Ergod. Th. & Dynam. Sys. (2008), **28**, 1291–1322 doi:10.1017/S0143385707000508
© *2008 Cambridge University Press*
Printed in the United Kingdom

Some counterexamples in topological dynamics

RONNIE PAVLOV

Department of Mathematics, The Ohio State University, Columbus, OH 43210, USA
(e-mail: rpavlov@math.ubc.ca)

(*Received* 26 *December* 2006 *and accepted in revised form* 31 *May* 2007)

Abstract. In this paper, we exhibit, for any sparse-enough increasing sequence $\{p_n\}$ of integers, totally minimal, totally uniquely ergodic, and topologically mixing systems (X, T) and (X', T') and $f \in C(X)$ for which the averages $(1/N)\sum_{n=0}^{N-1} f(T^{p_n}x)$ fail to converge on a residual set in X, and where there exists $x' \in X'$ with $x' \notin \overline{\{T'^{p_n}x'\}}$.

1. *Introduction*

For a measure-preserving transformation T of a probability space $(X, \mathcal{B}, \mu)$, Birkhoff's ergodic theorem guarantees the existence of the limit of ergodic averages $\lim_{n\to\infty}(1/n)\sum_{i=0}^{n-1} f(T^i x)$ for μ-almost every $x \in X$ and for any $f \in L^1(X)$. Several results have been proved about the convergence of such averages when one averages not along all powers of T, but only along some distinguished subset of the integers [**3, 4, 9**]. In particular, when one averages along $\{p(n)\}_{n\in\mathbb{N}}$ for a polynomial $p(n)$ with integer coefficients, there is the following result of Bourgain.

THEOREM 1.1. [**3**, p. 7, Theorem 1] *For any measure-preserving system $(X, \mathcal{B}, \mu, T)$, for any polynomial $q(t) \in \mathbb{Z}[t]$, and for any $f \in L^p(X, \mathcal{B}, \mu)$ with $p > 1$, $\lim_{N\to\infty}(1/N)\sum_{n=1}^{N} f(T^{q(n)}x)$ exists μ-almost everywhere.*

Theorem 1.1 can be interpreted as follows: for any polynomial $q(t) \in \mathbb{Z}[t]$, any measure-preserving system $(X, \mathcal{B}, \mu, T)$, and any measure-theoretically 'nice' function f, the set of points x where $\lim_{N\to\infty}(1/N)\sum_{n=1}^{N} f(T^{q(n)}x)$ does not converge is of measure zero, or negligible measure-theoretically. It is then natural to wonder whether or not there is a topological parallel to this result using topological notions of 'niceness' (continuity) and negligibility (first category), and in fact such a question was posed by Bergelson.

QUESTION 1.1. [**1**, p. 51, Question 5] *Assume that a topological dynamical system (X, T) is uniquely ergodic, and suppose $p \in \mathbb{Z}[t]$ and $f \in C(X)$. Is it true that, for all but a first-category set of points, $\lim_{n\to\infty}(1/n)\sum_{i=0}^{n-1} f(T^{p(i)}x)$ exists?*

To properly discuss and motivate Question 1.1, we need some definitions.

Definition 1.1. A *topological dynamical system* (X, T) consists of a compact topological space X and a continuous map $T : X \to X$.

https://doi.org/10.1017/S0143385707000508 Published online by Cambridge University Press

Definition 1.2. A topological dynamical system (X, T) is *minimal* if for any closed set K with $T^{-1}K \subseteq K$, $K = \varnothing$ or $K = X$. The system (X, T) is *totally minimal* if (X, T^n) is minimal for every $n \in \mathbb{N}$.

Definition 1.3. A topological dynamical system (X, T) is *uniquely ergodic* if there is only one Borel measure μ on X such that $\mu(A) = \mu(T^{-1}A)$ for every Borel set $A \subseteq X$. (X, T) is *totally uniquely ergodic* if (X, T^n) is uniquely ergodic for every $n \in \mathbb{N}$.

Definition 1.4. A topological dynamical system (X, T) is *topologically mixing* if for any non-empty open sets U, $V \subseteq X$, there exists $N \in \mathbb{N}$ such that, for any $n > N$, $U \cap T^n V \neq \varnothing$.

Definition 1.5. For any set $A \subseteq \mathbb{N}$, the *upper Banach density* of A is defined as

$$d^*(A) = \limsup_{n\to\infty} \sup_{m\in\mathbb{N}} \frac{|\{m, m+1, \ldots, m+n-1\} \cap A|}{n}.$$

Definition 1.6. For any set $A \subseteq \mathbb{N}$, the *upper density* of A is defined as

$$\overline{d}(A) = \limsup_{n\to\infty} \frac{|\{1, \ldots, n\} \cap A|}{n}.$$

Definition 1.7. For a set $A \subseteq \mathbb{N}$, the *density* of A is defined as

$$d(A) = \lim_{n\to\infty} \frac{|\{1, \ldots, n\} \cap A|}{n}$$

if this limit exists.

Definition 1.8. Given a topological dynamical system (X, T) and a T-invariant Borel probability measure μ, a point $x \in X$ is (T, μ)-*generic* if, for every $f \in C(X)$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i x) = \int f \, d\mu.$$

Bergelson added the hypothesis of unique ergodicity because it is a classical result that a system (X, T) is uniquely ergodic with unique T-invariant measure μ if and only if for every $x \in X$ and $f \in C(X)$, $\lim_{n\to\infty}(1/n) \sum_{i=0}^{n-1} f(T^i x) = \int_X f \, d\mu$, and so this is a natural assumption to make about (X, T) in Question 1.1.

However, Bergelson was particularly interested in the convergence of these averages to the 'correct limit', i.e. $\int_X f \, d\mu$ where μ is the unique T-invariant measure on X. To have any hope for such a result, it also becomes necessary to assume ergodicity of all powers of T in order to avoid some natural counterexamples related to the distribution (mod k) of $p(n)$ for positive integers k. For example, if $p(n) = n^2$, T is the permutation on $X = \{0, 1, 2\}$ defined by $Tx = x + 1 \pmod 3$, μ is the normalized counting measure on X, and $f = \chi_{\{0\}}$, then T is obviously uniquely ergodic with unique invariant measure $\mu = (\delta_0 + \delta_1 + \delta_2)/3$, but

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^{p(i)} x) = \begin{cases} \frac{1}{3} & \text{if } x = 0, \\ 0 & \text{if } x = 1, \text{ and} \\ \frac{2}{3} & \text{if } x = 2. \end{cases}$$

https://doi.org/10.1017/S0143385707000508 Published online by Cambridge University Press

To avoid such examples, we would need T to be totally ergodic as well as uniquely ergodic, and so it makes sense to assume total unique ergodicity to encompass both properties. Bergelson's revised question then looks like the following.

QUESTION 1.2. *Assume that a topological dynamical system (X, T) is totally uniquely ergodic with unique T-invariant measure μ, and suppose $p \in \mathbb{Z}[t]$ and $f \in C(X)$. Is it true that, for all but a first-category set of points,* $\lim_{n\to\infty}(1/n)\sum_{i=0}^{n-1} f(T^{p(i)}x) = \int_X f\,d\mu$*?*

We answer Questions 1.1 and 1.2 negatively in the case where the degree of p is at least two, and in fact prove some slightly more general results. The level of generality depends on what hypotheses we place on the space X. In particular, we can exhibit more counterexamples in the case where X is a totally disconnected space than we can in the case where X is a connected space. Here are our main results.

THEOREM 1.2. *For any increasing sequence $\{p_n\}$ of integers with upper Banach density zero, there exists a totally minimal, totally uniquely ergodic, and topologically mixing topological dynamical system (X, T) and a continuous function f on X with the property that for a residual set of $x \in X$, $(1/N)\sum_{n=0}^{N-1} f(T^{p_n}x)$ does not converge.*

THEOREM 1.3. *For any increasing sequence $\{p_n\}$ of integers with the property that for some integer d, $p_{n+1} < (p_{n+1} - p_n)^d$ for all sufficiently large n, there exists a totally minimal, totally uniquely ergodic, and topologically mixing topological dynamical system (X, T) and a continuous function f on X with the property that for a residual set of $x \in X$, $(1/N)\sum_{n=0}^{N-1} f(T^{p_n}x)$ does not converge. In addition, the space X is a connected $(2d + 9)$-manifold.*

We note that Theorems 1.2 and 1.3 answer Questions 1.1 and 1.2 negatively for nonlinear $p \in \mathbb{Z}[t]$ with positive leading coefficient. In fact, it is not hard to modify the constructions contained in this paper to make T invertible; this yields a negative answer to these questions for all nonlinear p. We address this issue at the end of the paper.

Theorems 1.2 and 1.3 are about non-convergence of ergodic averages along certain sequences of powers of x. We also prove two similar results about non-recurrence of points. As motivation, we note that a minimal system has the property that every point is recurrent. In other words, if (X, T) is minimal, then, for all $x \in X$, it is the case that $x \in \overline{\{T^n x\}_{n\in\mathbb{N}}}$. If (X, T) is totally minimal, then all points are recurrent even along infinite arithmetic progressions: for any non-negative integers a, b, and for all $x \in X$, $x \in \overline{\{T^{an+b}x\}_{n\in\mathbb{N}}}$. It is then natural to wonder if the same is true for other sequences of powers of T, and in this regard there is the following result of Bergelson and Leibman, which is a corollary to their polynomial van der Waerden theorem.

THEOREM 1.4. [**2**, p. 14, Corollary 1.8] *For any minimal system (X, T) and any polynomial $q[t] \in \mathbb{Z}[t]$ with $q(0) = 0$, for a residual set of $x \in X$ it is the case that x is a limit point of $\{T^{q(n)}x\}_{n\in\mathbb{N}}$.*

The following two results proved in this paper show that this is the best that can be hoped for. In other words, it is not the case that for every minimal system every point is recurrent under polynomial powers of T.

https://doi.org/10.1017/S0143385707000508 Published online by Cambridge University Press

THEOREM 1.5. *For any increasing sequence $\{p_n\}$ of integers with upper Banach density zero, there exists a totally minimal, totally uniquely ergodic, and topologically mixing topological dynamical system (X, T) and an uncountable set $A \subset X$ such that, for every $x \in A$, the sequence $\{T^{p_n}x\}$ does not have x as a limit point, i.e. there is no sequence of positive integers $\{n_i\}$ such that $T^{p_{n_i}}x$ converges to x.*

THEOREM 1.6. *For any increasing sequence $\{p_n\}$ of integers with the property that for some integer d, $p_{n+1} < (p_{n+1} - p_n)^d$ for all sufficiently large n, there exists a totally minimal, totally uniquely ergodic, and topologically mixing topological dynamical system (X, T) and a point $x \in X$ such that the sequence $\{T^{p_n}x\}$ does not have x as a limit point, i.e. there is no sequence of positive integers $\{n_i\}$ such that $T^{p_{n_i}}x$ converges to x. In addition, the space X is a connected $(2d + 7)$-manifold.*

The following simple lemma shows that Theorems 1.5 and 1.6 cannot be improved too much, i.e. we cannot exhibit topologically mixing examples with a second-category set of such non-recurrent points.

LEMMA 1.1. *If a topological dynamical system (X, T) is topologically mixing, then for any increasing sequence $\{p_n\}$, the set of $x \in X$ for which x is not a limit point of $\{T^{p_n}x\}_{n\in\mathbb{N}}$ is of first category.*

Proof. For any $\epsilon > 0$, define $C_\epsilon = \{x : d(x, T^{p_n}x) \geq \epsilon \ (\forall n \in \mathbb{N})\}$. It is clear that all C_ϵ are closed. We claim that C_ϵ contains no non-empty open set; this implies that it is nowhere dense, so that that $C = \bigcup_{n=1}^{\infty} C_{1/n}$, the set of points x for which x is not a limit point of $\{T^{p_n}x\}_{n\in\mathbb{N}}$, is of first category. Suppose, for a contradiction, that there is a non-empty open set U with $U \subseteq C_\epsilon$ for some ϵ. Then, there exists V with $\operatorname{diam}(V) < \epsilon$ such that $V \subseteq U \subseteq C_\epsilon$. By topological mixing, there exists n such that $V \cap T^{-p_n}V \neq \varnothing$. This implies that there exists $x \in V$ such that $T^{p_n}x \in V$. Since $\operatorname{diam}(V) < \epsilon$, $d(x, T^{p_n}x) < \epsilon$. However, $x \in V \subseteq C_\epsilon$, so we have a contradiction. □

However, Theorem 1.5 shows that it is possible for this set of points non-recurrent along p_n to be uncountable.

We mention that some mixing condition is necessary for a statement like Lemma 1.1: as a simple example, consider an irrational circle rotation $T : x \mapsto x + \alpha$ on the circle $\mathbb{T}$. There is clearly some increasing sequence of integers $\{p_n\}$ such that $p_n\alpha \pmod 1 \to 1/2$. Then, for any $x \in \mathbb{T}$, $T^{p_n}x \to x + (1/2)$, and so for every $x \in X$, $\{T^{p_n}x\}$ does not have x as a limit point.

Before proceeding with the proofs, we now give a brief description of the content of this paper. In §2, we will describe some general symbolic constructions of topological dynamical systems with particular mixing properties. At the end of this section, we will arrive at a construction of a system that is totally minimal, totally uniquely ergodic, and topologically mixing, and that has as a parameter a sequence of integers $\{n_k\}$.

In §3, by taking this sequence $\{n_k\}$ to grow very quickly, we will show that the examples constructed in §2 are sufficient to prove Theorems 1.2 and 1.5. Some interesting questions also arise, pertaining to the upper Banach density of countable unions of sets of upper Banach density zero, and these are answered in §3.

https://doi.org/10.1017/S0143385707000508 Published online by Cambridge University Press

In §4, we create a flow under a function with base transformation a skew product that acts on a connected manifold, and that is totally minimal, totally uniquely ergodic, and topologically mixing. This transformation has as a parameter a function $f \in C(\mathbb{T})$. We use conditions of Fayad [**5**] on flows under functions to achieve topological mixing, and some conditions of Furstenberg [**6**] on skew products to prove total minimality and total unique ergodicity.

In §5, by a judicious choice of f, we use the examples of §4 to prove Theorems 1.3 and 1.6.

Finally, in §6 we give some open questions about strengthening our results.

2. *Some general symbolic constructions*

Our proofs of Theorems 1.2 and 1.5 will use symbolic topological dynamical systems. Every symbolic topological dynamical system (X, T) in this paper is constructed as follows: T is always the left-shift map on $\{0, 1\}^{\mathbb{N}}$, defined by $Tx[n] = x[n + 1]$ for every $n \in \mathbb{N}$ and $x \in \{0, 1\}^{\mathbb{N}}$. We choose $x \in \{0, 1\}^{\mathbb{N}}$, and the space X is the orbital closure of x: $X = \overline{\{T^n x\}_{n\in\mathbb{N}}}$, endowed with the induced topology from $\{0, 1\}^{\mathbb{N}}$, which has the discrete product topology. Alternatively, the topology of X is defined by the metric $d(x, y) = 2^{-n}$, where n is minimal such that $x[n] \neq y[n]$.

We will outline three constructions that algorithmically create x for which T will act in a certain way on the orbital closure X of x. (Here the 'certain way' in question depends on which construction is used.) To describe the constructions, a few more definitions are necessary.

Definition 2.1. An *alphabet* is any finite set, whose elements are called *letters*.

Definition 2.2. A *word* w on an alphabet A is any element of A^n for some positive integer n, which is called the *length* of w and written $|w|$. Equivalently, a word of length n on A is a string of n letters of A: $w = w[1]w[2] \dots w[n]$.

For any words v of length m and w of length n, we denote by vw their concatenation, i.e. the word $v[1]v[2] \dots v[m]w[1]w[2] \dots w[n]$ of length $m + n$. We denote by w^k the word $ww \dots w$ given by the concatenation of k copies of w.

Definition 2.3. A word w of length n is a *subword* of a sequence $u \in A^{\mathbb{N}}$ if there exists $k > 0$ such that $u[i + k] = w[i]$ for $1 \leq i \leq n$. Analogously, w is a *subword* of a word v of length m if there exists $0 < k < m - n$ such that $v[i + k] = w[i]$ for $1 \leq i \leq n$.

Definition 2.4. Given any closed shift-invariant set $X \subseteq A^{\mathbb{N}}$, the *language* of X, denoted by $L(X)$, is the set of all words that appear as subwords of elements of X.

In the case where X is the orbital closure of a single point x, $L(X)$ is just the set of subwords of x. We may now describe our first construction. It should also be mentioned that many ideas from these constructions are taken from the work of Hahn and Katznelson [**7**], who also algorithmically constructed symbolic topological dynamical systems with certain ergodicity and mixing properties.

Construction 1 (minimal). We define inductively n_k, w_k, and A_k, which are, respectively, sequences of positive integers, words on the alphabet $\{0, 1\}$, and sets of words on the

https://doi.org/10.1017/S0143385707000508 Published online by Cambridge University Press

alphabet $\{0, 1\}$. Each word in A_k is of length n_k, and w_k is a member of A_k. (We will use the term 'A_k-word' to refer to a member of A_k from now on.) We define these as follows. Always define $n_1 = 1$, $w_1 = 0$, $A_1 = \{0, 1\}$. Then, for any $k \geq 1$, n_{k+1} is defined to be any integer greater than or equal to $n_k|A_k|$ that is also a multiple of n_k, and then A_{k+1} is chosen to be the set of words of length n_{k+1} that are concatenations of A_k-words, containing each A_k-word in the concatenation at least once. The word w_{k+1} is taken to be any A_{k+1}-word that has w_k as a prefix.

In this way, a list of words $\{w_k\}_{k\in\mathbb{N}}$ is created, each of which is a prefix of the next. This means that one can define x to be the limit of the w_k, i.e. for any $m < n_k$, $x[m] := w_k[m]$. The claim is that, regardless of the choice of the integers n_k, as long as n_k divides n_{k+1} and $n_{k+1} \geq n_k|A_k|$, T will act minimally on the orbital closure of x. We then need to show that for, any $y \in X$, $\overline{\{T^n y\}_{n\in\mathbb{N}}} = X$. Choose any $y \in X$ and $w \in L(X)$. By the definition of x, there exists k such that w is a subword of w_k. But w_k is an A_k-word, so by definition every A_{k+1}-word contains w_k, and therefore w, as a subword. Finally, note that, again by the definition of Construction 1, x is an infinite concatenation of A_{k+1}-words. This implies that any $2n_{k+1}$-letter subword of x contains some complete A_{k+1}-word, and therefore w, as a subword. In particular, since $y \in X$, $y[1] \ldots y[2n_{k+1}]$ contains w as a subword, and so there exists $n \in \mathbb{N}$ such that $T^n y$ begins with w. Since w was an arbitrary subword of x, this implies that $\overline{\{T^n y\}_{n\in\mathbb{N}}} = X$, and so (X, T) is minimal. □

So, we have now demonstrated a way of constructing an x with minimal orbital closure. We will now make this construction a bit more complex in order to construct an x with a totally minimal orbital closure.

Construction 2 (totally minimal). We define inductively n_k, w_k, and A_k, which are, respectively, sequences of positive integers, words on the alphabet $\{0, 1\}$, and sets of words on the alphabet $\{0, 1\}$. Each word in A_k is of length n_k, and w_k is a member of A_k. We define these as follows. Always define $n_1 = 1$, $w_1 = 0$, $A_1 = \{0, 1\}$. Then, for any $k \geq 1$, n_{k+1} is defined to be any integer greater than or equal to $(k!)^2 n_k|A_k| + k! + n_k^2$, and then A_{k+1} is chosen to be the set of words w' of length n_{k+1} that are concatenations of A_k-words and the word 1 with the following properties: the word 1 does not appear at the beginning or end of w'; only a single 1 can be concatenated between two A_k-words; and for every $w \in A_k$ and every $0 \leq i < k!$, w appears in w' at an $i \pmod{k!}$-indexed place. That is, there exists $m \equiv i \pmod{k!}$ with $w'[m]w'[m+1] \ldots w'[m+n_k-1] = w$. From now on, to refer to this third condition, we say that every $w \in A_k$ occurs in w' at places indexed by all residue classes modulo $k!$. The word w_{k+1} is taken to be any element of A_{k+1} that begins with w_k.

Since this construction is a bit complicated, a few quick examples may be in order. Suppose that $n_2 = 6$, $|A_2| = 4$, and we choose $n_3 = 134$. Say that $A_2 = \{a, b, c, d\}$. Then $w = abcd1abcd1dabcdabcabcdab$ is an A_3-word: each A_2-word appears at least once beginning with a letter of w with an odd index, and at least once beginning with a letter of w with an even index. Examples of words that would not be A_3-words include $abcd11abcddabcdababcabcd$ (1 is concatenated twice between d and a), $abcda1bcd1dbcdaabcbbbbdc$ (occurrences of the word a begin only with even-indexed letters), or $abcd1dcbabcdabcdabcdadb$ (wrong number of letters).

https://doi.org/10.1017/S0143385707000508 Published online by Cambridge University Press

For this definition to make sense, it must be shown that if A_k is non-empty and contains at least one word w_k, then A_{k+1} is non-empty and contains at least one word w_{k+1} beginning with w_k. For any k, assume that $w_k \in A_k$. Then, enumerate the elements of A_k by $w_k = a_1, a_2, \ldots, a_{|A_k|}$, and define the words $u_{k+1} = a_1^{k!} a_2^{k!} \ldots a_{|A_k|}^{k!}$ and $w' = (u_{k+1}1)^{k!}(a_1 1)^i a_1^{(n_{k+1}-k!(k!n_k|A_k|+1)-i(n_k+1))/n_k}$, where $i \equiv n_{k+1} - k! \pmod{n_k}$. Since $n_{k+1} > (k!)^2 n_k |A_k| + k! + n_k^2$, w' exists, and is a concatenation of A_k-words and the word 1 with length n_{k+1}. In w', at most a single 1 is concatenated between any two A_k-words, and 1 does not appear at the beginning or end of w'. Also, since the length of u_{k+1} is divisible by $k!$, and since all A_k-words are subwords of u_{k+1}, all A_k-words appear in $(u_{k+1}1)^{k!}$ at places indexed by all residue classes modulo $k!$, and so all A_k-words appear in w' at places indexed by all residue classes modulo $k!$ as well. Therefore, $w' \in A_{k+1}$, and w' is a possible choice for w_{k+1} since it begins with w_k.

Since for every k, w_k is a prefix of w_{k+1}, we can define the limit of the w_k to be our sequence x. The claim is that every x constructed in this way will have orbital closure totally minimal with respect to T. Let us verify this. Fix any $m > 0$. We wish to show that for any $y \in X$, $\overline{\{T^{mn} y\}_{n\in\mathbb{N}}} = X$. Choose any such y, and fix any word w that is a subword of x. Since x is the limit of the w_k, there exists k such that w is a subword of w_k. Without loss of generality, we assume that $k > m$. By the construction, w_k occurs in every A_{k+1}-word, and it occurs at places indexed by every residue class modulo $k!$. Since $k > m$, in particular this implies that w_k, and therefore w, occurs in every A_{k+1}-word at places indexed by every residue class modulo m. Since x is a concatenation of A_{k+1}-words and single ones, every $(2n_{k+1} + 2)$-letter subword of x contains w at places indexed by every residue class modulo m. In particular, since $y \in X$, the word $y[1] \ldots y[2n_{k+1} + 2]$ must have this property, and so there exists n such that $T^{mn} y$ begins with w. Since w was an arbitrary subword of x, this shows that $\overline{\{T^{mn} y\}_{n\in\mathbb{N}}} = X$, and since m was arbitrary, that (X, T) is totally minimal. □

We now define one more general type of construction, again more complex than the last, so that the system created will always be totally uniquely ergodic and topologically mixing, in addition to being totally minimal. For this last construction, we first need a couple of definitions.

Definition 2.5. For any integers $0 \le i < m$ and k, and $w \in A_{k-1}$ and $w' \in A_k$, we define $fr^*_{i,m}(w, w')$ to be the ratio of the number of occurrences of w as a concatenated A_{k-1}-word at $i \pmod m$-indexed places in w' to the total number of A_{k-1}-words concatenated in w'.

We consider any positive integer to be equal to 0 (mod 1) for the purposes of this definition. An example is clearly in order: if $A_1 = \{01, 10\}$ (in Constructions 2 and 3, A_1 is always taken to be $\{0, 1\}$, but here we deviate from this for illustrative purposes), $w = 01$, and w' is the A_2-word 01|10|1|01|10 (here vertical bars illustrate where breaks in the concatenation occur), then w occurs twice out of four A_1-words, so $fr^*_{0,1}(w, w') = 1/2$. Since one of these occurrences begins at $w'[1]$ and one begins at $w'[6]$, $fr^*_{0,2}(w, w') = fr^*_{1,2}(w, w') = 1/4$. We make a quick note here that there could be some ambiguity here if an A_{k+1}-word could be decomposed as a concatenation of A_k words and ones in more than one way. For this reason, we just assume that when computing $fr^*_{i,j}(w, w')$, the definition

https://doi.org/10.1017/S0143385707000508 Published online by Cambridge University Press

of the A_{k+1}-word w' includes its representation as a concatenation of A_k-words and ones. (i.e. in the example given, w' is defined as the concatenation 01|10|1|01|10 of A_1-words and ones, rather than the nine-letter word 011010110.)

Definition 2.6. Given any words w' of length n' and w of length $n \leq n'$, and any integers $0 \leq i < m$, define $fr_{i,m}(w, w')$ to be the number of occurrences of w at $i \pmod m$-indexed places in w', divided by $n' - n + 1$.

Taking the previous example again, $fr_{0,1}(w, w') = 3/8$, since 01 occurs three times as a subword of 011010110. Since two of these occurrences begin at letters of w' with even indices and one begins at a letter of w' with odd index, $fr_{0,2}(w, w') = 2/8$ and $fr_{1,2}(w, w') = 1/8$.

Construction 3 (totally minimal, totally uniquely ergodic, and topologically mixing). We define inductively n_k, w_k, and A_k, which are, respectively, sequences of positive integers, words on the alphabet {0, 1}, and sets of words on the alphabet {0, 1}. Each word in A_k is of length n_k, and w_k is a member of A_k. We define these as follows. Always define $n_1 = 1$, $w_1 = 0$, and $A_1 = \{0, 1\}$. Then, we fix any sequence $\{d_k\}$ of positive reals such that $\sum_{k=1}^{\infty} d_k < \infty$, and define, for each $k \geq 1$, some $n_{k+1} = C_k(k+1)!|A_k|n_k + p$ for any integer $C_k > n_k > 1/d_k$ and prime $n_k < p \leq 2n_k$. (We may choose such a p by Bertrand's postulate [**8**].) Note that this implies that $(n_k, k!) = 1$ for all $k \in \mathbb{N}$. We then define A_{k+1} to be the set of words w' of length n_{k+1} with all of the same properties as in Construction 2, along with the property that, for any $w \in A_k$, and for any $0 \leq i < k!$, $fr^*_{i,k!}(w, w') \in [(1 - d_k)/k!|A_k|, (1 + d_k)/k!|A_k|]$. The word w_{k+1} is taken to be any element of A_{k+1} that begins with w_k.

For this definition to make sense, it must again be shown that if A_k is non-empty and contains at least one word w_k, then A_{k+1} is non-empty and contains at least one word w_{k+1} beginning with w_k. For any k, assume that $w_k \in A_k$. Then, enumerate the elements of A_k by $w_k = a_1, a_2, \ldots, a_{|A_k|}$, and define the words $u_{k+1} = a_1^{k!} a_2^{k!} \ldots a_{|A_k|}^{k!}$ and $w' = (u_{k+1})^{C_k(k+1)-p}(u_{k+1}1)^p$. Clearly for large k, w' exists, and is a concatenation of A_k-words and the word 1 with length n_{k+1}. In w', at most a single 1 is concatenated between any two A_k-words, and 1 does not appear at the beginning or end of w'. Since $(n_k, k!) = 1$, for every $0 \leq i < k!$ and $x \in A_k$, x appears in u_{k+1} exactly once as a concatenated A_k-word at an $i \pmod{k!}$-indexed place. Therefore, x appears in w' exactly $C_k(k+1)$ times as a concatenated A_k-word at $i \pmod{k!}$-indexed places, and so $fr^*_{i,k!}(x, w') = 1/|A_k|n_k k!$. Since i and x were arbitrary, $w' \in A_{k+1}$. Also, w' is a possible choice for w_{k+1} since it begins with w_k.

Since any x created using Construction 3 could be said to have been created using Construction 2 as well, it will automatically have totally minimal orbital closure X. We claim that X will, in addition, be totally uniquely ergodic. Take any word $w \in L(X)$, and any fixed integer j. We define two sequences $\{m_k^{(j)}\}$ and $\{M_k^{(j)}\}$ as follows: $m_k^{(j)}$ is the minimum value of $fr_{i,j}(w, w')$, where $0 \leq i < j$ and w' ranges over all A_k-words, and $M_k^{(j)}$ is the maximum value of $fr_{i,j}(w, w')$, where $0 \leq i < j$ and w' ranges over all A_k-words.

https://doi.org/10.1017/S0143385707000508 Published online by Cambridge University Press

Suppose that $m_k^{(j)}$ and $M_k^{(j)}$ are known, and that $k > j$. We wish to show that $m_{k+1}^{(j)}$ and $M_{k+1}^{(j)}$ are quite close to each other. Let us consider any element w' of A_{k+1} and, for any fixed $0 \le i < j$, see how few occurrences of w there could possibly be at $i \pmod j$-indexed places in w'. By the definition of Construction 3, for every $w'' \in A_k$, and $0 \le i' < k!$, the ratio of the number of times w'' occurs as a concatenated A_k-word in w' whose first letter is a letter of w' whose index is equal to $i' \pmod{k!}$ to the total number of A_k-words concatenated in w' is at least $(1 - d_k)/k!|A_k|$. Since j divides $k!$, for any $0 \le i' < j$, the ratio of the number of times that w'' occurs as a concatenated A_k-word at $i' \pmod j$-indexed places in w' to the total number of A_k-words concatenated in w' is at least $(1 - d_k)/j|A_k|$. Since the total number of A_k-words concatenated in w' is at least $n_{k+1}/(n_k + 1)$, this implies that the number of such occurrences of w'' in w' is at least $((1 - d_k)/j|A_k|)\, n_{k+1}/(n_k + 1)$ for any i' and w''. For any w'' and i', the number of times that w occurs at $i \pmod j$-indexed places in w' as a subword of an occurrence of w'' that occurs at an $i' \pmod j$-indexed place in w is then at least

$$\frac{1 - d_k}{j|A_k|} \frac{n_{k+1}}{n_k + 1}(n_k - |w| + 1) fr_{i - i' \pmod j, j}(w, w'').$$

Summing over all $w'' \in A_k$ and $0 \le i' < j$, the number of occurrences of w in w' at $i \pmod j$-indexed places is at least

$$(1 - d_k) n_{k+1} \frac{n_k - |w| + 1}{n_k + 1} \frac{\sum_{w'' \in A_k} \sum_{m=0}^{j-1} fr_{m,j}(w, w'')}{j|A_k|}.$$

Since w' was arbitrary in A_{k+1} and $0 \le i < j$ was arbitrary,

$$m_{k+1}^{(j)} \ge (1 - d_k) \frac{n_{k+1}}{n_{k+1} - |w| + 1} \frac{n_k - |w| + 1}{n_k + 1} \frac{\sum_{w'' \in A_k} \sum_{m=0}^{j-1} fr_{m,j}(w, w'')}{j|A_k|}.$$

Let us now bound from above the number of occurrences of w in w' at $i \pmod j$-indexed places. For precisely the same reasons as above, for any $0 \le i < j$, the number of occurrences of w at $i \pmod j$-indexed places that lie entirely within a concatenated A_k-word in w' is not more than

$$(1 + d_k) n_{k+1} \frac{n_k - |w| + 1}{n_k} \frac{\sum_{w'' \in A_k} \sum_{m=0}^{j-1} fr_{m,j}(w, w'')}{j|A_k|}.$$

(The denominator of the first fraction has changed because there are at most n_{k+1}/n_k A_k-words concatenated in w'.) However, it is possible that there are occurrences of w in w' that do not lie entirely within a concatenated A_k-word in w'. The number of such occurrences of w is not more than $|w| + 1$ times the number of concatenated A_k-words in w', which in turn is less than or equal to $(|w| + 1)(n_{k+1}/n_k)$. This means that the number of occurrences of w at $i \pmod j$-indexed places in w' is bounded from above by

$$(1 + d_k) n_{k+1} \frac{n_k - |w| + 1}{n_k} \frac{\sum_{w'' \in A_k} \sum_{m=0}^{j-1} fr_{m,j}(w, w'')}{j|A_k|} + (|w| + 1) \frac{n_{k+1}}{n_k},$$

https://doi.org/10.1017/S0143385707000508 Published online by Cambridge University Press

and since $0 \le i < j$ was arbitrary, this implies that

$$M_{k+1}^{(j)} \le (1+d_k)\frac{n_{k+1}}{n_{k+1}-|w|+1}\frac{n_k-|w|+1}{n_k}\frac{\sum_{w''\in A_k}\sum_{m=0}^{j-1} fr_{m,j}(w,w'')}{j|A_k|} + \frac{n_{k+1}}{n_{k+1}-|w|+1}\frac{|w|+1}{n_k}.$$

This implies that

$$M_{k+1}^{(j)} - m_{k+1}^{(j)} \le 2d_k\frac{n_{k+1}}{n_{k+1}-|w|+1}\frac{n_k-|w|+1}{n_k}\frac{\sum_{w''\in A_k}\sum_{m=0}^{j-1} fr_{m,j}(w,w'')}{j|A_k|} + \frac{n_{k+1}}{n_{k+1}-|w|+1}\frac{|w|+1}{n_k}.$$

Since $fr_{m,j}(w,w'') \le 1$ for every $0 \le m < j$ and $w'' \in A_k$, for large k this shows that $M_{k+1}^{(j)} - m_{k+1}^{(j)} \le 2d_k + 2(|w|+1)/n_k$, which clearly approaches zero as $k \to \infty$. We now note that since

$$m_{k+1}^{(j)} \ge (1-d_k)\frac{n_{k+1}}{n_{k+1}-|w|+1}\frac{n_k-|w|+1}{n_k+1}\frac{\sum_{w''\in A_k}\sum_{m=0}^{j-1} fr_{m,j}(w,w'')}{j|A_k|},$$

and since by definition $fr_{m,j}(w,w'') \ge m_k^{(j)}$ for all $w'' \in A_k$,

$$m_{k+1}^{(j)} \ge (1-d_k)\frac{n_{k+1}}{n_{k+1}-|w|+1}\frac{n_k-|w|+1}{n_k+1}m_k^{(j)},$$

which implies that

$$\begin{aligned} m_{k+1}^{(j)} - m_k^{(j)} &\ge m_k^{(j)}\left[(1-d_k)\left(1+\frac{|w|-1}{n_{k+1}-|w|+1}\right)\left(1-\frac{|w|}{n_k+1}\right)-1\right] \\ &\ge -m_k^{(j)}\left(d_k+\frac{|w|}{n_k+1}\right) \ge -\left(d_k+\frac{|w|}{n_k}\right). \end{aligned}$$

By almost completely analogous reasoning, for large k

$$\begin{aligned} M_{k+1}^{(j)} - M_k^{(j)} &\le M_k^{(j)}\left[(1+d_k)\left(1+\frac{|w|-1}{n_{k+1}-|w|+1}\right)\left(1-\frac{|w|-1}{n_k}\right)-1\right] \\ &\quad + \frac{n_{k+1}}{n_{k+1}-|w|+1}\frac{|w|+1}{n_k} \le M_k^{(j)}d_k + \frac{2(|w|+1)}{n_k} \le d_k + \frac{2|w|+1}{n_k}. \end{aligned}$$

Therefore,

$$\begin{aligned} m_{k+1}^{(j)} \le M_{k+1}^{(j)} \le M_k^{(j)} + d_k + \frac{2|w|+1}{n_k} &\le m_k^{(j)} + 2d_{k-1} + \frac{2(|w|+1)}{n_{k-1}} + d_k + \frac{2|w|+1}{n_k} \\ &\le m_k^{(j)} + 2d_{k-1} + d_k + \frac{4|w|+3}{n_{k-1}}, \end{aligned}$$

so

$$|m_{k+1}^{(j)} - m_k^{(j)}| \le d_k + 2d_{k-1} + \frac{4|w|+3}{n_{k-1}}.$$

https://doi.org/10.1017/S0143385707000508 Published online by Cambridge University Press

In a completely analogous fashion, we find that

$$|M_{k+1}^{(j)} - M_k^{(j)}| \leq d_k + 2d_{k-1} + \frac{3|w|+2}{n_{k-1}}.$$

We know that $\sum_{k=1}^{\infty} d_k$ converges, and since $n_k \geq 2^k$ for all k, $\sum_{k=1}^{\infty}(1/n_k)$ converges as well. Therefore, we see that the sequences $\{m_k^{(j)}\}$ and $\{M_k^{(j)}\}$ are Cauchy, and converge. Since we also showed that $M_k^{(j)} - m_k^{(j)} \to 0$, we know that they have the same limit: call it α.

This implies that for very large k, $|fr_{i,j}(w, w') - \alpha|$ is very small for every $0 \leq i < j$ and $w' \in A_k$. We claim that this, in turn, implies that for very large N, $|fr_{i,j}(w, w'') - \alpha|$ is very small for every word w'' of length N that is a subword of x: fix any $\epsilon > 0$, and take k such that $|fr_{i,j}(w, w') - \alpha| < \epsilon/2$ for every $0 \leq i < j$ and $w' \in A_k$, and such that $(1 + |w|)/n_k < \epsilon/4$. Then for any word $w'' \in L(X)$ of length at least $8n_k/\epsilon$, w'' is a subword of a concatenation of A_k-words and copies of the word 1. The number of full A_k-words appearing in the concatenation forming w'' will be at least $(|w''|/(n_k+1)) - 2$, and at most $|w''|/n_k$. So, the number of occurrences of w at $i \pmod j$-indexed places in w'' that are contained entirely within a concatenated A_k-word is at least

$$\begin{aligned}\left(\frac{(n_k - |w| + 1)|w''|}{n_k + 1} - 2(n_k - |w| + 1)\right)\left(\alpha - \frac{\epsilon}{2}\right) &\geq |w''|\left(\left(1 - \frac{\epsilon}{4}\right) - \frac{\epsilon}{4}\right)\left(\alpha - \frac{\epsilon}{2}\right) \\ &\geq |w''|(\alpha - \epsilon),\end{aligned}$$

and at most

$$|w''|\frac{n_k - |w| + 1}{n_k}\left(\alpha + \frac{\epsilon}{2}\right) \leq |w''|\left(\alpha + \frac{\epsilon}{2}\right).$$

Since there are at most $(|w|+1)|w''|/n_k < |w''|\epsilon/4$ occurrences of w not contained entirely within a concatenated A_k-word, this implies that $fr_{i,j}(w, w'')$ is at least $\alpha - \epsilon$, and at most $\alpha + \epsilon$.

Since for any $\epsilon > 0$, this statement is true for any long enough word $w'' \in L(X)$ and $0 \leq i < j$, we see that $(1/n)\sum_{i=0}^{n-1} \chi_{[w]}(T^{ij}y) \to \alpha$ uniformly for $y \in X$. Since w was arbitrary, and since characteristic functions of cylinder sets are dense in $C(X)$, $(1/n)\sum_{i=0}^{n-1} f(T^{ij}y)$ approaches a uniform limit for all $f \in C(X)$, and so (X, T^j) is uniquely ergodic for every $j \in \mathbb{N}$. Since an invariant measure for (X, T) would be invariant for any (X, T^j) as well, the unique invariant measure is the same for every j.

Finally, we claim that the orbital closure of any x constructed in this way is also topologically mixing. Consider any words $w, w' \in L(X)$. By construction, there exists k such that there are A_k-words y, y' with w a subword of y and w' a subword of y'. We also claim that for any $n_{k+1}/6 < i < 5n_{k+1}/6$, there exists an A_{k+1}-word b_i where $b_i[i+1]b_i[i+2]\ldots b_i[i+n_k] = y$, and similarly $b_i' \in A_{k+1}$ with $b_i'[i+1]b_i'[i+2]\ldots b_i'[i+n_k] = y'$. We show only the existence of b_i, as the proof for b_i' is trivially similar. Consider any $n_{k+1}/6 < i < 5n_{k+1}/6$, and take $j = i \pmod{n_k}$. Then, if we enumerate the elements of A_k by $a_1, a_2, \ldots, a_{|A_k|}$, first define the word $u_{k+1} = a_1^{k!}a_2^{k!}\ldots a_{|A_k|}^{k!}$, and then define the word $y_i = (u_{k+1}1)^j(u_{k+1})^{C_k(k+1)-p}(u_{k+1}1)^{p-j}$. Then y_i has the property that $y_i[i+1]y_i[i+2]\ldots y_i[i+n_k]$ is a concatenated A_k-word in y_i as long as $y_i[i+1]y_i[i+2]\ldots y_i[i+n_k]$ lies in the subword $(u_{k+1})^{C_k(k+1)-p}$ of y_i, which will

https://doi.org/10.1017/S0143385707000508 Published online by Cambridge University Press

be true for large k and $i \in (n_{k+1}/6, 5n_{k+1}/6)$. This means that if we reorder $a_1, \ldots, a_{|A_k|}$ in the definition of u_{k+1}, we may create a word b_i where $b_i[i+1]b_i[i+2]\ldots b_i[i+n_k] = y$. We have $b_i \in A_{k+1}$, since for every $0 \le i < k!$ and $x \in A_k$, x appears exactly $C_k(k+1)$ times in b_i as a concatenated A_k-word at $i \pmod{k!}$-indexed places, implying that $fr^*_{i,k!}(x, b_i) = 1/|A_k|n_k k!$. (This uses the fact that $(n_k, k!) = 1$, which has already been shown.) We create b'_i in the same way for each i. Since w is a subword of y and w' is a subword of y', for every $n_{k+1}/6 + n_k \le i \le 5n_{k+1}/6 - n_k$, it is easy to choose a word z_i to be b_j for properly chosen j so that $z_i[i+1]\ldots z_i[i+|w|] = w$, and similarly z'_i so that $z'_i[i+1]\ldots z'_i[i+|w'|] = w'$. For large k, this means that we can construct such z_i and z'_i for any $i \in [n_{k+1}/5, 4n_{k+1}/5]$.

We will now use these z_i and z'_i to prove that for any $n > |w| + n_{k+1}$, there exists a word $x \in L(X)$ of length n such that $wxw' \in L(X)$. We do this by proving a lemma.

LEMMA 2.1. *For any $t > k+1$, and for any $0 \le i, j < n_t$ such that there exists an A_t-word x where $x[i+1]x[i+2]\ldots x[i+n_{k+1}]$ and $x[j+1]x[j+2]\ldots x[j+n_{k+1}]$ are concatenated A_{k+1}-words in x, and for any two A_{k+1}-words z and z', there exists an A_t-word x' where $x'[i+1]x'[i+2]\ldots x'[i+n_{k+1}] = z$ and $x'[j+1]x'[j+2]\ldots x'[j+n_{k+1}] = z'$.*

Proof. We prove this by induction. First we prove the base case $t = k+2$. Take an A_{k+2}-word x where $x[i+1]x[i+2]\ldots x[i+n_{k+1}]$ and $x[j+1]x[j+2]\ldots x[j+n_{k+1}]$ are concatenated A_{k+1}-words in x: call them a and b respectively. Since x is an A_{k+2}-word, there exists an occurrence of z at an $(i \pmod{(k+1)!}) \pmod{(k+1)!}$-indexed place, i.e. there exists $i' \equiv i \pmod{(k+1)!}$ such that $x[i'+1]x[i'+2]\ldots x[i'+n_{k+1}] = z$. Similarly, there exists $j' \equiv j \pmod{(k+1)!}$ such that $x[j'+1]x[j'+2]\ldots x[j'+n_{k+1}] = z'$. We now create x' by leaving almost all of x alone, but defining $x'[i+1]\ldots x'[i+n_{k+1}] = z$, $x'[j+1]\ldots x'[j+n_{k+1}] = z'$, $x'[i'+1]\ldots x'[i'+n_{k+1}] = a$, and $x'[j'+1]\ldots x'[j'+n_{k+1}] = b$. This new word x' is still a concatenation of A_{k+1}-words and ones, and since we switched two pairs of A_{k+1}-words which occurred at indices with the same residue class modulo $(k+1)!$, $fr^*_{i,(k+1)!}(w, x) = fr^*_{i,(k+1)!}(w, x')$ for all $0 \le i < (k+1)!$ and $w \in A_{k+1}$. Therefore, x' is an A_{k+2}-word, with z and z' occurring at the proper places, completing our proof of the base case.

Now, let us assume that the inductive hypothesis is true for a certain value of t, and prove it for $t+1$. Consider an A_{t+1}-word x where $x[i+1]\ldots x[i+n_{k+1}]$ and $x[j+1]\ldots x[j+n_{k+1}]$ are concatenated A_{k+1}-words in x, call them a and b. Call the concatenated A_t-word of which $x[i+1]\ldots x[i+n_{k+1}]$ is a subword a', and denote the corresponding A_t-word for $x[j+1]\ldots x[j+n_{k+1}]$ by b'. From now on, when we speak of these words a, b, a' and b', we are talking about the pertinent occurrences at the places within x already described. There are two cases: either a' and b' are the same, i.e. the same A_t-word in x, occurring at the same place; or they are not. If a' and b' do occur at the same place, then by the inductive hypothesis, there exists an A_t-word c with an occurrence of z at the same place as a occurs in $a' = b'$, and an occurrence of z' at the same place as b occurs in $a' = b'$. If we can replace $a' = b'$ by c in x, then we will be done. If a' and b' do not occur at the same place, then since a is a concatenated A_{k+1}-word in a',

https://doi.org/10.1017/S0143385707000508 Published online by Cambridge University Press

by the inductive hypothesis there exists an A_t-word a'' such that a'' has z occurring at the same place where a occurs in a'. Similarly, there exists an A_t-word b'' such that b'' has an occurrence of z' at the same place where b occurs in b'. If we replace a' by a'' and b' by b'' in x, then we will be done. So regardless of which case we are in, our goal is to replace one or two chosen A_t-words within x with one or two other A_t-words. We will show how to replace two, which clearly implies that replacing one is possible. We wish to replace a' by a'' and b' by b''. We do this in exactly the same way as in the base case. Say that $a' = x[i'+1]\dots x[i'+n_t]$ and $b' = x[j'+1]\dots x[j'+n_t]$. Since $a'' \in A_t$, there exists $i'' = i' \pmod{t!}$ and $j'' = j' \pmod{t!}$ such that $x[i''+1]\dots x[i''+n_t] = a''$ and $x[j''+1]\dots x[j''+n_t] = b''$. As in the base case, we create x' by making $x'[i'+1]\dots x'[i'+n_t] = a''$ and $x'[i''+1]\dots x'[i''+n_t] = a'$, $x'[j'+1]\dots x'[j'+n_t] = b''$, and $x'[j''+1]\dots x'[j''+n_t] = b'$. Then x' is an A_{t+1}-word, and by construction $x'[i+1]\dots x'[i+n_{k+1}] = z$ and $x'[j+1]\dots x'[j+n_{k+1}] = z'$. □

Choose any sequence $\{v_m\}$ of A_m-words for all $m > k+1$. For any such m, take $P_m = \{n : v_m[n+1]\dots v_m[n+n_{k+1}]$ is a concatenated A_{k+1}-word in $v_m\}$. Since v_m is a concatenation of A_{k+1}-words and ones, if we write the elements of P_m as $p_1^{(m)} < p_2^{(m)} < \cdots < p_t^{(m)}$, then for any $1 \le \ell < t$, $p_{\ell+1}^{(m)} - p_\ell^{(m)} \le n_{k+1}+1$. For any $1 < \ell < t$, and $i, j \in [n_{k+1}/5, 4n_{k+1}/5]$, by Lemma 2.1, there exists an A_m-word v with the property that $v[p_1^{(m)}+1]\dots v[p_1^{(m)}+n_{k+1}] = z_i$ and $v[p_\ell^{(m)}+1]\dots v[p_\ell^{(m)}+n_{k+1}] = z'_j$. This implies that there is a subword of v of the form wxw' where the length of x is $p_\ell^{(m)} - p_1^{(m)} + (j-i) - |w|$. We note that $j-i$ can take any integer value between $-3n_{k+1}/5$ and $3n_{k+1}/5$ inclusive. Therefore, the set of possible lengths of x for which $wxw' \in L(X)$ contains

$$\bigcup_{\ell=2}^{t}\left(|w| + p_\ell^{(m)} - p_1^{(m)} + \left[-\frac{3n_{k+1}}{5}, \frac{3n_{k+1}}{5}\right]\right).$$

When ℓ is increased by one, $p_\ell^{(m)}$ is increased by at most $n_{k+1}+1$. This, along with the fact that the intervals $[-3n_{k+1}/5, 3n_{k+1}/5]$ have length $6n_{k+1}/5$, which for large k exceeds $n_{k+1}+1$, implies that this set of possible lengths of x contains

$$\left[|w| + p_2^{(m)} - p_1^{(m)} - \frac{3n_{k+1}}{5}, |w| + p_t^{(m)} - p_1^{(m)} + \frac{3n_{k+1}}{5}\right]$$
$$\supseteq [|w| + n_{k+1}, |w| + n_m - 2n_{k+1}].$$

Since this entire argument could be made for any m, we see that for any $n > |w| + n_{k+1}$, there exists $x \in L(X)$ of length n such that $wxw' \in L(X)$. Then, for any non-empty open sets $U, V \subseteq X$, there exist w and w' such that $[w] \subseteq U$ and $[w'] \subseteq V$. By the above arguments, there exists N such that for any $n > N$, $[w] \cap T^n[w'] \neq \varnothing$, implying that $U \cap T^n V \neq \varnothing$. This shows that (X, T) is topologically mixing. □

3. *Some symbolic counterexamples*

Proof of Theorem 1.2. We take the continuous function $f(y) = y[1]$ for all $y \in X$, and first note that

https://doi.org/10.1017/S0143385707000508 Published online by Cambridge University Press

$$\left\{y \in X : \frac{1}{N}\sum_{n=0}^{N-1} f(T^{p_n} y) \text{ does not converge}\right\}$$
$$\supseteq \left(\bigcap_{n>0}\bigcup_{k>n}\left\{y \in X : fr_{0,1}(0,\, y[p_1+1]y[p_2+1]\ldots y[p_k+1]) < \frac{1}{4}\right\}\right)$$
$$\cap \left(\bigcap_{n>0}\bigcup_{k>n}\left\{y \in X : fr_{0,1}(0,\, y[p_1+1]y[p_2+1]\ldots y[p_k+1]) > \frac{3}{4}\right\}\right),$$

and that the latter set, call it B, is clearly a G_δ. We will choose x so that B is dense in X. This will imply that B is a dense G_δ, and since X is a complete metric space, by the Baire category theorem, that B is residual, which will prove Theorem 1.2. Now let us describe the construction of x.

Recall that we have assumed that the sequence $\{p_n\}$ has upper Banach density zero. We define $A := \{p_n : n \in \mathbb{N}\}$. We also define the intervals of integers $B_j = [2j!, (j+1)!] \cap \mathbb{N}$ for every $j \in \mathbb{N}$, and take any partition of $\mathbb{N}$ into infinitely many disjoint infinite sets in $\mathbb{N}$, call them $C_1, C_2, \ldots$. Define the set $D_1 = \bigcup_{j\in C_1} B_j$, and then define the set $A_1 = \{p_n : n \in D_1\} + 1$. Next, choose some r_2 large enough so that $(\min C_{r_2})! > 2 \cdot 2$, and define $D_2 = \bigcup_{j\in C_{r_2}} B_j$, and then define $A_2 = \{p_n : n \in D_2\} + 2$. Continuing in this way, we may inductively define A_k and D_k for all $k \in \mathbb{N}$ so that for all k, $D_k = \bigcup_{j\in C_{r_k}} B_j$ for some r_k with the property that $(\min C_{r_k})! > 2k$, and $A_k = \{p_n : n \in D_k\} + k$. We will verify some properties of these sets. Most importantly, we denote by H the union $\bigcup_{n=1}^{\infty} A_n$, and claim that $d^*(H) = 0$. We show this by noting that H has a certain structure: H consists of shifted subintervals of A, separated by gaps that approach infinity. More precisely, we have the following lemma.

LEMMA 3.1. *There exist intervals $I_k = [a_k, b_k] \cap \mathbb{N}$ and integers j_k such that $H = \bigcup_{k=1}^{\infty}((A \cap I_k) + j_k)$ and such that*

$$\lim_{k\to\infty} (\min((A \cap I_{k+1}) + j_{k+1}) - \max((A \cap I_k) + j_k)) = \infty.$$

Proof. Take the set $Q = \bigcup_{k=1}^{\infty} C_{r_k}$, and denote its members by $q_1 < q_2 < \ldots$. Then, for any k, B_{q_k} is a subset of some D_s. The interval I_k is then defined to be $[p_{2(q_k)!}, p_{(q_k+1)!)}]$, (which means $a_k = p_{2(q_k)!}$ and $b_k = p_{(q_k+1)!)}$) and j_k is defined to be s. It is just a rewriting of the definition of the A_k that $H = \bigcup_{k=1}^{\infty}((A \cap I_k) + j_k)$ with these notations. All that must be checked is that $\lim_{k\to\infty}(\min((A \cap I_{k+1}) + j_{k+1}) - \max((A \cap I_k) + j_k)) = \infty$. We will show that $a_{k+1} + j_{k+1} - b_k - j_k \to \infty$, which implies the desired result. Since $q_{k+1} > q_k$, $a_{k+1} - b_k = (q_{k+1})! > (q_k)!$. So, we must simply show that $(q_k)! - j_k \to \infty$. Suppose that B_{q_k} is a subset of D_s. Then $j_k = s$. We also note that by construction, $(\min C_{r_s})! > 2s$. But, since $B_{q_k} \subset D_s$, $q_k \in C_{r_s}$, and so $\min C_{r_s} \le q_k$. Therefore, $(q_k)! > 2s$, and so $(q_k)! - j_k = (q_k)! - s > (q_k)!/2$, which clearly shows that this quantity approaches ∞, since $\{q_k\}$ is an increasing sequence of integers. □

We will now prove a general lemma that implies, in particular, that $d^*(H) = 0$.

https://doi.org/10.1017/S0143385707000508 Published online by Cambridge University Press

LEMMA 3.2. *If $d^*(A) = 0$, and if there exist intervals $I_k = [a_k, b_k] \cap \mathbb{N}$ and integers j_k such that*

$$\lim_{k\to\infty} (\min((A \cap I_{k+1}) + j_{k+1}) - \max((A \cap I_k) + j_k)) = \infty,$$

then the set $B = \bigcup_{k=1}^{\infty} (A \cap I_k) + j_k$ has upper Banach density zero.

Proof. Fix $\epsilon > 0$. By the fact that $d^*(A) = 0$, there exists N such that for any interval J of integers of length at least N, $|A \cap J|/|J| < \epsilon$. Take J to be any interval of integers of length exactly N. Since

$$\lim_{k\to\infty} (\min((A \cap I_{k+1}) + j_{k+1}) - \max((A \cap I_k) + j_k)) = \infty,$$

there is some K such that if J has non-empty intersection with $(A \cap I_k) + j_k$ for some $k > K$, it is disjoint from $(A \cap I_{k'}) + j_{k'}$ for every $k' \neq k$. Therefore, for intervals J of integers of length N with large enough minimum element, $J \cap B$ consists of a subset of a shifted copy of $J \cap A$, and so $|B \cap J|/|J| \leq |A \cap J'|/|J'|$ for some interval J' of integers whose length is also N. This means that in this case, $|B \cap J|/|J| < \epsilon$. We have then shown that for every ϵ, there exist N and M such that for any interval of integers J of length N with $\min J > M$, $|B \cap J|/|J| < \epsilon$. We will show that this slightly modified definition still implies that $d^*(B) = 0$. Again fix $\epsilon > 0$, and define M and N as was just done. Now consider any interval of integers I with length at least $(N + M)/\epsilon$. Then, partition I into subintervals: define $I_0 = I \cap \{1, \ldots, M\}$, and then break $I \setminus I_0$ into consecutive subintervals of length N, called $I_1, I_2, \ldots, I_k$. There may be one last subinterval left over of length less than N; call it I_{k+1} (which may be empty). Note that $|I| \geq Nk$, or $N/|I| \leq 1/k$. We see that

$$\begin{aligned}\frac{|B \cap I|}{|I|} &= \frac{|B \cap I_0|}{|I|} + \sum_{i=1}^{k} \frac{|B \cap I_i|}{|I|} + \frac{|B \cap I_{k+1}|}{|I|} \\ &\leq \frac{M}{|I|} + \frac{N}{|I|}\left(\sum_{i=1}^{k} \frac{|B \cap I_i|}{|I_i|}\right) + \frac{N}{|I|} \\ &\leq \frac{M+N}{|I|} + \frac{1}{k}(k\epsilon) < 2\epsilon.\end{aligned}$$

Since ϵ was arbitrary, $d^*(B) = 0$. □

By combining Lemmas 3.1 and 3.2, $d^*(H) = 0$. We will now create n_k, A_k, and w_k to use for Construction 3. We note that this part of our construction will use only the fact that $d^*(H) = 0$, and no other properties. We take $n_1 = 1$, $A_1 = \{0, 1\}$, and $w_1 = 0$. We recall that $\{n_k\}$ must be a sequence of integers with the following properties: for all $k > 1$, $n_{k+1} = C_k(k + 1)!|A_k|n_k + p$ for some positive integer $C_k > n_k > 1/d_k$ and prime $n_k < p \leq 2n_k$. We also require n_k to grow quickly enough so that for all k, and for any interval of integers I of length at least n_{k+1}, $|I \cap H|/|I| < d_k/2k!|A_k|n_k$. That we may choose such n_k is a consequence of the fact that $d^*(H) = 0$. Using these n_k, we define A_k as in Construction 3. We now prove a lemma.

LEMMA 3.3. *For any $k, m \in \mathbb{N}$, and for any sequence of letters $u \in \{0, 1\}^{\mathbb{N}}$, there exists an A_k-word $v_{u,k,m}$ such that $v_{u,k,m}[i - m] = u[i]$ for all $i \in H \cap [m + 1, m + n_k]$.*

https://doi.org/10.1017/S0143385707000508 Published online by Cambridge University Press

Proof. This is proved by induction on k. Clearly the hypothesis is true for $k = 1$ and for any u and m. Now suppose it to be true for a particular k. We will show that it is true for $k+1$ and every u and m. We again construct an auxiliary word u_{k+1}. Enumerate the elements of A_k by $a_1, a_2, \ldots, a_{|A_k|}$. Then, we again define the word $u_{k+1} = a_1^{k!} a_2^{k!} \ldots a_{|A_k|}^{k!}$. Define $v'_{k+1} = (u_{k+1}1)^p (u_{k+1})^{C_k(k+1)-p}$, where $n_{k+1} = C_k(k+1)!|A_k|n_k + p$ as above. We note that for any $0 \le i < k!$ and $w \in A_k$, w occurs exactly $C_k(k+1)$ times as a concatenated A_k-word at $i \pmod{k!}$-indexed places in v'_{k+1}. (This uses the fact that $(n_k, k!) = 1$ for all k, which has already been shown.) Now, fix $m \in \mathbb{N}$. We wish to construct an A_{k+1}-word $v_{u,k+1,m}$ such that $v_{u,k+1,m}[i-m] = u[i]$ for all $i \in H \cap [m+1, m+n_{k+1}]$. We begin with the A_{k+1}-word v'_{k+1}. Clearly it is not necessarily true that $v'_{k+1}[i-m] = u[i]$ for all $i \in H \cap [m+1, m+n_{k+1}]$. We force this condition to be true by changing some of the A_k-words concatenated in v'_{k+1}. We show that this is possible: for any concatenated A_k-word in v'_{k+1}, say $v'_{k+1}[j]v'_{k+1}[j+1]\ldots v'_{k+1}[j+n_k-1]$, the necessary condition is that ones or zeroes (depending on u) be introduced at digits whose indices are of the form $i-m$ for all $i \in H \cap [m+j+1, \ldots, m+j+n_k-1]$. To do this, we replace this A_k-word by $v_{u,k,m+j}$, which by the inductive hypothesis has the correct digits of u at the desired places. So, we may change v'_{k+1} into a concatenation of A_k-words and ones, call it $v_{u,k+1,m}$, which has the proper digits of u in all desired places. This may be done by changing at most $|H \cap [m+1, \ldots, m+n_{k+1}]| < n_{k+1}d_k/2k!|A_k|n_k \le d_k C_k(k+1)$ A_k-words. Therefore, since for any $0 \le i < k!$ and $w \in A_k$, w occurred in v'_{k+1} as a concatenated A_k-word at $i \pmod{k!}$-indexed places exactly $C_k(k+1)$ times, w occurs in $v_{u,k+1,m}$ as a concatenated A_k-word at $i \pmod{k!}$-indexed places between $C_k(k+1)(1-d_k)$ and $C_k(k+1)k(1+d_k)$ times. This implies that $fr^*_{i,k!}(w, v_{k+1,m}) \in [(1-d_k)/k!|A_k|, (1+d_k)/k!|A_k|]$, and since i and w were arbitrary, that $v_{k+1,m}$ is an A_{k+1}-word. By induction, the lemma is proved. □

This implies in particular that for every u, k there exists an A_k-word $v_{u,k,0}$ with $v_{u,k,0}[i] = u[i]$ for all $i \in H \cap [1, \ldots, n_k]$. By a standard diagonalization argument, there exists a sequence $\{k_j\}$ and $x \in \{0,1\}^{\mathbb{N}}$ such that x is the limit of $v_{u,k_j,0}$ as $j \to \infty$. Then, for every k, choose $k_j > k$, and take w_k to be the A_k-word that is a prefix of x. This allows us to define $w_k \in A_k$ for all k, and to see that x is their limit as well. Since for every j, $v_{u,k_j,0}[i] = 1$ for all $i \in H \cap [1, n_{k_j}]$, clearly $x[i] = u[i]$ for all $i \in H$. As mentioned above, this entire construction could be done with any set of zero upper Banach density in place of H, which lets us state the following corollary.

COROLLARY 3.1. *For any $C \subset \mathbb{N}$ with $d^*(C) = 0$, there exists $x \in \{0,1\}^{\mathbb{N}}$ such that $X = \overline{\{T^k x\}_{k\in\mathbb{N}}}$ is totally minimal, totally uniquely ergodic, topologically mixing, and with the property that for any sequence $u \in \{0,1\}^{\mathbb{N}}$, there exists $x_u \in X$ such that $x_u[i] = u[i]$ for all $i \in C$.*

Proof. Given the set C, fix $\{n_k\}$ as in the proof of Lemma 3.3. Then note that the reasoning from the lemma yields that for any $u \in \{0,1\}^{\mathbb{N}}$, there exists a limit x_u of A_k-words such that $x_u[i] = u[i]$ for all $i \in C$. Fix any one of these x_u and call it x. Note that since x contains every A_k-word as a subword, $x_u \in \overline{\{T^k x\}_{k\in\mathbb{N}}} = X$ for all u. Since x was created using Construction 3, X is totally minimal, totally uniquely ergodic, and topologically mixing. □

https://doi.org/10.1017/S0143385707000508 Published online by Cambridge University Press

We use Corollary 3.1 to create the x that will prove Theorem 1.2. Recall that $H = \bigcup_{k=1}^{\infty} A_k$, where $A_k = \{p_n + k : n \in D_k\}$ for all k, $D_k = \bigcup_{j \in C_{r_k}} B_j$ for some r_k, and $B_j = [2j!, (j+1)!] \cap \mathbb{Z}$ for all j. For each k, we write the elements of C_{r_k} in increasing order as $c_{r_k}^{(1)}, c_{r_k}^{(2)}, \ldots$. We now decompose H into two disjoint subsets: define $H_o = \{m \in H : m = p_n + k$ for some $n \in B_j$ where $j = c_{r_k}^{(i)}$ for odd $i\}$ and $H_e = \{m \in H : m = p_n + k$ for some $n \in B_j$ where $j = c_{r_k}^{(i)}$ for even $i\}$. Since $d^*(H) = 0$, we use Corollary 3.1 to create x with totally minimal, totally uniquely ergodic, and topologically mixing orbital closure X, and for which $x[n] = 0$ for $n \in H_o$ and $x[n] = 1$ for $n \in H_e$. Recall that we wish to show that for the continuous function $f : y \mapsto y[1]$ from X to $\{0, 1\}$, the set of points y such that $(1/N) \sum_{n=0}^{N-1} f(T^{p_n} y)$ fails to converge is residual. We showed earlier that it is sufficient to show that the set

$$B = \left(\bigcap_{n>0} \bigcup_{k>n} \left\{ y \in X : fr_{0,1}(0, y[p_1 + 1]y[p_2 + 1] \ldots y[p_k + 1]) < \frac{1}{4} \right\} \right)$$
$$\cap \left(\bigcap_{n>0} \bigcup_{k>n} \left\{ y \in X : fr_{0,1}(0, y[p_1 + 1]y[p_2 + 1] \ldots y[p_k + 1]) > \frac{3}{4} \right\} \right)$$

is dense in X. By definition, for any $w \in L(X)$, there is some j such that $x[j+1]x[j+2] \ldots x[j+|w|] = w$. By the construction of x, $\{n : j + 1 + p_n \in H\} = D_k$ for some k, $x[i] = 0$ for all $i \in H_o$, and $x[i] = 1$ for all $i \in H_e$. In particular, $x[j + 1 + p_n] = 0$ for all n in $B_{c_{r_k}^{(i)}}$ for odd i and $x[j + 1 + p_n] = 1$ for all n in $B_{c_{r_k}^{(i)}}$ for even i. But then for any odd integer i, $(T^j x)\,[p_n + 1] = 0$ for all integers $n \in [2(c_{r_k}^{(i)})!, (c_{r_k}^{(i)} + 1)!]$, and so

$$fr_{0,1}(0, (T^j x)\,[p_1 + 1](T^j x)\,[p_2 + 1] \ldots (T^j x)[p_{(c_{r_k}^{(i)}+1)!} + 1]) > \frac{c_{r_k}^{(i)} - 1}{c_{r_k}^{(i)} + 1},$$

which is clearly larger than $3/4$ for sufficiently large k. Similarly, for any even integer i, $(T^j x)\,[p_n + 1] = 1$ for all integers $n \in [2(c_{r_k}^{(i)})!, (c_{r_k}^{(i)} + 1)!]$, and so

$$fr_{0,1}(0, (T^j x)\,[p_1 + 1](T^j x)\,[p_2 + 1] \ldots (T^j x)\,[p_{(c_{r_k}^{(i)}+1)!} + 1]) < \frac{2}{c_{r_k}^{(i)} + 1},$$

which is clearly less than $1/4$ for sufficiently large k. Therefore, $T^j x \in B$, and $T^j x$ begins with the word w. Since w was an arbitrary word in $L(X)$, this shows that B is dense in X, completing the proof of Theorem 1.2. □

We note that this proof in fact shows that the set

$$\left(\bigcap_{n>0} \bigcup_{k>n} \left\{ y \in X : fr_{0,1}(0, y[p_1 + 1]y[p_2 + 1] \ldots y[p_{k!} + 1]) < \frac{2}{k} \right\} \right)$$
$$\cap \left(\bigcap_{n>0} \bigcup_{k>n} \left\{ y \in X : fr_{0,1}(0, y[p_1 + 1]y[p_2 + 1] \ldots y[p_{k!} + 1]) > \frac{k-2}{k} \right\} \right)$$

is a residual set in X, and so we can also say that for a residual set of x,

$$\liminf_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^{p_n} y) = 0 = \inf_{x \in X} f(x)$$

and

https://doi.org/10.1017/S0143385707000508 Published online by Cambridge University Press

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^{p_n} y) = 1 = \sup_{x\in X} f(x).$$

Proof of Theorem 1.5. We use Corollary 3.1. Consider any set $C = \{p_n\}_{n\in\mathbb{N}}$ with $d^*(C) = 0$. Choose any set C' with $(C+1) \subset C'$, $1 \in C'$, $d^*(C') = 0$, and $|C' \setminus (C+1)| = \infty$. Denote the elements of $C' \setminus (C+1)$ by $b_1 < b_2 < \dots$. By Corollary 3.1, we may construct x such that x has totally minimal, totally uniquely ergodic, topologically mixing orbital closure, and with the property that for every $u \in \{0,1\}^{\mathbb{N}}$, there exists $x_u \in X$ with $x_u[i] = u[i]$ for all $i \in C'$. For every $v \in \{0,1\}^{\mathbb{N}}$, define some $u_v \in \{0,1\}^{\mathbb{N}}$ by $u_v[1] = 0$, $u_v[a] = 1$ for all $a \in (C+1)$, and $u_v[b_k] = v[k]$ for all $k \in \mathbb{N}$. Then, for any such v, $x_{u_v}[i] = 1$ for all $i \in (C+1)$, $x_{u_v}[1] = 0$, and $x_{u_v}[b_k] = v[k]$ for all $k \in \mathbb{N}$. Since for all $n \in \mathbb{N}$, $p_n + 1 \in C + 1$, $(T^{p_n} x_{u_v})[1] = x_{u_v}[p_n + 1] = 1$, whereas $x_{u_v}[1] = 0$. It is then clear that x_{u_v} is not a limit point of $\{T^{p_n} x_{u_v}\}_{n\in\mathbb{N}}$. Since $x_{u_v}[b_k] = v[k]$ for all $k \in \mathbb{N}$, $x_{u_v} \neq x_{u_{v'}}$ for $v \neq v'$, and so the set $\{x_{u_v}\}_{v\in\{0,1\}^{\mathbb{N}}}$ is uncountable. □

It is natural to wonder about one aspect of the proof: why is it that in our construction, we only force certain digits to occur along shifted subsets of A, rather than along entire shifted copies of A? The reason comes from a combinatorial fact that is somewhat interesting in its own right.

Example. There exists a set $D \subseteq \mathbb{N}$ with $d^*(D) = 0$ with the property that for any infinite set G of integers, the set $D + G = \{d + g : d \in D,\ g \in G\}$ has upper Banach density one.

Proof. We begin with the sequence $d_n = 3^{|n|_2}$, where $|n|_2$ is the maximal integer k such that $2^k | n$. So, $\{d_n\}$ begins $1, 3, 1, 9, 1, 3, 1, 27, 1, 3, 1, 9, 1, 3, 1, \dots$. Then, define $c_n = \sum_{i=1}^{n} d_n$. This $\{c_n\}$ is then an increasing sequence of integers, with the property that the nth gap $c_{n+1} - c_n$ is d_{n+1} for all n. We first claim that $d^*(\{c_n\}) = 0$. Choose any positive integer k, and any $2^k + 1$ consecutive elements $c_i, \dots, c_{i+2^k}$ of the sequence $\{c_n\}$. There must be some integer $j \in [0, 2^k - 1]$ such that $2^k | i + j$. This means that $3^k | d_{i+j}$, and so that $c_{i+2^k} - c_i = \sum_{m=i}^{i+2^k-1} d_m > d_{i+j} \geq 3^k$. This means that any interval of integers of length less than 3^k can contain at most 2^k elements of $\{c_n\}$ for any k, which implies that $d^*(\{c_n\}) = 0$ since $\lim_{k\to\infty}(2^{k+1}/3^k) = 0$. Therefore, if we construct a new sequence by increasing the gaps $\{d_n\}$, it will still have upper Banach density zero. We will change $\{d_n\}$ countably many times, never decreasing any element. In other words, we inductively construct, for every $k \in \mathbb{N}$, a sequence $\{d_n^{(k)}\}$, so that these sequences are non-decreasing in k, i.e. $d_n^{(k)} \geq d_n^{(k-1)}$ for all n and k. We add the additional hypothesis that for every k, $d(\{n : d_n^{(k)} > d_n\}) = 0$, in other words that only a density-zero subset of the elements of $\{d_n\}$ have been changed after any step.

Step 1. We change d_n for some infinite, but zero-density, set of n, so that for every positive integer m, there exists n such that $d_n = m$. This is clearly possible: for example, by increasing d_n for a zero-density set of odd n. Call the resulting sequence $\{d_n^{(1)}\}$.

Step k ($k > 1$). Assume that we have already defined $\{d_n^{(k-1)}\}$, a sequence of integers with the property that $d_n^{(k-1)} \geq d_n$ for all n, and that $d(\{n : d_n^{(k-1)} > d_n\}) = 0$. Define the set $R_k \subset \mathbb{N}^k$ by $R_k = \{(a_1, \dots, a_k) : a_i \in \mathbb{N},\ a_i > 3^k\}$. Since $\{n : d_n^{(k-1)} > d_n\}$ has density zero, there exist infinitely many intervals of integers I_j such that $|I_j| > 2^{k+1}$ for

https://doi.org/10.1017/S0143385707000508 Published online by Cambridge University Press

all j, and such that $d_n^{(k-1)} = d_n$ for all $n \in I_j$ for any j. Therefore, by the construction of the sequence $\{d_n\}$, each I_j contains a subinterval I_j' of integers of length 2^k such that for every j, and for all $n \in I_j'$, $d_n^{(k-1)} \leq 3^k$. We may also assume, by passing to a subset if necessary, that the union of all I_j' has density zero. We now take any bijection ϕ from $\mathbb{N}$ to R_{2^k}, and for every j, if $I_j' = \{s, s+1, \ldots, s+2^k-1\}$, and if $\phi(j) = (a_1, \ldots, a_{2^k}) \in R_{2^k}$, define $d_s^{(k)} = a_1$, $d_{s+1}^{(k)} = a_2, \ldots, d_{s+2^k-1}^{(k)} = a_{2^k}$. After making these changes on each I_j', for any m not in any I_j', define $d_m^{(k)} = d_m^{(k-1)}$. This defines $d_n^{(k)}$ for all n. Since for every n, $d_n^{(k)} \geq d_n^{(k-1)}$, by the inductive hypothesis we see that $d_n^{(k)} \geq d_n$ for all n. Also, since $d(\{n : d_n^{(k)} > d_n^{(k-1)}\}) = 0$, and since by the inductive hypothesis, $d(\{n : d_n^{(k-1)} > d_n\}) = 0$, we see that $d(\{n : d_n^{(k)} > d_n\}) = 0$, completing the inductive step.

It is a consequence of this construction that if we define $H_k = \{n : d_n^{(k)} > d_n^{(k-1)}\}$ for every k, the sets H_k are pairwise disjoint. Therefore, the sequences $\{d_n^{(k)}\}$ have a pointwise limit, call it $\{e_n\}$. By the construction, for any k, and for any k-tuple $(a_1, \ldots, a_k)$ of integers all greater than 3^k, there exists m such that $d_{m+i-1}^{(k)} = a_i$ for $1 \leq i \leq k$. And since the H_k are disjoint, this means that $e_{m+i-1} = a_i$ for $1 \leq i \leq k$. Now, define the sequence $f_n = \sum_{k=1}^{n} e_n$ for all n. As already noted, since $e_n \geq d_n$ for all n, $D = \{f_n\}$ has upper Banach density zero. We claim that for any infinite set G of integers, $D + G$ contains arbitrarily long intervals of integers. Fix any infinite $G = \{g_n\}$, with $g_1 < g_2 < \cdots$. For any k, there exist $m_1, m_2, \ldots, m_{2^k}, m_{2^k+1}$ such that $g_{m_{j+1}} - g_{m_j} > 3^k$ for every $1 \leq j \leq 2^k$. Then, by construction, there exists m such that $e_{m+j} = g_{m_{2^k-j+2}} - g_{m_{2^k-j+1}} + 1$ for $1 \leq j \leq 2^k$. This means that

$$\begin{aligned} f_{m+i} &= f_m + \sum_{j=1}^{i} e_{m+j} \\ &= f_m + \sum_{j=1}^{i} (g_{m_{2^k-j+2}} - g_{m_{2^k-j+1}} + 1) = (f_m + g_{m_{2^k+1}}) - g_{m_{2^k+1-i}} + i \end{aligned}$$

for $1 \leq i \leq 2^k$. But then for each such i, $f_{m+i} + g_{m_{2^k+1-i}} = f_m + g_{m_{2^k+1}} + i \in D + G$. This implies that $\{f_m + g_{m_{2^k+1}} + 1, \ldots, f_m + g_{m_{2^k+1}} + 2^k\}$ is an interval of integers of length 2^k that is a subset of $D + G$. Since k was arbitrary, $D + G$ contains arbitrarily long intervals and so has upper Banach density one. □

This answers our question: if we had, in the proof of Theorem 1.2, tried to force ones to occur along infinitely many shifted copies of our set A of upper Banach density zero, it's possible that no matter what set G of shifts we used, we would be attempting to force $x[i] = 1$ for arbitrarily long intervals of integers i, which would imply that $1^\infty \in X$, yielding the closed invariant set $\{1^\infty\} \subsetneq X$ and contradicting the minimality of X.

4. *Some general constructions on connected manifolds*

We will now construct some totally minimal, totally uniquely ergodic, and topologically mixing transformations which act on a connected manifold, and in §5 we will use such examples to prove Theorems 1.3 and 1.6. The constructions in question will use both skew

https://doi.org/10.1017/S0143385707000508 Published online by Cambridge University Press

products and flows under functions. We will be repeatedly using the topological space $\mathbb{T}$, which can most easily be considered as the half-open interval $[0, 1)$ with 0 and 1 identified and the operation of addition (mod 1). (Whenever we refer to the addition of elements of $\mathbb{T}$, it should be understood to be addition (mod 1).) We also note that $\mathbb{T}^n$ is a metric space for any n, with metric d defined by $d(x, y) = \min_{u,v\in\mathbb{Z}^n} \widetilde{d}(x + u, y + v)$, where $\widetilde{d}$ is the Euclidean metric in $\mathbb{R}^n$. We will denote by λ_k Lebesgue measure on $\mathbb{T}^k$ for any $k > 0$.

For any $k > 1$, irrational $\alpha \in \mathbb{T}$ and continuous self-map f of $\mathbb{T}$, we define the transformation $S = S_{k,\alpha,f}$ on $\mathbb{T}^k$ as follows: for any $v = (v_1, \ldots, v_k) \in \mathbb{T}^k$, $(Sv)_1 = v_1 + \alpha$, $(Sv)_2 = f(v_1) + v_2$, and for every $j > 2$, $(Sv)_j = v_{j-1} + v_j$.

THEOREM 4.1. *For any f differentiable with $1/2 < f'(x) < 3/2$ for all $x \in \mathbb{T}$, $S_{k,\alpha,f}$ is totally minimal and totally uniquely ergodic with respect to λ_k.*

Proof. During the proof, since k, α, and f are taken to be fixed, we suppress notational dependence and refer to $S_{k,\alpha,f}$ simply as S. Fix any rectangles $R = \prod_{i=1}^k R_i$ and $R' = \prod_{i=1}^k R'_i$ in $\mathbb{T}^k$, where R_i and R'_i are intervals of length C for every $1 \le i \le k$. Now, fix any positive integer ℓ such that $\lambda_1((R_1 + \ell\alpha) \cap R'_1) > C/2$. Fix any $r_2 \in R_2, \ldots, r_k \in R_k$, and define the set

$$E_1 = \{x_1 \in R_1 : \pi_1(S^\ell(x_1, r_2, \ldots, r_k)) \in R'_1\}.$$

(Here and elsewhere, $\pi_i : \mathbb{T}^k \to \mathbb{T}$ is the usual projection map $v \mapsto v_i$ for $1 \le i \le k$.) Since $\pi_1(S^\ell(x_1, r_2, \ldots, r_k)) = x_1 + \ell\alpha$, by choice of ℓ we know that E_1 is an interval with $\lambda_1(E_1) > C/2$. Now, define the set

$$E_2 = \{x_1 \in R_1 : \pi_i(S^\ell(x_1, r_2, \ldots, r_k)) \in R'_i,\ 1 \le i \le 2\}.$$

We will examine the structure of E_2 by bounding $\partial\pi_2(S^\ell(x_1, r_2, \ldots, r_k))/\partial x_1$ from above and below. For this, we note that $\pi_2(S^\ell(x_1, r_2, \ldots, r_k)) = r_2 + \sum_{i=0}^{\ell-1} f(x_1 + i\alpha)$, and make the observation that for any i, $f(x_1 + i\alpha)$ is equal modulo one to an increasing function in x_1 whose slope is between $1/2$ and $3/2$. Therefore, considered as a function of x_1, $\pi_2(S^\ell(x_1, r_2, \ldots, r_k))$ is equal modulo one to an increasing function from $[0, 1)$ to $\mathbb{R}$ whose derivative is in $(\ell/2, 3\ell/2)$ for all x_1. This implies that E_2 is a union of many intervals separated by gaps of length less than $1/(\ell/2)$, where the length of all intervals but the first and last is greater than $C/(3\ell/2)$. For large ℓ, this implies that E_2 contains some set F_2, a union of intervals of length D_2C/ℓ for some constant D_2, where $\lambda_1(F_2) > B_2C^2$ for some constant B_2.

We proceed inductively: for any $2 \le i < k$, assume that we are given F_i, a union of intervals whose lengths are $D_iC\ell^{-(i-1)}$ for some constant D_i, where $\lambda_1(F_i) > B_iC^i$ for some constant B_i, and such that for any $x_1 \in F_i$, $\pi_j(S^\ell(x_1, r_2, \ldots, r_k)) \in R'_j$ for $1 \le j \le i$. We now wish to define F_{i+1}. Define the set

$$E_{i+1} = \{x_1 \in F_i : \pi_{i+1}(S^\ell(x_1, r_2, \ldots, r_k)) \in R'_{i+1}\}.$$

For any interval I of length $D_iC\ell^{-(i-1)}$ in F_i, let's examine $I \cap E_{i+1}$. We note that

$$\pi_{i+1}(S^\ell(x_1, r_2, \ldots, r_k)) = \eta_{i+1,\ell}(r_2, \ldots, r_{i+1}) + \sum_{j=0}^{\ell-i} \binom{\ell - j - 1}{i - 1} f(x_1 + j\alpha),$$

https://doi.org/10.1017/S0143385707000508 Published online by Cambridge University Press

where $\eta_{i+1,\ell}$ is some function of $r_2, \ldots, r_{i+1}$ that does not depend on x_1. So, as a function of x_1, $\pi_{i+1}(S^\ell(x_1, r_2, \ldots, r_k))$ is equal modulo one to an increasing function from $[0, 1)$ to $\mathbb{R}$ whose derivative is between $A\ell^i$ and $B\ell^i$ for some constants $A, B > 0$. This implies that for large ℓ, $I \cap E_{i+1}$ is a union of many intervals separated by gaps of length less than $(1/B)\ell^{-i}$, where the length of all but the first and last is greater than $(C/A)\ell^{-i}$. For large ℓ, this implies that $I \cap E_{i+1}$ contains $F_{I,i+1}$, a union of intervals of length $DC\ell^{-i}$ where $\lambda_1(F_{I,i+1}) > EC\lambda_1(I)$ for some constants D, E. By taking F_{i+1} to be the union of all $F_{I,i+1}$, we see that F_{i+1} is a union of intervals of length $D_{i+1}C\ell^{-i}$, where $\lambda_1(F_{i+1}) > B_{i+1}C^{i+1}$ for some constants D_{i+1} and B_{i+1}. Since $F_{i+1} \subset E_{i+1}$, for any $x_1 \in F_{i+1}$, $\pi_j(S^\ell(x_1, r_2, \ldots, r_k)) \in R'_j$ for $1 \le j \le i+1$.

By inductively proceeding in this way, we will eventually arrive at a set F_k where $\lambda_1(F_k) > B_kC^k$ for some constant B_k, and where $S^\ell(x_1, r_2, \ldots, r_k) \in R'$ for every $x_1 \in F_k$. By integrating over all possible $r_2, \ldots, r_k$, we see that $\lambda_k(S^\ell R \cap R') > B_kC^{2k-1}$. We have then shown that for any ℓ with $\lambda_1((R_1 + k\alpha) \cap R'_1) > C/2$, $\lambda_k(S^\ell R \cap R') > B_kC^{2k-1}$. Denote by L the set of such ℓ. Then if we define R_α to be the transformation $x \mapsto x + \alpha$ on $\mathbb{T}$, then $L = \{n : R_\alpha^n(0) \in J\}$, where $J = \{x \in \mathbb{T} : \lambda_1((R + x) \cap R') > C/2\}$. It is easily checked that $\lambda_1(J) = C$. This means that for any $M, N \in \mathbb{N}$,

$$\frac{1}{N}|L \cap \{0, M, \ldots, M(N-1)\}| = \frac{1}{N}\sum_{i=0}^{N-1} \chi_J((R_\alpha^M)^i 0),$$

which approaches $\lambda_1(J) = C$ as $N \to \infty$ by total unique ergodicity of R_α with respect to λ_1. Then, for large N,

$$\frac{1}{N}\sum_{\ell=0}^{N-1} \lambda_k(S^{M\ell}R \cap R') \ge \frac{1}{N}|L \cap \{0, M, \ldots, M(N-1)\}|B_kC^{2k-1} \to B_kC^{2k}.$$

Therefore,

$$\liminf_{N\to\infty} \frac{1}{N}\sum_{\ell=0}^{N-1} \lambda_k(S^{M\ell}R \cap R') \ge B_k\lambda_k(R)\lambda_k(R'). \tag{1}$$

We have shown that equation (1) holds for R and R' arbitrary congruent cubes in $\mathbb{T}^k$. It is clear that it also holds for R and R' disjoint unions of congruent cubes. Suppose that for some $M \in \mathbb{N}$, there exists a Lebesgue-measurable S^M-invariant set $A \subseteq \mathbb{T}^k$ with $\lambda_k(A) \in (0, 1)$. By taking complements if necessary, without loss of generality we may assume that $\lambda_k(A) \le 1/2$. By regularity of Lebesgue measure, there exists ϵ and A', a union of cubes with side length ϵ, such that $(A')^c$ is also a union of cubes of side length ϵ, $\lambda_k(A) \le \lambda_k(A') \le 1/2$, and $\lambda_k(A \triangle A') < (B_k/2)\lambda_k(A)^2$. Then, for any $\ell \in \mathbb{N}$, since $S^{M\ell}A = A$ and $\lambda_k(A \triangle A') < (B_k/2)\lambda_k(A)^2$, $\lambda_k(A \triangle S^{M\ell}A') < (B_k/2)\lambda_k(A)^2$ as well. Similarly, $\lambda_k(A^c \triangle S^{M\ell}(A')^c) < (B_k/2)\lambda_k(A)^2$. Therefore, $\lambda_k(S^{M\ell}A' \cap S^{M\ell}(A')^c) < B_k\lambda_k(A)^2$ for all $\ell \in \mathbb{N}$. Since $\lambda_k(A) \le \lambda_k(A') \le \lambda_k((A')^c)$, this contradicts equation (1).

So, S^M is ergodic with respect to λ_k for every $M > 0$. We claim that this also implies unique ergodicity of S^M for every $M > 0$, which follows from an argument of Furstenberg, and rests on the fact that S^M is a skew product over an irrational circle rotation. The following fact is shown in [**6**, Proof of Lemma 2.1 on p. 578].

https://doi.org/10.1017/S0143385707000508 Published online by Cambridge University Press

Fact. For any minimal system (X_0, T_0) that is uniquely ergodic with respect to a measure μ_0, and any skew product T that acts on $X = X_0 \times \mathbb{T}$ by $T(x_0, y) = (T_0 x_0, y + h(x_0))$ where $h : X_0 \to \mathbb{T}$ is a continuous function, if T is ergodic with respect to $\mu_0 \times m$, then T is minimal and uniquely ergodic with respect to $\mu_0 \times m$.

Denote by $(S^M)^{(i)}$ the action of S on its first i coordinates for any $1 \le i \le k$. Since they are factors of S^M, each $(S^M)^{(i)}$ is ergodic with respect to λ_i. Also, for each $1 \le i < k$, $(S^M)^{(i+1)}$ is a skew product as described above with $T_0 = (S^M)^{(i)}$. We may then use Furstenberg's result and the fact that $(S^M)^{(1)}$ is minimal and uniquely ergodic with respect to λ_1 (since it is an irrational circle rotation) to see that $(S^M)^{(2)}$ is minimal and uniquely ergodic with respect to λ_2. We can continue inductively in this fashion to arrive at the fact that S^M is minimal and uniquely ergodic with respect to λ_k. Since M was arbitrary, S is totally minimal and totally uniquely ergodic with respect to λ_k. □

We will now use these skew products to define flows under functions that have all of the previous properties and are also topologically mixing. Define the continuous function $g : \mathbb{T}^2 \to \mathbb{R}$ by

$$g(x, y) = 2 + \mathrm{Re}\left(\sum_{k=2}^{\infty} \frac{e^{2\pi i k x}}{e^k} + \sum_{l=2}^{\infty} \frac{e^{2\pi i l y}}{e^l}\right).$$

Note that $1 < g(x, y) < 3$ for all x, y. We then define the space $X = \{(v, x, y, t) : v \in \mathbb{T}^k, x, y \in \mathbb{T}, 0 \le t \le g(x, y)\}$ where $(v, x, y, g(x, y))$ and $(S_{k,\alpha,f} v, x + \gamma, y + \gamma', 0)$ are identified for all v, x, y. X is then homeomorphic to the mapping torus of $\mathbb{T}^{k+2}$ and a continuous map, and so is a connected $(k + 3)$-manifold. For any irrational $\gamma, \gamma' \in \mathbb{T}$, we then define the continuous map $T_{k+3,\alpha,\gamma,\gamma',f} : X \to X$ by

$$T_{k+3,\alpha,\gamma,\gamma',f}(v, x, y, t) = \begin{cases} (v, x, y, t + 1) & \text{if } t + 1 < g(x, y), \\ (S_{k,\alpha,f} v, x + \gamma, y + \gamma', t + 1 - g(x, y)) & \text{if } t + 1 \ge g(x, y). \end{cases}$$

Finally, we define $\mu = (\int_{\mathbb{T}^{k+2}} g \, d\lambda_{k+2})^{-1} \lambda_{k+3} = (1/2)\lambda_{k+3}$, a $T_{k+3,\alpha,\gamma,\gamma',f}$-invariant Borel probability measure on X. We will prove the following theorem.

THEOREM 4.2. *For any f differentiable with $1/2 < f'(x) < 3/2$ for all $x \in \mathbb{T}$ and any irrational $\alpha, \gamma, \gamma' \in \mathbb{T}$ that are linearly independent and satisfy $q'_n > e^{3q_n}$ and $q_{n+1} > e^{3q'_n}$, where $\{q_n\}$ and $\{q'_n\}$ are the digits in the continued-fraction expansions of γ and γ' respectively, $T_{k+3,\alpha,\gamma,\gamma',f}$ is totally minimal, totally uniquely ergodic with respect to μ, and topologically mixing.*

Again, since α, γ, γ', and f are fixed, for now we will suppress the dependence on these quantities in notation and denote the transformations $T_{k+3,\alpha,\gamma,\gamma',f}$ and $S_{k,\alpha,f}$ by T and S respectively. We also use the notation $g_\ell(x, y) = \sum_{i=0}^{\ell-1} g(x + i\gamma, y + i\gamma')$ for any integer $\ell > 0$, and define $g_0(x, y) = 0$. The proof of Theorem 4.2 rests mostly on the following lemma, which is essentially taken from [**5**].

https://doi.org/10.1017/S0143385707000508 Published online by Cambridge University Press

LEMMA 4.1. *For any sufficiently large integer $n > 0$, $y \in \mathbb{T}$, $\ell \in [(1/2)e^{2q_n}, 2e^{2q'_n}]$, and any $x_0 \in \mathbb{T}$ with $q_n x_0 \in [1/6, 1/3]$,*

$$\frac{\ell q_n}{e^{q_n}} < \left|\frac{\partial g_\ell(x, y)}{\partial x}(x_0)\right| < \frac{7\ell q_n}{e^{q_n}}.$$

Proof. We have

$$g_\ell(x, y) = 2\ell + \mathrm{Re}\left(\sum_{l=2}^{\infty} \frac{X(\ell, l)}{e^l} e^{2\pi i l x} + \sum_{m=2}^{\infty} \frac{Y(\ell, m)}{e^m} e^{2\pi i m y}\right),$$

where

$$X(\ell, l) = \frac{1 - e^{2\pi i \ell l \gamma}}{1 - e^{2\pi i l \gamma}},$$
$$Y(\ell, m) = \frac{1 - e^{2\pi i \ell m \gamma'}}{1 - e^{2\pi i m \gamma'}}.$$

The following facts are proved in [**5**, p. 454]:

$$\text{For all } l \in \mathbb{N} \setminus \{0\},\ \ell \in \mathbb{N},\ |X(\ell, l)| \le \ell. \tag{2}$$
$$\text{For all } n \in \mathbb{N},\ l < q_n,\ \ell \in \mathbb{N},\ |X(\ell, l)| \le q_n. \tag{3}$$
$$\text{For all } n \in \mathbb{N},\ l \in (q_n, 2q_n),\ \ell \in \mathbb{N},\ |X(\ell, l)| \le 2q_n. \tag{4}$$
$$\text{For any } \ell \le \frac{q_{n+1}}{2},\ |X(\ell, q_n)| \ge \frac{2\ell}{\pi}. \tag{5}$$
$$\text{For any } \ell \le \frac{q_{n+1}}{2},\ |\arg(X(\ell, q_n))| \le \pi \frac{\ell - 1}{q_{n+1}}. \tag{6}$$

We will use these to prove our lemma. It is easy to check that

$$\begin{aligned}
\frac{\partial g_\ell(x, y)}{\partial x}(x_0) &= \mathrm{Re}\left(\sum_{l=2}^{\infty} 2\pi i l \frac{X(\ell, l)}{e^l} e^{2\pi i l x_0}\right) \\
&= \mathrm{Re}\left(2\pi i q_n \frac{|X(\ell, q_n)|}{e^{q_n}} e^{2\pi i q_n x_0}\right) + \mathrm{Re}\left(\sum_{l=2}^{q_n - 1} 2\pi i l \frac{X(\ell, l)}{e^l} e^{2\pi i l x_0}\right) \\
&\quad + \mathrm{Re}\left(\sum_{l=q_n+1}^{2q_n - 1} 2\pi i l \frac{X(\ell, l)}{e^l} e^{2\pi i l x_0}\right) + \mathrm{Re}\left(\sum_{l=2q_n}^{\infty} 2\pi i l \frac{X(\ell, l)}{e^l} e^{2\pi i l x_0}\right) \\
&\quad + \mathrm{Re}\left(2\pi i q_n \frac{X(\ell, q_n) - |X(\ell, q_n)|}{e^{q_n}} e^{2\pi i q_n x_0}\right).
\end{aligned}$$

We bound the first term from above and below and the rest from above. We have

$$\left|\mathrm{Re}\left(2\pi i q_n \frac{|X(\ell, q_n)|}{e^{q_n}} e^{2\pi i q_n x_0}\right)\right| = \frac{|X(\ell, q_n)|}{e^{q_n}} 2\pi q_n |\sin(2\pi q_n x_0)|,$$

and since $\ell \in [(1/2)e^{2q_n}, 2e^{2q'_n}]$, $\ell \le q_{n+1}/2$. By (2) and (5), $(2/\pi)\ell \le |X(\ell, q_n)| \le \ell$. Since $q_n x_0 \in [1/6, 1/3]$, $1/2 \le |\sin(2\pi q_n x_0)| \le 1$. Therefore,

$$\frac{2\ell q_n}{e^{q_n}} \le \left|\mathrm{Re}\left(2\pi i q_n \frac{|X(\ell, q_n)|}{e^{q_n}} e^{2\pi i q_n x_0}\right)\right| \le \frac{2\pi \ell q_n}{e^{q_n}}.$$

https://doi.org/10.1017/S0143385707000508 Published online by Cambridge University Press

Next, by (3),

$$\left|\mathrm{Re}\left(\sum_{l=2}^{q_n-1} 2\pi i l \frac{X(\ell, l)}{e^l} e^{2\pi i l x_0}\right)\right| \le 2\pi \sum_{l=2}^{q_n-1} l \frac{|X(\ell, l)|}{e^l} \le 2\pi \sum_{l=2}^{q_n-1} \frac{q_n l}{e^l} \le 2\pi q_n^2.$$

We similarly have

$$\left|\mathrm{Re}\left(\sum_{l=q_n+1}^{2q_n-1} 2\pi i l \frac{X(\ell, l)}{e^l} e^{2\pi i l x_0}\right)\right| \le 4\pi q_n^2.$$

Also, from (2), we can conclude that for large n

$$\left|\mathrm{Re}\left(\sum_{l=2q_n}^{\infty} 2\pi i l \frac{X(\ell, l)}{e^l} e^{2\pi i l x_0}\right)\right| \le 2\pi \sum_{l=2q_n}^{\infty} l \frac{|X(\ell, l)|}{e^\ell} \le 2\pi \ell \sum_{l=2q_n}^{\infty} \frac{l}{e^l} \le \frac{2\pi \ell}{e^{1.5q_n}}.$$

Finally, from (6) and (2), we see that

$$|X(\ell, q_n) - |X(\ell, q_n)|| \le 2|X(\ell, q_n)| \arg(X(\ell, q_n)) \le 2\pi \frac{\ell - 1}{q_{n+1}} \ell,$$

and so since $\ell \le 2e^{2q_n'}$ and $q_{n+1} \ge e^{3q_n'}$, this implies that

$$\left|\mathrm{Re}\left(2\pi i q_n \frac{X(\ell, q_n) - |X(\ell, q_n)|}{e^{q_n}} e^{2\pi i q_n x_0}\right)\right| \le 8\pi^2 e^{-q_n'} q_n \frac{\ell}{e^{q_n}}.$$

By combining all of these bounds,

$$\frac{2\ell q_n}{e^{q_n}} - 2\pi q_n^2 - 4\pi q_n^2 - \frac{2\pi \ell}{e^{1.5q_n}} - 8\pi^2 e^{-q_n'} \frac{k\ell q_n}{e^{q_n}} \le \left|\frac{\partial g_\ell(x, y)}{\partial x}(x_0)\right| \le \frac{2\pi \ell q_n}{e^{q_n}} + 2\pi q_n^2 + 4\pi q_n^2 + \frac{2\pi \ell}{e^{1.5q_n}} + 8\pi^2 e^{-q_n'} \frac{\ell q_n}{e^{q_n}},$$

and since $\ell \ge (1/2)e^{2q_n}$, for large n we have

$$\frac{\ell q_n}{e^{q_n}} < \left|\frac{\partial g_\ell(x, y)}{\partial x}(x_0)\right| < \frac{7\ell q_n}{e^{q_n}}. \qquad \square$$

The proof of the following fact is trivially similar.

LEMMA 4.2. *For any sufficiently large integer $n > 0$, $x \in \mathbb{T}$, $\ell \in [(1/2)e^{2q_n'}, 2e^{2q_{n+1}}]$, and any $y_0 \in \mathbb{T}$ with $q_n' y_0 \in [1/6, 1/3]$,*

$$\frac{\ell q_n'}{e^{q_n'}} < \left|\frac{\partial g_\ell(x, y)}{\partial y}(y_0)\right| < \frac{7\ell q_n'}{e^{q_n'}}.$$

Proof of Theorem 4.2. Consider any $m \in [(5/3)e^{2q_n}, 2e^{2q_n'}]$, and any cubes $R = \prod_{i=1}^{k+3} R_i$ and $R' = \prod_{i=1}^{k+3} R_i'$ where R_i and R_i' are intervals of length C for $1 \le i \le k+3$. Take intervals Q_{k+1} and Q_{k+2} of length $C/2$ central in R_{k+1} and R_{k+2} respectively. Define $\widetilde{R} = \prod_{i=1}^{k} R_i$. We also make the following definition: $\ell(m, x, y, t)$ is the integer ℓ such that $g_\ell(x, y) \le m + t < g_{\ell+1}(x, y)$. Alternately, for any $v \in \mathbb{T}^k$,

$$T^m(v, x, y, t) = (S^{\ell(m,x,y,t)}v, x + \ell(m, x, y, t)\gamma, y + \ell(m, x, y, t)\gamma', m + t - g_{\ell(m,x,y,t)}(x, y)).$$

https://doi.org/10.1017/S0143385707000508 Published online by Cambridge University Press

Now, fix any $y \in Q_{k+2}$ and $t \in Q_{k+3}$. We first define the set

$$E_1' = \left\{ x_1 \in Q_{k+1} : q_n x_1 \in \left[\frac{1}{6}, \frac{1}{3} \right] \right\}.$$

For large m, $\lambda_1(E_1') > D_1 C$ for some constant $D_1 > 0$, and E_1' is a union of many intervals, where all but the first and last have length $1/6q_n$. By removing those, we can define E_1'', a union of intervals of length $1/6q_n$ where $\lambda_1(E_1'') > D_2 C$ for some constant $D_2 > 0$. We then define the set

$$E_2 = \left\{ x_1 \in E_1'' : \lambda_1(R_1 + \ell(m, x_1, y, t)\alpha \cap R_1') > \frac{C}{2}, \right.$$
$$\left. Q_{k+1} + \ell(m, x_1, y, t)\gamma \subset R_{k+1}', Q_{k+2} + \ell(m, x_1, y, t)\gamma' \subset R_{k+2}' \right\}.$$

We wish to use Lemma 4.1 to analyze the structure of E_2. First we note that since $1 < g(x, y) < 3$ for all x, y, by definition of $\ell(m, x, y, t)$, $\ell(m, x, y, t) \le m + t < 3(\ell(m, x, y, t) + 1)$, and so for large enough n, $\ell(m, x, y, t) \le m \le (10/3)\ell(m, x, y, t)$, or $\ell(m, x, y, t) \in [0.3m, m]$. By our assumption on m, this means that for any x, y, t, $\ell(m, x, y, t) \in [(1/2)e^{2q_n}, 2e^{2q_n'}]$. Now, fix any interval I of length $1/6q_n$ in E_1'', and let us examine $E_2 \cap I$. By the preceding remarks and Lemma 4.1,

$$\frac{\ell(m, x_1, y, t)q_n}{e^{q_n}} < \left| \frac{\partial \pi_{k+3} T^m(v, x, y, t)}{\partial x}(x_1) \right| < \frac{7\ell(m, x_1, y, t)q_n}{e^{q_n}}$$

for every $x_1 \in I$. Since $\ell(m, x, y, t) \in [0.3m, m]$,

$$\frac{0.3mq_n}{e^{q_n}} < \left| \frac{\partial \pi_{k+3} T^m(v, x, y, t)}{\partial x}(x_1) \right| < \frac{7mq_n}{e^{q_n}} \tag{7}$$

for every $x_1 \in I$. This means that $\partial \pi_{k+3} T^m(v, x, y, t)/\partial x$ has the same sign for all $x_1 \in I$, and without loss of generality we assume it to be positive. Let us define L to be the set of possible values for $\ell(m, x_1, y, t)$ for $x_1 \in I$. (Since g_ℓ is continuous for every ℓ, L is an interval of integers.) Then for any fixed $\ell \in L$, define $I_\ell = \{x_1 \in I : \ell(m, x_1, y, t) = \ell\}$ $= \{x_1 \in I : g_\ell(x_1, y) \le m + t < g_{\ell+1}(x_1, y)\} = \{x_1 \in I : g_\ell(x_1, y) \in [m + t - g(x_1 + \ell\gamma, y + \ell\gamma'), m + t]\}$. Since $1 < g(x_1 + \ell\gamma, y + \ell\gamma') < 3$, by (7) we have

$$\frac{1}{(7mq_n/e^{q_n})} < m(I_\ell) < \frac{3}{(0.3mq_n/e^{q_n})} \quad \text{or} \quad m(I_\ell) \in \left(\frac{1}{7}\frac{e^{q_n}}{mq_n}, 10\frac{e^{q_n}}{mq_n} \right)$$

for all $\ell \in L$ except the smallest and largest, for which $m(I_\ell)$ could be smaller.

Since $m > e^{2q_n}$, this means that the number of elements in L approaches infinity as m does. Now, we note that $E_2 \cap I = \bigcup_{\ell \in L'} I_\ell$, where L' is the set of $\ell \in L$ where $\lambda_1((R_1 + \ell\alpha) \cap R_1') > C/2$, $Q_{k+1} + \ell\gamma \subset R_{k+1}'$, and $Q_{k+2} + \ell\gamma' \subseteq R_{k+2}'$. Then

$$\frac{|L'|}{|L|} = \frac{1}{|L|} \sum_{\ell \in L} \chi_J(R^\ell_{\alpha,\gamma,\gamma'}(\mathbf{0})),$$

where $R_{\alpha,\gamma,\gamma'}$ is the rotation on $\mathbb{T}^3$ given by $(a, b, c) \mapsto (a + \alpha, b + \gamma, c + \gamma')$ and $J = \{(a, b, c) \in \mathbb{T}^3 : \lambda_1((R_1 + a) \cap R_1') > C/2, Q_{k+1} + b \subset R_{k+1}', Q_{k+2} + c \subset R_{k+2}'\}$.

https://doi.org/10.1017/S0143385707000508 Published online by Cambridge University Press

Since α, γ, and γ' are rationally independent, $R_{\alpha,\gamma,\gamma'}$ is uniquely ergodic, and so as $m \to \infty$, $|L'|/|L| \to \lambda_3(J) = C^3/4$.

Because of the already established bounds on $\lambda_1(I_\ell)$ for $\ell \in L$, this implies that there is a constant $D_3 > 0$ such that $\lambda_1(E_2 \cap I) > D_3 C^3 \lambda_1(I)$ for every interval I in E_1''. By removing the possibly shorter first and last subintervals of $E_2 \cap I$, we have $F_I \subseteq E_2 \cap I$ a union of intervals of length greater than $(1/7)$ (e^{q_n}/mq_n) with $\lambda_1(F_I) > D_4 C^3 \lambda_1(I)$ for some constant $D_4 > 0$. We take F_2 to be the union of all F_I, and then $\lambda_1(F_2) > D_5 C^4$ for some constant $D_5 > 0$. Finally, we define

$$E_3 = \{x_1 \in F_2 : \pi_{k+3}(T^m(v, x_1, y, t)) \in R'_{k+3} \ (\forall v \in \mathbb{T}^k)\}.$$

Note that F_2 is a union of intervals I_ℓ, and fix any such interval I_ℓ of length greater than $(1/7)$ (e^{q_n}/mq_n). By definition, $\pi_{k+3}(T^m(v, x_1, y, t)) = m + t - g_\ell(x_1, y)$ for any $x_1 \in I_\ell$, and $\pi_{k+3}(T^m(v, x_1, y, t))$ ranges monotonically from 0 to $g(x_1 + \ell\gamma, y + \ell\gamma')$ as x_1 increases over I_ℓ. Since, by Lemma 4.1,

$$\frac{\ell q_n}{e^{q_n}} < \left| \frac{\partial g_\ell(x, y)}{\partial x}(x_1) \right| < \frac{7\ell q_n}{e^{q_n}},$$

$\lambda_1(E_3 \cap I_\ell) > D_6 C \lambda_1(I_\ell)$ for some constant $D_6 > 0$. By taking the union over all I_ℓ, $\lambda_1(E_3) > D_7 C^5$ for some $D_7 > 0$.

Consider any $x_1 \in E_3$. We know that $\lambda_1((R_1 + \ell(m, x_1, y, t)\alpha) \cap R'_1) > C/2$, and by the proof of Theorem 4.1, this implies that if we define the set $A_{x_1} = \{v \in \widetilde{R} : \pi_i(T^m(v, x_1, y, t)) \in R'_i, \ 1 \le i \le k\}$, then $\lambda_k(A_{x_1}) > B_k C^{2k-1}$ for some constant $B_k > 0$. But this means that for any $v \in A_{x_1}$,

$$T^m(v, x_1, y, t) = (S^{\ell(m,x_1,y,t)}v, x_1 + \ell(m, x_1, y, t)\gamma, y + \ell(m, x_1, y, t)\gamma', \\ m + t - g_{\ell(m,x_1,y,t)}(x_1, y))$$

is in R', by the definitions of E_3 and A_{x_1}. So, there exists a constant $D_8 > 0$ such that $\lambda_{k+1}(\{(v, x_1) : T^m(v, x_1, y, t) \in R')\} > D_8 C^{2k+4}$. By integrating over all possible $y \in Q_{k+2}$ and $t \in Q_{k+3}$, we see that there is a constant $D_9 > 0$ such that $\mu((T^m R) \cap R') > (D_9/4)C^{2k+6} = D_9 \mu(R)\mu(R')$.

This argument works for any large enough $m \in [(5/3)e^{2q_n}, 2e^{2q'_n}]$ for some n. An analogous argument using Lemma 4.2, which involves varying y instead of x, shows that the same is true for any large enough $m \in [(5/3)e^{2q'_n}, 2e^{q_{n+1}}]$. However, this implies that for all sufficiently large m and any congruent cubes R and R', $\mu((T^m R) \cap R') > D_9 \mu(R)\mu(R')$, which implies that T is totally ergodic with respect to μ and topologically mixing. It remains to show that T is in fact totally uniquely ergodic.

Our proof is similar to the argument of Furstenberg used earlier. However, since this is about a flow under a function and his argument was about skew products, we will present the proof in its entirety here. Fix any $M \in \mathbb{N}$. Since T^M is ergodic with respect to μ, μ-almost every point of X is (T^M, μ)-generic. Since μ is shift-invariant, and since shifts in the last coordinate commute with T^M, if a point $(v, x, y, t) \in X$ is (T^M, μ)-generic, (v, x, y, t') is as well for all $0 \le t' < g(x, y)$. This implies that for λ_{k+2}-almost all (v, x, y), the fiber $\{(v, x, y, t)\}_{0 \le t < g(x,y)}$ consists of (T^M, μ)-generic points. Denote by G this set of (v, x, y) that give rise to (T^M, μ)-generic fibers. Choose any

https://doi.org/10.1017/S0143385707000508 Published online by Cambridge University Press

$(v, x, y, t) \in \mathbb{T}^{k+2}$. We know that for large n, $\ell(nM, x, y, t) \in [0.3nM, nM]$, and that $\ell(nM, x, y, t)$ is increasing in n. Therefore, the set $\{\ell(nM, x, y, t) : n \in \mathbb{N}\}$ has positive density. The skew product U on $\mathbb{T}^{k+2}$ defined by $U(v, x, y) = (S_{k,\alpha,f}v, x + \gamma, y + \gamma')$ is totally uniquely ergodic with respect to λ_{k+2} for the same reason as in the proof of Theorem 4.1. (The only difference is that here the base case is the rotation on $\mathbb{T}^3$ given by $(x, y, z) \mapsto (x + \alpha, y + \gamma, z + \gamma')$ instead of an irrational rotation on $\mathbb{T}$. However, since α, γ, and γ' are rationally independent, this rotation is also totally uniquely ergodic and totally minimal.) This implies that the set $\{\ell : U^{\ell}(v, x, y) \in G\}$ has density one. Together, these facts imply that there exists n such that $U^{\ell(nM,x,y,t)}(v, x, y) \in G$, or that $T^{nM}(v, x, y, t) = (g, s)$ for some $g \in G$ and $s \in \mathbb{R}$. This means that $T^{nM}(v, x, y, t)$ is (T^M, μ)-generic, and so (v, x, y, t) is as well. Since $(v, x, y, t) \in X$ was arbitrary, every point in X is (T^M, μ)-generic, and so T^M is uniquely ergodic with respect to μ. Since $\mu(U) > 0$ for every non-empty open set U, T^M is minimal as well. Since M was arbitrary, T is totally uniquely ergodic, totally minimal, and topologically mixing. □

5. *Some counterexamples on connected manifolds*

Proof of Theorem 1.6. Our transformation is $T_{2d+7,\alpha,\gamma,\gamma',f}$ for properly chosen α, γ, γ', and f. We always take α to be the golden ratio $(\sqrt{5} - 1)/2$ because of a classical fact from the theory of continued fractions.

LEMMA 5.1. *For any $n \in \mathbb{N}$, the distance from $n\alpha$ to the nearest integer is greater than $1/3n$.*

γ and γ' can be any irrational elements of $\mathbb{T}$ satisfying the hypotheses of Theorem 4.2. What remains is to define f. Before doing so, we will use our sequence $\{p_n\}$ to construct another increasing sequence of integers. For any n, take $w_n = \ell(p_n, 0, 0, 0)$. In other words, for any $v \in \mathbb{T}^{2d+4}$, $T^{p_n}(v, 0, 0, 0) = (S^{w_n}(v), w_n\gamma, w_n\gamma', p_n - g_{w_n}(0, 0))$. (Here $S = S_{2d+4,\alpha,f}$ and $T = T_{2d+7,\alpha,\gamma,\gamma',f}$.) As before, for large n, $w_n \in [0.3p_n, p_n]$. We claim that $w_{n+1} < (w_{n+1} - w_n)^{d+1}$ for all large enough n. This is because $w_{n+1} - w_n = \ell(p_{n+1} - p_n, w_n\gamma, w_n\gamma', p_n - g_{w_n}(0, 0))$:

$$\begin{aligned}
&(S^{w_{n+1}}v, w_{n+1}\gamma, w_{n+1}\gamma, p_{n+1} - g_{w_{n+1}}(0, 0)) \\
&\quad = T^{p_{n+1}}(v, 0, 0, 0) = T^{p_{n+1}-p_n}(T^{p_n}(v, 0, 0, 0)) \\
&\quad = T^{p_{n+1}-p_n}(S^{w_n}v, w_n\gamma, w_n\gamma', p_n - g_{w_n}(0, 0)) \\
&\quad = (S^{w_n+\ell(p_{n+1}-p_n, w_n\gamma, w_n\gamma', p_n-g_{w_n}(0,0))}(S^{w_n}(v)), \\
&\qquad (w_n + \ell(p_{n+1} - p_n, w_n\gamma, w_n\gamma', p_n - g_{w_n}(0, 0)))\gamma, \\
&\qquad (w_n + \ell(p_{n+1} - p_n, w_n\gamma, w_n\gamma', p_n - g_{w_n}(0, 0)))\gamma', \\
&\qquad p_{n+1} - g_{w_{n+1}-1}(0, 0)).
\end{aligned}$$

Therefore, since $p_{n+1} - p_n \to \infty$, $w_{n+1} - w_n \in [0.3(p_{n+1} - p_n), p_{n+1} - p_n]$ for large n for the same reasons as before. This implies for large n that

$$\begin{aligned}
w_{n+1} \le p_{n+1} < (p_{n+1} - p_n)^d &\le \left(\tfrac{10}{3}(w_{n+1} - w_n)\right)^d \le \left(\tfrac{10}{3}\right)^d (w_{n+1} - w_n)^d \\
&< (w_{n+1} - w_n)^{d+1}
\end{aligned}$$

https://doi.org/10.1017/S0143385707000508 Published online by Cambridge University Press

since $w_{n+1} - w_n \to \infty$. We wish to choose f so that $S^{w_n}(\mathbf{0})$ is bounded away from $\mathbf{0}$, where $\mathbf{0} \in \mathbb{T}^{2d+4}$ is the zero vector.

We will define f as an infinite sum: $F(v_1) = v_1 + \sum_{i=1}^{\infty} c_i s_{x_i,\epsilon_i}(v_1)$ is a function from $\mathbb{T}$ to $\mathbb{R}$, and $f(v_1) = F(v_1) \pmod 1$ is a self-map of $\mathbb{T}$. In this sum, $c_i \in \mathbb{R}^+$ with $\sum_{i=1}^{\infty} c_i < 1$, $x_i \in \mathbb{T}$ and $\epsilon_i > 0$ will be chosen later, and the function $s_{x,\epsilon}(y)$ for any $x \in \mathbb{T}$ and $\epsilon > 0$ is the function defined by

$$s_{x,\epsilon}(y) = \begin{cases} \dfrac{\epsilon}{2\pi}\left[\cos\left(\dfrac{\pi}{\epsilon}(y-x)\right) + 1\right] & \text{if } x - \epsilon \le y \le x + \epsilon, \\ 0 & \text{otherwise.} \end{cases}$$

The pertinent properties of $s_{x,\epsilon}$ are that it is non-zero only on the interval $[x - \epsilon, x + \epsilon]$, it attains a maximum of ϵ/π at $y = x$, and its derivative is bounded from above in absolute value by $1/2$. Since each term $c_i s_{x_i,\epsilon_i}$ in the definition of F is a differentiable function with derivative bounded from above in absolute value by $c_i/2$, and since $\sum_{i=1}^{\infty} c_i < 1$ and the identity function has derivative one everywhere, F is a differentiable function with $F'(v_1) \in (1/2, 3/2)$ for all $v_1 \in \mathbb{T}$. This shows that for any choice of c_i with $\sum_{i=1}^{\infty} c_i < 1$, and for any choice of x_i, ϵ_i, by Theorem 4.2, T is totally minimal, totally uniquely ergodic, and topologically mixing.

We wish to choose f so that $S^{w_n}(\mathbf{0})$ is bounded away from $\mathbf{0}$. The only quantities still to be chosen are c_i, x_i and ϵ_i. We note that for any $1 < k \le 2d + 4$ and any $j \ge k - 1$,

$$\pi_k(S^j(\mathbf{0})) = \sum_{i=0}^{j-k+1} \binom{j-i-1}{k-2} f(i\alpha).$$

This can be proved by a quick induction, and is left to the reader. In particular, if we use the notation $y_n = \pi_{2d+4}(S^{w_n}(\mathbf{0}))$ for any $w_n \ge 2d + 3$, then

$$y_n = \sum_{i=0}^{w_n-2d-3} \binom{w_n-i-1}{2d+2} f(i\alpha).$$

Our goal is to choose c_i, x_i, and ϵ_i so that $y_{w_n} = 1/3$ for all sufficiently large n. To do this, we choose $x_n = w_n\alpha$ for all n, and $\epsilon_n = \inf_{0 \le i < w_{n+1}, i \ne w_n} |x_n - i\alpha| > 1/3w_{n+1}$ by Lemma 5.1. This guarantees that $s_{x_n,\epsilon_n}(i\alpha) = 0$ for any $0 \le i < w_{n+1}$, $i \ne w_n$. This means that each choice of c_i that we make will change the values of y_n for only $n > i$, and allows us to finally inductively define c_i.

Recall that our goal is to ensure that

$$y_n = \sum_{i=0}^{w_n-2d-3} \binom{w_n-i-1}{2d+2} f(i\alpha) = 1/3$$

for all sufficiently large n. Note that since $\{w_n\}$ is an increasing sequence of integers, $w_n \ge n$ for all n. We have already shown that $w_n < (w_n - w_{n-1})^{d+1}$ for all large n, and so $w_n - w_{n-1} > n^{1/(d+1)}$, and so

$$w_n = w_1 + \sum_{i=2}^{n}(w_i - w_{i-1}) > \sum_{i=2}^{n} i^{1/(d+1)} > \frac{n}{2}\left(\frac{n}{2}\right)^{1/(d+1)} > \frac{1}{4}n^{1+(1/(d+1))}$$

https://doi.org/10.1017/S0143385707000508 Published online by Cambridge University Press

for all large n, and so $\sum_{n=1}^{\infty} w_n^{-1}$ converges, a fact that will be important in a moment. We choose N so that $\sum_{n=N+1}^{\infty} w_n^{-1} < 1/6\pi(2d+2)!$. The procedure for defining the sequence c_n is then as follows. We take $c_i = 0$ for $1 \le i \le N$. For any $n > N$, assume that $y_i = 1/3$ for $N+1 < i \le n$. Then, we choose c_n so that $y_{n+1} = 1/3$. Note that for any $n > 1$,

$$y_n = h_n(c_1, \ldots, c_{n-2}) + c_{n-1}\frac{1}{\pi}\epsilon_{n-1}\binom{w_n - w_{n-1} - 1}{2d+2}$$

for some $h_n : \mathbb{T}^{n-2} \to \mathbb{T}$. This means that taking

$$c_n = \pi \frac{(\frac{1}{3} - h_{n+1}(c_1, \ldots, c_{n-1})) \pmod 1}{\epsilon_n \binom{w_{n+1}-w_n-1}{2d+2}}$$

gives $y_{n+1} = 1/3$. Note that

$$\binom{w_{n+1} - w_n - 1}{2d+2} > \frac{1}{2(2d+2)!}(w_{n+1} - w_n)^{2d+2}$$

for large n, and recall that $\epsilon_n > 1/3w_{n+1}$ for all n by Lemma 5.1. This means that $c_n < 6\pi(2d+2)!w_{n+1}(w_{n+1} - w_n)^{-(2d+2)}$, which, by the hypothesis on the sequence $\{w_n\}$, is less than $6\pi(2d+2)!(w_{n+1})^{-1}$, again for sufficiently large n. This means that $\sum_{n=1}^{\infty} c_n < 6\pi(2d+2)! \sum_{n=N+1}^{\infty} w_{n+1}^{-1} < 1$ by the definition of N. We have then chosen c_n so that $d(S^{w_n}(\mathbf{0}), \mathbf{0}) \ge d(y_n, 0) = 1/3$ for all $n > N$. For $n \le N$, note that $\pi_1(S^{w_n}(\mathbf{0})) = w_n\alpha \ne 0$, since $\alpha \notin \mathbb{Q}$. This means that for all n, $d(S^{w_n}(\mathbf{0}), \mathbf{0}) \ge \min(1/3, \min_{1\le n\le N} d(w_n\alpha, 0)) > 0$.

However, by definition, $\pi_{2d+4}(T^{p_n}(\mathbf{0}, 0, 0, 0)) = \pi_{2d+4}(S^{w_n}(\mathbf{0}))$ for all n. Therefore, $T^{p_n}(\mathbf{0}, 0, 0, 0)$ is bounded away from $(\mathbf{0}, 0, 0, 0)$, and so since T is totally uniquely ergodic, totally minimal, and topologically mixing, we are done. □

We note that there was nothing special about the number $1/3$ in this proof, and so the proof of the following corollary is trivially similar.

COROLLARY 5.1. *For any increasing sequence $\{w_n\}$ of integers with the property that for some integer d, $w_{n+1} < (w_{n+1} - w_n)^{d+1}$ for all sufficiently large n, and for any sequence $\{z_n\} \subseteq \mathbb{T}$, there exists f satisfying the hypotheses of Theorem 4.1 such that for all sufficiently large n, $\pi_{2d+4}((S_{2d+4,\alpha,f})^{w_n}(\mathbf{0})) = z_n$.*

Proof of Theorem 1.3. Our transformation is $T_{2d+9,\alpha,\gamma,\gamma',f}$ for the same α, γ, and γ' as before. We use the same strategy as we did to prove Theorem 1.2; in other words, we will be forcing certain types of non-recurrence behavior along a set comprised of a union of infinitely many shifted subsequences of $\{p_n\}$.

We now proceed roughly as we did in proving Theorem 1.2. We will define a sequence $\{t_n\}$ by shifting different p_n by different amounts. First, define the intervals of integers $B_j = [j! + 1, (j+1)!] \cap \mathbb{N}$ for every $j \in \mathbb{N}$, and take any partition of $\mathbb{N}$ into infinitely many disjoint infinite sets $C_1, C_2, \ldots$. We denote the elements of C_i, written in increasing order, by $c_i^{(1)}, c_i^{(2)} \ldots$. Choose some s_1 large enough so that $p_{n+1} - p_n > 2 \cdot 1$ for $n \ge (\inf C_{s_1})!$, define the set $D_1 = \bigcup_{j\in C_{s_1}} B_j$, and then define $t_n = p_n + 1$ for all $n \in D_1$. Next, choose some s_2 large enough so that $p_{n+1} - p_n > 2 \cdot 2$ for $n \ge (\inf C_{s_2})!$, and define

https://doi.org/10.1017/S0143385707000508 Published online by Cambridge University Press

$D_2 = \bigcup_{j\in C_{s_2}} B_j$, and then define $t_n = p_n + 2$ for all $n \in D_2$. Continuing in this way, we may inductively define D_k for all $k \in \mathbb{N}$ so that $D_k = \bigcup_{j\in C_{s_k}} B_j$ for some s_k with the property that $p_{n+1} - p_n > 2k$ for $n \geq (\inf C_{s_k})!$, and then define $t_n = p_n + k$ for all $n \in D_k$. For any $n \notin \bigcup_{k=1}^{\infty} D_k$, $t_n = p_n$. Note that by the construction, for any n, k where $t_n = p_n + k$, it must be the case that $n \in D_k$, and therefore that $n > (\inf C_{s_k})!$, and so that $p_{n+1} - p_n > 2k$. Therefore, since $t_{n+1} \geq p_{n+1}$ for all n, $t_{n+1} - t_n \geq p_{n+1} - p_n - k > (p_{n+1} - p_n)/2$, and so $\{t_n\}$ is increasing. Since $n - 1 \geq (\inf C_{s_k})!$, $p_n - p_{n-1} > 2k$, and so in particular $p_n > 2k$. This implies that $t_n = p_n + k < 2p_n$ for all n. Finally, we see that $t_{n+1} < 2p_{n+1} < 2(p_{n+1} - p_n)^d < 2^{d+1}(t_{n+1} - t_n)^d$ for all large enough n. Since $t_{n+1} - t_n > (p_{n+1} - p_n)/2 \to \infty$ as $n \to \infty$, this means that $t_{n+1} < (t_{n+1} - t_n)^{d+1}$ for large enough n.

We again define a sequence $\{w_n\}$: for any n, take $w_n = \ell(t_n, 0, 0, 0)$. In other words, for any $v \in \mathbb{T}^{2d+6}$, $T^{t_n}(v, 0, 0, 0) = (S^{w_n}(v), w_n\gamma, w_n\gamma', t_n - g_{w_n}(0, 0))$. (Here $S = S_{2d+6,\alpha,f}$ and $T = T_{2d+9,\alpha,\gamma,\gamma',f}$.) For exactly the same reasons as in the proof of Theorem 1.6, $w_{n+1} < (w_{n+1} - w_n)^{d+2}$ for large n.

Therefore, by using Corollary 5.1, for any sequence $\{z_n\} \subseteq \mathbb{T}$, we may choose f such that $\pi_{2d+6}(T^{t_n}(\mathbf{0}, 0, 0, 0)) = \pi_{2d+6}(S^{w_n}(\mathbf{0})) = z_n$ for all n. We define $z_n = 1/3$ for any $n \in D_k$ where $n \in B_j$ for $j = c_{s_k}^{(i)}$ with odd i, and $z_n = 2/3$ for any $n \in D_k$ where $n \in B_j$ for $j = c_{s_k}^{(i)}$ with even i. For any z_n not defined by these conditions, z_n may be anything. (We can take $z_n = 0$ for such n if it is convenient.) Take $h \in C(X)$ such that $h(v, x, y, t) = 0$ if $v_{2d+6} = 2/3$, $h(v, x, y, t) = 1$ if $v_{2d+6} = 1/3$, $\inf_{x\in X} h(x) = 0$, and $\sup_{x\in X} h(x) = 1$. Now, we note that

$$\begin{aligned}
&\left\{z \in X : \frac{1}{N}\sum_{n=0}^{N-1} h(T^{p_n}z) \text{ does not converge}\right\} \\
&\quad \supseteq \left(\bigcap_{n>0}\bigcup_{k>n}\left\{z \in X : \frac{|\{i : 1 \leq i \leq k, \pi_{2d+6}(T^{p_i}(z)) = \frac{1}{3}\}|}{k} > \frac{3}{4}\right\}\right) \\
&\quad \cap \left(\bigcap_{n>0}\bigcup_{k>n}\left\{z \in X : \frac{|\{i : 1 \leq i \leq k, \pi_{2d+6}(T^{p_i}(z)) = \frac{2}{3}\}|}{k} > \frac{3}{4}\right\}\right),
\end{aligned}$$

and that the latter set, call it B, is clearly a G_δ. We will show that B is dense in X. Choose any non-empty open set $U \subset X$. By minimality of T, there is some k such that $T^k(\mathbf{0}, 0, 0, 0) \in U$. By construction, $t_n = p_n + k$ for all $n \in D_k$. Also by construction, $D_k = \bigcup_{j\in C_{s_k}} B_j$, and $\pi_{2d+6}(T^{t_n}(\mathbf{0}, 0, 0, 0)) = \pi_{2d+6}(T^{p_n}(T^k(\mathbf{0}, 0, 0, 0))) = 1/3$ for $j = c_{s_k}^{(i)}$ and i odd. But then for any odd integer i, $\pi_{2d+6}(T^{p_n}(T^k(\mathbf{0}, 0, 0, 0))) = 1/3$ for any $n \in [(c_{s_k}^{(i)})! + 1, (c_{s_k}^{(i)} + 1)!]$, and so

$$\frac{|\{i : 1 \leq i \leq (c_{s_k}^{(i)} + 1)!, \pi_{2d+6}(T^{p_i}(T^k(\mathbf{0}, 0, 0, 0)) = \frac{1}{3}\}|}{(c_{s_k}^{(i)} + 1)!} \geq \frac{c_{s_k}^{(i)}}{c_{s_k}^{(i)} + 1},$$

which is clearly larger than $3/4$ for sufficiently large k. Similarly, for any even integer i, $\pi_{2d+6}(T^{p_n}(T^k(\mathbf{0}, 0, 0, 0))) = 2/3$ for any $n \in [(c_{s_k}^{(i)})! + 1, (c_{s_k}^{(i)} + 1)!]$, and so

$$\frac{|\{i : 1 \leq i \leq (c_{s_k}^{(i)} + 1)!, \pi_{2d+6}(T^{p_i}(T^k(\mathbf{0}, 0, 0, 0))) = \frac{2}{3}\}|}{(c_{s_k}^{(i)} + 1)!} \geq \frac{c_{s_k}^{(i)}}{c_{s_k}^{(i)} + 1},$$

https://doi.org/10.1017/S0143385707000508 Published online by Cambridge University Press

which is also greater than 3/4 for sufficiently large k. This implies that $T^k(\mathbf{0}, 0, 0, 0) \in B$, and so that $B \cap U$ is non-empty. Since U was arbitrary, this shows that B is dense, and so a dense G_δ. Therefore, B is a residual set by the Baire category theorem, and for every $z \in B$, $(1/N) \sum_{n=0}^{N-1} h(T^{p_n} z)$ does not converge. □

We note that, exactly as in the proof of Theorem 1.2, this in fact shows that for a residual set of $z \in X$,

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} h(T^{p_n} z) = 0 = \inf_{x\in X} h(x)$$

and

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} h(T^{p_n} z) = 1 = \sup_{x\in X} h(x).$$

6. *Questions*

There are some natural questions motivated by these results. For any totally minimal and totally uniquely ergodic system (X, T), any non-empty open set $U \subseteq X$ with $\mu(U) \in (0, 1)$, and any $x \notin \overline{U}$, take the set $A = \{n \in \mathbb{N} : T^n x \in U\}$. Since (X, T) is uniquely ergodic, the density $d(A) := \lim_{n\to\infty} |\{1, 2, \ldots, n\} \cap A|/n$ of A equals $\mu(U) > 0$, and so the sequence $\{a_n\}$ of the elements of A written in increasing order does not satisfy the hypotheses of any of our theorems. However, since $x \notin \overline{U}$, and since $T^{a_n} x \in U$ for all n, $T^{a_n} x$ is bounded away from x.

For a similar example, take T to be a totally minimal and totally uniquely ergodic isometry of a complete metric space (X, d) with diameter greater than 2. Take $x, y \in X$ with $d(x, y) > 2$, and define $f \in C(X)$ where $f(z) = 0$ for all $z \in B_1(x)$ and $f(z) = 1$ for all $z \in B_1(y)$. Then, take the sets $A = \{n \in \mathbb{N} : T^n x \in B_{(1/2)}(y)\}$ and $B = \{n \in \mathbb{N} : T^n x \in B_{(1/2)}(x)\}$. Since (X, T) is uniquely ergodic, $d(A) = \mu(B_{(1/2)}(y)) > 0$ and $d(B) = \mu(B_{(1/2)}(x)) > 0$. For any $z \in B_{(1/2)}(x)$, $d(T^{a_n} z, y) \le d(T^{a_n} z, T^{a_n} x) + d(T^{a_n} x, y) = d(z, x) + d(T^{a_n} x, y) < (1/2) + (1/2) = 1$, and so $f(T^{a_n} z) = 1$. Also for any $z \in B_{(1/2)}(x)$, $d(T^{b_n} z, x) \le d(T^{b_n} z, T^{b_n} x) + d(T^{b_n} x, x) = d(z, x) + d(T^{b_n} x, x) < (1/2) + (1/2) = 1$, and so $f(T^{b_n} z) = 0$. This means that by making a sequence $\{c_n\}$ by alternately choosing longer and longer subsequences of $\{a_n\}$ and $\{b_n\}$, we could create such $\{c_n\}$ with $d(\{c_n\}) > 0$ (so $\{c_n\}$ does not satisfy the hypotheses of any of our theorems) where $\lim_{N\to\infty}(1/N) \sum_{n=1}^{N} f(T^{c_n} z)$ does not converge for any z in the second-category set $B_{(1/2)}(x)$.

The point of these examples is to show that our hypotheses are certainly not the only ones under which examples of the types constructed in this paper exist. This brings up the following questions.

QUESTION 6.1. *For what increasing sequences $\{p_n\}$ of integers does there exist a totally minimal and totally uniquely ergodic system (X, T) and a point $x \in X$ such that $x \notin \overline{\{T^{p_n} x\}}$?*

QUESTION 6.2. *For what increasing sequences $\{p_n\}$ of integers does there exist a totally minimal and totally uniquely ergodic system (X, T) and a function $f \in C(X)$ such that $\lim_{N\to\infty}(1/N) \sum_{n=1}^{N} f(T^{p_n} x)$ fails to converge for a set of x of second category?*

https://doi.org/10.1017/S0143385707000508 Published online by Cambridge University Press

Also, it is interesting that we could create examples for a wider class of sequences $\{p_n\}$ when X was not connected. We would like to know whether or not this is necessary, i.e. the answer to the following question.

QUESTION 6.3. *Given an increasing sequence $\{p_n\}$ of integers and a totally minimal totally uniquely ergodic topological dynamical system (X, T) and $x \in X$ such that $x \notin \overline{\{T^{p_n}x\}}$, must there exist a system with the same properties where X is a connected space?*

QUESTION 6.4. *Given an increasing sequence $\{p_n\}$ of integers and a totally minimal totally uniquely ergodic topological dynamical system (X, T) and $f \in C(X)$ such that $\lim_{N\to\infty}(1/N)\sum_{n=1}^{N} f(T^{p_n}x)$ fails to converge for a set of x of second category, must there exist a system with the same properties where X is a connected space?*

Finally, we briefly address one more issue about generalizing our results. In our symbolic examples, the dynamical systems considered were not invertible, since every x considered was in $\{0, 1\}^{\mathbb{N}}$ rather than $\{0, 1\}^{\mathbb{Z}}$. It is, however, not hard to extend our results to the invertible case. The rough idea is to define each w_{k+1} to have w_k occurring somewhere in the middle rather than as a prefix. Then, w_k approaches a limit x in $\{0, 1\}^{\mathbb{Z}}$, and the orbital closure of x is again taken to be X. The remaining parts of the proofs go through in a similar fashion.

Acknowledgements. The author would like to express his gratitude to Vitaly Bergelson for suggesting this question and for numerous helpful comments and suggestions in the preparation of this paper, and also to the anonymous referee for useful suggestions and corrections.

REFERENCES

[1] V. Bergelson. Ergodic Ramsey theory — an update. *Ergodic Theory of $\mathbb{Z}^d$-Actions (London Math. Soc. Lecture Notes Series, 228)*. Eds. M. Pollicott and K. Schmidt. Cambridge University Press, Cambridge, 1996. pp. 1–61.

[2] V. Bergelson and A. Leibman. Polynomial extensions of van der Waerden's and Szemeredi's theorems. *J. Amer. Math. Soc.* **9**(3) (1996), 725–753.

[3] J. Bourgain. Pointwise ergodic theorems for arithmetic sets. *Publ. Math. IHES* **69**(3) (1989), 5–45.

[4] J. Bourgain. On pointwise ergodic theorems for arithmetic sets. *C. R. Acad. Sci. Paris Sér. I Math.* **305**(10) (1987), 397–402.

[5] B. Fayad. Analytic mixing reparametrizations of irrational flows. *Ergod. Th. & Dynam. Sys.* **22**(2) (2002), 437–468.

[6] H. Furstenberg. Strict ergodicity and transformation of the torus. *Amer. J. Math.* **83** (1961), 573–601.

[7] F. Hahn and Y. Katznelson. On the entropy of uniquely ergodic transformations. *Trans. Amer. Math. Soc.* **126** (1967), 335–360.

[8] G. H. Hardy and E. M. Wright. *An Introduction to the Theory of Numbers*. Clarendon Press, Oxford, 1979.

[9] M. Wierdl. Pointwise ergodic theorem along the prime numbers. *Israel J. Math.* **64**(3) (1988), 315–336.

https://doi.org/10.1017/S0143385707000508 Published online by Cambridge University Press