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PAPER

Inquisitive split and structural completeness

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Abstract

In this paper, the notion of structural completeness is explored in the context of a generalized class of superintuitionistic logics that also involve systems that are not closed under uniform substitution. We just require that each logic must be closed under D-substitutions assigning to atomic formulas only $\vee$-free formulas. For these systems, we introduce four different notions of structural completeness and study how they are related. We focus on superintuitionistic inquisitive logics that validate a schema called Split and have the disjunction property. In these logics, disjunction can be interpreted in the sense of inquisitive semantics as a question-forming operator. It is shown that a logic is structurally complete with respect to D-substitutions if and only if it validates Split. Various consequences of this result are explored. For example, it is shown that every superintuitionistic inquisitive logic can be characterized by a Kripke model built from D-substitutions. We also formulate an algebraic counterpart of this result that says that the Lindenbaum–Tarski algebra $\mathscr{H}$ of any inquisitive logic can be embedded into the Heyting algebra formed from left ideals of endomorphisms on $\mathscr{H}$. Additionally, we resolve a conjecture concerning superintuitionistic inquisitive logics due to Miglioli et al. and show that a false conjecture about superintuitionistic logics due to Minari and Wroński becomes true in the broader space of regular generalized superintuitionistic logics.

Keywords: Inquisitive logic; superintuitionistic logics; structural completeness; substitution; Kripke semantics

1. Introduction

The property of *structural completeness* of a logic Λ is satisfied when every rule admissible in Λ is also derivable in Λ. Since its introduction by Pogorzelski (1971), the property has been studied in a number of contexts, in fuzzy logics in Cintula and Metcalfe (2009) or more generally in substructural logics in Olson et al. (2008), but most exhaustively in the context of superintuitionistic (also known as intermediate) logics, for example, in Citkin (1978, 2018).

Research on structural completeness has largely assumed the requirement of closure under uniform substitutions. A notable exception to this requirement is found in *inquisitive logic* (Ciardelli and Roelofsen, 2011). Inquisitive logic makes up a framework in which declarative propositions (asserting information) are distinguished from inquisitive propositions (in which a question is posed). In order to preserve the distinction, uniform substitution must be relaxed.

As Punčochář (2016) documents, there is an interesting class of superintuitionistic inquisitive logics that inhabit the space of theories extending intuitionistic logic but are not closed under all substitutions. Both Miglioli et al. (1989) and Iemhoff and Yang (2016) touch on properties concerning structural completeness relevant to inquisitive logics. Miglioli et al. (1989) consider variants of structural completeness over a wide range of theories in which substitutivity might

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fail, emphasizing two particular inquisitive logics, while Iemhoff and Yang (2016) consider the inquisitive logic InqB among related systems of logics of dependency. While both studies have lessons for the structure of inquisitive logics neither focuses on the whole class of these logics.

In what follows, we investigate structural completeness from the perspective of intuitionistic inquisitive logic and its extensions. Several interesting facts become clear from this standpoint, including the fact that all superintuitionistic inquisitive logics enjoy a property of hereditary structural completeness and that hereditary structural completeness coincides with structural completeness *simpliciter* in the space of extensions of intuitionistic inquisitive logic. We apply these results to expose a relationship with extensions of Gödel-Dummett logic LC and to explore the features of Kripke models for inquisitive logics built from substitutions. Along the way, we resolve a two-part conjecture concerning inquisitive logics due to Miglioli et al. made in Miglioli et al. (1989) and show that a false conjecture about superintuitionistic logics due to Minari and Wroński (1988) becomes true in the broader space of "regular" generalized superintuitionistic logics.

This paper is a revised and extended version of the conference paper (Ferguson and Punčochář, 2023). Some parts of the text are completely new, for example, Theorems 4 and 5, including the proceeding discussion in Section 3 and the algebraic part of Section 5. Other parts are significantly revised. Most importantly, we corrected a mistake in the formulation of the crucial Lemma 9, corresponding to Lemma 2 in Ferguson and Punčochář (2023), by slightly strengthening the condition expressed in the antecedent of the statement.

2. Intuitionistic Inquisitive Logic and Its Extensions

The standard inquisitive logic (Ciardelli, 2009, 2016; Ciardelli and Roelofsen, 2011; Ciardelli et al., 2019) uses an "information-based" semantics for classical logic that allows us to add to the language and characterize semantically some question-forming operators, like inquisitive disjunction on the propositional level and an inquisitive existential quantifier on the first-order level. Since we will focus in this paper on the propositional level, we will be concerned only with inquisitive disjunction. This operator, when applied to statements S_1, S_2, forms the question *whether* S_1 *or* S_2, which can be contrasted with the statement *that* S_1 *or* S_2. Interestingly, this construction can be embedded under other operators, thus allowing one to form, for example, conjunctive questions and conditional questions. It is also possible to form disjunctive questions with more than two alternatives (*whether* S_1, S_2, *or* S_3, and so on) and polar (yes/no) questions as a special kind of disjunctive question (*whether* S_1 *or not* S_1).

Inquisitive disjunction has some constructive features and resembles intuitionistic disjunction, though, as we will see, it is not identical with it. Standard inquisitive logic can thus be viewed as classical logic extended with this constructive operator. It is possible to vary almost arbitrarily the background logic of declarative sentences, while keeping fixed the most characteristic features of inquisitive semantics. In this way, we obtain nonclassical inquisitive logics. A general semantic and syntactic theory of these logics is developed in Punčochář (2019). An important example of such a logic is intuitionistic inquisitive logic (Ciardelli et al., 2020; Punčochář, 2016, 2021). It can be presented in two different ways. One can take the standard language of intuitionistic logic and add inquisitive disjunction to this language as an extra operator, as in Punčochář (2017) and Ciardelli et al. (2020). Alternatively, one can work just with the standard language involving only one disjunction, which is however interpreted in the inquisitive way, as in Ciardelli (2009), Ciardelli and Roelofsen (2011), and Punčochář (2021). In this paper, we will take the latter approach, which has the advantage that the axiomatization of the corresponding logic is much simpler and more elegant, and it can be compared directly with other logical systems in the same language that have been already studied in the literature on superintuitionistic logics.

However, the main reason for the decision to work with only one disjunction is that our main results concerning structural completeness hold only for this restricted language and they cease to be valid if the second disjunction is added. A precise formulation of this fact would require a

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lot of extra space because we would need to fully describe the system with the two disjunctions. Nevertheless, we can provide some explanation without going into the details by pointing out that intuitionistic logic is not structurally complete, and if we add intuitionistic declarative disjunction to the language, structural completeness fails for the same reasons for which it fails in the standard language of intuitionistic logic. For example, the Harrop rule with declarative disjunction, which is often used as a counterexample to structural completeness in intuitionistic logic, would be admissible but not derivable in the system.

Note that this is not in conflict with the results in Iemhoff and Yang (2016), which show structural completeness for dependence logic considering languages that involve inquisitive as well as declarative disjunction (called tensor in this context). Dependence logic is based on classical logic, which is structurally complete, and so the presence of the second disjunction does not cause any problems. Our main results point to an interesting discrepancy between intuitionistic and inquisitve disjunction. While inquisitive disjunction preserves the constructive features of intuitionistic disjunction, its addition does not, unlike the addition of intuitionistic disjunction, lead to a violation of structural completeness.

In this paper, we will work with this basic language:

$$\varphi ::= p \mid \bot \mid \varphi \to \varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi$$

where $\vee$ will be viewed as an inquisitive disjunction. Negation, equivalence, and a constant for validity are defined in the following usual way:

$$\neg\varphi =_{def} \varphi \to \bot, \varphi \leftrightarrow \psi =_{def} (\varphi \to \psi) \wedge (\psi \to \varphi), \top =_{def} \bot \to \bot.$$

Fix any Hilbert-style axiomatization of intuitionistic logic in this language, which has modus ponens as the only rule of inference. Let IL denote the set of its derivable formulas. We write $\vdash_{\mathsf{IL}} \varphi$ instead of $\varphi \in \mathsf{IL}$, $\varphi \vdash_{\mathsf{IL}} \psi$ instead of $\varphi \to \psi \in \mathsf{IL}$, and $\varphi \equiv_{\mathsf{IL}} \psi$ instead of $\varphi \leftrightarrow \psi \in \mathsf{IL}$. (The same notation will be used also for other logics.)

An axiomatization of intuitionistic inquisitive logic is obtained by extending the system for intuitionistic logic with the following schema called *Split*:

$$(\alpha \to (\psi \vee \chi)) \to ((\alpha \to \psi) \vee (\alpha \to \chi)),$$

where α ranges over $\vee$-free formulas.[1] The set of derivable formulas will be denoted as InqIL. The Split schema can be viewed as a piece that is missing in intuitionistic logic in order to prove inductively the following disjunctive form theorem, which is a cornerstone of propositional inquisitive logic that has been proved and used for many of its incarnations.

Theorem 1. *For any formula φ, there are $\vee$-free formulas $\alpha_1, \ldots, \alpha_n$ such that*

$$\varphi \equiv_{\mathsf{InqIL}} \alpha_1 \vee \ldots \vee \alpha_n.$$

Proof. One can proceed by induction. We show just the inductive step for implication. Assume that $\psi \equiv_{\mathsf{InqIL}} \beta_1 \vee \ldots \vee \beta_m$ and $\chi \equiv_{\mathsf{InqIL}} \gamma_1 \vee \ldots \vee \gamma_n$. Let $I = \{1, \ldots, m\}$ and $J = \{1, \ldots, n\}$. Then (using Split in the equivalence between the second and the third line):

$$\begin{aligned}
\psi \to \chi &\equiv_{\mathsf{InqIL}} \bigvee_{i\in I} \beta_i \to \bigvee_{j\in J} \gamma_j \\
&\equiv_{\mathsf{InqIL}} \bigwedge_{i\in I} (\beta_i \to \bigvee_{j\in J} \gamma_j) \\
&\equiv_{\mathsf{InqIL}} \bigwedge_{i\in I} \bigvee_{j\in J} (\beta_i \to \gamma_j) \\
&\equiv_{\mathsf{InqIL}} \bigvee_{f: I\to J} \bigwedge_{i\in I} (\beta_i \to \gamma_{f(i)}).
\end{aligned}$$

□

The Split schema can be formulated in a stronger form, using the notion of a Harrop formula. A formula is called a *Harrop formula* if disjunction occurs in it only within antecedents

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of implications. So, the only allowed occurrences of disjunction are in the following context: $(\ldots((\ldots\vee\ldots)\to\ldots)\ldots)$. It can be observed that for every Harrop formula φ, there is a $\vee$-free formula α such that $\varphi\equiv_{\mathsf{InqIL}}\alpha$. To see this, assume that φ contains a subformula $\psi\to\chi$. The antecedent ψ may possibly involve a disjunction, but all such disjunctions can be eliminated, since, according to Theorem 1, there are $\vee$-free formulas $\beta_1,\ldots,\beta_n$ such that $\psi\equiv_{\mathsf{InqIL}}\beta_1\vee\ldots\vee\beta_m$, and so $\psi\to\chi\equiv_{\mathsf{InqIL}}\bigvee_i\beta_i\to\chi\equiv_{\mathsf{InqIL}}\bigwedge_i(\beta_i\to\chi)$. As a consequence of this observation, InqIL can be equivalently axiomatized by extending intuitionistic logic with the schema that we can call *H-Split* and which is like Split except that α ranges over arbitrary Harrop formulas.

Nevertheless, the restriction on the antecedents in Split (or H-Split) is important and cannot be completely avoided. Not all substitutional instances of Split hold in InqIL. For example:

$$((p\vee q)\to(p\vee q))\to(((p\vee q)\to p)\vee((p\vee q)\to q))\notin\mathsf{InqIL}.$$

In this respect, InqIL is an unusual logic because it is not closed under uniform substitution. This fact is however well-motivated, given that the logic deals with two different categories of propositions: statements and questions. Questions are generated only by the inquisitive operator, and so every $\vee$-free formula represents a statement. It is not surprising that questions behave differently and in some contexts cannot be substituted for statements.

In this paper, we want to explore InqIL within the context of superintuitionistic logics. However, since superintuitionistic logics are required to be closed under uniform substitution, we need to introduce a more general notion that will encompass also InqIL (and other inquisitive logics). For this purpose, we use the following definitions.

Definition 1. *A substitution is a function that assigns a formula to each atomic formula. An H-substitution is a substitution that assigns to each atomic formula a Harrop formula. A D-substitution is a substitution that assigns to each atomic formula a $\vee$-free formula.*[2] *If s is a substitution and φ is a formula, then $s(\varphi)$ denotes the formula that is obtained from φ by replacing simultaneously every occurrence of each atomic formula p in φ with the formula $s(p)$.*

Definition 2. *A gsi-logic (generalized superintuitionistic logic) is any set of formulas that*

(i) *contains all intuitionistically valid formulas,*
(ii) *does not contain $\bot$,*
(iii) *is closed under every D-substitution,*
(iv) *is closed under modus ponens.*[3]

We say that a gsi-logic is regular if it is closed under every H-substitution and it is standard if it is closed under all substitutions.

Note that the notion of a standard gsi-logic coincides with the standard notion of a superintuitionistic (also called intermediate) logic. It is clear that InqIL is a gsi-logic, though not standard. Another example of a nonstandard gsi-logic is the classical inquisitive logic, often called basic inquisitive logic and denoted as InqB (see Ciardelli (2016)), which can be obtained by adding to InqIL the restricted double negation law *DN*: $\neg\neg\alpha\to\alpha$, where α ranges over $\vee$-free (or, equivalently, Harrop) formulas. Note that the $\vee$-free fragment of InqB is identical with classical logic, while it can be shown that the $\vee$-free fragment of InqIL is identical with (the $\vee$-free fragment of) intuitionistic logic.

Next we define, in accordance with Punčochář (2021), the general notion of an inquisitive gsi-logic.

Definition 3. *We say that a gsi-logic Λ is inquisitive if it (a) contains all instances of Split and (b) has the disjunction property, that is, $\alpha\vee\beta\in\Lambda$ only if $\alpha\in\Lambda$ or $\beta\in\Lambda$. Instead of saying that a gsi-logic Λ contains all instances of Split, we will sometimes say that Split is valid in Λ.*

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The notion of an inquisitive gsi-logic coincides with the notion of a G-logic introduced in Punčochář (2016). It can be shown that both InqIL and InqB have the disjunction property and thus are indeed inquisitive in the sense of our general definition. In fact, InqIL is the weakest, and InqB is the strongest inquisitive gsi-logic. Every inquisitive gsi-logic is regular, but no inquisitive gsi-logic is an intermediate logic in the usual sense; that is, in our terminology, no inquisitive gsi-logic is standard. However, for any standard gsi-logic Λ, there is exactly one inquisitive gsi-logic that conservatively extends the $\vee$-free fragment of Λ. As a consequence, there are uncountably many inquisitive gsi-logics. For a justification of these claims, see Punčochář (2016).

To the best of our knowledge, InqIL and InqB were both studied for the first time in Miglioli et al. (1989), under the names $\mathsf{F}_{\mathsf{int}}$ and F_{cl}, where it was proved, for instance, that the schematic fragment of InqB (called the standardization of F_{cl} in Miglioli et al., (1989)), that is, the set $S(\mathsf{InqB}) = \{\varphi \mid s(\varphi) \in \mathsf{InqB}, \text{for every substitution } s\}$, is identical with Medvedev's logic of finite problems ML. Interestingly, the same was stated without proof also for InqIL. That this quite nontrivial claim is true follows from the main result of Grilletti (2022).

The system of InqB was later rediscovered in Ciardelli (2009) and Ciardelli and Roelofsen (2011) and proved to be sound and complete with respect to the modern version of inquisitive semantics. This logic was applied to linguistic phenomena (see Ciardelli et al. (2019) for an overview) but also studied from algebraic and topological perspectives (Roelofsen, 2013; Frittella et al., 2016; Bezhanishvili et al., 2019). There is also an extensive literature on various modal extensions of InqB (Ciardelli and Roelofsen, 2015; van Gessel, 2020, 2022; Punčochář, 2015; Punčochář and Sedlár, 2021).

A generalization of inquisitive semantics that corresponds to InqIL was introduced in Punčochář (2016). Intuitionistic inquisitive logic was further studied in Ciardelli et al. (2020), Punčochář (2017), Sano (2022), and Punčochář and Pezlar (2024) and from an algebraic perspective in Punčochář (2021) and Quadrellaro (2022). A slightly different approach to intuitionistic inquisitive logic was proposed in Holliday (2022). This approach is based on a more general framework that does not validate Split.

A recent interesting result also shows that the system of InqIL provides a sound and complete axiomatization of proof-theoretic semantics (Stafford, 2021). This connection to proof-theoretic semantics nicely stresses the significance of this logic.

3. Structural Completeness

In this section, we show that the Split schema is quite intimately connected to the notion of structural completeness. Structural completeness is usually studied in relation to logics that are closed under arbitrary substitutions. For a more general notion of a logic, we need a more flexible notion of structural completeness. In particular, we can consider various kinds of structural completeness defined in terms of some restricted classes of substitutions. Such notions were studied in depth in Miglioli et al. (1989), and they were employed also in Iemhoff and Yang (2016), where it was shown that InqB and some related propositional dependence logics are structurally complete with respect to a suitably adapted sense of the term. In Miglioli et al. (1989), the authors considered, besides other options, structural completeness defined in terms of H-substitutions, and they called the corresponding notion *H-smoothness*. We will call it *SH-completeness*. Besides that, we introduce three other notions of structural completeness.

Definition 4. *Let Λ be a gsi-logic. By $sub(\Lambda)$, we denote the set of all substitutions under which Λ is closed, that is, $s \in sub(\Lambda)$ iff $s(\varphi) \in \Lambda$, for every $\varphi \in \Lambda$. We say that Λ is SF-complete (structurally fully complete) if it holds:*

(a) $\varphi \vdash_\Lambda \psi$ *iff for any substitution s, if $\vdash_\Lambda s(\varphi)$, then $\vdash_\Lambda s(\psi)$.*

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We say that Λ *is SG-complete (structurally generally complete) if it holds:*

(b) $\varphi \vdash_\Lambda \psi$ *iff for any* $s \in sub(\Lambda)$, *if* $\vdash_\Lambda s(\varphi)$, *then* $\vdash_\Lambda s(\psi)$.

We say that Λ *is SH-complete (structurally Harrop complete) if it holds:*

(c) $\varphi \vdash_\Lambda \psi$ *iff for any H-substitution* s, *if* $\vdash_\Lambda s(\varphi)$, *then* $\vdash_\Lambda s(\psi)$.

We say that Λ *is SD-complete (structurally declaratively complete) if it holds:*

(d) $\varphi \vdash_\Lambda \psi$ *iff for any D-substitution* s, *if* $\vdash_\Lambda s(\varphi)$, *then* $\vdash_\Lambda s(\psi)$.

We say that Λ *is hereditarily SF-complete (SG-complete, SH-complete, SD-complete) if every gsi-logic* $\Lambda' \supseteq \Lambda$ *is SF-complete (SG-complete, SH-complete, SD-complete).*

Note that the notions of *SF*-completeness and *SG*-completeness both generalize, in a clear sense, the standard notion of structural completeness. For standard gsi-logics (i.e., intermediate logics in the usual sense), both these notions are equivalent and coincide with the usual definition. Even though the notion of *SF*-completeness preserves the literal formulation of the usual definition and just applies it to a broader context, we think that *SG*-completeness provides a more natural generalization of structural completeness. This should be clear from the following observation that indicates that the notion of *SF*-completeness is reasonably applicable only to standard gsi-logics. We can also immediately observe that *SD*-completeness is stronger than *SG*-completeness.

Proposition 5. *Let* Λ *be a gsi-logic. Then:*

(a) *if* Λ *is SD-complete, then it is SG-complete;*
(b) Λ *is SF-complete if and only if it is standard and SG-complete.*

Proof. (a) Assume that Λ is *SD*-complete. Obviously, if $\varphi \vdash_\Lambda \psi$, then for any $s \in sub(\Lambda)$, if $\vdash_\Lambda s(\varphi)$, then $\vdash_\Lambda s(\psi)$. For the opposite direction, assume that for any $s \in sub(\Lambda)$, if $\vdash_\Lambda s(\varphi)$, then $\vdash_\Lambda s(\psi)$. Then also for any *D*-substitution s, if $\vdash_\Lambda s(\varphi)$, then $\vdash_\Lambda s(\psi)$, and thus $\varphi \vdash_\Lambda \psi$.

(b) Assume that Λ is *SF*-complete and $\vdash_\Lambda \varphi$. Then also $\vdash_\Lambda \top \to \varphi$. Take any substitution s. Since $\vdash_\Lambda s(\top)$, by *SF*-completeness, we obtain $\vdash_\Lambda s(\varphi)$. We have shown that *SF*-complete gsi-logics are standard. The rest follows from the observation that for standard gsi-logics, the notions of *SF*-completeness and *SG*-completeness coincide. □

Our aim in this section is to show that Split is closely related to *SH*-completeness and *SD*-completeness. Through this connection, we will also be able to see that these two notions of structural completeness are in fact equivalent. First, we can observe the following.

Proposition 6. *Every gsi-logic in which Split is valid is regular, that is, closed under all H-substitutions.*

Proof. Assume that Λ is a gsi-logic in which Split is valid, and take any *H*-substitution s. We have already observed that Split guarantees that every Harrop formula is equivalent to a $\vee$-free formula. In particular, for any p, there is a $\vee$-free formula α_p such that $s(p) \equiv_\Lambda \alpha_p$. Now we can define a *D*-substitution s^* fixing $s^*(p) = \alpha_p$, for each atomic p. If $\vdash_\Lambda \varphi$, then also $\vdash_\Lambda s^*(\varphi)$, since Λ is closed under *D*-substitutions. It follows that also $\vdash_\Lambda s(\varphi)$. □

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Let Λ be a gsi-logic that validates Split. Proposition 6 says that if s is an H-substitution, then $s \in sub(\Lambda)$. One can ask whether the converse also holds. If we formulate the converse directly, it is false for trivial reasons. If a substitution s assigns formulas that are not Harrop but are all equivalent to Harrop formulas, then Λ must also be closed under s. A more interesting question is whether Λ may also be closed under substitutions that are not equivalent to any H-substitution. Let us formulate it more precisely. We say that two substitutions, s and t, are Λ-*equivalent* if $s(p) \equiv_\Lambda t(p)$, for every atomic formula p. Then the converse of Proposition 6 would state that if $s \in sub(\Lambda)$, then s is Λ-equivalent to an H-substitution. Interestingly, this is a property that distinguishes InqIL from InqB.

Proposition 7. *Every $s \in sub(\mathsf{InqB})$ is* InqB*-equivalent to an H-substitution (and even D-substitution). In contrast, there is an $s \in sub(\mathsf{InqIL})$ that is not* InqIL*-equivalent to any H-substitution.*

Proof. For the first part, let s be a substitution under which InqB is closed, and let p be an arbitrary atomic formula. Then, as InqB proves $\neg\neg p \leftrightarrow p$, we obtain $\neg\neg s(p) \equiv_{\mathsf{InqB}} s(p)$. By Theorem 1, $s(p) \equiv_{\mathsf{InqB}} \alpha_1 \vee \ldots \vee \alpha_n$, where each α_i is $\vee$-free, whence $s(p) \equiv_{\mathsf{InqB}} \neg\neg(\alpha_1 \vee \ldots \vee \alpha_n)$. But as $(\neg\varphi \wedge \neg\psi) \leftrightarrow \neg(\varphi \vee \psi)$ holds intuitionistically, $s(p) \equiv_{\mathsf{InqB}} \neg(\neg\alpha_1 \wedge \ldots \wedge \neg\alpha_n)$, where $\neg(\neg\alpha_1 \wedge \ldots \wedge \neg\alpha_n)$ is $\vee$-free and thus Harrop.

For the second part of the proposition, fix an atomic formula p, and take the substitution s that assigns to every atomic formula the formula $p \vee \neg p$. Assume, for the sake of contradiction, that $p \vee \neg p \equiv_{\mathsf{InqIL}} \alpha$, for some Harrop formula α. Then, by Split and disjunction property, we would obtain $\vdash_{\mathsf{InqIL}} \alpha \to p$ or $\vdash_{\mathsf{InqIL}} \alpha \to \neg p$, and thus $\vdash_{\mathsf{InqIL}} (p \vee \neg p) \to p$ or $\vdash_{\mathsf{InqIL}} (p \vee \neg p) \to \neg p$. This leads to a contradiction because none of these formulas is valid in classical logic, which extends InqIL. So, s is not InqIL-equivalent to any H-substitution.

It remains to be shown that $s \in sub(\mathsf{InqIL})$. In order to show that a logic generated by modus ponens from a set of axioms is closed under a substitution s, it is sufficient to show that $s(\chi)$ is provable for every axiom χ. So, we have to show $\vdash_{\mathsf{InqIL}} s(\chi)$, for our specific s and every instance χ of Split. First, observe that for every formula χ, $s(\chi)$ is intuitionistically equivalent to one of these formulas: $\bot, p \vee \neg p, \top$. To see this, note that $\bot$ and $\top$ are generated from $p \vee \neg p$ by negation and the set $\{\bot, p \vee \neg p, \top\}$ is, up to intuitionistic equivalence, closed under the operations $\to, \wedge, \vee$. But then, if $s(\alpha), s(\varphi), s(\psi)$ are intuitionistically equivalent to any of the formulas $\bot, p \vee \neg p, \top$, the formula

$$(s(\alpha) \to (s(\varphi) \vee s(\psi))) \to ((s(\alpha) \to s(\varphi)) \vee (s(\alpha) \to s(\varphi)))$$

is intuitionistically valid. Hence, if we apply s to any instance of Split, we always obtain a formula that is valid in InqIL. □

We are not sure how to interpret philosophically the previous rather surprising result. What does it mean in terms of the inquisitive interpretation of the system that associates formulas equivalent to $\vee$-free formulas with statements and other formulas with questions? We do not have a clear explanation. Technically speaking, Proposition 7 shows that H-substitutions do not necessarily cover the space of all substitutions under which a logic validating Split is closed. Nevertheless, it will be clear from our main result, Theorem 2 below, that these substitutions play a special role in these logics.

We will rely heavily on a standard technique for proving structural completeness developed by Prucnal (1973), or more precisely, on its refinement introduced later by Minari and Wroński (1988). Let α be a Harrop formula and v a classical valuation (i.e., a function assigning to each formula 0 or 1 in accordance with the semantics of classical logic). Following Minari and Wroński (1988), we define, relative to α and v, the following H-substitution:

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$$s^{\nu}_{\alpha}(p) = \begin{cases} \alpha \to p & \text{if } \nu(p) = 1 \\ \neg\neg\alpha \wedge (\alpha \to p) & \text{otherwise} \end{cases}$$

Note that if α is a $\vee$-free formula, then s^{ν}_{α} is a D-substitution. We need the following two features of this substitution.

Lemma 8. *Let φ be any formula, α a Harrop formula, and ν a classical valuation. Then:*

(a) $s^{\nu}_{\alpha}(\varphi) \vdash_{\mathsf{IL}} \alpha \to \varphi$,
(b) *if* $\nu(\alpha) = 1$ *then* $\vdash_{\mathsf{IL}} s^{\nu}_{\alpha}(\alpha)$.

Proof. The claim (a) follows from Lemma 1 in Minari and Wroński (1988), which can be proved by induction and which states that for any formula φ:

$$\alpha \vdash_{\mathsf{IL}} \varphi \leftrightarrow s^{\nu}_{\alpha}(\varphi).$$

The claim (b) follows from Lemma 3 in Minari and Wroński (1988) that states that the property by which s^{ν}_{α} was defined is preserved by every Harrop formula β in this sense:

$$s^{\nu}_{\alpha}(\beta) \equiv_{\mathsf{IL}} \begin{cases} \alpha \to \beta & \text{if } \nu(\beta) = 1 \\ \neg\neg\alpha \wedge (\alpha \to \beta) & \text{otherwise} \end{cases}$$

□

This lemma implies the following one that generalizes the main result of Minari and Wroński (1988), which was originally formulated only for the usual intermediate logics, that is, in our terminology, standard gsi-logics.

Lemma 9. *Let Λ be a gsi-logic and α a Harrop formula. If Λ is regular or α is $\vee$-free, then for all formulas ψ, χ:*

$$\text{if } \alpha \to (\psi \vee \chi) \in \Lambda \text{ then } (\alpha \to \psi) \vee (\alpha \to \chi) \in \Lambda.$$

Proof. Assume $\alpha \to (\psi \vee \chi) \in \Lambda$. We want to prove $(\alpha \to \psi) \vee (\alpha \to \chi) \in \Lambda$. If $\neg\alpha \in \Lambda$, then the required conclusion is immediate. Assume $\neg\alpha \notin \Lambda$. Then, due to Glivenko's theorem, $\neg\alpha$ is not classically valid, and so there is a valuation ν such that $\nu(\alpha) = 1$. Then, by Lemma 8-b, $s^{\nu}_{\alpha}(\alpha) \in \Lambda$. If Λ is regular or α is $\vee$-free, then Λ is closed under s^{ν}_{α}. So, $s^{\nu}_{\alpha}(\alpha) \to (s^{\nu}_{\alpha}(\psi) \vee s^{\nu}_{\alpha}(\chi)) \in \Lambda$, and hence also $s^{\nu}_{\alpha}(\psi) \vee s^{\nu}_{\alpha}(\chi) \in \Lambda$. Thus, by Lemma 8-a, $(\alpha \to \psi) \vee (\alpha \to \chi) \in \Lambda$. □

The previous lemma plays a crucial role in the proof of one part of the next theorem, which is our main result relating Split with the notions of *SH*-completeness and *SD*-completeness.

Theorem 2. *For every gsi-logic Λ, the following claims are equivalent:*

(a) *Λ is SD-complete,*
(b) *Λ is SH-complete,*
(c) *Split is valid in Λ.*

Proof. First, assume that Λ is *SH*-complete. Then Λ must be regular ($\vdash_{\Lambda} \varphi$ implies $\vdash_{\Lambda} \top \to \varphi$, which, by *SH*-completeness, implies $\vdash_{\Lambda} s(\varphi)$, for every H-substitution s). Let α be a $\vee$-free formula and φ, ψ arbitrary formulas. By Lemma 9, for any H-substitution s, if $s(\alpha \to (\psi \vee \chi)) \in \Lambda$, then $s((\alpha \to \psi) \vee (\alpha \to \chi)) \in \Lambda$. It follows from *SH*-completeness that Split is valid in Λ. In a similar way, one can prove that *SD*-completeness implies the validity of Split.

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Second, assume that Split is valid in Λ. We show that $\varphi \vdash_\Lambda \psi$ iff for any H-substitution s, if $\vdash_\Lambda s(\varphi)$, then $\vdash_\Lambda s(\psi)$. The left-to-right direction follows immediately from Proposition 6. We prove the right-to-left direction. Assume that

(i) for any H-substitution s, if $\vdash_\Lambda s(\varphi)$, then $\vdash_\Lambda s(\psi)$.

We have to show that $\varphi \vdash_\Lambda \psi$. If $\vdash_\Lambda \neg\varphi$, we are done, so we can assume that $\nvdash_\Lambda \neg\varphi$. As Split is valid in Λ, we can take, due to the disjunctive form theorem, Harrop formulas $\alpha_1, \ldots, \alpha_n$ such that

(ii) $\varphi \equiv_\Lambda \alpha_1 \vee \ldots \vee \alpha_n$.

Let us assume that the disjunction is minimal; that is, φ is not equivalent with the disjunction of any proper subset of $\{\alpha_1, \ldots, \alpha_n\}$. It follows that for every α_i $(1 \leq i \leq n)$, $\nvdash_{\mathsf{IL}} \neg\alpha_i$ (otherwise the disjunction in (ii) would not be minimal). Thus, due to Glivenko's theorem, for each $1 \leq i \leq n$, $\nvdash_{\mathsf{CL}} \neg\alpha_i$, and so there is a classical valuation v_i such that $v_i(\alpha_i) = 1$. Then the following holds:

(1) $\vdash_\Lambda s^{v_i}_{\alpha_i}(\alpha_i)$ (by Lemma 8-b),
(2) $s^{v_i}_{\alpha_i}(\alpha_i) \vdash_\Lambda s^{v_i}_{\alpha_i}(\varphi)$ ($\alpha_i \vdash_\Lambda \varphi$, $s^{v_i}_{\alpha_i}$ is a H-substitution and Λ is closed under H-substitutions),
(3) $\vdash_\Lambda s^{v_i}_{\alpha_i}(\varphi)$ (from 1 and 2),
(4) $\vdash_\Lambda s^{v_i}_{\alpha_i}(\psi)$ (from 3 and the assumption (i)),
(5) $s^{v_i}_{\alpha_i}(\psi) \vdash_\Lambda \alpha_i \to \psi$ (by Lemma 8-a),
(6) $\alpha_i \vdash_\Lambda \psi$ (from 4 and 5).

Since $\alpha_i \vdash_\Lambda \psi$ holds for all $1 \leq i \leq n$, we obtain $\varphi \vdash_\Lambda \psi$ as required. In a similar way, one can prove that the validity of Split implies that the logic is *SD*-complete. □

In the rest of this paper, we will explore some consequences of this result. The first one is immediate, and it shows an interesting difference between *SF*-completeness/*SG*-completeness (on the one hand) and *SH*-completeness/*SD*-completeness (on the other).

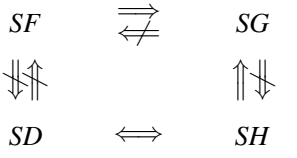
Corollary 10. *Every SH-complete (SD-complete) gsi-logic is hereditarily SH-complete* (*SD-complete).*

In contrast to this corollary, there are examples of gsi-logics that are *SF*- and *SG*-complete but neither hereditarily *SF*-complete nor hereditarily *SG*-complete. For instance, the logic of finite problems ML has this property (see Citkin (1978); Dzik and Wroński (1973)). The next corollary is based on the observation that *SD*-completeness is a stronger property than *SG*-completeness.

Corollary 11. InqIL *is hereditarily SG-complete.*

In other words, every gsi-logic that validates Split is *SG*-complete. The converse does not hold. ML does not validate Split, but it is *SG*-complete.

The relations between different kinds of structural completeness are summarized in the following picture:



ML is a counterexample to $SF \Longrightarrow SD(H)$ and $SG \Longrightarrow SD(H)$, and any inquisitive gsi-logic is a counterexample to $SD(H) \Longrightarrow SF$ and $SG \Longrightarrow SF$.

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We will conclude this section by another observation related to our main result, Theorem 2. In its proof, we relied heavily on the result from Minari and Wroński (1988) that states that for every standard gsi-logic Λ, for every Harrop formula α, and for all formulas ψ, χ:

$$\text{if } \alpha \to (\psi \vee \chi) \in \Lambda \text{ then } (\alpha \to \psi) \vee (\alpha \to \chi) \in \Lambda.$$

In Lemma 9, we had to formulate this claim more generally, which gave us two implications in the equivalences of Theorem 2 (i.e., (a) $\Rightarrow$ (c) and (b) $\Rightarrow$ (c)). Minari and Wroński (1988) concluded their paper by an open problem asking whether their result is best possible in the sense that the following two conditions are equivalent for every formula φ:

(a) for every standard gsi-logic Λ and for all formulas ψ, χ, it holds that if $\varphi \to (\psi \vee \chi) \in \Lambda$, then $(\varphi \to \psi) \vee (\varphi \to \chi) \in \Lambda$.

(b) $\varphi \equiv_{\mathsf{IL}} \alpha$, for some Harrop formula α.

Minari and Wroński's result is the implication from (b) to (a), and the question they posed is the implication from (a) to (b). This question is resolved negatively by Wojtylak (2004), who formulated a class of sentences satisfying (a) but not generally (b).

Definition 12. *A Wojtylak sentence is a sentence of the form* $\bigwedge_{1 \leq i \leq k} (p_i \leftrightarrow \varphi_i)$, *where* $\varphi_1, \ldots, \varphi_k$ *are arbitrary formulas and the atoms* $p_1, \ldots p_k$ *do not appear in* $\varphi_1, \ldots, \varphi_k$.

It is clear that not every Wojtylak sentence is intuitionistically equivalent to a Harrop formula, the most simple example being $p \leftrightarrow (q \vee r)$. Nevertheless, Wojtylak (2004) proved the following theorem.

Theorem 3. *Let* φ *be a Wojtylak sentence. Then for every standard gsi-logic* Λ *and for all formulas* ψ, χ, *it holds that if* $\varphi \to (\psi \vee \chi) \in \Lambda$, *then* $(\varphi \to \psi) \vee (\varphi \to \chi) \in \Lambda$.

This provides a remarkable feature distinguishing the usual intermediate logics, that is, standard gsi-logics, from inquisitive gsi-logics. One can show that no inquisitive gsi-logic has the property expressed in Theorem 3. Indeed, for every inquisitive gsi-logic Λ, we have (due to Split and the laws of intuitionistic logic):

$$(p \leftrightarrow (q \vee r)) \to ((p \to q) \vee (p \to r)) \in \Lambda.$$

However,

$$((p \leftrightarrow (q \vee r)) \to (p \to q)) \vee ((p \leftrightarrow (q \vee r)) \to (p \to r)) \notin \Lambda.$$

To see this, assume for contradiction that the disjunction is in Λ. Then, by the disjunction property of Λ, at least one of its disjuncts would be in Λ. But none of the disjuncts is valid in classical logic and thus cannot be valid in Λ.

Interestingly, we can resolve Minari and Wroński's open question positively if we do not restrict ourselves to the standard gsi-logics and instead expand the claim to all regular gsi-logics.

Theorem 4. *The following two conditions are equivalent for every formula* φ:

(a) *for every regular gsi-logic* Λ *and for all formulas* ψ, χ, *if it holds that* $\varphi \to (\psi \vee \chi) \in \Lambda$, *then it also holds that* $(\varphi \to \psi) \vee (\varphi \to \chi) \in \Lambda$.

(b) $\varphi \equiv_{\mathsf{IL}} \alpha$, *for some Harrop formula* α.

Proof. The implication from (b) to (a) follows from Lemma 9. For the implication from (a) to (b), assume that φ is not intuitionistically equivalent to any Harrop formula. We need to specify

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a concrete counterexample to (a). Let Λ be the logic InqIL (any inquisitive gsi-logic could be used here). Then $\varphi \equiv_\Lambda \alpha_1 \vee \ldots \vee \alpha_n$, for some Harrop formulas $\alpha_1, \ldots, \alpha_n$, where $n \geq 2$. We can assume that the disjunction is minimal in the sense that if we remove any disjunct, the equivalence will not hold any more. Then we can arbitrarily split the disjunction into two parts ψ, χ such that $\psi \vee \chi = \alpha_1 \vee \ldots \vee \alpha_n$, and we have $\varphi \to (\psi \vee \chi) \in \Lambda$, but $(\varphi \to \psi) \vee (\varphi \to \chi) \notin \Lambda$ because of the disjunction property of Λ. □

We can formulate an analogous result related to the class of all gsi-logics.

Theorem 5. *The following two conditions are equivalent for every formula* φ:

(a) *for every gsi-logic* Λ *and for all formulas* ψ, χ, *if it holds that* $\varphi \to (\psi \vee \chi) \in \Lambda$, *then it also holds that* $(\varphi \to \psi) \vee (\varphi \to \chi) \in \Lambda$.

(b) $\varphi \equiv_{\mathsf{IL}} \alpha$, *for some* $\vee$*-free formula* α.

Proof. Similar to the proof of Theorem 4. □

4. Schematic Closures of Inquisitive gsi-logics

In this section, we discuss some issues concerning *SF*-completeness, and for this purpose, we employ the following notation. If Λ is a gsi-logic and Δ a set of formulas, then $\Lambda \oplus \Delta$ will denote the set of formulas derivable from $\Lambda \cup \Delta$ by modus ponens. More precisely,

$$\Lambda \oplus \Delta = \{\varphi \mid \psi \wedge \chi_1 \wedge \ldots \wedge \chi_n \vdash_{\mathsf{IL}} \varphi \text{ for some } \psi \in \Lambda \text{ and } \chi_1, \ldots, \chi_n \in \Delta\}.$$

Note that if Δ is closed under *D*-substitutions, then so is $\Lambda \oplus \Delta$, and if, moreover, $\Lambda \cup \Delta$ is consistent, that is, $\bot \notin \Lambda \oplus \Delta$, then $\Lambda \oplus \Delta$ is the smallest gsi-logic including $\Lambda \cup \Delta$.

Let us recall that LC is the Gödel–Dummett fuzzy logic (Gödel, 1932; Dummett, 1959). It is often presented as an extension of intuitionistic logic by the prelinearity schema:

$$(\varphi \to \psi) \vee (\psi \to \varphi).$$

Let us denote the set of all instances of this schema as *PreLin*, and let *FullSplit* denote the set of instances of the schema (where φ is not restricted)

$$(\varphi \to (\psi \vee \chi)) \to ((\varphi \to \psi) \vee (\varphi \to \chi)).$$

An obvious connection of LC to the logics we are focused on in this paper is given by the following observation made in Dummett (1959).

Proposition 13. $\mathsf{LC} = \mathsf{IL} \oplus \mathit{PreLin} = \mathsf{IL} \oplus \mathit{FullSplit}$.

So, LC is a standard extension of InqIL (= IL $\oplus$ *Split*). It is well-known that the logic LC is hereditarily structurally complete in the class of standard gsi-logics (shown in Dzik and Wroński (1973)). This result can also be obtained as a direct consequence of our Corollary 11, if we recall that *SG*-completeness generalizes the usual structural completeness. A natural question arises whether LC is also hereditarily *SF*-complete over the space of gsi-logics in general.

When one considers results over the case of standard gsi-logics, there is reason to be cautious before importing them to our general setting. That a logic is hereditarily structurally complete over the standard gsi-logics does not *a priori* entail hereditary *SF*-completeness over the broader space of gsi-logics. To see that LC is indeed hereditarily *SF*-complete, we need to use the following observation that is due to Dummett (1959).

Lemma 14. $\varphi \vee \psi \equiv_{\mathsf{LC}} ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi)$.

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This result shows that disjunction is definable in LC. In fact, as pointed out in Minari and Wroński (1988), every (standard) gsi-logic in which disjunction is definable includes LC. The previous lemma implies the following one.

Lemma 15. *Every gsi-logic that includes* LC *is standard.*

Proof. Take any gsi-logic Λ that includes LC, any $\varphi \in \Lambda$, and any substitution s. It follows from Lemma 14 that for every formula ψ, there is a $\vee$-free formula α_ψ such that $\psi \equiv_\Lambda \alpha_\psi$. We define a D-substitution s^* fixing $s^*(p) = \alpha_{s(p)}$, for every atom p. Note that s^* is Λ-equivalent to s. Since $s^*(\varphi) \in \Lambda$, we also have $s(\varphi) \in \Lambda$. Hence, Λ is standard. □

Using this lemma, we can prove the following result.

Theorem 6. LC *is hereditarily SF-complete over all gsi-logics.*

We can even strengthen this result in the following way.

Theorem 7. *Let* Λ *be a gsi-logic including* InqIL. *Then the following claims are equivalent:*

(a) Λ *is hereditarily SF-complete,*
(b) Λ *is SF-complete,*
(c) Λ *is standard,*
(d) Λ *includes all instances of FullSplit*,
(e) *if* $\vdash_\Lambda \varphi \to (\psi \vee \chi)$, *then* $\vdash_\Lambda (\varphi \to \psi) \vee (\varphi \to \chi)$,
(f) $\mathsf{LC} \subseteq \Lambda$.

Proof. $(a) \Rightarrow (b)$ is immediate by definition. $(b) \Rightarrow (c)$ is from Proposition 5. $(c) \Rightarrow (d)$: As a standard extension of InqIL, Λ must contain all instances of *FullSplit*. $(d) \Rightarrow (e)$ is immediate. $(e) \Rightarrow (f)$: As an extension of IL, Λ includes all instances of $(\psi \vee \varphi) \to (\psi \vee \varphi)$. By the assumption (e), we obtain $((\psi \vee \varphi) \to \psi) \vee ((\psi \vee \varphi) \to \varphi) \in \Lambda$. Since $\varphi \to (\psi \vee \varphi)$, $\psi \to (\psi \vee \varphi)$ are theorems, we obtain $(\varphi \to \psi) \vee (\psi \to \varphi) \in \Lambda$. By Proposition 13, $\mathsf{LC} \subseteq \Lambda$. $(f) \Rightarrow (a)$: By Theorem 6, LC is hereditarily *SF*-complete, whence in virtue of the inclusion of LC in Λ, Λ is hereditarily *SF*-complete. □

It is clear that the logic LC is what we obtain if we close InqIL under all substitutions. Let us consider this operation of *schematic closure* from a more general perspective. We have already indicated that any gsi-logic Λ determines its *schematic fragment*:

$$S(\Lambda) = \{\varphi \mid s(\varphi) \in \Lambda, \text{for every substitution } s\},$$

which is a standard gsi-logic. It is a rather surprising fact that all inquisitive gsi-logics have the same schematic fragment, namely, ML. This follows from a combination of results in Ciardelli and Roelofsen (2011), showing that ML is the schematic fragment of the strongest inquisitive gsi-logic InqB and the main result of Grilletti (2022), which implies that ML is also the schematic fragment of the weakest inquisitive gsi-logic InqIL.

The operation of schematic closure that we denote as C can be generally defined as follows: $\varphi \in C(\Lambda)$ iff $s_1(\psi_1) \wedge \ldots \wedge s_n(\psi_n) \vdash_{\mathsf{IL}} \varphi$, for some substitutions $s_1, \ldots, s_n$ and some $\psi_1, \ldots, \psi_n \in \Lambda$.

Note that $C(\Lambda)$ is indeed a standard gsi-logic. In general, it holds that $S(\Lambda) \subseteq \Lambda \subseteq C(\Lambda)$. In fact, $S(\Lambda)$ is the greatest standard gsi-logic below Λ, and $C(\Lambda)$ is the smallest standard gsi-logic above Λ. So, if Λ is itself standard, we obtain $S(\Lambda) = \Lambda = C(\Lambda)$. In the rest of this section, we explore the schematic closures of inquisitive gsi-logics.

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It is clear that $C(\mathsf{InqIL}) = \mathsf{LC}$ and $C(\mathsf{InqB}) = \mathsf{CL}$. What do the schematic closures of the other inquisitive gsi-logics look like? The standard gsi-logics that include LC form a chain consisting of the multivalued logics G_n, $n \geq 3$, plus the classical logic CL on the top. If we denote classical logic as G_2 and the logic LC as G_ω, then we have (see Gödel (1932)):

$$\mathsf{G}_\omega \subseteq \ldots \subseteq \mathsf{G}_5 \subseteq \mathsf{G}_4 \subseteq \mathsf{G}_3 \subseteq \mathsf{G}_2.$$

The schematic closure of every gsi-logic extending InqIL will be one of the G-logics. For those gsi-logics that include InqIL and have the disjunction property, that is, for the inquisitive gsi-logics, we can characterize the schematic closures in an elegant and systematic way.

For any set of formulas Δ, let Δ^{df} denote the $\vee$-free fragment of Δ, that is, $\Delta^{df} = \{\varphi \in \Delta \mid \varphi \text{ is } \vee\text{-free}\}$. The disjunction property of inquisitive logics is crucial for the proof of the following lemma.

Lemma 16. *Let Λ be any inquisitive gsi-logic and G_n, where $n \in \{2, 3, 4, \ldots, \omega\}$, any gsi-logic including* LC. *Then:*

(a) $\Lambda = \mathsf{InqIL} \oplus \Lambda^{df}$,

(b) $\mathsf{G}_n = \mathsf{LC} \oplus \mathsf{G}_n^{df}$.

Proof. (a) Let Λ be any inquisitive gsi-logic. Clearly, it holds that $\mathsf{InqIL} \oplus \Lambda^{df} \subseteq \Lambda$. Assume $\varphi \in \Lambda$. Let $\varphi \equiv_{\mathsf{InqIL}} \alpha_1 \vee \ldots \vee \alpha_n$, where each α_i is $\vee$-free. By the disjunction property, for some i, $\alpha_i \in \Lambda$ and thus $\alpha_i \in \Lambda^{df}$. Hence, $\varphi \in \mathsf{InqIL} \oplus \Lambda^{df}$. (b) Clearly, $\mathsf{LC} \oplus \mathsf{G}_n^{df} \subseteq \mathsf{G_n}$. Assume $\varphi \in \mathsf{G}_n$. Then, by Lemma 14, there is $\vee$-free α such that $\varphi \equiv_{\mathsf{LC}} \alpha$. Then $\alpha \in \mathsf{G}_n^{df}$ and thus $\varphi \in \mathsf{LC} \oplus \mathsf{G}_n^{df}$. □

With the help of this lemma, we can characterize the schematic closures of inquisitive gsi-logics in the following way.

Theorem 8. *Let Λ be an inquisitive gsi-logic. Then*

$$C(\Lambda) = \mathsf{LC} \oplus \Lambda^{df} = \mathsf{G}_n, \text{ for } n = \max\{m \mid \Lambda^{df} \subseteq \mathsf{G}_m^{df}\}.$$

Proof. Assume that Λ is an inquisitive gsi-logic. Clearly, it holds that $\mathsf{LC} \oplus \Lambda^{df} \subseteq C(\Lambda)$. By Lemma 16-a, $\Lambda = \mathsf{InqIL} \oplus \Lambda^{df}$ and thus $\Lambda \subseteq \mathsf{LC} \oplus \Lambda^{df}$. By Lemma 15, $\mathsf{LC} \oplus \Lambda^{df}$ is standard, and since $C(\Lambda)$ is the smallest standard gsi-logic extending Λ, we obtain $C(\Lambda) \subseteq \mathsf{LC} \oplus \Lambda^{df}$. So, we have proved that $C(\Lambda) = \mathsf{LC} \oplus \Lambda^{df}$.

Clearly, $C(\Lambda) = \mathsf{G}_n$, for some n. Then it must hold $\Lambda^{df} \subseteq \mathsf{G}_n^{df}$. Assume, for the sake of contradiction, that $n \neq \omega$ and $\Lambda^{df} \subseteq \mathsf{G}_{n+1}^{df}$. Then, using Lemma 16-b, we obtain $C(\Lambda) = \mathsf{LC} \oplus \Lambda^{df} \subseteq \mathsf{LC} \oplus \mathsf{G}_{n+1}^{df} = \mathsf{G}_{n+1}$. But $C(\Lambda) \subseteq \mathsf{G}_{n+1}$ contradicts the assumption $C(\Lambda) = \mathsf{G}_n$. Thus $n = \max\{m \mid \Lambda^{df} \subseteq \mathsf{G}_m^{df}\}$. □

So, while $\Lambda = \mathsf{InqIL} \oplus \Lambda^{df}$, $C(\Lambda) = \mathsf{LC} \oplus \Lambda^{df}$. This result shows that the schematic closure operation collapses the uncountable space of inquisitive gsi-logics to a countable linear order of standard gsi-logics of order type $1 + \omega^*$. Moreover, this mapping to G_n-logics is surjective as the following proposition shows.

Proposition 17. *For any n, $\mathsf{InqIL} \oplus \mathsf{G}_n^{df}$ is inquisitive and $C(\mathsf{InqIL} \oplus \mathsf{G}_n^{df}) = \mathsf{G}_n$.*

Proof. Let $\Lambda = \mathsf{InqIL} \oplus \mathsf{G}_n^{df}$. We will show that Λ has the disjunction property, and so it is inquisitive. Assume $\varphi \vee \psi \in \Lambda$. Then there is $\alpha \in \mathsf{G}_n^{df}$ such that $\alpha \to (\varphi \vee \psi) \in \mathsf{InqIL}$. Applying Split

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and the disjunction property of InqIL, we obtain $\alpha \to \varphi \in \mathsf{InqIL}$ or $\alpha \to \psi \in \mathsf{InqIL}$. It follows that $\varphi \in \Lambda$ or $\psi \in \Lambda$. For the second part of the proposition, observe that $\Lambda^{df} = \mathsf{G}_n^{df}$, and so, by Theorem 8, $C(\Lambda) = \mathsf{G}_n$. □

5. Characteristic Kripke Models Built from Substitutions

Structural completeness is an important property of classical logic that intuitionistic logic lacks. On the other hand, the disjunction property is an important property of intuitionistic logic that classical logic violates. An interesting question is whether there are logics that have both these properties.

Definition 18. *A gsi-logic is optimal if it is SG-complete and has the disjunction property.*

The only standard gsi-logic that is known to be optimal is ML (see Wojtylak (2004)). As another direct consequence of Theorem 2, we obtain the following result showing that among nonstandard gsi-logics, there are uncountably many optimal logics.

Theorem 9. *Every inquisitive gsi-logic is optimal.*

We can also consider alternative notions of optimality involving the other kinds of structural completeness. For instance, we can say that a gsi-logic is *H-optimal* if it is *SH*-complete and has the disjunction property. It follows from our previous observations that every *H*-optimal gsi-logic is optimal. However, the inverse implication does not hold. The logic ML is not *H*-optimal because it does not validate Split and thus is not *SH*-complete. Theorem 2 immediately implies that inquisitive gsi-logics are exactly those gsi-logics that are *H*-optimal.

Let us point out that at the end of Miglioli et al. (1989) a conjecture is formulated, which, when translated in our terminology, says that (a) InqIL is *H*-optimal, and (b) InqIL and InqB are the only *H*-optimal gsi-logics.[4] Our results verify the first half of the conjecture but falsify the latter half.

In this section, we show that Theorem 9 is related to the possibility of an interesting construction of characteristic models for inquisitive gsi-logics. For any gsi-logic, one can construct, in the usual way, a canonical Kripke model built out of prime theories. A peculiar feature of inquisitive gsi-logics is that they can also be characterized by a Kripke model built directly out of consistent $\vee$-free formulas.[5] We will briefly formulate this construction and compare it with another unusual characteristic model construction that we obtain as an application of our main results. For the rest of this section, let us fix an inquisitive gsi-logic Λ. Moreover, let us say that α is *consistent* if $\nvdash_\Lambda \neg\alpha$.

Now we introduce the Kripke semantics for intuitionistic logic. A *Kripke frame* is a pair $\langle S, \leq\rangle$, where $\leq$ is a preorder, that is, a reflexive and transitive relation on S. A Kripke model is a Kripke frame equipped with a valuation V, that is, a function that assigns to each atomic formula an upward closed subset of S (i.e., if $w \in V(p)$ and $w \leq v$, then $v \in V(p)$). Given any Kripke model, the relation $\Vdash$ between states of the model and formulas is defined in the usual recursive way. For the atomic formulas, we set $w \Vdash p$ iff $w \in V(p)$. For the constant $\bot$ and complex formulas, the relation is determined as follows:

(a) $w \nVdash \bot$,
(b) $w \Vdash \varphi \to \psi$ iff for any $v \geq w$, if $v \Vdash \varphi$, then $v \Vdash \psi$,
(c) $w \Vdash \varphi \wedge \psi$ iff $w \Vdash \varphi$ and $w \Vdash \psi$,
(d) $w \Vdash \varphi \vee \psi$ iff $w \Vdash \varphi$ or $w \Vdash \psi$.

The relation is persistent: if $w \Vdash \varphi$ and $w \leq v$, then $v \Vdash \varphi$. We say that φ is valid in a model $\langle S, \leq, V\rangle$ if $w \Vdash \varphi$ holds in that model for every $w \in S$. It is well-known that a formula is intuitionistically valid iff it is valid in all Kripke models (Kripke, 1965). The recursive clauses for $\Vdash$ actually

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mirror some characteristic properties of $\vdash$ in inquisitive gsi-logics, as is shown in the following proposition.

Proposition 19. *Let α be a consistent $\vee$-free formula and φ, ψ arbitrary formulas. Then*

(a) $\alpha \nvdash_\Lambda \bot$,
(b) $\alpha \vdash_\Lambda \varphi \to \psi$ *iff for any consistent $\vee$-free β s.t. $\beta \vdash_\Lambda \alpha$, if $\beta \vdash_\Lambda \varphi$, then $\beta \vdash_\Lambda \psi$*,[6]
(c) $\alpha \vdash_\Lambda \varphi \wedge \psi$ *iff* $\alpha \vdash_\Lambda \varphi$ *and* $\alpha \vdash_\Lambda \psi$,
(d) $\alpha \vdash_\Lambda \varphi \vee \psi$ *iff* $\alpha \vdash_\Lambda \varphi$ *or* $\alpha \vdash_\Lambda \psi$.

Proof. (a) and (c) are immediate. Due to Lemma 9, (d) holds for every gsi-logic, which has the disjunction property. Let us prove (b). The left-to-right direction is immediate. For the right-to-left direction, assume that for any consistent $\vee$-free β, if $\beta \vdash_\Lambda \alpha$ and $\beta \vdash_\Lambda \varphi$, then $\beta \vdash_\Lambda \psi$. Obviously, $\beta \vdash_\Lambda \psi$ holds also in the case that β is not consistent. Assume that $\varphi \equiv_\Lambda \gamma_1 \vee \ldots \vee \gamma_n$, where $\gamma_1, \ldots, \gamma_n$ are consistent and $\vee$-free. Note that $\gamma_i \wedge \alpha \vdash_\Lambda \alpha$ and $\gamma_i \wedge \alpha \vdash_\Lambda \varphi$, for each $1 \leq i \leq n$. Our assumption implies that $\gamma_i \wedge \alpha \vdash_\Lambda \psi$. So, for all $1 \leq i \leq n$, $\gamma_i \vdash_\Lambda \alpha \to \psi$, and thus $\varphi \vdash_\Lambda \alpha \to \psi$. It follows that $\alpha \vdash_\Lambda \varphi \to \psi$. □

This observation motivates the following construction of a characteristic Kripke model $\mathscr{M}_\Lambda = \langle S_\Lambda, \leq_\Lambda, V_\Lambda \rangle$, where $S_\Lambda = \{\alpha \mid \alpha \text{ is } \vee\text{-free and consistent}\}$; $\alpha \leq_\Lambda \beta$ iff $\beta \vdash_\Lambda \alpha$; $\alpha \in V_\Lambda(p)$ iff $\alpha \vdash_\Lambda p$. Then Proposition 19 directly implies that $\Vdash$ in $\mathscr{M}_\Lambda$ coincides with $\vdash_\Lambda$.

Theorem 10. *For every inquisitive gsi-logic Λ, each φ and each consistent $\vee$-free α, $\alpha \Vdash \varphi$ in $\mathscr{M}_\Lambda$ if and only if $\alpha \vdash_\Lambda \varphi$. As a consequence, $\varphi \in \Lambda$ if and only if φ is valid in $\mathscr{M}_\Lambda$.*

Note that the previous result is intuitively related to the fact that $\vee$-free formulas are associated with statements and as such describe specific states of affairs (pieces of information), as opposed to questions. The logic of questions is determined by the logic of statements.

There is also a remarkable correspondence between the semantic clauses of Kripke semantics and the behavior of D-substitutions in inquisitive gsi-logics. This correspondence is based on Theorem 9. To make it more visible, let us write $s \succ \varphi$ instead of $\vdash_\Lambda s(\varphi)$, and let us define for any D-substitutions s, t that $s \leq t$ iff there is a D-substitution u such that $t = u \circ s$ (where $\circ$ is the composition of substitutions).

Proposition 20. *Let s be a D-substitution and φ, ψ arbitrary formulas. Then*

(a) $s \not\succ \bot$,
(b) $s \succ \varphi \to \psi$ *iff for any D-substitution $t \geq s$, if $t \succ \varphi$, then $t \succ \psi$*,
(c) $s \succ \varphi \wedge \psi$ *iff* $s \succ \varphi$ *and* $s \succ \psi$,
(d) $s \succ \varphi \vee \psi$ *iff* $s \succ \varphi$ *or* $s \succ \psi$.

Proof. All cases are straightforward. We will just comment on the case (b). This case can be reformulated in this way: $s(\varphi) \vdash s(\psi)$ iff for any D-substitution u, if $\vdash u(s(\varphi))$, then $\vdash u(s(\psi))$. But this is exactly structural D-completeness applied to the implication $s(\varphi) \to s(\psi)$. □

Using this observation, we can build from D-substitutions a particular characteristic Kripke model $\mathscr{M}^\Lambda = \langle S^\Lambda, \leq^\Lambda, V^\Lambda \rangle$, where S^Λ is the set of all D-substitutions; $s \leq^\Lambda t$ iff there is a D-substitution u such that $t = u \circ s$; $s \in V^\Lambda(p)$ iff $s \succ p$.[7]

Let id be the identity function on atomic formulas. Then id is a D-substitution and $id \leq^\Lambda s$, for every D-substitution s. So, φ is valid in $\mathscr{M}^\Lambda$ iff $id \Vdash \varphi$ in $\mathscr{M}^\Lambda$. Proposition 20 implies the following result.

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Theorem 11. *For every inquisitive gsi-logic* Λ*, for each* φ *and each D-substitution* s*,* $s \Vdash \varphi$ *in* $\mathscr{M}^{\Lambda}$ *if and only if* $s \succ \varphi$*. As a consequence,* $\varphi \in \Lambda$ *if and only if* φ *is valid in* $\mathscr{M}^{\Lambda}$*.*

The previous result follows from *SD*-completeness of Λ. Note that since *SD*-completeness implies *SG*-completeness, and thus Λ is also *SG*-complete, we could build the characteristic model also from all substitutions in $sub(\Lambda)$ rather than just from *D*-substitutions. For inquisitive gsi-logics, *D*-substitutions are equivalent to *H*-substitutions because every Harrop formula is equivalent to a $\vee$-free formula. However, we demonstrated in Proposition 7 that $sub(\Lambda)$ may be substantially different from the classes of *D*-substitutions and *H*-substitutions.

In the rest of this section, we will look at the previous result from a different angle and spell out its meaning in purely algebraic terms. For this purpose, we will need some basic algebraic terminology. Recall that a *Heyting algebra* is a structure $\mathscr{H} = \langle H, \sqcap, \sqcup, \Rightarrow, 0\rangle$, where $\mathscr{A} = \langle A, \sqcap, \sqcup\rangle$ is a distributive lattice; 0 is its least element with respect to the lattice ordering $\sqsubseteq$; and $\Rightarrow$ is a binary operation on A satisfying

$$a \sqsubseteq b \Rightarrow c \text{ iff } a \sqcap b \sqsubseteq c.$$

Let $\mathscr{H}_1 = \langle H_1, \sqcap_1, \sqcup_1, \Rightarrow_1, 0_1\rangle$, $\mathscr{H}_2 = \langle H_2, \sqcap_2, \sqcup_2, \Rightarrow_2, 0_2\rangle$ be Heyting algebras. A *homomorphism* from $\mathscr{H}_1$ to $\mathscr{H}_2$ is a function $h : H_1 \to H_2$ satisfying: $h(a \sqcap_1 b) = h(a) \sqcap_2 h(b)$, $h(a \sqcup_1 b) = h(a) \sqcup_2 h(b)$, $h(a \Rightarrow_1 b) = h(a) \Rightarrow_2 h(b)$, and $h(0_1) = 0_2$. An *embedding* of $\mathscr{H}_1$ into $\mathscr{H}_2$ is an injective homomorphism form $\mathscr{H}_1$ to $\mathscr{H}_2$. An *endomorphism* on $\mathscr{H}$ is a homomorphism from $\mathscr{H}$ to $\mathscr{H}$.

We keep fixed a gsi-logic Λ. For any formula φ, we define its equivalence class $[\varphi]_{\Lambda} = \{\psi \mid \varphi \equiv_{\Lambda} \psi\}$. The *Lindenbaum–Tarski algebra* of Λ is the following Heyting algebra:

$$\mathscr{H}_{\Lambda} = \langle Prop(\Lambda), \sqcap, \sqcup, \Rightarrow, 0_{\Lambda}\rangle,$$

where $Prop(\Lambda)$ contains all the equivalence classes $[\varphi]_{\Lambda}$, for every formula φ, and the operations are defined as follows:

$$[\varphi]_{\Lambda} \sqcap [\psi]_{\Lambda} = [\varphi \wedge \psi]_{\Lambda}, [\varphi]_{\Lambda} \sqcup [\psi]_{\Lambda} = [\varphi \vee \psi]_{\Lambda}, [\varphi]_{\Lambda} \Rightarrow [\psi]_{\Lambda} = [\varphi \to \psi]_{\Lambda}, 0_{\Lambda} = [\bot]_{\Lambda}.$$

Now we can observe that we can identify (a) substitutions under which Λ is closed with arbitrary endomorphisms on the Lindenbaum–Tarski algebra of Λ and (b) *D*-substitutions with endomorphisms of a special kind.

Let us describe this idea more carefully. Recall that two substitutions s, t are Λ-*equivalent* if $s(p) \equiv_{\Lambda} t(p)$, for every atomic formula p. In that case, we write $s \sim_{\Lambda} t$. The equivalence class of all substitutions equivalent to a substitution s is denoted as $[s]_{\Lambda}$. If clear from the context, we will be omitting the subscript Λ.

Let $[s]$ be an equivalence class of substitutions. We say that $[s]$ is *admissible* if $[s] \subseteq sub(\Lambda)$. Note that if $s \in sub(\Lambda)$, then $[s]$ is admissible. We say that $[s]$ is *declarative* if there is a *D*-substitution $t \in [s]$. Moreover, we say that an endomorphism h on $\mathscr{H}_{\Lambda}$ is *declarative* if for every atomic formula p, there is a $\vee$-free formula $\alpha \in h([p])$.

Now we define a function assigning an endomorphism to any admissible equivalence class of substitutions. For any substitution $s \in sub(\Lambda)$, we define:

$$[s]^*([\varphi]) = [s(\varphi)].$$

Note that this is a well defined function since if $s \sim t$ and $\varphi \equiv \psi$, then $[s(\varphi)] = [t(\psi)]$ because Λ is closed under s. Moreover, we can observe that $[s]^*$ is indeed an endomorphism on $\mathscr{H}_{\Lambda}$.

On the other hand, we define, for any endomorphism h on $\mathscr{H}_{\Lambda}$, a set of substitutions h° in this way:

$$h^{\circ} = \{s \in sub(\Lambda) \mid s(p) \in h([p]), \text{for every atomic formula } p\}.$$

Let us demonstrate that h° is an admissible equivalence class of substitutions. We need to show that $h^{\circ} = [s]$, for some substitution $s \in sub(\Lambda)$. Take any substitution s such that $s(p) \in h([p])$, for

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every atomic formula p. Assume $\varphi \in \Lambda$. Then $s(\varphi) \in h([\varphi]) = h([\top]) = [\top]$, and so $s(\varphi) \in \Lambda$. This means that $s \in sub(\Lambda)$. Hence, $s \in h^\circ$. Moreover, it holds that (a) if $t \in h^\circ$, then $s \sim t$, and (b) if $s \sim t$, then $t \in h^\circ$. Hence $[s] = h^\circ$.

Proposition 21. *Let h be an endomorphism on $\mathscr{H}_\Lambda$ and $s \in sub(\Lambda)$. Then*

(a) $h^{\circ *} = h$,
(b) $[s]^{*\circ} = [s]$,
(c) *if h is a declarative endomorphism, then h° is a declarative class of substitutions,*
(d) *if $[s]$ is a declarative class of substitutions, then $[s]^*$ is a declarative endomorphism.*

Proof.

(a) $h^{\circ *}[\varphi] = [t(\varphi)]$, for some $t \in h^\circ$. But $t \in h^\circ$ implies $[t(\varphi)] = h([\varphi])$.
(b) $t \in [s]^{*\circ}$ iff $t(p) \in [s]^*([p])$, for every atomic formula p, iff $t(p) \in [s(p)]$, for every atomic formula p, iff $t \in [s]$.
(c) Assume that h is declarative. Then for every atomic formula p, there is a $\vee$-free formula $\alpha_p \in h([p])$. We can define a D-substitution t by fixing $t(p) = \alpha_p$. Since $t \in h^\circ$, we conclude that h° is declarative.
(d) Assume that $[s]$ is declarative. Then there is a D-substitution $t \in [s]$. So, for every atomic formula p, $[s]^*([p]) = [t]^*([p]) = [t(p)]$. Since $t(p)$ is $\vee$-free and $t(p) \in [s]^*([p])$, for every p, we obtain that $[s]^*$ is declarative. □

Proposition 21 shows that the two functions $*$ and $\circ$ are mutually inverse bijections between endomorphisms and all admissible classes of substitutions and, when suitably restricted, between declarative endomorphisms and declarative classes of substitutions.

A monoid is an algebraic structure $\mathscr{A} = \langle A, \cdot, e\rangle$, where $\cdot$ is an associative operation on A, and e is a neutral element, that is, $e \cdot a = a \cdot e = a$, for every $a \in A$. Note that the Lindenbaum–Tarski algebra of Λ determines the monoid of its endomorphisms $\mathscr{E}_\Lambda = \langle E_\Lambda, \circ, id\rangle$, where E_Λ is the set of all endomorphisms on $\mathscr{H}_\Lambda$, $\circ$ is composition, and id is the identity endomorphism. In the same way, $\mathscr{H}_\Lambda$ determines the monoid of its declarative endomorphisms $\mathscr{E}^D_\Lambda = \langle E^D_\Lambda, \circ, id\rangle$.

Let $\mathscr{A} = \langle A, \cdot, e\rangle$ be a monoid. A set $I \subseteq A$ is called a *left ideal* in A if $a \cdot i \in I$ for all $a \in A$ and $i \in I$. Left ideals in $\mathscr{A}$ form a Heyting algebra $LeId(\mathscr{A}) = \langle LeId(A), \cap, \cup, \Rightarrow, \emptyset\rangle$, where $LeId(A)$ is the set of all left ideals in $\mathscr{A}$; $\cap, \cup, \emptyset$ are just intersection, union, and the empty set; and $\Rightarrow$ is defined as follows:

$$I \Rightarrow J = \{a \in A \mid \text{ for all } b \in A, \text{if } b \cdot a \in I \text{ then } b \cdot a \in J\}.$$

Note that left ideals in $\mathscr{A}$ are precisely the downward closed sets of the so-called Green's preorder on $\mathscr{A}$ defined as follows: $a \leq_{\mathscr{L}} b$ iff $a = c \cdot b$, for some $c \in A$. The algebra $LeId(\mathscr{A})$ is the Heyting algebra of such downward closed sets. Note that when defined on substitutions, the Green's preorder $\leq_{\mathscr{L}}$ is the converse of the preoder considered above in the construction of the characteristic model $\mathscr{M}^\Lambda$. For more details on these notions, see Grillet (1995). We define functions F_Λ from $\mathscr{H}_\Lambda$ to $LeId(\mathscr{E}_\Lambda)$ and F^D_Λ from $\mathscr{H}_\Lambda$ to $LeId(\mathscr{E}^D_\Lambda)$:

$$F_\Lambda([\varphi]_\Lambda) = \{h \in E_\Lambda \mid h([\varphi]_\Lambda) = [\top]_\Lambda\} \text{ and } F^D_\Lambda([\varphi]_\Lambda) = \{h \in E^D_\Lambda \mid h([\varphi]_\Lambda) = [\top]_\Lambda\}.$$

The algebraic fact behind Theorem 11 can now be spelled out in the following way.

Theorem 12. *For any inquisitive gsi-logic Λ, the function F_Λ is an embedding of $\mathscr{H}_\Lambda$ into $LeId(\mathscr{E}_\Lambda)$, and the function F^D_Λ is an embedding of $\mathscr{H}_\Lambda$ into $LeId(\mathscr{E}^D_\Lambda)$.*

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Proof. We will consider the case of F_Λ. To simplify the notation, we will be omitting the subscript Λ, and we will write $h[\varphi]$ and $F[\varphi]$ instead of $h([\varphi])$ and $F([\varphi])$. First, we will prove that F is a homomorphism.

$$\begin{aligned}
F([\varphi] \sqcap [\psi]) &= \{h \in E \mid h([\varphi] \sqcap [\psi]) = [\top]\} \\
&= \{h \in E \mid h[\varphi] \sqcap h[\psi] = [\top]\} \\
&= \{h \in E \mid h[\varphi] = [\top] \text{ and } h[\psi] = [\top]\} \\
&= \{h \in E \mid h[\varphi] = [\top]\} \cap \{h \in E \mid h[\psi] = [\top]\} \\
&= F[\varphi] \cap F[\psi]
\end{aligned}$$

$$\begin{aligned}
F([\varphi] \sqcup [\psi]) &= \{h \in E \mid h([\varphi] \sqcup [\psi]) = [\top]\} \\
&= \{h \in E \mid h[\varphi] \sqcup h[\psi] = [\top]\} \\
&\overset{(a)}{=} \{h \in E \mid h[\varphi] = [\top] \text{ or } h[\psi] = [\top]\} \\
&= \{h \in E \mid h[\varphi] = [\top]\} \cup \{h \in E \mid h[\psi] = [\top]\} \\
&= F[\varphi] \cup F[\psi]
\end{aligned}$$

$$\begin{aligned}
F([\varphi] \Rightarrow [\psi]) &= \{h \in E \mid h([\varphi] \Rightarrow [\psi]) = [\top]\} \\
&= \{h \in E \mid h[\varphi] \Rightarrow h[\psi] = [\top]\} \\
&\overset{(b)}{=} \{h \in E \mid \text{ for all } g \in E, \text{if } g \circ h[\varphi] = [\top] \text{then } g \circ h[\psi] = [\top]\} \\
&= \{h \in E \mid \text{ for all } g \in E, \text{ if } g \circ h \in F[\varphi] \text{ then } g \circ h \in F[\psi]\} \\
&= F[\varphi] \Rrightarrow F[\psi]
\end{aligned}$$

The equation (a) is obtained by the disjunction property and the equation (b) by SG-completeness of Λ. Moreover, it holds that $F[\bot] = \{h \in E \mid h[\bot] = [\top]\} = \emptyset$, which finishes the proof that F is a homomorphism. It remains to be proved that F is injective. Assume $F[\varphi] = F[\psi]$. This means that for every $h \in E$, $h[\varphi] = [\top]$ iff $h[\psi] = [\top]$. It follows, by SG-completeness, that $\varphi \equiv \psi$, that is, $[\varphi] = [\psi]$.

The case of F^D_Λ is strongly analogous. Only in the justification of the equation (b) and in the justification of injectivity we have to refer to SD-completeness instead of SG-completeness. □

The construction that we described in this section is analogous to the familiar canonical model construction, where we represent every proposition by a set of prime theories that make the proposition true. Our untypical construction shows that in the case of inquisitive gsi-logics, we can similarly represent every proposition by a set of substitutions that make the proposition logically true. Unfortunately, we do not have a deeper philosophical explanation of this fact, but we find it quite intriguing. In purely algebraic terms, we have shown that every inquisitive gsi-logic has the property that its Lindenbaum–Tarski algebra can be embedded into the algebra of left ideals on the monoid of its endomorphisms.

6. Conclusion

Let us summarize the main results of this paper. We have studied four notions of structural completeness (*SF*-, *SG*-, *SH*-, and *SD*-completeness) in a class of generalized superintuitionistic logics that are not required to be closed under all substitutions but only under substitutions assigning

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disjunction-free formulas. We have shown a connection between these notions and the schema Split that axiomatizes intuitionistic inquisitive logic InqIL. A characteristic feature of Split is that it allows us to transform every formula to a disjunctive normal form (Theorem 1).

Our main result (Theorem 2) shows that *SH*-completeness is equivalent to *SD*-completeness, and these properties hold exactly for those logics that validate Split. As a consequence, *SH*(*D*)-completeness is hereditary (Corollary 10). We have also shown that InqIL is hereditarily *SG*-complete (Corollary 11) and its closure under substitutions, that is, the Gödel-Dummett logic LC, remains hereditarily *SF*-complete (Theorem 6 and its extension Theorem 7).

We have further studied inquisitive logics, that is, those logics that include InqIL and have the disjunction property. We have proved that the operation that closes every such logic under substitutions maps the uncountably large class of inquisitive logics onto the countably infinite chain of those logics that include LC (Theorem 8). It follows directly from our main result that inquisitive logics are optimal; that is, they are structurally complete and have disjunction property (Theorem 9). They can be characterized by a Kripke model built from consistent disjunction-free formulas (Theorem 10). Interestingly, their optimality means that they can be alternatively characterized by a Kripke model built from substitutions assigning disjunction-free formulas (Theorem 11). We also formulated a corresponding purely algebraic claim that says that the Lindenbaum–Tarski algebra $\mathscr{H}$ of every inquisitive logic can be embedded into the Heyting algebra of left ideals on the monoid of endomorphisms on $\mathscr{H}$ (Theorem 12).

In future work, we would like to study structural completeness in the more general context of substructural inquisitive logics described in Punčochář (2019). We also plan to explore the notion of structural completeness for these logics in the setting of multi-conclusion consequence relation considered in Iemhoff (2016).

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Notes

1 For a formulation of intuitionistic inquisitive logic as a system of natural deduction, see, for example, Punčochář (2017).

2 Intuitively, given the inquisitive interpretation of disjunction, *D*-substitutions assign only declarative sentences to atomic formulas.

3 In Punčochář (2021), these sets of formulas were called superintuitionistic logics*, using the star to indicate that the notion of a logic is used in a nonstandard way, since closure under uniform substitution is not generally required.

4 In the notation of Miglioli et al. (1989), this means that (a) $\mathsf{F}_{\mathsf{int}}$ is optimal and (b) $\mathsf{F}_{\mathsf{int}}$ and F_{cl} are the only optimal gsi-logics.

5 This is related to a fact that was already observed in Punčochář (2016), namely, that any inquisitive gsi-logic Λ can be characterized by a Kripke model built up from the Lindenbaum–Tarski algebra of the $\vee$-free fragment of Λ.

6 The word "consistent" could be omitted here, and the equivalence would still hold, but in a moment, we will need this particular form of the statement that quantifies over consistent formulas.

7 An analogous construction was described in Wojtylak (2004) for the standard optimal gsi-logic ML.

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