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# THE EFFECTS OF DIFFERENCING ON THE GAUSSIAN LIKELIHOOD OF MODELS WITH UNOBSERVABLE STOCHASTIC TRENDS: A SIMPLE EXAMPLE

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EDITORS' NOTE: In the course of preparing the final version of Bergstrom (1997) "Gaussian estimation of mixed-order continuous-time dynamic models with unobservable stochastic trends from mixed stock and flow data" (*Econometric Theory* 13, 467–505), Rex Bergstrom submitted a handwritten twenty-six page document. This document responded to an issue raised in the review and provided an illustration that the Gaussian likelihood remained invariant under differencing. The notes are published here because the material is of interest in its own right and illustrates the care and clarity of vision with which Rex Bergstrom approached all of his work.

## THE MODEL

We assume the model

$$x_{1t} = a_{11}x_{1,t-1} + a_{12}x_{2,t-1} + u_{1t}, \tag{1}$$

$$x_{2t} = a_{21}x_{1,t-1} + a_{22}x_{2,t-1} + \mu_t + u_{2t}, \tag{2}$$

$$\Delta\mu_t = \lambda + u_{3t}, \tag{3}$$

where $x_{1t}$ and $x_{2t}(t = 1, 2, \ldots, T)$ are observable random variables, $x_{10}$ and $x_{20}$ are observable nonrandom quantities, $u_{1t}, u_{2t}$, $u_{3t}$, and $\mu_t(t = 1, \ldots, T)$ are unobservable random variables with $\mathrm{E}(u_{it}) = 0$ $(i = 1, 2, 3;\ t = 1, \ldots, T), \mathrm{E}(u_{it}^2) = \sigma_i^2$ $(i = 1, 2, 3;\ t = 1, \ldots, T), \mathrm{E}(u_{1t}u_{2t}) = \sigma_{12}$ $(t = 1, \ldots, T), \mathrm{E}(u_{1t}u_{3t}) = \mathrm{E}(u_{2t}u_{3t}) = 0$ $(t = 1, \ldots, T), \mathrm{E}(u_{is}u_{jt}) = 0$ $(i, j = 1, 2, 3;\ s \neq t)$, and $a_{11}, a_{12}, a_{21}, a_{22}, \lambda, \sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_{12}$ are unknown parameters. We also impose the restriction $\mu_0 = 0$. This could, for example, be a model of supply and demand for a stock commodity with $x_{1t}$ being the logarithm of the price and $x_{2t}$ the logarithm of the quantity produced. Then (1) represents the adjustment of the price in response to the quantity produced, while (2) represents the adjustment of the quantity produced in response to the price. The speeds of adjustment are $(1 - a_{11})$ and $(1 - a_{22})$, the long-run demand elasticity is $(1 - a_{11})/a_{12}$, and the long-run

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supply elasticity is $a_{21}/(1-a_{22})$, implied a priori restrictions on the parameters being $a_{12}<0, a_{21}>0, 0<a_{11}<1, 0<a_{22}<1$. The stochastic trend variable $\mu_t$ represents cumulative technical progress and the drift parameter $\lambda$ is the average long-run rate of technical progress.

We shall now describe two alternative methods of deriving the Gaussian likelihood function for the parameters of the above model. In Method 1, we difference both (1) and (2), eliminate $\Delta\mu_t$ using (3), and then apply a simplified version of the algorithm developed in my paper [Bergstrom, 1997] in order to obtain the likelihood function. In Method 2 we difference (2) only, eliminate $\Delta\mu_t$ using (3), and then apply the same algorithm to obtain the likelihood. We shall then show, with a numerical illustration, that the two methods yield exactly the same values of the Gaussian likelihood.

## METHOD 1

Putting $t=1$ in (1) and (2) and using (3) together with the assumption $\mu_0=0$ to eliminate $\mu_1$ we obtain

$$x_{11}=a_{11}x_{10}+a_{12}x_{20}+\eta_{11}, \tag{4}$$

$$x_{21}=a_{21}x_{10}+a_{22}x_{20}+\lambda+\eta_{21}, \tag{5}$$

where

$$\eta_{11}=u_{11}, \tag{6}$$

$$\eta_{21}=u_{31}+u_{21}. \tag{7}$$

Differencing (1) and (2) and using (3) to eliminate $\Delta\mu_t$, we obtain

$$\Delta x_{1t}=a_{11}\Delta x_{1,t-1}+a_{12}\Delta x_{2,t-1}+\eta_{1t} \quad (t=2,\ldots,T), \tag{8}$$

$$\Delta x_{2t}=a_{21}\Delta x_{1,t-1}+a_{22}\Delta x_{2,t-1}+\lambda+\eta_{2t} \quad (t=2,\ldots,T), \tag{9}$$

where

$$\eta_{1t}=u_{1t}-u_{1,t-1} \quad (t=2,\ldots,T), \tag{10}$$

$$\eta_{2t}=u_{3t}+u_{2t}-u_{2,t-1} \quad (t=2,\ldots,T). \tag{11}$$

Now let $\eta_t=[\eta_{1t},\eta_{2t}]'$ and let $\eta=[\eta_1',\eta_2',\ldots,\eta_T']'$. Then

$$\mathrm{E}(\eta\eta')=\Omega, \tag{12}$$

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where

$$\Omega = \begin{bmatrix} \Omega_{11} & \Omega_1' & 0 & 0 & \dots & 0 \\ \Omega_1 & \Omega_0 & \Omega_1' & 0 & \dots & 0 \\ 0 & \Omega_1 & \Omega_0 & \Omega_1' & \dots & 0 \\ 0 & 0 & \Omega_1 & \Omega_0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \Omega_0 \end{bmatrix}, \tag{13}$$

$$\Omega_{11} = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 + \sigma_3^2 \end{bmatrix}, \tag{14}$$

$$\Omega_0 = \begin{bmatrix} 2\sigma_1^2 & 2\sigma_{12} \\ 2\sigma_{12} & 2\sigma_2^2 + 2\sigma_3^2 \end{bmatrix}, \tag{15}$$

$$\Omega_1 = \begin{bmatrix} -\sigma_1^2 & -\sigma_{12} \\ -\sigma_{12} & -\sigma_2^2 \end{bmatrix}. \tag{16}$$

Now let $M$ be the lower triangular matrix, with positive elements along the diagonal, defined by the Cholesky factorization

$$\Omega = MM'. \tag{17}$$

Then $M$ has the form

$$M = \begin{bmatrix} M_{11} & 0 & 0 & \dots & 0 & 0 \\ M_{21} & M_{22} & 0 & \dots & 0 & 0 \\ 0 & M_{32} & M_{33} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & M_{T,T-1} & M_{TT} \end{bmatrix}, \tag{18}$$

where $M_{11}$, $M_{21}$, $M_{22}$, $M_{t,t-1}$, $M_{tt}$ $(t = 3, \dots, T)$ can be obtained, recursively, from the equations

$$M_{11}M_{11}' = \Omega_{11}, \tag{19}$$

$$M_{t,t-1} = \Omega_1[M_{t-1,t-1}']^{-1} \quad (t = 2, \dots, T), \tag{20}$$

$$M_{tt}M_{tt}' = \Omega_0 - M_{t,t-1}M_{t,t-1}' \quad (t = 2, \dots, T). \tag{21}$$

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Let $\epsilon = [\epsilon_1', \epsilon_2', \ldots, \epsilon_T']'$ be the $2T \times 1$ vector defined by

$$\epsilon = M^{-1}\eta, \tag{22}$$

so that

$$\eta = M\epsilon. \tag{23}$$

Then, from (12), (17), and (22), we have

$$\mathrm{E}(\epsilon) = 0, \quad \mathrm{E}(\epsilon\epsilon') = I, \tag{24}$$

and, from (4)–(11) and (23),

$$\begin{bmatrix} x_{11} - a_{11}x_{10} - a_{12}x_{20} \\ x_{21} - a_{21}x_{10} - a_{22}x_{20} - \lambda \end{bmatrix} = M_{11}\epsilon_1, \tag{25}$$

$$\begin{bmatrix} \Delta x_{1t} - a_{11}\Delta x_{1,t-1} - a_{12}\Delta x_{2,t-1} \\ \Delta x_{2t} - a_{21}\Delta x_{1,t-1} - a_{22}\Delta x_{2,t-1} - \lambda \end{bmatrix}$$
$$= M_{tt}\epsilon_t + M_{t,t-1}\epsilon_{t-1} \quad (t = 2, \ldots, T), \tag{26}$$

and, from (25) and (26),

$$\epsilon_1 = M_{11}^{-1}\begin{bmatrix} x_{11} - a_{11}x_{10} - a_{12}x_{20} \\ x_{21} - a_{21}x_{10} - a_{22}x_{20} - \lambda \end{bmatrix}, \tag{27}$$

$$\epsilon_t = M_{tt}^{-1}\begin{bmatrix} \Delta x_{1t} - a_{11}\Delta x_{1,t-1} - a_{12}\Delta x_{2,t-1} \\ \Delta x_{2t} - a_{21}\Delta x_{1,t-1} - a_{22}\Delta x_{2,t-1} - \lambda \end{bmatrix}$$
$$- M_{tt}^{-1}M_{t,t-1}\epsilon_{t-1} \quad (t = 2, \ldots, T). \tag{28}$$

Then, letting $L$ denote minus twice the logarithm of the Gaussian likelihood, we have

$$L = \sum_{t=1}^{T} \left(\epsilon_t'\epsilon_t + 2\log|M_{tt}|\right). \tag{29}$$

## METHOD 2

Putting $t = 1$ in (1) and (2) and using (3) together with the assumption $\mu_0 = 0$ to eliminate $\mu_1$ we obtain

$$x_{11} = a_{11}x_{10} + a_{12}x_{20} + \eta_{11}^*, \tag{30}$$

$$x_{21} = a_{21}x_{10} + a_{22}x_{20} + \lambda + \eta_{21}^*, \tag{31}$$

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where

$$\eta_{11}^* = u_{11}, \tag{32}$$

$$\eta_{21}^* = u_{31} + u_{21}. \tag{33}$$

Differencing (2), using (3) to eliminate $\Delta\mu_t$, and combining with (1), we obtain

$$x_{1t} = a_{11}x_{1,t-1} + a_{12}x_{2,t-1} + \eta_{1t}^* \quad (t = 2, \ldots, T), \tag{34}$$

$$\Delta x_{2t} = a_{21}\Delta x_{1,t-1} + a_{22}\Delta x_{2,t-1} + \lambda + \eta_{2t}^* \quad (t = 2, \ldots, T), \tag{35}$$

where

$$\eta_{1t}^* = u_{1t} \quad (t = 2, \ldots, T), \tag{36}$$

$$\eta_{2t}^* = u_{3t} + u_{2t} - u_{2,t-1} \quad (t = 2, \ldots, T). \tag{37}$$

Now let $\eta_t^* = [\eta_{1t}^*, \eta_{2t}^*]'$ and let $\eta^* = [\eta_1^{*\prime}, \eta_2^{*\prime}, \ldots, \eta_T^{*\prime}]'$. Then

$$\mathrm{E}(\eta^*\eta^{*\prime}) = \Omega^*, \tag{38}$$

where

$$\Omega = \begin{bmatrix} \Omega_{11}^* & \Omega_1^{*\prime} & 0 & 0 & \ldots & 0 \\ \Omega_1^* & \Omega_0^* & \Omega_1^{*\prime} & 0 & \ldots & 0 \\ 0 & \Omega_1^* & \Omega_0^* & \Omega_1^{*\prime} & \ldots & 0 \\ 0 & 0 & \Omega_1^* & \Omega_0^* & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & \Omega_0^* \end{bmatrix}, \tag{39}$$

$$\Omega_{11}^* = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 + \sigma_3^2 \end{bmatrix}, \tag{40}$$

$$\Omega_0^* = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & 2\sigma_2^2 + \sigma_3^2 \end{bmatrix}, \tag{41}$$

$$\Omega_1^* = \begin{bmatrix} 0 & 0 \\ -\sigma_{12} & -\sigma_2^2 \end{bmatrix}. \tag{42}$$

Now let $M^*$ be the lower triangular matrix, with positive elements along the diagonal, defined by the Cholesky factorization

$$\Omega^* = M^*M^{*\prime}. \tag{43}$$

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Then $M^*$ has the form

$$M^* = \begin{bmatrix} M_{11}^* & 0 & 0 & \dots & 0 & 0 \\ M_{21}^* & M_{22}^* & 0 & \dots & 0 & 0 \\ 0 & M_{32}^* & M_{33}^* & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & M_{T,T-1}^* & M_{TT}^* \end{bmatrix}, \tag{44}$$

where $M_{11}^*$, $M_{21}^*$, $M_{22}^*$, $M_{t,t-1}^*$, $M_{tt}^* (t = 3, \dots, T)$ can be obtained, recursively, from the equations

$$M_{11}^* M_{11}^{*\prime} = \Omega_{11}^*, \tag{45}$$

$$M_{t,t-1}^* = \Omega_1^* [M_{t-1,t-1}^{*\prime}]^{-1} \quad (t = 2, \dots, T), \tag{46}$$

$$M_{tt}^* M_{tt}^{*\prime} = \Omega_0^* - M_{t,t-1}^* M_{t,t-1}^{*\prime} \quad (t = 2, \dots, T). \tag{47}$$

Let $\epsilon^* = [\epsilon_1^{*\prime}, \epsilon_2^{*\prime}, \dots, \epsilon_T^{*\prime}]'$ be the $2T \times 1$ vector defined by

$$\epsilon^* = [M^*]^{-1} \eta^*, \tag{48}$$

so that

$$\eta^* = M^* \epsilon^*. \tag{49}$$

Then, from (38), (43), and (48), we have

$$\mathrm{E}(\epsilon^*) = 0, \quad \mathrm{E}(\epsilon^* \epsilon^{*\prime}) = I, \tag{50}$$

and, from (30)–(37) and (49),

$$\begin{bmatrix} x_{11} - a_{11} x_{10} - a_{12} x_{20} \\ x_{21} - a_{21} x_{10} - a_{22} x_{20} - \lambda \end{bmatrix} = M_{11}^* \epsilon_1^*, \tag{51}$$

$$\begin{bmatrix} x_{1t} - a_{11} x_{1,t-1} - a_{12} x_{2,t-1} \\ \Delta x_{2t} - a_{21} \Delta x_{1,t-1} - a_{22} \Delta x_{2,t-1} - \lambda \end{bmatrix}$$

$$= M_{tt}^* \epsilon_t^* + M_{t,t-1}^* \epsilon_{t-1}^* \quad (t = 2, \dots, T), \tag{52}$$

and, from (51) and (52),

$$\epsilon_1^* = [M_{11}^*]^{-1} \begin{bmatrix} x_{11} - a_{11} x_{10} - a_{12} x_{20} \\ x_{21} - a_{21} x_{10} - a_{22} x_{20} - \lambda \end{bmatrix}, \tag{53}$$

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$$\epsilon_t^* = [M_{tt}^*]^{-1} \begin{bmatrix} x_{1t} - a_{11}x_{1,t-1} - a_{12}x_{2,t-1} \\ \Delta x_{2t} - a_{21}\Delta x_{1,t-1} - a_{22}\Delta x_{2,t-1} - \lambda \end{bmatrix}$$

$$-[M_{tt}^*]^{-1} M_{t,t-1}^* \epsilon_{t-1}^* \quad (t = 2, \ldots, T). \tag{54}$$

Then, letting $L^*$ denote minus twice the logarithm of the Gaussian likelihood, we have

$$L^* = \sum_{t=1}^{T} \left( \epsilon_t^{*\prime} \epsilon_t^* + 2 \log |M_{tt}^*| \right). \tag{55}$$

## NUMERICAL ILLUSTRATION OF METHOD 1

Assume the following values of the parameters: $\sigma_1^2 = 4, \sigma_2^2 = 4, \sigma_{12} = 2, \sigma_3^2 = 1$. Then

$$\Omega_{11} = \begin{bmatrix} 4 & 2 \\ 2 & 5 \end{bmatrix}, \quad \Omega_1 = \begin{bmatrix} -4 & -2 \\ -2 & -4 \end{bmatrix}, \quad \Omega_0 = \begin{bmatrix} 8 & 4 \\ 4 & 9 \end{bmatrix}, \quad M_{11} = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix},$$

$$M_{21} = \Omega_1 [M_{11}']^{-1} = \begin{bmatrix} -4 & -2 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 0 \\ -1 & -1.5 \end{bmatrix},$$

$$M_{22} M_{22}' = \Omega_0 - M_{21} M_{21}' = \begin{bmatrix} 8 & 4 \\ 4 & 9 \end{bmatrix} - \begin{bmatrix} -2 & 0 \\ -1 & -1.5 \end{bmatrix} \begin{bmatrix} -2 & -1 \\ 0 & -1.5 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 5.75 \end{bmatrix},$$

$$M_{22} = \begin{bmatrix} 2 & 0 \\ 1 & 2.18 \end{bmatrix}, \quad M_{32} = \Omega_1 [M_{22}']^{-1} = \begin{bmatrix} -4 & -2 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 2.18 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} -2 & 0 \\ -1 & -1.38 \end{bmatrix},$$

$$M_{33} M_{33}' = \Omega_0 - M_{32} M_{32}' = \begin{bmatrix} 8 & 4 \\ 4 & 9 \end{bmatrix} - \begin{bmatrix} -2 & 0 \\ -1 & -1.38 \end{bmatrix} \begin{bmatrix} -2 & -1 \\ 0 & -1.38 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 2 \\ 2 & 6.10 \end{bmatrix},$$

$$M_{33} = \begin{bmatrix} 2 & 0 \\ 1 & 2.26 \end{bmatrix}, \quad \text{etc.}$$

It is interesting to see that $M_{tt}$ is converging rapidly as $t$ increases.

$$\epsilon_1 = M_{11}^{-1} \begin{bmatrix} x_{11} - a_{11}x_{10} - a_{12}x_{20} \\ x_{21} - a_{21}x_{10} - a_{22}x_{20} - \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} x_{11} - a_{11}x_{10} - a_{12}x_{20} \\ x_{21} - a_{21}x_{10} - a_{22}x_{20} - \lambda \end{bmatrix} = \begin{bmatrix} \epsilon_{11} \\ \epsilon_{21} \end{bmatrix},$$

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where

$$\epsilon_{11} = 0.50(x_{11} - a_{11}x_{10} - a_{12}x_{20}),$$

$$\epsilon_{21} = 0.50(x_{21} - a_{21}x_{10} - a_{22}x_{20} - \lambda) - 0.25(x_{11} - a_{11}x_{10} - a_{12}x_{20}),$$

$$\epsilon_2 = M_{22}^{-1}\begin{bmatrix} \Delta x_{12} - a_{11}\Delta x_{11} - a_{12}\Delta x_{21} \\ \Delta x_{22} - a_{21}\Delta x_{11} - a_{22}\Delta x_{21} - \lambda \end{bmatrix} - M_{22}^{-1}M_{21}\epsilon_1$$

$$= \begin{bmatrix} 2 & 0 \\ 1 & 2.18 \end{bmatrix}^{-1}\begin{bmatrix} \Delta x_{12} - a_{11}\Delta x_{11} - a_{12}\Delta x_{21} \\ \Delta x_{22} - a_{21}\Delta x_{11} - a_{22}\Delta x_{21} - \lambda \end{bmatrix}$$

$$- \begin{bmatrix} 2 & 0 \\ 1 & 2.18 \end{bmatrix}^{-1}\begin{bmatrix} -2 & 0 \\ -1 & -1.5 \end{bmatrix}\begin{bmatrix} \epsilon_{11} \\ \epsilon_{21} \end{bmatrix} = \begin{bmatrix} \epsilon_{12} \\ \epsilon_{22} \end{bmatrix},$$

where

$$\epsilon_{12} = 0.50(x_{12} - a_{11}x_{11} - a_{12}x_{21}),$$

$$\epsilon_{22} = 0.46(\Delta x_{22} - a_{21}\Delta x_{11} - a_{22}\Delta x_{21} - \lambda) - 0.23(\Delta x_{12} - a_{11}\Delta x_{11} - a_{12}\Delta x_{21})$$

$$+ 0.345(x_{21} - a_{21}x_{10} - a_{22}x_{20} - \lambda) - 0.1725(x_{11} - a_{11}x_{10} - a_{12}x_{20}), \quad \text{etc.}$$

## NUMERICAL ILLUSTRATION OF METHOD 2

Assume, again, the parameter values $\sigma_1^2 = 4$, $\sigma_2^2 = 4$, $\sigma_{12} = 2$, $\sigma_3^2 = 1$. Then

$$\Omega_{11}^* = \begin{bmatrix} 4 & 2 \\ 2 & 5 \end{bmatrix}, \quad \Omega_1^* = \begin{bmatrix} 0 & 0 \\ -2 & -4 \end{bmatrix}, \quad \Omega_0^* = \begin{bmatrix} 4 & 2 \\ 2 & 9 \end{bmatrix}, \quad M_{11}^* = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix},$$

$$M_{21}^* = \Omega_1^*[M_{11}^{*\prime}]^{-1} = \begin{bmatrix} 0 & 0 \\ -2 & -4 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 \\ -1 & -1.5 \end{bmatrix},$$

$$M_{22}^* M_{22}^{*\prime} = \Omega_0^* - M_{21}^* M_{21}^{*\prime} = \begin{bmatrix} 4 & 2 \\ 2 & 9 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ -1 & -1.5 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 0 & -1.5 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 5.75 \end{bmatrix},$$

$$M_{22}^* = \begin{bmatrix} 2 & 0 \\ 1 & 2.18 \end{bmatrix},$$

$$M_{32}^* = \Omega_1^*[M_{22}^{*\prime}]^{-1} = \begin{bmatrix} 0 & 0 \\ -2 & -4 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & 2.18 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 \\ -1 & -1.38 \end{bmatrix},$$

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$$M_{33}^* M_{33}^{*\prime} = \Omega_0^* - M_{32}^* M_{32}^{*\prime}$$

$$= \begin{bmatrix} 4 & 2 \\ 2 & 9 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ -1 & -1.38 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 0 & -1.38 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 6.10 \end{bmatrix},$$

$$M_{33}^* = \begin{bmatrix} 2 & 0 \\ 1 & 2.26 \end{bmatrix}, \quad \text{etc.}$$

We note that for $t = 1, 2, 3$ $M_{tt}^* = M_{tt}$ but $M_{t,t-1}^* \neq M_{t,t-1}$. This pattern will obviously continue as $t$ increases.

$$\epsilon_1^* = [M_{11}^*]^{-1} \begin{bmatrix} x_{11} - a_{11}x_{10} - a_{12}x_{20} \\ x_{21} - a_{21}x_{10} - a_{22}x_{20} - \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} x_{11} - a_{11}x_{10} - a_{12}x_{20} \\ x_{21} - a_{21}x_{10} - a_{22}x_{20} - \lambda \end{bmatrix} = \begin{bmatrix} \epsilon_{11}^* \\ \epsilon_{21}^* \end{bmatrix},$$

where

$$\epsilon_{11}^* = 0.50(x_{11} - a_{11}x_{10} - a_{12}x_{20}),$$

$$\epsilon_{21}^* = 0.50(x_{21} - a_{21}x_{10} - a_{22}x_{20} - \lambda) - 0.25(x_{11} - a_{11}x_{10} - a_{12}x_{20}),$$

$$\epsilon_2^* = [M_{22}^*]^{-1} \begin{bmatrix} x_{12} - a_{11}x_{11} - a_{12}x_{21} \\ \Delta x_{22} - a_{21}\Delta x_{11} - a_{22}\Delta x_{21} - \lambda \end{bmatrix} - [M_{22}^*]^{-1} M_{21}^* \epsilon_1^*$$

$$= \begin{bmatrix} 2 & 0 \\ 1 & 2.18 \end{bmatrix}^{-1} \begin{bmatrix} x_{12} - a_{11}x_{11} - a_{12}x_{21} \\ \Delta x_{22} - a_{21}\Delta x_{11} - a_{22}\Delta x_{21} - \lambda \end{bmatrix}$$

$$- \begin{bmatrix} 2 & 0 \\ 1 & 2.18 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 \\ -1 & -1.5 \end{bmatrix} \begin{bmatrix} \epsilon_{11}^* \\ \epsilon_{21}^* \end{bmatrix} = \begin{bmatrix} \epsilon_{12}^* \\ \epsilon_{22}^* \end{bmatrix},$$

where

$$\epsilon_{12}^* = 0.50(x_{12} - a_{11}x_{11} - a_{12}x_{21}),$$

$$\epsilon_{22}^* = 0.46(\Delta x_{22} - a_{21}\Delta x_{11} - a_{22}\Delta x_{21} - \lambda) - 0.23(\Delta x_{12} - a_{11}\Delta x_{11} - a_{12}\Delta x_{21})$$

$$+ 0.345(x_{21} - a_{21}x_{10} - a_{22}x_{20} - \lambda) - 0.1725(x_{11} - a_{11}x_{10} - a_{12}x_{20}), \quad \text{etc.}$$

We see from the above results that $\epsilon_1^* = \epsilon_1$ and $\epsilon_2^* = \epsilon_2$. Moreover, since $M_{33}^* = M_{33}$ and $M_{32}^*$ is related to $M_{32}$ in the same way as $M_{21}^*$ is related to $M_{21}$, it is

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fairly obvious that we will have $\epsilon_3^* = \epsilon_3$ and that this pattern will continue for higher values of $t$.

## COMPARISON OF METHODS

The above results show, for a particular set of parameter values and $t \leq 3$, that $M_{tt} = M_{tt}^*$ and $\epsilon_t = \epsilon_t^*$. It is easy to show, by a more general argument, that these equalities must hold for any set of parameter values and all positive integer values of $t$.

It follows from (26) and (52) that $M_{tt}\epsilon_t$ is the deviation of the vector $[\Delta x_{1t}, \Delta x_{2t}]'$ from its conditional mean, conditional on all information up to time $t-1$, while $M_{tt}^*\epsilon_t^*$ is the deviation of the vector $[x_{1t}, \Delta x_{2t}]'$ from its conditional mean, conditional on all information up to time $t-1$. Since these deviations are obviously equal we have

$$M_{tt}\epsilon_t = M_{tt}^*\epsilon_t^* \quad (t = 1, \ldots, T). \tag{56}$$

It follows, using (24) and (50), that $M_{tt}M_{tt}'$ is the conditional covariance matrix of the vector $[\Delta x_{1t}, \Delta x_{2t}]'$ conditional on information up to time $t-1$, while $M_{tt}^*M_{tt}^{*\prime}$ is the conditional covariance matrix of $[x_{1t}, \Delta x_{2t}]'$ conditional on information up to time $t-1$. Since these are also equal, we have

$$M_{tt} = M_{tt}^* \quad (t = 1, \ldots, T). \tag{57}$$

It follows from (56) and (57) that

$$\epsilon_t = \epsilon_t^* \quad (t = 1, \ldots, T), \tag{58}$$

and, hence, from (29) and (55) that

$$L = L^*. \tag{59}$$

## IMPLICATIONS FOR A CONTINUOUS TIME MODEL

The results obtained for the above example have obvious implications for the first order continuous time model

$$dx(t) = [Ax(t) + c\mu(t)]\,dt + \zeta(dt), \tag{60}$$

$$d\mu(t) = \lambda dt + \zeta_3(dt), \tag{61}$$

where $x(t)$ is a $2 \times 1$ vector of observable variables that are observable at equispaced points of time, $A$ is a $2 \times 2$ matrix of parameters, $\mu(t)$ is an unobservable stochastic trend variable, $c$ is the $2 \times 1$ vector $[0, 1]'$, and $\zeta(dt)$ is a $2 \times 1$ vector of white noise innovations. The exact discrete model is

$$x(t) = e^A x(t-1) + A^{-1}(e^A - I)\,c\mu(t-1) + \eta_t, \tag{62}$$

$$\mu(t) - \mu(t-1) = \lambda + \eta_{3t}. \tag{63}$$

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We can transform (rotate) (62) in such a way that $\mu(t-1)$ occurs in only one equation of the transformed system. Then it is clear from the results obtained for the preceding discrete time model that it makes no difference to the computed value of the Gaussian likelihood whether we apply Method 1 or 2 to the transformed system, i.e., whether we start by differencing both equations of the transformed system or just the equation containing $\mu(t-1)$. Moreover, it obviously makes no difference to the computed value of the Gaussian likelihood whether we apply Method 1 directly to (62) or to the transformed system, and the direct application to (62) is, computationally, simpler.

## *REFERENCE*

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