We shall call the regular trapezium which results from halving a regular hexagon a 'half-hexagon'. By virtue of the half-turn symmetry of the tiling any half-hexagon centred on the edge of, and lying within, the region already claimed, implies a matching half-hexagon outside. Thus $SQRU$ implies $UQ'R'S$. We proceed outwards, always claiming the biggest possible external half-hexagons. The figure shows how additions are made to alternate sides of the hexagonal region with each iteration. This guarantees that the whole plane can be covered. At no step in the argument have we specified k. So, by construction, we have confirmed the general proposition.

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100.14 Optimum choice of interpolation points in Newton's divided difference formula

The widely-used Newton's divided difference interpolation formula is given by

$$f(x) = (x - x_0)[01] + (x - x_0)(x - x_1)[012] + \dots$$
$$\dots + (x - x_0)(x - x_1)\dots(x - x_{n-1})[012\dots n] + R_n, \tag{1}$$

where the truncation error R_n is given by

$$\frac{(x - x_0)(x - x_1)\dots(x - x_n)}{(n + 1)!} f^{(n+1)}(\xi) \tag{2}$$

with

$$x_0 < \xi < x_n \tag{3}$$

and the $[012\dots n]$ are divided differences, defined recursively (see [1]). (Note that the above applies only if the points x_i are arranged in increasing order.)

Let us define

$$\prod_n = (x - x_0)(x - x_1)\dots(x - x_n). \tag{4}$$

Then, if we have available a table of known values of f_i at X_i where

$$i = -k, -k + 1, \dots, \ell - 1, \ell \tag{5}$$

and wish to approximate $f(x)$ at a given value of x, then we have a wide choice of points $x_0, x_1, \dots, x_n$ from among the tabular points $X_{-k}, X_{-k+1}, \dots, X_\ell$. These points $x_0, x_1, \dots, x_n$ need not, and usually should not, be consecutive.

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Decades ago, in preparing to teach a course on numerical methods, this author was impressed by the statement in the book [1] that we should choose the points $x_0, x_1, \ldots, x_n$ to be as close as possible to x. For example, we should take x_0 as the overall closest, x_1 as the next closest, x_2 as the third closest, and so on. Since then, I have been teaching the desirability of this strategy, i.e. in decreasing the error by minimising the product $\prod_n$. (We cannot control the size of $f^{(n+1)}$ in practice, or even know what it is.) But I never saw a *proof* of this proposition; instead I regarded it as intuitively obvious.

Recently however I started to wonder if any other author(s) had given a rigorous or quantitative proof. After consulting about 30 books on numerical methods, I found that only four mentioned the issue even qualitatively. These were [1, 2, 3] and [4]. One or two, e.g. [5], discuss 'Chebyshev points', i.e. the fact that having tabular points closer together near the ends of the range $(x_0, x_1, \ldots, x_n)$ leads to a smaller error; but in general, if the table is pre-written we may not be able to do this. Also those authors do not discuss how we should choose the x_i relative to x (the point where we seek the value of $f(x)$).

I have found that my students, even although I teach the mentioned strategy, often take x_0 as the first point in the table, x_1 as the next in sequence and so on, regardless of where x may be. For example in a recent test, 3 out of 12 students who answered the relevant question did just that. Now in homework assignments and exams, this author (and probably most others in academia) gives a table with a rather limited number of tabular points, so that in fact x_0 etc. are reasonably close to x, and the truncation error is still reasonably small. But in 'real life' a scientist or engineer doing interpolation may use a table with a great many entries, and the start of the table may indeed be very far from x. Such a person, if not properly educated on this issue, may come up with a very erroneous result, as we will show by example in the sequel.

I have conducted numerical experiments as follows: I take 11 points $x_i = (0, 0.1, 0.2, \ldots, 0.9, 1.0) = (i \times .1)$ $(i = 0, 1, 2, \ldots, 10)$, and select just the first n $(n = 2, \ldots, 10)$. Then I take three different values of x:

(1) $x = \frac{n}{2.0} \times .1$ (n odd), e.g. if $n = 5, x = .25$
$x = \frac{n+1}{2.0} \times .1$ (n even), e.g. if $n = 6, x = .35$

(2) $x = .05$

(3) $x = -2.0$.

We will call these cases 1, 2 and 3 respectively. They correspond to what an expert, an inexpert student, or an inexpert 'real-life' practitioner might use (respectively). We wrote short Fortran programs which evaluated $\frac{\prod_n}{(n+1!}$ for $n = 2, 3, \ldots, 10$ and the values of x mentioned above. A sample of the results is shown in the table below:

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n	case 1 x in middle	case 2 x near start	case1/case2 improvement factor	case 3 x far away	case1/case3 improvement factor
2	.6E-4	.6E-4	1.0	1.5	.4E-4
5	.5E-8	.2E-7	.25	.18	2.6E-8
10	1.0E-15	.8E-13	.012	.5E-3	2E-12

Clearly, the 'keep x in the middle of tabular points used' strategy is much more accurate than the 'naive' alternative of taking the points near the start of the table.

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