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Linear Stability of Hyperbolic Moment Models for Boltzmann Equation

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Dedicated to Professor Zhenhuan Teng on the occasion of his 80th birthday

Abstract. Grad's moment models for Boltzmann equation were recently regularized to globally hyperbolic systems and thus the regularized models attain local well-posedness for Cauchy data. The hyperbolic regularization is only related to the convection term in Boltzmann equation. We in this paper studied the regularized models with the presentation of collision terms. It is proved that the regularized models are linearly stable at the local equilibrium and satisfy Yong's first stability condition with commonly used approximate collision terms, and particularly with Boltzmann's binary collision model.

AMS subject classifications: 82C40, 58J45, 35B35

Key words: Boltzmann equation, Grad's moment method, hyperbolic moment equation, linear stability.

1. Introduction

Boltzmann equation [3] is the most important kinetic equation, governing the movement of a particle system, particularly the gas particles. Since the distribution function in the Boltzmann equation is in very high dimension, Grad [13] purposed the famous moment method for gas kinetic theory to reduce the kinetic equation into low-dimensional models. In more than half a century, Grad's moment equations were suffered by the lack of hyperbolicity [6,15]. Only very recently, in [4,5], the authors revealed the underlying reason that Grad's moment equations lost its hyperbolicity during the model reduction and purposed new reduced models of Boltzmann equation. The

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new models are referred to as globally Hyperbolic Moment Equations (HME) hereafter, which are symmetric quasi-linear systems [7] with global hyperbolicity.

As new models for fluid dynamics, one may prefer to carry out studies on some fundamental mathematical properties on HME before further numerical applications. Among these fundamental mathematical properties, linear stability is one of the most important points [2, 16, 19] for a system to be applied in numerical experiments. It should be noted that the linear stability is not automatically attained for models in fluid dynamics. For instance, famous Burnett equations and super-Burnett equations are discovered not linearly stable [2,18] and thus are ill-posed and rarely have practical applications.

Except for linear stability, Yong proposed a stability condition [21,22], called Yong's first stability condition, for nonlinear first-order hyperbolic systems with source term. With this stability condition, a formal asymptotic approximation of the initial-layer solution to the nonlinear problem has been constructed [22]. Furthermore, with some regularity assumption of the solution, the existence of classical solutions is guaranteed in the uniform time interval. The stability condition is essential for the nonlinear first-order hyperbolic system. And in [21,22], several classical models have been verified to satisfy the stability condition.

In this paper, we focus on the linear stability analysis of HME at local equilibrium and Yong's first stability condition. The collision term under consideration includes the commonly used approximate formations, such as BGK model [1], ES-BGK model [14], Shakhov model [17] and the original Boltzmann's collision term [3], particularly the binary collision term [9, 12]. We prove that both HME and Ordered globally Hyperbolic Moment Equations(OHME) are linearly stable at local equilibrium for all the four collision models and satisfy Yong's first stability condition.

We start with a brief review of HME and the collision term to be considered. The globally hyperbolic regularization enables us to write HME into an elegant quasi-linear form. It is essential to expand the distribution function at the local equilibrium, where the collision term vanishes. This property provides us some additional equalities which significantly simplify the linear stability analysis. For the binary collision model, the symmetry of the collision plays an important role, which indicates some induced symmetry in the Jacobian of the collision term. With some linear algebra, we proved that HME is linear stable at local equilibrium for all the four collision models. This proof is not trivial noticing that HME we are studying is for arbitrary order.

For Yong's first stability condition, the third inequality plays a major role. We verified this inequality by applying the results in the linear stability analysis, together with some linear algebraic technique. In such sense, Yong's first stability condition can be regarded as an enhanced version of linear stability for nonlinear balance laws.

OHME, first proposed in [10], is the hyperbolic version of ordered Grad's moment system, which includes the well-known Grad's 13 moment system. Since OHME can be derived from HME, the linear stability of OHME at the local equilibrium is deduced from that of HME, as well as Yong's first stability condition.

The rest of the paper is organized as following. Section 2 presents a brief intro-

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duction of the linear stability and some useful linear algebraic results. The Boltzmann equation and Grad's moment method, together with the globally hyperbolic moment system are reviewed in Section 3. In Section 4, four Boltzmann collision terms are studied and the linear stability of HME at local equilibrium is rigorous proved. In Section 5, Yong's first stability condition is verified for HME. We extend the results in Section 4 and Section 5 to OHME and prove that OHME is also linearly stable at local equilibrium and satisfies Yong's first stability condition in Section 6. The paper ends with a conclusion.

2. Preliminaries

2.1. Linear stability

Let us consider the linear PDEs with source term as

$$\frac{\partial \boldsymbol{U}}{\partial t} + \sum_{d=1}^{D} \mathbf{A}_d \frac{\partial \boldsymbol{U}}{\partial x_d} = \mathbf{Q}\boldsymbol{U}, \tag{2.1}$$

where the matrices $\mathbf{A}_d$, $d = 1, \cdots, D$ and $\mathbf{Q}$ are constant. Following [18], we assume the solution is plane waves of the form

$$\boldsymbol{U} = \boldsymbol{U}_* \exp\left(\mathbf{i}(\Omega t - \boldsymbol{k}^T \boldsymbol{x})\right), \tag{2.2}$$

where $\mathbf{i}$ is the imaginary unit, $\boldsymbol{U}_*$ is the complex amplitude of the wave, Ω is its frequency and $\boldsymbol{k}$ is its wave number. Here we use complex variables for convenience and only the real parts of the expressions for the $\boldsymbol{U}$ are relevant. The Eq. (2.1) can be rewritten as

$$\left(\mathbf{i}\Omega\mathbf{I} - \sum_{d=1}^{D} \mathbf{i}k_d \mathbf{A}_d - \mathbf{Q}\right) \boldsymbol{U}_* = 0, \tag{2.3}$$

where $\mathbf{I}$ is the identity matrix. The existence of a nontrivial solution $\boldsymbol{U}_*$ of the equation requires the coefficient matrix to be singular

$$\det\left(\Omega\mathbf{I} - \sum_{d=1}^{D} k_d \mathbf{A}_d + \mathbf{i}\mathbf{Q}\right) = 0. \tag{2.4}$$

This gives us the dispersion relation between Ω and $\boldsymbol{k}$.

Considering a disturbance in space, the wave number $\boldsymbol{k}$ is real and the frequency is complex

$$\Omega = \Omega_r(\boldsymbol{k}) + \mathbf{i}\Omega_i(\boldsymbol{k}).$$

Then the plane wave solutions have the form

$$\boldsymbol{U} = \boldsymbol{U}_* \exp(-\Omega_i(\boldsymbol{k})t) \exp(\mathbf{i}(\Omega_r(\boldsymbol{k})t - \boldsymbol{k}^T \boldsymbol{x})).$$

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Note that $\boldsymbol{U}_* \exp(-\Omega_i(\boldsymbol{k})t)$ is the local amplitude of $\boldsymbol{U}$ as a function of time and stability requires the local amplitude to be non-increasing, thus $\Omega_i(\boldsymbol{k}) \geq 0$.

If we consider a disturbance in time at a given location, the frequency Ω is real and the wave number is complex

$$k = k_r(\Omega) + \mathbf{i}k_i(\Omega),$$

where we consider this problem for one-dimensional processes following [18,19]. Then the plane wave solutions is

$$\boldsymbol{U} = \boldsymbol{U}_* \exp(k_i(\Omega)x) \exp(\mathbf{i}(\Omega t - k_r(\Omega)x)).$$

Here $\boldsymbol{U}_* \exp(k_i(\Omega)x)$ is the amplitude of $\boldsymbol{U}$ at the point x. To be a stable solution, which is a wave traveling in positive x direction($k_r > 0$), it requires a non-increasing amplitude ($k_i \leq 0$) and vice versa, thus $k_r k_i \leq 0$.

Definition 2.1 (Stability). *The system* (2.1) *is stable in time if* $\Omega_i(\boldsymbol{k}) \geq 0$ *for each* $\boldsymbol{k} \in \mathbb{R}^D$*; it is stable in space for one-dimensional processes if* $k_r(\Omega)k_i(\Omega) \leq 0$ *for each* $\Omega \in \mathbb{R}^+$.

2.2. Yong's first stability condition

In [22], Yong developed a singular perturbation theory for initial-value problems of nonlinear first-order hyperbolic system with stiff source term in several space variables and proposed the called Yong's first stability condition. Under the stability condition, a formal asymptotic approximation of the initial-layer solution to the nonlinear problem are constructed. Moreover, with some regularity assumption on the solution, the existence of classical solutions is guaranteed in uniform time interval. The stability condition is fundamental for the nonlinear first-order hyperbolic system with the form

$$\frac{\partial \boldsymbol{U}}{\partial t} + \sum_{d=1}^{D} \mathbf{A}_d(\boldsymbol{U}) \frac{\partial \boldsymbol{U}}{\partial x_d} = \boldsymbol{S}(\boldsymbol{U}), \quad \boldsymbol{U} \in \mathbb{G} \subset \mathbb{R}^n. \tag{2.5}$$

Let

$$\mathbf{Q} = \frac{\partial \boldsymbol{S}}{\partial \boldsymbol{U}},$$

and define the equilibrium manifold

$$\mathcal{E} := \{\boldsymbol{U} \in \mathbb{G} \mid \boldsymbol{S}(\boldsymbol{U}) = 0\}.$$

The stability condition in [22] reads

1. There is an invertible $n \times n$ matrix $\mathbf{P}(\boldsymbol{U})$ and an invertible $r \times r$ matrix $\hat{\mathbf{Q}}(\boldsymbol{U})$, defined on the equilibrium manifold $\mathcal{E}$, such that

$$\mathbf{P}(\boldsymbol{U})\mathbf{Q}(\boldsymbol{U}) = \begin{pmatrix} 0 & 0 \\ 0 & \hat{\mathbf{Q}}(\boldsymbol{U}) \end{pmatrix} \mathbf{P}(\boldsymbol{U}) \quad \text{for } \boldsymbol{U} \in \mathcal{E}. \tag{2.6}$$

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2. There is a symmetric positive definite matrix $\mathbf{A}_0(\boldsymbol{U})$ such that

$$\mathbf{A}_0(\boldsymbol{U})\mathbf{A}_d(\boldsymbol{U}) = \mathbf{A}_d^T(\boldsymbol{U})\mathbf{A}_0(\boldsymbol{U}), \quad \boldsymbol{U} \in \mathbb{G}, \quad d = 1, \cdots, D. \tag{2.7}$$

3. The hyperbolic part and the source term are coupled in the sense

$$\mathbf{A}_0(\boldsymbol{U})\mathbf{Q}(\boldsymbol{U}) + \mathbf{Q}(\boldsymbol{U})^T\mathbf{A}_0(\boldsymbol{U}) \leq -\mathbf{P}(\boldsymbol{U})^T \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{I}_r \end{pmatrix} \mathbf{P}(\boldsymbol{U}). \tag{2.8}$$

The first condition requires that the source term is dissipation or relaxation and the second condition guarantees that the hyperbolic part is a symmetric hyperbolic system. The third condition specifies how the hyperbolic part and the source term can be coupled, which is the key condition to the stability of the solution.

2.3. Two lemmas

At the end of the section, we give two useful lemmas in linear algebra for usage later on.

Lemma 2.1. *Matrices* $\mathbf{A}$, $\mathbf{B} \in \mathbb{R}^{n\times n}$ *are symmetric and* $\mathbf{B}$ *is negative semi-definite, then each eigenvalue of the matrix* $\mathbf{A} - \mathbf{i}\mathbf{B}$ *has a non-negative imaginary part.*

Proof. We prove it by contradiction. Suppose that $\lambda = a + b\mathbf{i}$, a, $b \in \mathbb{R}$, is an eigenvalue of matrix $\mathbf{A} - \mathbf{i}\mathbf{B}$ and $b < 0$, with the corresponding eigenvector $\boldsymbol{v} \in \mathbb{C}^n$, then

$$[(\mathbf{A} - a\mathbf{I}) - \mathbf{i}(\mathbf{B} + b\mathbf{I})]\boldsymbol{v} = \mathbf{0}.$$

Denote $\boldsymbol{v}$ by $\boldsymbol{v} = \boldsymbol{v}_r + \mathbf{i}\boldsymbol{v}_i$, $\boldsymbol{v}_r$, $\boldsymbol{v}_i \in \mathbb{R}^n$. Multiplying the upper equation by $\overline{\boldsymbol{v}} = \boldsymbol{v}_r - \mathbf{i}\boldsymbol{v}_i$, we obtain

$$\overline{\boldsymbol{v}}^T(\mathbf{A} - a\mathbf{I})\boldsymbol{v} - \mathbf{i}\overline{\boldsymbol{v}}^T(\mathbf{B} + b\mathbf{I})\boldsymbol{v} = 0.$$

Noticing that $\overline{\boldsymbol{v}}^T(\mathbf{A} - a\mathbf{I})\boldsymbol{v}$, $\overline{\boldsymbol{v}}^T(\mathbf{B} + b\mathbf{I})\boldsymbol{v} \in \mathbb{R}$, we have

$$\overline{\boldsymbol{v}}^T(\mathbf{B} + b\mathbf{I})\boldsymbol{v} = 0.$$

Direct calculations yield that

$$\boldsymbol{v}_r^T(\mathbf{B} + b\mathbf{I})\boldsymbol{v}_r + \boldsymbol{v}_i^T(\mathbf{B} + b\mathbf{I})\boldsymbol{v}_i = 0.$$

Since $\mathbf{B}+b\mathbf{I}$ is symmetric negative definite, $\boldsymbol{v}_r$ and $\boldsymbol{v}_i$ have to vanish and thus $\boldsymbol{v} = \mathbf{0}$. This contradiction ends the proof. □

Lemma 2.2. *Matrices* $\mathbf{A}$, $\mathbf{B} \in \mathbb{R}^{n\times n}$ *are symmetric and* $\mathbf{B}$ *is negative semi-definite. Let* $k = k_r + ik_i \in \mathbb{C}$, k_r, $k_i \in \mathbb{R}$ *be the solution of* $\det(k\mathbf{A} - \mathbf{i}\mathbf{B} - \lambda\mathbf{I}) = 0$, *for any given* $0 < \lambda \in \mathbb{R}$, *then* $k_r k_i \leq 0$.

Proof. Let $\boldsymbol{v} \in \mathbb{C}^n$, $\|\boldsymbol{v}\| \neq 0$ be an vector s.t. $(k\mathbf{A} - \mathbf{i}\mathbf{B} - \lambda\mathbf{I})\boldsymbol{v} = \mathbf{0}$, then $\overline{\boldsymbol{v}}^T(k_r\mathbf{A} - \lambda\mathbf{I})\boldsymbol{v} + \mathbf{i}\overline{\boldsymbol{v}}^T(k_i\mathbf{A} - \mathbf{B})\boldsymbol{v} = 0$, which indicates $k_r\overline{\boldsymbol{v}}^T\mathbf{A}\boldsymbol{v} = \lambda\overline{\boldsymbol{v}}^T\boldsymbol{v} > 0$ and $k_i\overline{\boldsymbol{v}}^T\mathbf{A}\boldsymbol{v} = \overline{\boldsymbol{v}}^T\mathbf{B}\boldsymbol{v} \leq 0$. Thus, $k_r k_i \leq 0$. □

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3. HME for Boltzmann equation

Let us denote the distribution function in gas kinetic theory by $f(t, \boldsymbol{x}, \boldsymbol{\xi})$ describing the probability density to find a particle at space point $\boldsymbol{x}$ and the time t with velocity $\boldsymbol{\xi}$ in D-dimensional space. The macroscopic density ρ, flow velocity $\boldsymbol{u}$, temperature T, pressure tensor $P = (p_{ij})_{D\times D}$, stress tensor $\Sigma = (\sigma_{ij})_{D\times D}$ and heat flux $\boldsymbol{q}$ are related to the distribution function by

$$\rho(t, \boldsymbol{x}) = \int_{\mathbb{R}^D} f(t, \boldsymbol{x}, \boldsymbol{\xi})\, \mathrm{d}\boldsymbol{\xi}, \tag{3.1a}$$

$$\rho(t, \boldsymbol{x})\boldsymbol{u}(t, \boldsymbol{x}) = \int_{\mathbb{R}^D} \boldsymbol{\xi} f(t, \boldsymbol{x}, \boldsymbol{\xi})\, \mathrm{d}\boldsymbol{\xi}, \tag{3.1b}$$

$$D\rho RT = \int_{\mathbb{R}^D} |\boldsymbol{\xi} - \boldsymbol{u}|^2 f(t, \boldsymbol{x}, \boldsymbol{\xi})\, \mathrm{d}\boldsymbol{\xi}, \tag{3.1c}$$

$$P = \int_{\mathbb{R}^D} (\boldsymbol{\xi} - \boldsymbol{u}) \otimes (\boldsymbol{\xi} - \boldsymbol{u}) f(t, \boldsymbol{x}, \boldsymbol{\xi})\, \mathrm{d}\boldsymbol{\xi}, \tag{3.1d}$$

$$\boldsymbol{q} = \frac{1}{2} \int_{\mathbb{R}^D} |\boldsymbol{\xi} - \boldsymbol{u}|^2 (\boldsymbol{\xi} - \boldsymbol{u}) f(t, \boldsymbol{x}, \boldsymbol{\xi})\, \mathrm{d}\boldsymbol{\xi}, \tag{3.1e}$$

$$\Sigma = P - p\mathbf{I}, \tag{3.1f}$$

where

$$p = \frac{1}{D} \sum_{d=1} p_{dd} = \rho RT$$

is pressure and the constant R stands for the gas constant. For convenience, use $\theta(t, \boldsymbol{x}) = RT(t, \boldsymbol{x})$ to simplify the notations.

The distribution function $f(t, \boldsymbol{x}, \boldsymbol{\xi})$ is governed by the Boltzmann equation [3]

$$\frac{\partial f}{\partial t} + \boldsymbol{\xi} \cdot \nabla_{\boldsymbol{x}} f = Q(f, f), \tag{3.2}$$

where the right hand side $Q(f, f)$ is the collision term, which models the interaction among particles at the position $\boldsymbol{x}$ and time t. The collision term is assumed to have only 1, $\boldsymbol{\xi}$ and $|\boldsymbol{\xi}|^2$ as locally conserved quantities, saying

$$\int_{\mathbb{R}^D} Q(f, f)(1, \boldsymbol{\xi}, |\boldsymbol{\xi}|^2)^T\, \mathrm{d}\boldsymbol{\xi} = \mathbf{0}, \tag{3.3}$$

and if

$$\int_{\mathbb{R}^D} Q(f, f)\psi(\boldsymbol{\xi})\, \mathrm{d}\boldsymbol{\xi} = 0 \ \text{ for all } f, \ \text{ then } \psi(\boldsymbol{\xi}) = a + \boldsymbol{b}^T\boldsymbol{\xi} + c|\boldsymbol{\xi}|^2. \tag{3.4}$$

The collision term is also assumed that

$$Q(f, f) = 0 \Rightarrow f = f_{eq}, \tag{3.5}$$

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where f_{eq} is the local equilibrium

$$f_{eq}(t,\boldsymbol{x},\boldsymbol{\xi}) = \frac{\rho(t,\boldsymbol{x})}{[2\pi\theta(t,\boldsymbol{x})]^{D/2}} \exp\left(-\frac{|\boldsymbol{\xi}-\boldsymbol{u}(t,\boldsymbol{x})|^2}{2\theta(t,\boldsymbol{x})}\right). \tag{3.6}$$

The binary collision term [9, 13] is commonly used to model the dilute gas and has a quadratic form

$$Q(f,f) = \int_{\mathbb{R}^D}\int_{S_+^{D-1}} (f'f_1' - ff_1)B(|\boldsymbol{\xi}-\boldsymbol{\xi}_1|,\sigma)\,\mathrm{d}\boldsymbol{n}\,\mathrm{d}\boldsymbol{\xi}_1, \tag{3.7}$$

where S_+^{D-1} is the upper half sphere, $B(|\boldsymbol{\xi}-\boldsymbol{\xi}_1|,\sigma)$ is the collision kernel and σ is a function of $\boldsymbol{n}$, $\boldsymbol{\xi}$ and $\boldsymbol{\xi}_1$, depending on the type of particles. In (3.7),

$$\begin{aligned} f &= f(t,\boldsymbol{x},\boldsymbol{\xi}), & f_1 &= f(t,\boldsymbol{x},\boldsymbol{\xi}_1),\\ f' &= f(t,\boldsymbol{x},\boldsymbol{\xi}'), & f_1' &= f(t,\boldsymbol{x},\boldsymbol{\xi}_1'), \end{aligned}$$

where $\boldsymbol{\xi}$ and $\boldsymbol{\xi}_1$ are the velocities of two particles before collision, $\boldsymbol{\xi}'$ and $\boldsymbol{\xi}_1'$ are their velocities after collision and $\boldsymbol{n}$ is the direction between their centers of mass. The specific expressions of $B(|\boldsymbol{\xi}-\boldsymbol{\xi}_1|,\sigma)$ and σ are not concerned in this paper.

As simplifications of the binary collision, researchers proposed some alternative collision models to approximate the binary collision model, such as BGK model [1], Shakhov model [17] and ES-BGK model [14]. We list these models below for later usage:

- **Bhatnagar-Gross-Krook(BGK) model [1]:**

$$Q(f,f) = \frac{1}{\tau}(f_{eq}-f), \tag{3.8}$$

  where τ is relaxation time.

- **Shakhov model [17]:**

$$Q(f,f) = \frac{1}{\tau}(f_S-f), \tag{3.9}$$

  where

$$f_S(t,\boldsymbol{x},\boldsymbol{\xi}) = f_{eq}(t,\boldsymbol{x},\boldsymbol{\xi})\left(1+\frac{(1-\mathrm{Pr})\boldsymbol{q}^T(\boldsymbol{\xi}-\boldsymbol{u})}{(D+2)\rho\theta^2}\left(\frac{|\boldsymbol{\xi}-\boldsymbol{u}|^2}{\theta}-(D+2)\right)\right),$$

  where Pr is the Prandtl number, which is $2/3$ for monatomic gas.

- **ES-BGK model [14]:**

$$Q(f,f) = \frac{\mathrm{Pr}}{\tau}(f_G-f), \tag{3.10}$$

  where

$$f_G = \frac{\rho}{\sqrt{\det(2\pi\boldsymbol{\Lambda})}}\exp\left(-\frac{1}{2}(\boldsymbol{\xi}-\boldsymbol{u})^T\boldsymbol{\Lambda}^{-1}(\boldsymbol{\xi}-\boldsymbol{u})\right),$$

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where $\mathbf{\Lambda} = (\lambda_{ij}) \in \mathbb{R}^{D\times D}$ is a symmetric positive definite matrix with entries as

$$\lambda_{ij} = \frac{\theta\delta_{ij}}{\mathrm{Pr}} + \left(1 - \frac{1}{\mathrm{Pr}}\right)\frac{p_{ij}}{\rho}, \quad i,j = 1,\cdots,D,$$

and δ_{ij} is Kronecker delta symbol.

All of the four collision models satisfy the relationship (3.3)-(3.5).

In 1949, Grad proposed the well-known Grad's moment method [13] to derive moment equations from the Boltzmann equation. The key point is to expand the distribution function around the local Maxwellian into Hermite series as

$$f(t,\boldsymbol{x},\boldsymbol{\xi}) = \sum_{\alpha\in\mathbb{N}^D} f_\alpha(t,\boldsymbol{x})\mathcal{H}_\alpha^{[\boldsymbol{u},\theta]}(\boldsymbol{\xi}), \tag{3.11}$$

where α is a D-dimensional multi-index, $\mathcal{H}_\alpha^{[\boldsymbol{u},\theta]}(\boldsymbol{\xi})$ is the basis function, defined by

$$\mathcal{H}_\alpha^{[\boldsymbol{u},\theta]}(\boldsymbol{\xi}) = \mathrm{He}_\alpha^{[\boldsymbol{u},\theta]}(\boldsymbol{\xi})\omega^{[\boldsymbol{u},\theta]}(\boldsymbol{\xi}), \tag{3.12a}$$

$$\omega^{[\boldsymbol{u},\theta]}(\boldsymbol{\xi}) = \frac{f_{eq}}{\rho} = \frac{1}{[2\pi\theta]^{D/2}}\exp\left(-\frac{|\boldsymbol{\xi}-\boldsymbol{u}|^2}{2\theta}\right), \tag{3.12b}$$

where

$$\mathrm{He}_\alpha^{[\boldsymbol{u},\theta]}(\boldsymbol{\xi}) = \frac{1}{\omega^{[\boldsymbol{u},\theta]}(\boldsymbol{\xi})}\prod_{d=1}^{D}\frac{\partial^{\alpha_d}}{\partial\xi_d^{\alpha_d}}\omega^{[\boldsymbol{u},\theta]}(\boldsymbol{\xi}), \quad \alpha\in\mathbb{N}^D. \tag{3.13}$$

Due to the orthogonality of the basis function, we have [8, 11]

$$f_\alpha = \frac{\theta^{|\alpha|}}{\alpha!}\int_{\mathbb{R}^D} f\mathrm{He}_\alpha^{[\boldsymbol{u},\theta]}(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi}, \tag{3.14}$$

where

$$|\alpha| = \sum_{d=1}^{D}\alpha_d, \quad \alpha! = \prod_{d=1}^{D}\alpha_d!.$$

Particularly, we have for $i,j = 1,\cdots,D$

$$f_0 = \rho, \qquad f_{e_i} = 0, \qquad \sum_{d=1}^{D} f_{2e_d} = 0, \tag{3.15a}$$

$$p_{ij} = p\delta_{ij} + (1+\delta_{ij})f_{e_i+e_j}, \qquad q_i = 2f_{3e_i} + \sum_{d=1}^{D} f_{e_i+2e_d}, \tag{3.15b}$$

where e_i, $i = 1,\cdots,D$ is unit multi-index with its i-th entry to be 1.

Substituting the expansion (3.11) into the Boltzmann equation (3.2) and matching the coefficients of the basis function $\mathcal{H}_\alpha^{[\boldsymbol{u},\theta]}(\boldsymbol{\xi})$, we can obtain the governing equation of $\boldsymbol{u}$, θ and f_α, $\alpha\in\mathbb{N}^3$. However, the resulting system contains infinite number of

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equations. A cut-off and moment closure are required. Choosing a positive integer $3 \leq M \in \mathbb{N}$ and discarding all the equations including $\partial f_\alpha/\partial t$, $|\alpha| > M$ and setting $f_\alpha = 0$, $|\alpha| > M$ to closure the residual system, we can obtain M-th order Grad's moment system as

$$\begin{aligned}
&\frac{\partial f_\alpha}{\partial t} + \sum_{d=1}^{D}\left(\theta\frac{\partial f_{\alpha-e_d}}{\partial x_d} + u_d\frac{\partial f_\alpha}{\partial x_d} + (1-\delta_{|\alpha|,M})(\alpha_d+1)\frac{\partial f_{\alpha+e_d}}{\partial x_d}\right) + \sum_{k=1}^{D} f_{\alpha-e_k}\frac{\partial u_k}{\partial t} \\
&\quad + \sum_{k,d=1}^{D}\frac{\partial u_k}{\partial x_d}(\theta f_{\alpha-e_k-e_d} + u_d f_{\alpha-e_k} + (\alpha_d+1)f_{\alpha-e_k+e_d}) + \frac{1}{2}\sum_{k=1}^{D} f_{\alpha-2e_k}\frac{\partial\theta}{\partial t} \\
&\quad + \sum_{k,d=1}^{D}\frac{1}{2}\frac{\partial\theta}{\partial x_d}(\theta f_{\alpha-2e_k-e_d} + u_d f_{\alpha-2e_k} + (\alpha_d+1)f_{\alpha-2e_k+e_d}) \\
=&S_\alpha, \quad |\alpha| \leq M,
\end{aligned} \tag{3.16}$$

where

$$S_\alpha = \frac{\theta^{|\alpha|}}{\alpha!}\int_{\mathbb{R}^D} Q(f,f)\mathrm{He}_\alpha^{[\boldsymbol{u},\theta]}(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi}. \tag{3.17}$$

It is well-known that Grad's moment system lacks of global hyperbolicity [15] and it was found recently that it is not hyperbolic even around the local Maxwellian [6]. The globally hyperbolic regularization proposed in [4,5] essentially fixes this drawback and yields the globally hyperbolic moment equations (HME) as

$$\begin{aligned}
&\frac{\partial f_\alpha}{\partial t} + \sum_{d=1}^{D}\left(\theta\frac{\partial f_{\alpha-e_d}}{\partial x_d} + u_d\frac{\partial f_\alpha}{\partial x_d} + (1-\delta_{|\alpha|,M})(\alpha_d+1)\frac{\partial f_{\alpha+e_d}}{\partial x_d}\right) + \sum_{k=1}^{D} f_{\alpha-e_k}\frac{\partial u_k}{\partial t} \\
&\quad + \sum_{k,d=1}^{D}\frac{\partial u_k}{\partial x_d}(\theta f_{\alpha-e_k-e_d} + u_d f_{\alpha-e_k} + (1-\delta_{|\alpha|,M})(\alpha_d+1)f_{\alpha-e_k+e_d}) + \frac{1}{2}\sum_{k=1}^{D} f_{\alpha-2e_k}\frac{\partial\theta}{\partial t} \\
&\quad + \sum_{k,d=1}^{D}\frac{1}{2}\frac{\partial\theta}{\partial x_d}(\theta f_{\alpha-2e_k-e_d} + u_d f_{\alpha-2e_k} + (1-\delta_{|\alpha|,M})(\alpha_d+1)f_{\alpha-2e_k+e_d}) \\
=&S_\alpha, \quad |\alpha| \leq M,
\end{aligned} \tag{3.18}$$

where $(\cdot)_\alpha$ is taken as zero if any component of α is negative. To simplify the notations, we introduce the ordering relation on $\mathbb{N}^D$.

Definition 3.1 (Graded reverse lexicographic)**.** *An ordering relaxation on $\mathbb{N}^D$ is called graded reverse lexicographic ordering $\prec$ if for any $\alpha, \beta \in \mathbb{N}^D$,*

$$\begin{aligned}
\alpha \prec \beta \Longleftrightarrow &|\alpha| \leq |\beta| \text{ or} \\
&|\alpha| = |\beta|, \text{ and } \exists i\ (1 \leq i \leq D), \text{ s.t. } \alpha_i > \beta_i,\ \alpha_j = \beta_j\ (i < j \leq D).
\end{aligned}$$

With this ordering, we adopt the multi-indices as the subscripts of vectors and matrices since now on, sorting the multi-indeices by the graded reverse lexicographic ordering $\prec$. Let N to be all the multi-indices not greater than Me_D, which is the total

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number of the equations in M-th order Grad's moment system. For a vector $\boldsymbol{w} \in \mathbb{R}^N$, w_α stands for the entry with α as subscript and for a matrix $\mathbf{D} \in \mathbb{R}^{N\times N}$, $D_{\alpha,\beta}$ stands for the entry with row index α and column index β.

Following the notations in [5], define $\boldsymbol{w} \in \mathbb{R}^N$ and

$$w_\alpha = \begin{cases} \rho, & \alpha = 0, \\ u_i, & \alpha = e_i, \quad i = 1, \cdots, D, \\ \dfrac{p_{ij}}{1+\delta_{ij}}, & \alpha = e_i + e_j, \quad i, j = 1, \cdots, D, \\ f_\alpha, & 3 \le |\alpha| \le M. \end{cases} \tag{3.19}$$

The HME (3.18) can be written into quasi-linear form [7]:

$$\mathbf{D}\frac{\partial \boldsymbol{w}}{\partial t} + \sum_{d=1}^{D} \mathbf{M}_d \mathbf{D} \frac{\partial \boldsymbol{w}}{\partial x_d} = \boldsymbol{S}, \tag{3.20}$$

where the coefficient matrices $\mathbf{D}$, $\mathbf{M}_d$ are defined as [11]

$$\begin{aligned} \mathbf{D} = \mathbf{I} + \sum_{|\alpha|\le M} \Bigg(& \sum_{d=1}^{D} f_{\alpha-e_d} \mathbf{E}_{\alpha,e_d} - \frac{\theta}{2\rho} \sum_{d=1}^{D} f_{\alpha-2e_d} \mathbf{E}_{\alpha,0} \\ & + H(|\alpha|-3) \frac{1}{D\rho} \left(\sum_{d=1}^{D} f_{\alpha-2e_d} \right) \sum_{k=1}^{D} \mathbf{E}_{\alpha,2e_k} \Bigg) - \sum_{d=1}^{D} \mathbf{E}_{e_d,e_d}, \end{aligned} \tag{3.21a}$$

$$\mathbf{M}_d = \sum_{|\alpha|\le M} \left(\theta \mathbf{E}_{\alpha,\alpha-e_d} + u_d \mathbf{E}_{\alpha,\alpha} + (1-\delta_{|\alpha|,M})(\alpha_d+1)\mathbf{E}_{\alpha,\alpha+e_d} \right), \tag{3.21b}$$

where $\mathbf{I}$ is the identity matrix and $\mathbf{E}_{\alpha,\beta}$ is zero matrix if any component of α, β is negative or $|\beta| > M$ and is the matrix with all its entries to be 0, except for the only entry with row index α and column index β to be 1. The Heaviside step function $H(x)$ is defined as

$$H(x) = \begin{cases} 0, & x < 0, \\ 1, & x \ge 0. \end{cases}$$

As pointed out in [11], $\mathbf{D}$ is a lower triangular matrix with all diagonal entries nonzero thus invertible and its inverse is

$$\begin{aligned} \mathbf{D}^{-1} = \mathbf{I} - \sum_{|\alpha|\le M} & \left(\sum_{d=1}^{D} \frac{f_{\alpha-e_d}}{\rho} \mathbf{E}_{\alpha,e_d} + H(|\alpha|-3) \frac{1}{D\rho} \left(\sum_{d=1}^{D} f_{\alpha-2e_d} \right) \sum_{k=1}^{D} \mathbf{E}_{\alpha,2e_k} \right) \\ & + \sum_{d=1}^{D} \frac{1}{\rho} \mathbf{E}_{e_d,e_d} + \frac{\theta}{2} \sum_{d=1}^{D} \mathbf{E}_{2e_d,0}. \end{aligned} \tag{3.22}$$

Noticing (3.3) and (3.22), we obtain $\left(\mathbf{D}^{-1} - \mathbf{I}\right)\boldsymbol{S} = \mathbf{0}$, thus

$$\mathbf{D}^{-1}\boldsymbol{S} = \boldsymbol{S}. \tag{3.23}$$

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Hence, the HME (3.20) can be reformulated as

$$\frac{\partial \boldsymbol{w}}{\partial t} + \sum_{d=1}^{D} \mathbf{A}_d \frac{\partial \boldsymbol{w}}{\partial x_d} = \boldsymbol{S}, \tag{3.24}$$

where $\mathbf{A}_d = \mathbf{D}^{-1}\mathbf{M}_d\mathbf{D}$.

4. Linear stability of HME

Now we begin to investigate the linear stability of the HME at the thermodynamic equilibrium. First we linearize the HME into linear balance laws at a local Maxwellian given by ρ_0, $\boldsymbol{u}_0 = \mathbf{0}$ and θ_0. Let us introduce the dimensionless variables $\bar{\rho}$, $\bar{\theta}$, $\bar{\boldsymbol{u}}$, $\bar{p}$, $\bar{p}_{ij}$ and $\bar{f}_\alpha$ as

$$\rho = \rho_0(1+\bar{\rho}), \qquad u_i = \sqrt{\theta_0}\bar{u}_i, \qquad \theta = \theta_0(1+\bar{\theta}), \qquad p = p_0(1+\bar{p}), \tag{4.1a}$$

$$p_{ij} = p_0(\delta_{ij}+\bar{p}_{ij}), \qquad f_\alpha = \rho_0\theta_0^{\frac{|\alpha|}{2}}\cdot\bar{f}_\alpha, \qquad \boldsymbol{x} = L\cdot\bar{\boldsymbol{x}}, \qquad t = \frac{L}{\sqrt{\theta_0}}\bar{t}, \tag{4.1b}$$

where L is a characteristic length, $\bar{\boldsymbol{x}}$ and $\bar{t}$ are the dimensionless coordinates and time, respectively. Let

$$\boldsymbol{\Lambda}_0 = \sum_{|\alpha|\le M, |\alpha|\neq 1} \rho_0\theta_0^{|\alpha|/2}\mathbf{E}_{\alpha,\alpha} + \sqrt{\theta_0}\sum_{d=1}^{D}\mathbf{E}_{e_d,e_d}, \tag{4.2a}$$

$$\boldsymbol{w}_0 = \begin{cases} 1, \\ 0, \\ \dfrac{\delta_{ij}}{2}, \\ 0, \end{cases} \qquad \bar{\boldsymbol{w}} = \begin{cases} \bar{\rho}, & \alpha = 0, \\ \bar{u}_i, & \alpha = e_i, \quad i = 1,\cdots,D, \\ \dfrac{\bar{p}_{ij}}{1+\delta_{ij}}, & \alpha = e_i+e_j,\ i,j = 1,\cdots,D, \\ \bar{f}_\alpha, & 3 \le |\alpha| \le M, \end{cases} \tag{4.2b}$$

then $\boldsymbol{w} = \boldsymbol{\Lambda}_0(\boldsymbol{w}_0 + \bar{\boldsymbol{w}})$. All the dimensionless variables $\bar{\rho}$, $\bar{\theta}$, $\bar{\boldsymbol{u}}$, $\bar{p}$, $\bar{p}_{ij}$ and $\bar{f}_\alpha$ are small quantities. Substituting (4.1a), (4.2a) and (4.2b) into the globally hyperbolic moment system (3.20) and discarding all the high-order quantities, we obtain the linearized HME as

$$\mathbf{D}(\boldsymbol{\Lambda}_0\boldsymbol{w}_0)\boldsymbol{\Lambda}_0\frac{\partial \bar{\boldsymbol{w}}}{\partial \bar{t}}\frac{\sqrt{\theta_0}}{L} + \sum_{d=1}^{D}\mathbf{M}(\boldsymbol{\Lambda}_0\boldsymbol{w}_0)\mathbf{D}(\boldsymbol{\Lambda}_0\boldsymbol{w}_0)\boldsymbol{\Lambda}_0\frac{\partial \bar{\boldsymbol{w}}}{\partial \bar{x}_d}\frac{1}{L} = \mathbf{Q}(\boldsymbol{\Lambda}_0\boldsymbol{w}_0)\boldsymbol{\Lambda}_0\bar{\boldsymbol{w}}, \tag{4.3}$$

where $\boldsymbol{S}(\boldsymbol{\Lambda}_0\boldsymbol{w}_0) = 0$ is applied and $\mathbf{Q} = \partial\boldsymbol{S}/\partial\boldsymbol{w}$. Let $\boldsymbol{\Lambda}_1 = \sum_{|\alpha|\le M}\rho_0\theta_0^{|\alpha|/2}\mathbf{E}_{\alpha,\alpha}$, then some simplifications yield

$$\bar{\mathbf{D}}\frac{\partial \bar{\boldsymbol{w}}}{\partial \bar{t}} + \sum_{d=1}^{D}\bar{\mathbf{M}}_d\bar{\mathbf{D}}\frac{\partial \bar{\boldsymbol{w}}}{\partial \bar{x}_d} = \bar{\mathbf{Q}}\bar{\boldsymbol{w}}, \tag{4.4}$$

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where

$$\bar{\mathbf{D}} = \mathbf{\Lambda}_1^{-1}\mathbf{D}(\mathbf{\Lambda}_0\boldsymbol{w}_0)\mathbf{\Lambda}_0 = \mathbf{I} - \frac{1}{2}\sum_{d=1}^{D}\mathbf{E}_{2e_d,0}, \tag{4.5a}$$

$$\bar{\mathbf{M}}_d = \frac{1}{\sqrt{\theta_0}}\mathbf{\Lambda}_1^{-1}\mathbf{M}_d(\mathbf{\Lambda}_0\boldsymbol{w}_0)\mathbf{\Lambda}_1 = \sum_{|\alpha|\le M}\left(\mathbf{E}_{\alpha,\alpha-e_d} + (1-\delta_{|\alpha|,M})(\alpha_d+1)\mathbf{E}_{\alpha,\alpha+e_d}\right), \tag{4.5b}$$

$$\bar{\mathbf{Q}} = \frac{L}{\sqrt{\theta_0}}\mathbf{\Lambda}_1^{-1}\mathbf{Q}(\mathbf{\Lambda}_0\boldsymbol{w}_0)\mathbf{\Lambda}_0 = \frac{L}{\sqrt{\theta_0}}\mathbf{\Lambda}_1^{-1}\mathbf{Q}(\mathbf{\Lambda}_0\boldsymbol{w}_0)\mathbf{\Lambda}_1, \tag{4.5c}$$

where (3.3) is used in the last equation. The Eq. (3.23) indicates $\bar{\mathbf{D}}^{-1}\bar{\mathbf{Q}} = \bar{\mathbf{Q}}$, so we have

$$\frac{\partial\bar{\boldsymbol{w}}}{\partial\bar{t}} + \sum_{d=1}^{D}\bar{\mathbf{A}}_d\frac{\partial\bar{\boldsymbol{w}}}{\partial\bar{x}_d} = \bar{\mathbf{Q}}\bar{\boldsymbol{w}} \quad \text{with} \quad \bar{\mathbf{A}}_d = \bar{\mathbf{D}}^{-1}\bar{\mathbf{M}}_d\bar{\mathbf{D}}. \tag{4.6}$$

To investigate the linear stability of the HME (3.24) is to study the stability of the linearized HME (4.6). We first directly propose two lemmas on the properties of the linearized HME (4.6) and leave the proof to the following part of this section.

Lemma 4.1. *There exists a constant invertible matrix $\mathbf{T} \in \mathbb{R}^{N\times N}$ subject to $\mathbf{T}^{-1}\bar{\mathbf{M}}_d\mathbf{T}$, $d = 1, \cdots, D$ is symmetric, and $\mathbf{T}^{-1}\bar{\mathbf{Q}}\mathbf{T}$ is symmetric negative semi-definite.*

Lemma 4.2. *Matrices $\bar{\mathbf{D}}$ and $\bar{\mathbf{Q}}$ satisfy*

$$\bar{\mathbf{D}}^{-1}\bar{\mathbf{Q}}\bar{\mathbf{D}} = \bar{\mathbf{Q}}, \tag{4.7}$$

for all the four collision models, including BGK model, Shakhov model, ES-BGK model and binary collision model.

With the lemmas above, our main result of this section is the following theorem.

Theorem 4.1. *The HME* (3.24) *is linearly stable both in space and in time at the local Maxwellian, i.e., the linearized HME* (4.6) *is stable both in space and in time.*

Proof. We first prove the linear stability in time. Let $\mathbf{T}$ be the constant invertible matrix $\mathbf{T}$ in Lemma 4.1, then $\sum_{d=1}^{D}\boldsymbol{k}_d\mathbf{T}^{-1}\bar{\mathbf{M}}_d\mathbf{T}$ is symmetric, and $\mathbf{T}^{-1}\bar{\mathbf{Q}}\mathbf{T}$ is symmetric negative semi-definite. Due to Lemma 2.1, each eigenvalue of the matrix $\sum_{d=1}^{D}k_d\mathbf{T}^{-1}\bar{\mathbf{M}}_d\mathbf{T} - \mathbf{i}\mathbf{T}^{-1}\bar{\mathbf{Q}}\mathbf{T}$ has a non-negative imaginary part and thus each eigenvalue of

$$\sum_{d=1}^{D}k_d\bar{\mathbf{A}}_d - \mathbf{i}\bar{\mathbf{Q}} = \left(\mathbf{T}^{-1}\bar{\mathbf{D}}\right)^{-1}\left(\sum_{d=1}^{D}k_d\mathbf{T}^{-1}\bar{\mathbf{M}}_d\mathbf{T} - \mathbf{i}\mathbf{T}^{-1}\bar{\mathbf{Q}}\mathbf{T}\right)\left(\mathbf{T}^{-1}\bar{\mathbf{D}}\right)$$

has a non-negative imaginary part, i.e., $\Omega_i \ge 0$. Here Lemma 4.2 is used.

Analogously, the linear stability in space can be proved directly with Lemma 2.2, Lemma 4.1 and Lemma 4.2. □

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To finish the proof of theorem 4.1, we need to check the validity of Lemma 4.1 and Lemma 4.2. Below we construct a constant invertible $\mathbf{T}$ subject to $\mathbf{T}^{-1}\bar{\mathbf{M}}_d\mathbf{T}$, $d = 1, \cdots, D$ is symmetric at first and then we prove that $\mathbf{T}^{-1}\bar{\mathbf{Q}}\mathbf{T}$ is symmetric negative semi-definite for all the four collisions and Lemma 4.2.

It is easy to see that the construction of the matrix $\mathbf{T}$ is not unique. Actually, if the matrix $\mathbf{T}$ satisfies the constraints in Lemma 4.1, then for any orthogonal matrix $\mathbf{T}_1$, $\mathbf{T}_1\mathbf{T}$ also satisfies the constraints in Lemma 4.1. Here, we provide a direct construction. Precisely, if we define

$$\mathbf{T} = \sum_{|\alpha|\le M} \frac{1}{\sqrt{\alpha!}}\mathbf{E}_{\alpha,\alpha}, \tag{4.8}$$

then

$$\mathbf{T}^{-1}\bar{\mathbf{M}}_d\mathbf{T} = \sum_{|\alpha|\le M} \left(\sqrt{\alpha_d}\mathbf{E}_{\alpha,\alpha-e_d} + (1-\delta_{|\alpha|,M})\sqrt{\alpha_d+1}\mathbf{E}_{\alpha,\alpha+e_d}\right), \quad d = 1, \cdots, D,$$

is symmetric.

In Grad's expansion (3.11), the basis function $\mathcal{H}_\alpha^{[\boldsymbol{u},\theta]}(\boldsymbol{\xi})$ is orthogonal but not normalized. The construction of $\mathbf{T}$ here is equivalent to a normalization of the basis functions.

Lemma 4.2 can be directly proved if Lemma 4.1 is valid.

Proof of Lemma 4.2. Similarly as the derivative of (3.23), it is easy to check $\bar{\mathbf{D}}^T\bar{\mathbf{Q}} = \bar{\mathbf{Q}}$. Let $\mathbf{K} = \frac{1}{2}\sum_{d=1}^{D}\mathbf{E}_{2e_d,0}$, then

$$\bar{\mathbf{D}} = \mathbf{I} - \mathbf{K} \quad \text{and} \quad \bar{\mathbf{D}}^{-1} = \mathbf{I} + \mathbf{K},$$

and thus $\mathbf{K}^T\bar{\mathbf{Q}} = 0$.

It is easy again to check

$$\mathbf{T}\mathbf{K} = \frac{1}{\sqrt{2}}\mathbf{K} \quad and \quad \mathbf{K}^T\mathbf{T}^{-1} = \sqrt{2}\mathbf{K}^T,$$

thus $\mathbf{K}^T\mathbf{T}^{-1}\bar{\mathbf{Q}}\mathbf{T} = 0$. If Lemma 4.1 is valid, then $\mathbf{T}^{-1}\bar{\mathbf{Q}}\mathbf{T}$ is symmetric and thus

$$0 = \mathbf{T}^{-1}\bar{\mathbf{Q}}\mathbf{T}\mathbf{K} = \frac{1}{\sqrt{2}}\mathbf{T}^{-1}\bar{\mathbf{Q}}\mathbf{K}.$$

Since $\mathbf{T}^{-1}$ is invertible, $\bar{\mathbf{Q}}\mathbf{K} = 0$, which indicates $\bar{\mathbf{Q}}\bar{\mathbf{D}}^{-1} = \bar{\mathbf{Q}}$. This completes the proof. □

Now let us prove Lemma 4.1. This requires us to verify that $\mathbf{T}^{-1}\bar{\mathbf{Q}}\mathbf{T}$ is symmetric negative semi-definite. Due to the definition of $\bar{\mathbf{Q}}$ (4.5), we need only to show that

$$\frac{L}{\sqrt{\theta_0}}\mathbf{T}^{-1}\boldsymbol{\Lambda}_1^{-1}\mathbf{Q}(\boldsymbol{\Lambda}_0\boldsymbol{w}_0)\boldsymbol{\Lambda}_1\mathbf{T} \text{ is symmetric negative semi-definite.} \tag{4.9}$$

We check (4.9) case by case for the four collision models we are considering:

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BGK model: Direct calculation of (3.17) yields $S_\alpha^{BGK} = H(|\alpha| - 2)f_\alpha$, thus

$$\mathbf{Q}^{BGK}(\mathbf{\Lambda}_0 \boldsymbol{w}_0) = -\frac{1}{\tau}\left(\mathbf{I} - \sum_{|\alpha|\le 1}\mathbf{E}_{\alpha,\alpha} - \frac{1}{D}\sum_{i,j=1}^{D}\mathbf{E}_{2e_i,2e_j}\right).$$

It is then easy to check (4.9) is valid for BGK model.

Shakhov model: Direct calculation of (3.17) yields

$$S_\alpha = \begin{cases} 0, & |\alpha| \le 1, \\ \dfrac{1-\Pr}{(D+2)\tau} q_i - \dfrac{f_\alpha}{\tau}, & \alpha = e_i + 2e_k, \quad i,k = 1,\cdots,D, \\ -\dfrac{f_\alpha}{\tau}, & \text{otherwise}, \end{cases}$$

thus

$$\mathbf{Q}^{Shakhov}(\mathbf{\Lambda}_0 \boldsymbol{w}_0) = -\frac{1}{\tau}\left(\mathbf{I} - \sum_{|\alpha|\le 1}\mathbf{E}_{\alpha,\alpha} - \frac{1}{D}\sum_{i,j=1}^{D}\mathbf{E}_{2e_i,2e_j} - \frac{1-\Pr}{D+2}\sum_{i,j,k=1}^{D}(1+2\delta_{ij})\mathbf{E}_{e_i+2e_k,e_i+2e_j}\right).$$

It is easy again to check (4.9) is valid for Shakhov model.

ES-BGK model: Let

$$G_\alpha = \begin{cases} \rho, & \alpha = 0, \\ 0, & |\alpha| \text{ is odd}, \\ \dfrac{1-1/\Pr}{\alpha_i \rho}\displaystyle\sum_{d=1}^{D}\sigma_{id} G_{\alpha - e_i - e_d}, & |\alpha| \ge 2, \quad i = 1,\cdots,D \text{ and } \alpha_i > 0, \end{cases}$$

then $S_\alpha^{ES-BGK} = \frac{\Pr}{\tau}(G_\alpha - f_\alpha)$. Direct calculation yields

$$\begin{aligned} &\mathbf{Q}^{ES-BGK}(\mathbf{\Lambda}_0 \boldsymbol{w}_0) \\ &= -\frac{\Pr}{\tau}\left(\mathbf{I} - \sum_{|\alpha|\le 2}\mathbf{E}_{\alpha,\alpha}\right) - \frac{1}{\tau}\left(\sum_{d=1}^{D}\mathbf{E}_{2e_d,2e_d} - \frac{1}{D}\sum_{i,j=1}^{D}\mathbf{E}_{2e_i,2e_j}\right). \end{aligned} \tag{4.10}$$

One then may directly show (4.9) is valid for ES-BGK model.

Binary collision model: It it clear that the symmetry of the matrix $\mathbf{T}^{-1}\bar{\mathbf{Q}}\mathbf{T}$ is equivalent to

$$\frac{\alpha!}{\theta_0^{|\alpha|}} Q_{\alpha,\beta}(\mathbf{\Lambda}_0 \boldsymbol{w}_0) = \frac{\beta!}{\theta_0^{|\beta|}} Q_{\beta,\alpha}(\mathbf{\Lambda}_0 \boldsymbol{w}_0), \quad |\alpha|, |\beta| \le M, \tag{4.11}$$

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where

$$Q_{\alpha,\beta} = \frac{\partial S_\alpha}{\partial w_\beta}, \tag{4.12}$$

and S_α is defined in (3.17). Noticing that at the local Maxwellian

$$\int_{\mathbb{R}^D} \boldsymbol{Q}(f_{eq}, f_{eq}) \mathrm{He}_\alpha^{[\boldsymbol{u},\theta]}(\boldsymbol{\xi})\, \mathrm{d}\boldsymbol{\xi} = 0,$$

we have

$$Q_{\alpha,\beta}(\boldsymbol{\Lambda}_0 \boldsymbol{w}_0) = \frac{\theta^{|\alpha|}}{\alpha!} \frac{\partial(\frac{\alpha!}{\theta^{|\alpha|}} S_\alpha)}{\partial w_\beta}\Big|_{\boldsymbol{\Lambda}_0 \boldsymbol{w}_0}.$$

Let

$$\bar{S}_\alpha = \frac{\alpha!}{\theta^{|\alpha|}} S_\alpha,$$

then considering (3.17), we have

$$\bar{S}_\alpha = \int_{\mathbb{R}^D} \int_{\mathbb{R}^D} \int_{S_+^{D-1}} \mathrm{He}_\alpha^{[\boldsymbol{u},\theta]}(\boldsymbol{\xi})(f' f_1' - f f_1) B(|\boldsymbol{\xi} - \boldsymbol{\xi}_1|, \sigma)\, \mathrm{d}\boldsymbol{n}\, \mathrm{d}\boldsymbol{\xi}_1\, \mathrm{d}\boldsymbol{\xi}.$$

We denote the notations

$$\int_{\mathbb{R}^D} \int_{\mathbb{R}^D} \int_{S_+^{D-1}}, B(|\boldsymbol{\xi} - \boldsymbol{\xi}_1|, \sigma) \quad \text{and} \quad \mathrm{d}\boldsymbol{n}\, \mathrm{d}\boldsymbol{\xi}_1\, \mathrm{d}\boldsymbol{\xi}$$

in the last equation by $\int$, B and $\mathrm{d}\boldsymbol{\tau}$, respectively, hereafter for convenience. Let $\boldsymbol{V} \in \mathbb{R}^{N+D+1}$ and $v_\alpha = f_\alpha$, $|\alpha| \le M$ and $v_{N+d} = u_d$ and $v_{N+D+1} = \theta$, then $\boldsymbol{V}$ contains all the variables in $\boldsymbol{w}$, together with velocity and temperature. And thus

$$\mathbf{Q} = \frac{\partial \boldsymbol{S}}{\partial \boldsymbol{w}} = \frac{\partial \boldsymbol{S}}{\partial \boldsymbol{V}} \frac{\partial \boldsymbol{V}}{\partial \boldsymbol{w}}, \quad \mathbf{Q}(\boldsymbol{\Lambda}_0 \boldsymbol{w}_0) = \frac{\theta^{|\alpha|}}{\alpha!} \frac{\partial \bar{\boldsymbol{S}}}{\partial \boldsymbol{V}} \frac{\partial \boldsymbol{V}}{\partial \boldsymbol{w}}\Big|_{\boldsymbol{\Lambda}_0 \boldsymbol{w}_0},$$

where $\bar{\boldsymbol{S}} = (\bar{S}_\alpha)$, Since

$$\frac{\partial f}{\partial s}\Big|_{\boldsymbol{\Lambda}_0 \boldsymbol{w}_0} = \frac{\partial f_{eq}}{\partial s}, \quad s \in \{u_1, \cdots, u_d, \theta\},$$

and $f' f_1' - f f_1 \,|_{\boldsymbol{\Lambda}_0 \boldsymbol{w}_0} = 0$ hold, we have

$$\frac{\partial f' f_1' - f f_1}{\partial s}\Big|_{\boldsymbol{\Lambda}_0 \boldsymbol{w}_0} = \frac{\partial (f' f_1' - f f_1)|_{\boldsymbol{\Lambda}_0 \boldsymbol{w}_0}}{\partial s} = 0, \qquad s \in \{u_1, \cdots, u_D, \theta\}.$$

Hence, $\bar{S}_\alpha$ only depends on f_β, $|\beta| \le M$ and does NOT depend on $\boldsymbol{u}$ and θ. Direct

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calculations yield

$$\begin{aligned}\frac{\partial \bar{S}_\alpha}{\partial w_\beta}\Big|_{\boldsymbol{\Lambda}_0\boldsymbol{w}_0} &= \int \mathrm{He}_\alpha^{[\boldsymbol{u},\theta]}(\boldsymbol{\xi}) \Bigg[\mathcal{H}_\beta^{[\boldsymbol{u},\theta]}(\boldsymbol{\xi}') \sum_{|\gamma|\le M} f_\gamma \mathcal{H}_\gamma^{[\boldsymbol{u},\theta]}(\boldsymbol{\xi}_1') + \mathcal{H}_\beta^{[\boldsymbol{u},\theta]}(\boldsymbol{\xi}_1') \sum_{|\gamma|\le M} f_\gamma \mathcal{H}_\gamma^{[\boldsymbol{u},\theta]}(\boldsymbol{\xi}') \\ &\quad - \mathcal{H}_\beta^{[\boldsymbol{u},\theta]}(\boldsymbol{\xi}) \sum_{|\gamma|\le M} f_\gamma \mathcal{H}_\gamma^{[\boldsymbol{u},\theta]}(\boldsymbol{\xi}_1) - \mathcal{H}_\beta^{[\boldsymbol{u},\theta]}(\boldsymbol{\xi}_1) \sum_{|\gamma|\le M} f_\gamma \mathcal{H}_\gamma^{[\boldsymbol{u},\theta]}(\boldsymbol{\xi})\Bigg] B\,\mathrm{d}\boldsymbol{\tau}\Big|_{\boldsymbol{\Lambda}_0\boldsymbol{w}_0} \\ &= \int \mathrm{He}_\alpha^{[\boldsymbol{u},\theta]}(\boldsymbol{\xi})\rho\omega^{[\boldsymbol{u},\theta]}(\boldsymbol{\xi})\omega^{[\boldsymbol{u},\theta]}(\boldsymbol{\xi}_1)L(\beta)B\,\mathrm{d}\boldsymbol{\tau}\Big|_{\boldsymbol{\Lambda}_0\boldsymbol{w}_0} \\ &= -\frac{1}{4}\int \rho\omega^{[\boldsymbol{u},\theta]}(\boldsymbol{\xi})\omega^{[\boldsymbol{u},\theta]}(\boldsymbol{\xi}_1)L(\alpha)L(\beta)B\,\mathrm{d}\boldsymbol{\tau}\Big|_{\boldsymbol{\Lambda}_0\boldsymbol{w}_0},\end{aligned}$$

where

$$L(\alpha) = \mathrm{He}_\alpha^{[\boldsymbol{u},\theta]}(\boldsymbol{\xi}') + \mathrm{He}_\alpha^{[\boldsymbol{u},\theta]}(\boldsymbol{\xi}_1') - \mathrm{He}_\alpha^{[\boldsymbol{u},\theta]}(\boldsymbol{\xi}) - \mathrm{He}_\alpha^{[\boldsymbol{u},\theta]}(\boldsymbol{\xi}_1).$$

Here the third equality is due to the symmetry of $\boldsymbol{\xi}$, $\boldsymbol{\xi}'$ and $\boldsymbol{\xi}_1$, $\boldsymbol{\xi}_1'$ and the fact that the collision kernel B preserves its formation once exchanging the variables $(\boldsymbol{\xi},\boldsymbol{\xi}_1) \leftrightarrow (\boldsymbol{\xi}',\boldsymbol{\xi}_1')$ and $(\boldsymbol{\xi},\boldsymbol{\xi}') \leftrightarrow (\boldsymbol{\xi}_1,\boldsymbol{\xi}_1')$ (see [18]). Obviously, we have

$$\frac{\partial \bar{S}_\alpha}{\partial w_\beta}\Big|_{\boldsymbol{\Lambda}_0\boldsymbol{w}_0} = \frac{\partial \bar{S}_\beta}{\partial w_\alpha}\Big|_{\boldsymbol{\Lambda}_0\boldsymbol{w}_0}, \tag{4.13}$$

which indicates $\mathbf{T}^{-1}\bar{\mathbf{Q}}\mathbf{T}$ is symmetric. Since $\rho\omega^{[\boldsymbol{u},\theta]}(\boldsymbol{\xi})\omega^{[\boldsymbol{u},\theta]}(\boldsymbol{\xi}_1)B > 0$ holds, the matrix $\mathbf{T}^{-1}\bar{\mathbf{Q}}\mathbf{T}$ is symmetric and negative semi-definite. This proved Lemma 4.1, so did Theorem 4.1. □

5. Yong's first stability condition

Now we examine Yong's first stability condition [22] for HME (3.24). The equation (3.5) indicates that the equilibrium manifold, denoted by $\mathcal{E}$ hereafter, for HME is the local equilibrium, which is denoted by $\boldsymbol{w}_{eq}$ in this section. Since the momentum is conserved, flow velocity does not change the collision term. Due to the Galilean transformation invariance of the model, the variation in the flow velocity is only a translation of the system. Hence, the value of the flow velocity $\boldsymbol{u}$ does not matter in our discussion in this section, thus we let $\boldsymbol{u} = 0$ without loss of generality. Each state in $\mathcal{E}$ can be uniquely determined by the density ρ and the temperature θ, so if we let $\boldsymbol{\Lambda}_0\boldsymbol{w}_0 = \boldsymbol{w}_{eq}$, then all the results in Section 4 are still valid. In the following, let us directly verify Yong's first stability condition for HME:

Condition 1: Let

$$\hat{\mathbf{P}} = \mathbf{I} + \sum_{i=2}^{D} \mathbf{E}_{2e_1,2e_i}, \tag{5.1}$$

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then the conservation law (3.3) indicates that the first $D+2$ rows of $\hat{\mathbf{P}}\mathbf{Q}(\boldsymbol{w}_{eq})$ are zeros and the Eq. (3.4) indicates the other rows are full row rank. Hence, there exists an invertible $(N-D-2)\times(N-D-2)$ matrix $\hat{\mathbf{Q}}(\boldsymbol{w}_{eq})$ such that

$$\hat{\mathbf{P}}\mathbf{Q}(\boldsymbol{w}_{eq}) = \begin{pmatrix} 0 & 0 \\ 0 & \hat{\mathbf{Q}}(\boldsymbol{w}_{eq}) \end{pmatrix} \hat{\mathbf{P}}.$$

Condition 2: Since $\mathbf{M}_d$, $d=1,\cdots,D$ only depends on $\boldsymbol{w}_{eq}$, we have

$$\begin{aligned}\mathbf{M}_d(\boldsymbol{w}) = \mathbf{M}_d(\boldsymbol{w}_{eq}) =& \sqrt{\theta}\boldsymbol{\Lambda}_1\bar{\mathbf{M}}_d\boldsymbol{\Lambda}_1^{-1} \\ =& \sqrt{\theta}\boldsymbol{\Lambda}_1\mathbf{T}(\mathbf{T}^{-1}\bar{\mathbf{M}}_d\mathbf{T})(\boldsymbol{\Lambda}_1\mathbf{T})^{-1}, \quad d=1,\cdots,D. \end{aligned} \tag{5.2}$$

Let

$$\mathbf{A}_0(\boldsymbol{w}) = \left((\boldsymbol{\Lambda}_1\mathbf{T})^{-1}\mathbf{D}(\boldsymbol{w})\right)^T \left((\boldsymbol{\Lambda}_1\mathbf{T})^{-1}\mathbf{D}(\boldsymbol{w})\right), \tag{5.3}$$

then

$$\mathbf{A}_0\mathbf{A}_d = \sqrt{\theta}\left((\boldsymbol{\Lambda}_1\mathbf{T})^{-1}\mathbf{D}\right)^T(\mathbf{T}^{-1}\bar{\mathbf{M}}_d\mathbf{T})(\boldsymbol{\Lambda}_1\mathbf{T})^{-1}\mathbf{D}$$

is symmetric, thus (2.7) holds.

Condition 3: The definition of $\bar{\mathbf{D}}$ and the definition of $\bar{\mathbf{Q}}$ (4.5) indicate that

$$\mathbf{D}(\boldsymbol{w}_{eq}) = \boldsymbol{\Lambda}_1\bar{\mathbf{D}}\boldsymbol{\Lambda}_0^{-1}, \qquad \mathbf{Q}(\boldsymbol{w}_{eq}) = \frac{\sqrt{\theta}}{L}\boldsymbol{\Lambda}_1\bar{\mathbf{Q}}\boldsymbol{\Lambda}_1^{-1} = \frac{\sqrt{\theta}}{L}\boldsymbol{\Lambda}_1\mathbf{T}\left(\mathbf{T}^{-1}\bar{\mathbf{Q}}\mathbf{T}\right)(\boldsymbol{\Lambda}_1\mathbf{T})^{-1}.$$

Direct calculation yields

$$\begin{aligned}\mathbf{D}(\boldsymbol{w}_{eq})\mathbf{Q}(\boldsymbol{w}_{eq}) =& \frac{\sqrt{\theta}}{L}\boldsymbol{\Lambda}_1\bar{\mathbf{D}}\boldsymbol{\Lambda}_0^{-1}\boldsymbol{\Lambda}_1\bar{\mathbf{Q}}\boldsymbol{\Lambda}_1^{-1} \\ =& \frac{\sqrt{\theta}}{L}\boldsymbol{\Lambda}_1\bar{\mathbf{D}}\bar{\mathbf{Q}}\boldsymbol{\Lambda}_1^{-1} = \frac{\sqrt{\theta}}{L}\boldsymbol{\Lambda}_1\bar{\mathbf{Q}}\boldsymbol{\Lambda}_1^{-1} = \mathbf{Q}(\boldsymbol{w}_{eq}),\end{aligned}$$

where the first equality is obtained by $\boldsymbol{\Lambda}_0^{-1}\boldsymbol{\Lambda}_1\bar{\mathbf{Q}} = \bar{\mathbf{Q}}$ and the relation $\bar{\mathbf{D}}\bar{\mathbf{Q}} = \bar{\mathbf{Q}}$, derived in the proof of Lemma 4.2, is used in the second equality. Analogously, we have

$$\begin{aligned}\mathbf{Q}(\boldsymbol{w}_{eq})\mathbf{D}(\boldsymbol{w}_{eq}) =& \frac{\sqrt{\theta}}{L}\boldsymbol{\Lambda}_1\bar{\mathbf{Q}}\boldsymbol{\Lambda}_1^{-1}\boldsymbol{\Lambda}_1\bar{\mathbf{D}}\boldsymbol{\Lambda}_0^{-1} \\ =& \frac{\sqrt{\theta}}{L}\boldsymbol{\Lambda}_1\bar{\mathbf{Q}}\boldsymbol{\Lambda}_0^{-1} = \frac{\sqrt{\theta}}{L}\boldsymbol{\Lambda}_1\bar{\mathbf{Q}}\boldsymbol{\Lambda}_1^{-1} = \mathbf{Q}(\boldsymbol{w}_{eq}),\end{aligned}$$

due to Lemma 4.2 and $\bar{\mathbf{Q}}\boldsymbol{\Lambda}_0^{-1} = \bar{\mathbf{Q}}\boldsymbol{\Lambda}_1^{-1}$. Thus, we have

$$\begin{aligned}&\mathbf{A}_0(\boldsymbol{w}_{eq})\mathbf{Q}(\boldsymbol{w}_{eq}) = \left((\boldsymbol{\Lambda}_1\mathbf{T})^{-1}\mathbf{D}(\boldsymbol{w}_{eq})\right)^T\left((\boldsymbol{\Lambda}_1\mathbf{T})^{-1}\mathbf{D}(\boldsymbol{w}_{eq})\right)\mathbf{Q}(\boldsymbol{w}_{eq}) \\ =& \left((\boldsymbol{\Lambda}_1\mathbf{T})^{-1}\mathbf{D}(\boldsymbol{w}_{eq})\right)^T\left((\boldsymbol{\Lambda}_1\mathbf{T})^{-1}\right)\mathbf{Q}(\boldsymbol{w}_{eq}) = \frac{\sqrt{\theta}}{L}\mathbf{D}^T(\boldsymbol{w}_{eq})(\boldsymbol{\Lambda}_1\mathbf{T})^{-T}(\mathbf{T}^{-1}\bar{\mathbf{Q}}\mathbf{T})(\boldsymbol{\Lambda}_1\mathbf{T})^{-1} \\ =& \frac{\sqrt{\theta}}{L}\left((\boldsymbol{\Lambda}_1\mathbf{T})^{-T}(\mathbf{T}^{-1}\bar{\mathbf{Q}}\mathbf{T})(\boldsymbol{\Lambda}_1\mathbf{T})^{-1}\mathbf{D}(\boldsymbol{w}_{eq})\right)^T = \left((\boldsymbol{\Lambda}_1\mathbf{T})^{-T}(\boldsymbol{\Lambda}_1\mathbf{T})^{-1}\mathbf{Q}(\boldsymbol{w}_{eq})\mathbf{D}(\boldsymbol{w}_{eq})\right)^T \\ =& \left((\boldsymbol{\Lambda}_1\mathbf{T})^{-T}(\boldsymbol{\Lambda}_1\mathbf{T})^{-1}\mathbf{Q}(\boldsymbol{w}_{eq})\right)^T = (\boldsymbol{\Lambda}_1\mathbf{T})^{-T}(\mathbf{T}^{-1}\bar{\mathbf{Q}}\mathbf{T})(\boldsymbol{\Lambda}_1\mathbf{T})^{-1},\end{aligned}$$

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where $-T$ stands for transposition of inverse. It is clear that this is a symmetric matrix. Since $\mathbf{T}^{-1}\bar{\mathbf{Q}}\mathbf{T}$ is symmetric negative semi-definite, there exists an invertible matrix $\mathbf{P}_1$ subject to

$$\mathbf{T}^{-1}\bar{\mathbf{Q}}\mathbf{T} = -\mathbf{P}_1^T \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{I}_{N-D-2} \end{pmatrix} \mathbf{P}_1.$$

Therefore, there exists an invertible matrix $\mathbf{P}$ subject to both (2.6) and (2.8).

This gives us the following theorem to end this section:

Theorem 5.1. *HME satisfies Yong's first stability condition.*

6. Stability analysis of OHME

In Grad's moment method, there are two groups of moment systems. One is choosing the basis function as

$$\left\{\mathcal{H}_\alpha^{[\boldsymbol{u},\theta]}(\boldsymbol{\xi}) : |\alpha| \le M\right\},$$

which gives us the reduced models with 20, 35, 56, 84, $\cdots$ moments for $D = 3$. Grad's 20 moment system is the most popular one of them. HME are globally hyperbolic regularized version of this group of Grad's moment system. The other one is choosing the basis function as

$$\left\{\mathcal{H}_\alpha^{[\boldsymbol{u},\theta]}(\boldsymbol{\xi}) : |\alpha| \le M-1\right\} \bigcup \left\{\sum_{d=1}^{D} \mathcal{H}_{\alpha+2e_d}^{[\boldsymbol{u},\theta]}(\boldsymbol{\xi}) : |\alpha| = M-2\right\},$$

which gives us moment system with 13, 26, 45, 71, $\cdots$, moments for $D = 3$. In this group, Grad's 13 moment system is definitely the most famous one. Following [20], we called this set of moment system as ordered Grad's moment system.

As the most important Grad's moment system, Grad's 13 moment equations [13] draw a lot of authors' attention in the past six decades. Due to the lack of hyperbolicity, a globally hyperbolic regularization, similarly as that for M-order Grad's moment system, is required. In [7], the authors extended the globally hyperbolic regularization in [4, 5] into a framework to derive moment equations from kinetic equations. By applying the framework on Grad's 13 moment system, the authors proposed a globally hyperbolic 13 moment equations (HME13). In [10], the authors applied the globally hyperbolic regularization on ordered Grad's moment system to obtain the Ordered Hyperbolic Moment Equations(OHME) and pointed out that M-th order OHME can be derived from M-th order HME.

Denote N_O by the number of equations of M-th order OHME and let

$$\mathbf{P}_b = \sum_{|\alpha|\le M-1} \hat{\mathbf{E}}_{\alpha,\alpha} + \sum_{d=1}^{D} \sum_{|\alpha|=M-2} \hat{\mathbf{E}}_{\alpha+2e_1,\alpha+2e_d},$$

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where $\hat{\mathbf{E}}_{\alpha,\beta} \in \mathbb{R}^{N_O \times N}$ is the matrix with all its entries to be 0, except for the only entry with row index α and column index β to be 1. We define the diagonal matrix $\mathbf{T}_O \in \mathbb{R}^{N_O \times N_O}$ as

$$\mathbf{T}_O = \sum_{|\alpha| \le M-1} \frac{1}{\sqrt{\alpha!}} \mathbf{E}^O_{\alpha,\alpha} + \sum_{|\alpha|=M-2} \frac{1}{\sqrt{\sum_{d=1}^D (\alpha + 2e_d)!}} \mathbf{E}^O_{\alpha+2e_1,\alpha+2e_1},$$

where $\mathbf{E}^O_{\alpha,\alpha} \in \mathbb{R}^{N_O \times N_O}$ has the same definition as $\mathbf{E}_{\alpha,\alpha}$. Let

$$\mathbf{P}_p = \mathbf{T}_O^2 \mathbf{P}_b (\mathbf{T}^2)^{-1}, \tag{6.1}$$

then OHME can be written as [10]

$$\begin{aligned} &\mathbf{P}_p \mathbf{D}(\mathbf{P}_b{}^T \mathbf{P}_p \boldsymbol{w}) \mathbf{P}_b{}^T \frac{\partial \mathbf{P}_p \boldsymbol{w}}{\partial t} + \sum_{d=1}^D \mathbf{P}_p \mathbf{M}_d (\mathbf{P}_b{}^T \mathbf{P}_p \boldsymbol{w}) \mathbf{P}_b{}^T \mathbf{P}_p \mathbf{D}(\mathbf{P}_b{}^T \mathbf{P}_p \boldsymbol{w}) \mathbf{P}_b{}^T \frac{\partial \mathbf{P}_p \boldsymbol{w}}{\partial x_d} \\ =&\mathbf{P}_p \boldsymbol{S}(\mathbf{P}_b{}^T \mathbf{P}_p \boldsymbol{w}). \end{aligned} \tag{6.2}$$

Let

$$\begin{aligned} &\boldsymbol{w}_O = \mathbf{P}_p \boldsymbol{w}, && \mathbf{D}_O(\boldsymbol{w}_O) = \mathbf{P}_p \mathbf{D}(\mathbf{P}_b{}^T \boldsymbol{w}_O) \mathbf{P}_b{}^T, \\ &\boldsymbol{S}_O = \mathbf{P}_p \boldsymbol{S}(\mathbf{P}_b{}^T \boldsymbol{w}_O), && \mathbf{M}_{O,d}(\boldsymbol{w}_O) = \mathbf{P}_p \mathbf{M}_d(\mathbf{P}_b{}^T \boldsymbol{w}_O) \mathbf{P}_b{}^T, \quad d = 1, \cdots, D, \end{aligned}$$

then (6.2) can be reformulated as

$$\mathbf{D}_O(\boldsymbol{w}_O) \frac{\partial \boldsymbol{w}_O}{\partial t} + \sum_{d=1}^D \mathbf{M}_O(\boldsymbol{w}_O) \mathbf{D}_O(\boldsymbol{w}_O) \frac{\partial \boldsymbol{w}_O}{\partial x_d} = \boldsymbol{S}_O(\boldsymbol{w}_O). \tag{6.3}$$

We claim that for system (6.3), it is linearly stable and fulfils Yong's first stability condition, exactly the same as HME we studied in the last sections.

Using the same linearization procedure as that in Section 4 on OHME, we obtain the linearized OHME as

$$\bar{\mathbf{D}}^O \frac{\partial \bar{\boldsymbol{w}}_O}{\partial t} + \sum_{d=1}^D \bar{\mathbf{M}}_O \bar{\mathbf{D}}_O \frac{\partial \bar{\boldsymbol{w}}_O}{\partial x_d} = \bar{\mathbf{Q}}_O \bar{\boldsymbol{w}}_O, \tag{6.4}$$

where

$$\begin{aligned} &\bar{\boldsymbol{w}}_O = \mathbf{P}_p \bar{\boldsymbol{w}} && \bar{\mathbf{D}}_O = \mathbf{P}_p \bar{\mathbf{D}} \mathbf{P}_b{}^T, \\ &\bar{\mathbf{Q}}_O = \mathbf{P}_p \bar{\mathbf{Q}} \mathbf{P}_b{}^T, && \bar{\mathbf{M}}_{O,d} = \mathbf{P}_p \bar{\mathbf{M}}_d \mathbf{P}_b{}^T, \quad d = 1, \cdots, D. \end{aligned}$$

Noticing the discussion in Section 4, we can prove OHME is also linearly stable both in space and in time at the local Maxwellian, once Lemma 4.1 and Lemma 4.2 are valid for $\bar{\mathbf{D}}_O$, $\bar{\mathbf{M}}_{O,d}$ and $\bar{\mathbf{Q}}_O$.

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Actually, due to (6.1), we find that both

$$\mathbf{T}_O^{-1}\bar{\mathbf{M}}_{O,d}\mathbf{T}_O = (\mathbf{T}^{-1}{\mathbf{P}_b}^T\mathbf{T}_O)^T(\mathbf{T}^{-1}\bar{\mathbf{M}}_d\mathbf{T})(\mathbf{T}^{-1}{\mathbf{P}_b}^T\mathbf{T}_O), \quad d = 1, \cdots, D,$$
$$\mathbf{T}_O^{-1}\bar{\mathbf{Q}}_O\mathbf{T}_O = (\mathbf{T}^{-1}{\mathbf{P}_b}^T\mathbf{T}_O)^T(\mathbf{T}^{-1}\bar{\mathbf{Q}}\mathbf{T})(\mathbf{T}^{-1}{\mathbf{P}_b}^T\mathbf{T}_O),$$

are symmetric matrices. Noticing here $\mathbf{T}$ and $\mathbf{T}_O$ are diagonal matrices, we obtain that Lemma 4.1 is valid for $\bar{\mathbf{M}}_{O,d}$ and $\bar{\mathbf{Q}}_O$.

The equation $\bar{\mathbf{D}}^{-1}\bar{\mathbf{Q}} = \bar{\mathbf{Q}}$ is valid, since the collision operator has $D+2$ conserved quantities and all entries of $\bar{\mathbf{D}} - \mathbf{I}$ are zeroes except for some entries with row and column indices corresponding to these conserved quantities. Since $\mathbf{P}_p$ and $\mathbf{P}_b$ only change entries with row and column indices corresponding to $|\alpha| > M-1$, Lemma 4.2 is still valid for $\bar{\mathbf{D}}_O$ and $\bar{\mathbf{Q}}_O$. Furthermore, we have

$$\mathbf{D}_O(\boldsymbol{w}_{eq}^O)\mathbf{Q}_O(\boldsymbol{w}_{eq}^O)\mathbf{D}_O(\boldsymbol{w}_{eq}^O) = \mathbf{Q}_O(\boldsymbol{w}_{eq}^O), \tag{6.5}$$

where $\mathbf{Q}_O = \dfrac{\partial \boldsymbol{S}_O}{\partial \boldsymbol{w}_O}$. Hence, we have the following corollary.

Corollary 6.1. *The linearized OHME* (6.4) *is stable both in space and in time. OHME* (6.3) *is linearly stable both in space and in time at the local Maxwellian.*

Following Section 5, here we verify that Yong's first stability condition is satisfied for OHME, making use of the connections (6.3) between HME and OHME. Precisely, we have the following theorem.

Theorem 6.1. *OHME satisfies Yong's first stability condition.*

Proof. Let us verify all three equalities one by one:

Condition 1: Let $\boldsymbol{w}_{eq}^O = \mathbf{P}_p\boldsymbol{w}_{eq}$. Direct calculations yield

$$\mathbf{Q}_O(\boldsymbol{w}_O) := \frac{\partial \boldsymbol{S}_O(\boldsymbol{w}_O)}{\partial \boldsymbol{w}_O} = \mathbf{P}_p\frac{\partial \boldsymbol{S}({\mathbf{P}_b}^T\boldsymbol{w}_O)}{\partial \boldsymbol{w}}\frac{\partial {\mathbf{P}_p}^T\boldsymbol{w}_O}{\partial \boldsymbol{w}_O} = \mathbf{P}_p\mathbf{Q}(\boldsymbol{w}){\mathbf{P}_b}^T.$$

Let

$$\hat{\mathbf{P}}_O = \mathbf{I} + \sum_{i=2}^{D}\mathbf{E}_{2e_1,2e_i}^O.$$

Then we have $\hat{\mathbf{P}}_O\mathbf{P}_p = \mathbf{P}_p\hat{\mathbf{P}}$ and $\hat{\mathbf{P}}_O\mathbf{P}_b = \mathbf{P}_b\hat{\mathbf{P}}$ and thus

$$\begin{aligned}\hat{\mathbf{P}}_O\mathbf{Q}_O(\boldsymbol{w}_{eq}^O) =& \mathbf{P}_p\hat{\mathbf{P}}\mathbf{Q}(\boldsymbol{w}_{eq}^O){\mathbf{P}_b}^T = \mathbf{P}_p\begin{pmatrix}0 & 0\\ 0 & \hat{\mathbf{Q}}(\boldsymbol{w}_{eq})\end{pmatrix}{\mathbf{P}_b}^T\hat{\mathbf{P}}_O\\ =& \begin{pmatrix}0 & 0\\ 0 & \hat{\mathbf{Q}}_O(\boldsymbol{w}_{eq})\end{pmatrix}\hat{\mathbf{P}}_O,\end{aligned}$$

where $\hat{\mathbf{Q}}_O \in \mathbb{R}^{(N_O-D-2)\times(N_O-D-2)}$ is an invertible matrix.

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Condition 2: Let $\mathbf{\Lambda}_1^O = \mathbf{P}_p\mathbf{\Lambda}_1{\mathbf{P}_b}^T$, then one is easy to see that

$$\mathbf{M}_{O,d} = \mathbf{P}_p\mathbf{M}_d{\mathbf{P}_p}^T = \sqrt{\theta}\mathbf{\Lambda}_1^O\mathbf{T}_0{\mathbf{P}_b}^T(\mathbf{T}^{-1}\bar{\mathbf{M}}_d\mathbf{T})\mathbf{P}_b(\mathbf{\Lambda}_1^O\mathbf{T}_O)^{-1}.$$

Let

$$\mathbf{A}_0^O(\boldsymbol{w}_O) == ((\mathbf{\Lambda}_1^O\mathbf{T}_O)^{-1}\mathbf{D}_O(\boldsymbol{w}_O))^T((\mathbf{\Lambda}_1^O\mathbf{T}_O)^{-1}\mathbf{D}_O(\boldsymbol{w}_O)).$$

Then

$$\mathbf{A}_0^O\mathbf{A}_d^O = \sqrt{\theta}((\mathbf{\Lambda}_1^O\mathbf{T}_O)^{-1}\mathbf{D}_O)^T\mathbf{P}_b(\mathbf{T}^{-1}\bar{\mathbf{M}}_{O,d}\mathbf{T}){\mathbf{P}_b}^T(\mathbf{\Lambda}_1^O\mathbf{T}_O)^{-1}\mathbf{D}_O$$

is symmetric, thus (2.7) holds.

Condition 3: Noticing (6.5), we obtain that

$$\begin{aligned}
\mathbf{A}_0^O(\boldsymbol{w}_{eq}^O)\mathbf{Q}_O(\boldsymbol{w}_{eq}^O) =&((\mathbf{\Lambda}_1^O\mathbf{T}_O)^{-1}\mathbf{D}_O(\boldsymbol{w}_{eq}^O))^T((\mathbf{\Lambda}_1^O\mathbf{T}_O)^{-1}\mathbf{D}_O(\boldsymbol{w}_{eq}^O))\mathbf{Q}^O(\boldsymbol{w}_{eq}^O)\\
=&((\mathbf{\Lambda}_1^O\mathbf{T}_O)^{-1}\mathbf{D}_O(\boldsymbol{w}_{eq}^O))^T(\mathbf{\Lambda}_1^O\mathbf{T}_O)^{-1}\mathbf{Q}_O(\boldsymbol{w}_{eq}^O)\\
=&\frac{\sqrt{\theta}}{L}\mathbf{D}_O^T(\boldsymbol{w}_{eq}^O)(\mathbf{\Lambda}_1^O\mathbf{T}_O)^{-T}\mathbf{P}_b(\mathbf{T}^{-1}\bar{\mathbf{Q}}\mathbf{T}){\mathbf{P}_b}^T(\mathbf{\Lambda}_1^O\mathbf{T}_O)^{-1}\\
=&\frac{\sqrt{\theta}}{L}\left((\mathbf{\Lambda}_1^O\mathbf{T}_O)^{-T}\mathbf{P}_b(\mathbf{T}^{-1}\bar{\mathbf{Q}}\mathbf{T}){\mathbf{P}_b}^T(\mathbf{\Lambda}_1^O\mathbf{T}^O)^{-1}\mathbf{D}_O(\boldsymbol{w}_{eq}^O)\right)^T\\
=&\left((\mathbf{\Lambda}_1^O\mathbf{T}_O)^{-T}(\mathbf{\Lambda}_1^O\mathbf{T}_O)^{-1}\mathbf{Q}_O(\boldsymbol{w}_{eq}^O)\mathbf{D}_O(\boldsymbol{w}_{eq}^O)\right)^T\\
=&\left((\mathbf{\Lambda}_1^O\mathbf{T}_O)^{-T}(\mathbf{\Lambda}_1^O\mathbf{T}_O)^{-1}\mathbf{Q}_O(\boldsymbol{w}_{eq}^O)\right)^T\\
=&(\mathbf{\Lambda}_1^O\mathbf{T}_O)^{-T}\mathbf{P}_b(\mathbf{T}^{-1}\bar{\mathbf{Q}}\mathbf{T}){\mathbf{P}_b}^T(\mathbf{\Lambda}_1^O\mathbf{T}_O)^{-1}
\end{aligned}$$

is symmetric. Analogous to that in Sec. 5, there exists an invertible matrix $\mathbf{P}_O$ subject to both (2.6) and (2.8). This is the end of the proof. □

7. Conclusions

The linear stability at the local equilibrium of both HME and OHME has been proved with commonly used approximate collision terms and particularly with Boltzmann's binary collision model. Since HME and OHME contain almost all hyperbolic regularized Grad's moment system, the linear stability of almost all Grad-type moment system is clarified.

Yong's first stability condition is essential to the existence of the solution of nonlinear first-order hyperbolic with stiff source term. The positive results in this paper may be helpful for the future study on the existence of the solution of HME and OHME.

The linearized equation of HME is same as that of Grad's moment equations at the local equilibrium, so the linear stability at the local equilibrium can be shared with the Grad's moment equations. However, for Grad's moment equations, due to the lack of the hyperbolicity, even in the neighborhood of the local equilibrium, the linear stability can not ensure the existence of the solution. What's more, Yong's stability condition is stronger than linear stability, which is satisfied by HME, but not Grad's moment equations.

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