm{supp}(w \cdot \eta)$. The latter assertion is well known and it

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can be seen in the following way: if $\mathrm{supp}(w \cdot \eta) \neq I$, then there exists some $a \in I$ while $a \notin \mathrm{supp}(w \cdot \eta)$, and the connectedness of I implies that there must be some $b \in \mathrm{supp}(w \cdot \eta)$ such that $(a, b) < 0$, but then $(w \cdot \eta, b) \neq 0$, which contradicts that $w \cdot \eta \in \mathrm{Rad}(I)$. Now, suppose that $x \in \Phi^+$ such that $(x, w \cdot \eta) = 0$. The fact that $\mathrm{supp}(w \cdot \eta) = I$ readily implies that there is no $b \in \mathrm{supp}(x)$ for which $(b, a) \neq 0$ for some $a \in I$. Thus, all $\mathrm{supp}(wz_i)$, for all $i \geq N$, must either be disconnected from I or contained in I (since the support of a root must be itself connected). By our construction, $\mathrm{supp}(wy_i) = I$ for all i large enough. Now, if some $wz_j \notin \Phi(W_I)$, where j is large enough, then $\mathrm{supp}(wz_j)$ must be disconnected from I, but this contradicts that $wy_j \underset{\neq}{\mathrm{dom}}\, wz_j$. Thus,

$$wz_i \in \Phi(W_I) \quad \text{for all } i \text{ large enough.}$$

Since $E(W_I) = \{w \cdot \eta\}$, it follows that

$$w \cdot \eta' = \lim_{i \to \infty} \widehat{wz_i} = w \cdot \eta,$$

whence $\eta' = \eta$ and such a sequence has a unique limit. □

In a separate forthcoming paper, we obtain additional results establishing that the support of any accumulation point of a normalized sequence corresponding to an infinite injective sequence of roots totally ordered by dominance is connected (and remains connected upon the dot-action by the group, and therefore, cannot be of affine-type); further, we show that any two potential accumulation points η_1 and η_2 from the same normalized sequence corresponding to one infinite injective dominance chain have the same support, and $\mathrm{Pos}(\eta_1) = \mathrm{Pos}(\eta_2)$.

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