Probability in the Engineering and Informational Sciences, **15**, 2001, 335–349. Printed in the U.S.A.

OPTION PRICING VIA MONTE CARLO SIMULATION

A WEAK DERIVATIVE APPROACH

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Using a weak derivation approach to gradient estimation, we consider the problem of pricing an American call option on stock paying dividends at discrete times. Similar simulation-based sensitivity estimators were introduced earlier by Fu and Hu (1995) who used smoothed perturbation analysis. We improve upon their results by presenting an estimator with a uniformly lower variance. In addition, we reduced the multidimensional optimization problem of pricing an option with multiple ex-dividend dates to a one-dimensional one. Numerical examples indicate that this approach saves a considerable amount of computation time. Our estimator holds uniformly for a class of payoff functions, and applications to other types of options will be addressed in the article.

1. AN AMERICAN CALL ON A DIVIDEND-PAYING STOCK

The basic approach is to model the stock price at time t, denoted by S_t, via a Brownian motion, denoted by Z_t, in the following way:

$$S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma Z_t\right), \tag{1}$$

where S_0 is the initial stock price, $\mu > 0$ is the so-called *mean rate of return*, $\sigma > 0$ is the *volatility*, and Z_t is a standard Brownian motion, with $Z_0 = 0$.

An American call option is defined on a stock paying dividends D_j at times t_j, with $1 \le j \le \eta(T)$, where $t_{\eta(T)+1} = T$ is the expiration date of the option and $\eta(T)$

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https://doi.org/10.1017/S0269964801153040 Published online by Cambridge University Press

denotes the total number of ex-dividend dates. Following the standard models, we assume that after the ex-dividend date t_i, the stock price drops by the amount of the *dividend* D_i (i.e., $S_{t_i^+} = S_{t_i^-} - D_i$; see Hull [4] and Stoll and Whaley [6]). Under the assumption of a frictionless market, such an option should be exercised prematurely (if at all) only right before an ex-dividend date leading to the following exercise policy: Exercise the option just prior to an *ex-dividend date* t_j if and only if the stock price exceeds a *threshold price* s_j. We denote the exercise price by K and assume that the ex-dividend dates t_i and the dividend amounts D_i are deterministic and known. We set $t_0 = 0 = D_0$ and the *payoff function* of the American call option is then given by

$$J_T = e^{-rT}\left(\sum_{i=1}^{\eta(T)}\left[\prod_{j=1}^{i-1} 1_{\{S_{t_j^-} \le s_j\}}\right] 1_{\{S_{t_i^-} > s_i\}}(S_{t_i^-} - K)e^{r(T-t_i)} + \prod_{j=1}^{\eta(T)} 1_{\{S_{t_j^-} \le s_j\}}(S_T - K)^+\right). \tag{2}$$

In order to value the option, we follow Fu and Hu [1, 2] and maximize the payoff function using Monte Carlo simulation. Specifically, we apply the technique of *weak differentiation* (WD); see Pflug [5]:

- When a priori information is available on the probability that the stock price process stays below certain thresholds, we obtain a gradient estimator that has uniformly smaller variance than the *smoothed perturbation analysis* (SPA) estimator of Fu and Hu [1] and Fu et al. [3].
- If $\eta(T) > 1$, then maximizing the payoff function is a multidimensional optimization problem. However, if we assume that the threshold values s_i are functions in some parameter $\theta \in \mathbb{R}$ [i.e., $s_i = s_i(\theta)$] (in the following referred to as the *multivariational case*), then we may translate the multidimensional optimization problem into a one-dimensional one, which has the advantage that the optimization is sped up considerably. For the weak derivative estimator, the estimators for the individual thresholds $s_i(\theta)$ can be combined/mixed such that we obtain an unbiased estimator for the derivative of the option price with respect to θ. This is in contrast to sample-path-based approaches like SPA.
- Weak derivative estimators hold uniformly for the class of bounded payoff functions. More precisely, we may replace the factors $(S_{t_i^-} - K)e^{r(T-t_i)}$ in (2) by any bounded function of the stock price. Hence, the analysis provided in this article yields unbiasedness of the weak derivative estimator for more exotic options as well. This is in contrast to sample path approaches, like SPA, where different types of payoff function have to be handled differently.

The article is organized as follows. In the next section, we give a detailed description of the model. In Section 3, we derive our estimator(s). We provide a comparison of our estimator and the SPA estimator by Fu and Hu [1] and show that the

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modified WD estimator outperforms SPA. Finally, we address the multivariational case.

2. THE MARKOV CHAIN MODEL

Define the geometric Brownian motion $\tilde{S}_t$ by

$$\tilde{S}_t = \tilde{S}_0 e^{(\mu-\sigma^2/2)t+\sigma\sqrt{t}Z} =: h(Z;\tilde{S}_0,t,\mu,\sigma) \tag{3}$$

with initial value $\tilde{S}_0$ and Z a standard normal random variable. Let

$$S := \{S(i): 0 \le i \le \eta(T)+1\}$$

denote the stock price just prior to ex-dividend dates; that is, we set $S(i) := S_{t_i^-}$, for $1 \le i \le \eta(T)$, $S(0) = S_0$, $S(\eta(T)+1) = S_T$. Let $\tilde{S}(i) = \tilde{S}(t_i)$ be the value of $\tilde{S}_t$ at ex-dividend date t_i, for $1 \le i \le \eta(T)+1$. Note that continuity of $\tilde{S}(t)$ implies that $\tilde{S}(i) = \tilde{S}(t_i^-)$. Equation (3) implies

$$\tilde{S}(i) = h(Z_i;\tilde{S}(i-1), t_i - t_{i-1}, \mu, \sigma) \tag{4}$$

for $1 \le i \le \eta(T)+1$, where Z_i is a standard normal random variable that is independent of everything else.

We recover the stock price process S_t from $\tilde{S}_t$ by discounting the value of the jumps correctly over time. Specifically, we denote the discounted value of the dividend D_i at ex-dividend date t_i by

$$d_i := \sum_{j=i}^{\eta(T)} D_j e^{-r(t_j-t_i)}, \qquad 0 \le i \le \eta(T),$$

and we obtain the stock price just prior to t_i from

$$S(i) = \tilde{S}(i) + d_i, \qquad 0 \le i \le \eta(T)+1, \tag{5}$$

with $\tilde{S}(0) = S_0 + d_0$. Combining (5) with (4), we obtain

$$S(i) = h_i(Z_i; S(i-1)), \tag{6}$$

with $h_i(Z_i;S(i-1)) = h(Z_i;S(i-1)-d_{i-1}, t_i - t_{i-1}, \mu, \sigma) + d_i$. Hence, the mapping $h_i(\cdot;s)$ generates the stock price just prior to t_i from the one just prior to t_{i-1}. Because the Z_i are mutually independent, S is a Markovian chain.

Remark 1: Replacing $h(\cdot)$ in (4) by $h(Z_i;\tilde{S}(i-1), t_i - t_{i-1}, \mu-\delta, \sigma)$ models a stock prices process with continuous interest rates, and the payoff function (2) describes a Bermudan option; see Fu et al. [3].

We denote the inverse mapping of $h_i(\cdot;s)$ by $h_{i,s}^{-1}(\cdot)$; more precisely, we set

$$h_{i,s}^{-1}(r) := \frac{1}{\sigma\sqrt{t_i - t_{i-1}}}\left[\ln\left(\frac{r-d_i}{s-d_{i-1}}\right) - \left(\mu - \frac{\sigma^2}{2}\right)(t_i - t_{i-1})\right]$$

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for $r \in (d_i, \infty)$. For $S(i-1) = s$, the event that $S(i)$ is lower than or equal to r is completely determined through $h_{i,s}^{-1}(r)$; in formula,

$$\{S(i) \leq r\} = \{Z_i \leq h_{i,s}^{-1}(r)\} \quad \forall r \geq d_i. \tag{7}$$

We describe our model as a Markovian chain X. A typical state of X will denote the stock price just prior to ex-dividend date t_i supplemented by information about whether we exercise the option at t_i or not. More precisely, we consider an absorbing Markov chain. If we have not exercised the option during the first $i-1$ ex-dividend dates, then we calculate the stock price just prior to t_i according to (6) and check whether we exercise the option at t_i or not. In case we exercise the option, we set the *exercising indicator* $V(i)$ to i and otherwise to zero. Once we have exercised the option, we stop our observation. This is modeled by letting the Markovian chain X jump to an absorbing state $*$, where $*$ is different from any possible stock price value (e.g., $* = -1$). We now give a formal definition of the Markovian chain

$$X := \{X(i) := (S(i), V(i)) : 0 \leq i \leq \eta(T) + 1\}.$$

Initialize X to $X(0) = (S(0), 0)$, and for $1 \leq i \leq \eta(T) + 1$, set

$$X(i) = \begin{cases} (h_i(Z_i; S(i-1)), 0), & h_i(Z_i; S(i-1)) < s_i \text{ and } V(i-1) = 0 \\ (h_i(Z_i; S(i-1)), i), & h_i(Z_i; S(i-1)) \geq s_i \text{ and } V(i-1) = 0 \\ *, & V(i-1) > 0 \end{cases}$$

for Z_i standard normal and independent of everything else. The Markov chain X completely determines the payoff function J_T. To see this, take

$$g(s,v) := \begin{cases} (s-K)e^{r(T-t_v)}, & 0 < v \leq \eta(T) \\ (s-K)^+, & v = \eta(T)+1 \\ 0, & v = 0 \end{cases} \tag{8}$$

and $g(*) = 0$, which gives

$$J_T = \sum_{i=1}^{\eta(T)+1} g(X(i)). \tag{9}$$

Our construction implies that $S(i) > d_i$. Hence, the state space of $X(i)$ is given by $E_i = (d_i, \infty) \times \{0, i\} \cup \{*\}$. Let $\mathbb{R}$ have the usual topology, subsets have the relative topology, discrete sets have the discrete topology, and product spaces have the product topology; denote the Borel field of E_i by $\mathcal{E}_i$. Then, the state space of X is given by $(E_i, \mathcal{E}_i)$, and for $(x,v) \in E_i$ and $A \in \mathcal{E}_i$, we denote the transition kernel of X by

$$Q_i(A; x, v) := \mathcal{P}(X(i) \in A | X(i-1) = (x,v)).$$

We summarize our analysis as follows.

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LEMMA 1: *Let $X = \{X(i) : 0 \le i \le \eta(T) + 1\}$ be a Markovian chain with state space $(E_i, \mathcal{E}_i)$ and let, for $(x,v) \in E_i$ and $A \in \mathcal{E}_i$, the ith transition probability of X be given by*

$$Q_i(A;x,v) = \begin{cases} \mathcal{P}((h_i(Z_i;x), i1_{h_i(Z_i;x)>s_i}) \in A), & v = 0 \\ 1_A(*), & v > 0, \end{cases}$$

with Z_i $(1 \le i \le \eta(T))$ as a sequence of mutually independent standard normal random variables and h_i given as in (6); then,

$$E[J_T] = E\left[\sum_{i=1}^{\eta(T)+1} g(X(i))\right]. \tag{10}$$

3. WEAK DERIVATIVE ANALYSIS

Weak differentiation (WD) applies to the Markov kernel Q_i in Lemma 1 if Q_i is differentiable as a function in s_i. Then, the derivative of the Q_i can be represented as the difference between two transition kernels, say Q^+ and Q^- (i.e., $dQ/ds_i = Q^+ - Q^-$). We then estimate the directional derivatives as follows. If we are interested in the derivative with respect to the threshold price at the ith ex-dividend date, we simulate the stock prize process and perform the first $i-1$ transitions according to Q, the nominal transition kernel. At the ith transition, we split the sample path: For one variant, we perform the transition according to Q^+, and according to Q^- for the other variant. Thereafter, we generate the transitions of the two stock price processes according to Q until we either prematurely exercise the option or reach the expiration date. Taking the difference between the performance evaluated for both variants of the processes and rescaling it by a normalization factor yields an unbiased estimate for the derivative of the payoff function with respect to s_i.

For $(x,v) \in E_{i-1}$, $Q_i(\cdot;x,v)$ is a probability measure on $(E_i, \mathcal{E}_i)$, whereas its weak derivative with respect to s_i is the triple

$$(c_i(x,v), Q_i^+(\cdot;x,v), Q_i^-(\cdot;x,v)),$$

with $c_i(x,v) \in \mathbb{R}$ and probability measures $Q_i^+(\cdot;x,v)$ and $Q_j^-(\cdot;x,v)$ on $(E_i, \mathcal{E}_i)$, such that for all continuous bounded real-valued functions f on E_i,

$$\frac{d}{ds_i}\int_{E_i} f(\gamma) Q_i(d\gamma;x,v) = c_i(x,v)\left(\int_{E_i} f(\gamma) Q_i^+(d\gamma;x,v) - \int_{E_i} f(\gamma) Q_i^-(d\gamma;x,v)\right);$$

see Definition 3.72 in Pflug [5]. If $Q_j(\cdot;x,v)$ is independent of s_i, we set $c_i(x,v) = 0$ and $Q_i^+(\cdot;x,v) = Q_i(\cdot;x,v) = Q_i^-(\cdot;x,v)$.

The following theorem shows that the derivative of Q_i with respect to s_i can be represented by two rather simple transition kernels: For both kernels, Q_i^+ and Q_i^-,

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we force the stock price just prior to t_i to be exactly equal to s_i and we exercise the option (almost surely) for Q_i^-, whereas we do not exercise the option (almost surely) for Q_i^+. For a proof, see the Appendix.

THEOREM 1: *For all $(x,0) \in E_i$, the weak derivative of $Q_i(\cdot;x,0)$ with respect to s_i is given by*

$$c_i(x) := \frac{1}{\sigma\sqrt{t_i - t_{i-1}}(s_i - d_i)} \phi(h_{i,x}^{-1}(s_i)),$$

where $\phi(\cdot)$ denotes the density of Z_i,

$$Q_i^+(\cdot;x,0) := \delta_{(s_i,0)}(\cdot) \quad \textit{and} \quad Q_i^-(\cdot;x,0) := \delta_{(s_i,i)}(\cdot),$$

where $\delta_{(s,v)}(\cdot)$ denotes the Dirac measure on E_i (i.e., $\delta_{(s,v)}(e) = 1$ for $(s,v) = e$ and otherwise zero). For $(x,v) \in E_i$, with $v \neq 0$, the weak derivative is given by $(0, Q_i(\cdot;x,v), Q_i(\cdot;x,v))$, which is also the weak derivative of Q_i with respect to s_j for $j \neq i$.

We now establish unbiased derivative estimators for $dE[J_T]/ds_i$. For $1 \leq i \leq \eta(T)$, let β_i, with

$$\beta_i = 0 \quad \text{if and only if } \sum_{j=0}^{i-1} V(j) = 0,$$

indicate whether we have already exercised before i or not. The ith transition only depends on the threshold value s_i if $\beta_i = 0$ (i.e., if we have not exercised the option before t_i). Put another way, $\beta_i = 0$ indicates that $X(i-1)$ has not entered the absorbing state $*$ yet.

The following theorem, which is proven in the Appendix, states the main result of our analysis.

THEOREM 2: *Under the conditions in Theorem 1, it holds that*

$$\frac{d}{ds_i} E[J_T] = E\left[1_{\beta_i=0} c_i(X(i-1)) \left(E\left[\sum_{j=i}^{\eta(T)+1} g(X(j)) \middle| X(i) = (s_i,0) \right] - g(i,s) \right)\right].$$

Remark 2: Note that Theorem 2 holds for any bounded mapping g from the state space of X onto $\mathbb{R}$. Hence, Theorem 2 applies to any option for which the expected payoff can be written as in (10) and the corresponding estimator follows easily from the subsequent discussion.

We call the estimator in Theorem 2 the *direct estimator.* This estimator can be obtained by the following algorithm.

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Algorithm 1

1. Generate the first $i-1$ transition of X and compute $1_{\beta_i=0}$.
2. At the ith transition, set $X(i)$ to $(s_i,0)$ and resume simulation.

Put another way, we simulate the original Markov chain X up to $X(i-1)$, then we reset X to $(s_i,0)$ and simulate the (new) Markov chain $\hat{X}$ until it either hits the absorbing state or terminates (which happens after at most $\eta(T)+1-i$ transitions). More precisely, we consider the Markov chain

$$\hat{X}(j) = E[X(j)|X(i) = (s_i,0)], \qquad i \le j \le \eta(T)+1. \tag{11}$$

Set

$$G_T = \sum_{i=1}^{\eta(T)+1} g(X(i))$$

and let ω_k $(1 \le k \le n)$ be n mutually independent sample paths; then, the direct estimator is given by

$$\frac{d}{ds_i} J_T^D(n) = \frac{1}{n}\sum_{k=1}^{n} c_i(X(i-1,\omega_k))1_{\beta_i(\omega_k)=0}(G_T(\omega_k) - (s_i - K)e^{r(T-t_i)})$$

and Theorem 2 implies

$$\frac{d}{ds_i} E[J_T] = E\left[\frac{d}{ds_i} J_T^D(n)\right] \quad \text{for } n \ge 1.$$

Suppose that $P(\beta_i = 0)$ is known a priori. For example, in simple cases, we may be able to calculate $P(\beta_i = 0)$ analytically or $P(\beta_i = 0)$ may be available from a previous analysis. In what follows, we describe how this information can be used to decrease the variance of the estimator.

Observe that the direct estimator does not take all sample paths into account since a sample path with $\beta_i = 0$ bears no information for the estimator. To reduce the variance of the estimator, we now condition on the event $\beta_i = 0$. We first give an algorithmic description of our new estimator, called the *conditioned estimator*, and postpone the proof of its unbiasedness.

Algorithm 2

1. Generate i transitions of a version of X, denoted by Y, such that for Y we do not exercise the option before i; record $c_i(Y(i-1))$.
2. Initialize $X(i)$ to $(s_i,0)$; simulate X until termination.

Put another way, we simulate a Markov chain Y such that Y behaves like X, given that we do not exercise the option prematurely. The outcome of Y is then needed to

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determine $c_i(Y(i-1))$. We start the simulation of X at the ith transition with $X(i) = (s_i,0)$ and simulate X until it either hits the absorbing state or terminates [which happens after at most $\eta(T) + 1 - i$ transitions] [i.e., we simulate $\hat{X}$, see (11)]. We then take

$$\frac{d}{ds_i} J_T^C(n) = P(\beta_i = 0)\left(\frac{1}{n}\sum_{k=1}^{n} c_i(Y(i-1,\omega_k))(G_T(\omega_k) - (s_i - K)e^{r(T-t_i)})\right),$$

for mutually independent observations ω_k $(1 \le k \le n)$ as our estimator.

We have the following theorem.

THEOREM 3: *Suppose that the conditions in Theorem 1 are satisfied. Let* $1 \le i \le \eta(T)$, *if* Y *has the same distribution as* $X(i-1)$, *conditioned on the event* $\beta_i = 0$; *then,*

$$\frac{d}{ds_i} E[J_T] = P(\beta_i = 0)E[c_i(Y)(E[G_T|X(i) = (s_i,0)] - (s_i - K)e^{r(T-t_i)})].$$

PROOF: Observe that the event $\{\beta_i = 0\}$ is measurable with respect to $\sigma\{X(j): 0 \le j < i\}$, the σ-field generated by the history of X up to $i-1$. The Markov property of X implies that $\{X(j): i \le j \le \eta(T) + 1\}$ is independent of the event $\beta_i = 0$. Furthermore, the indicator $1_{\beta_i=0}$ is independent of $c_i(Y) = E[c_i(X(i-1))|\beta_i = 0]$. Combining these arguments, we obtain

$$\begin{aligned}
&E[c_i(X(i-1))1_{\beta_i=0}(E[G_T|X(i) = (s_i,0)] - (s_i - K)e^{r(T-t_i)})] \\
&\quad = E[1_{\beta_i=0}E[c_i(X(i-1))|\beta_i = 0](E[G_T|X(i) = (s_i,0)] - (s_i - K)e^{r(T-t_i)})] \\
&\quad = P(\beta_i = 0)E[c_i(Y)(E[G_T|X(i) = (s_i,0)] - (s_i - K)e^{r(T-t_i)})],
\end{aligned}$$

which concludes the proof. ■

The direct and the conditioned estimator are both unbiased. However, as the following theorem shows, the variance of the conditioned estimator is smaller than or equal to the variance of the direct one.

THEOREM 4: *Under the conditions of Theorem 1, it holds for all* $n \in \mathbb{N}$ *that*

$$\mathrm{Var}\left(\frac{d}{ds_i} J_T^C(n)\right) \le \mathrm{Var}\left(\frac{d}{ds_i} J_T^D(n)\right) \quad \text{for } 1 \le i \le \eta(T).$$

For $P(\beta_i = 0) < 1$, *we obtain a strict inequality uniformly in* n, *whereas for* $P(\beta_i = 0) = 1$, *we obtain equality for all* n.

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PROOF: By calculation, we obtain

$$\begin{aligned}
&\operatorname{Var}\left(\frac{d}{ds_i} J_T^C(n)\right) \\
&\quad = P^2(\beta_i = 0) \operatorname{Var}\left(\frac{1}{n}\sum_{k=1}^{n} c_i(Y(i-1,\omega_k))(G_T(\omega_k) - (s_i - K)e^{r(T-t_i)})\right) \\
&\quad = P(\beta_i = 0) \\
&\qquad \times E\left[\left(\frac{1}{n}\sum_{k=1}^{n} c_i(X(i-1,\omega_k))1_{\beta_i(\omega_k)=0}(G_T(\omega_k) - (s_i - K)e^{r(T-t_i)})\right)^2\right] \\
&\qquad - E\left[\frac{1}{n}\sum_{k=1}^{n} c_i(X(i-1,\omega_k))1_{\beta_i(\omega_k)=0}(G_T(\omega_k) - (s_i - K)e^{r(T-t_i)})\right]^2 \\
&\quad \le \operatorname{Var}\left(\frac{d}{ds_i} J_T^D(n)\right)
\end{aligned}$$

for all $n \in \mathbb{N}$. ■

For $P(\beta_i = 0) = 1$, we never exercise the option before t_i, which means that the threshold values s_j $(1 \le j < i)$ are not relevant. Hence, for any meaningful model, we may expect that $P(\beta_i = 0) < 1$ and that the conditioned estimator will have a significantly smaller variance than the direct one.

The decrease in variance obtained from conditioning on $P(\beta_i = 0)$ depends on the value of $P(\beta_i = 0)$. We illustrate the effect with the help of a series of experiments.

Example 1: In our experiments, we consider an American option with striking price $K = 50$ on a dividend paying stock with ex-dividend dates $t_1 = 0.3$, $t_2 = 0.6$, and expiration date $T = 0.9$. The initial stock price is $S_0 = 50$, the mean rate of return is $\mu = 0.1$, and the volatility is $\sigma = 0.3$. The initial threshold prices are $s_1 = s_2 = 50$ and the dividends are $D_1 = D_2 = 2$. Furthermore, we take $r = 0.1$ as the interest rate. This model resembles the one given by Fu and Hu [1]. We let the probability $P(\beta_1 = 0)$ vary from a small value, $P(\beta_1 = 0) = 0.2$, to a large value, $P(\beta_1 = 0) = 0.8$. Table 1

TABLE 1. The Influence of $P(\beta_1 = 0)$ on the Sample Variances

		$n = 10^3$		$n = 10^5$		$n = 10^7$	
$P(\beta_1 = 0)$	s_1	$\frac{d}{ds_2} J_T^D(n)$	$\frac{d}{ds_2} J_T^C(n)$	$\frac{d}{ds_2} J_T^D(n)$	$\frac{d}{ds_2} J_T^C(n)$	$\frac{d}{ds_2} J_T^D(n)$	$\frac{d}{ds_2} J_T^C(n)$
0.2	44.70	$1.45\ 10^{-3}$	$2.06\ 10^{-4}$	$1.44\ 10^{-3}$	$2.44\ 10^{-4}$	$1.37\ 10^{-3}$	$2.43\ 10^{-4}$
0.5	50.76	$1.18\ 10^{-2}$	$5.62\ 10^{-3}$	$1.44\ 10^{-2}$	$6.12\ 10^{-3}$	$1.43\ 10^{-3}$	$6.23\ 10^{-2}$
0.8	57.73	$2.75\ 10^{-2}$	$2.07\ 10^{-2}$	$3.06\ 10^{-2}$	$2.30\ 10^{-2}$	$3.08\ 10^{-2}$	$2.30\ 10^{-2}$

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lists the values of $P(\beta_1 = 0)$ we considered (together with the values for s_1, which can be calculated analytically) and the corresponding value of the estimated variance of the direct and the conditioned estimators, respectively. For each value of $P(\beta_1 = 0)$, we performed three experiments, where the number of observations varied from $n = 10^3$ to $n = 10^7$.

Table 1 shows that if $P(\beta_1 = 0)$ is small, the variance of the direct estimator is more than six times larger than that of the conditioned one. If we increase $P(\beta_1 = 0)$, the difference between the direct and the conditioned estimator gets smaller. However, even for a relatively large value, like $P(\beta_1 = 0) = 0.8$, the variance of the direct estimator is 60% higher than the variance of the conditioned one.

We return to the pricing of the option. We maximize the payoff function of the option by incorporating the gradient estimate with respect to the thresholds into the following stochastic approximation algorithm:

$$s_i^{m+1} = \Pi_{[d_i,\infty)}\left(s_i^m + a_m \frac{d}{ds_i}\tilde{J}_T\right),$$

where s_i^m is the threshold value at the beginning of iteration m, a_m is a (positive) sequence of step sizes, $d\tilde{J}_T/ds_i$ is an unbiased estimator for $dE[J_T]/ds_i$, and $\Pi_{[d_i,\infty)}$ is a projection onto the set of feasible values for s_i. The above algorithm terminates if $|s_i^{m+1} - s_i^m| \leq \epsilon$ for a predefined precision ϵ. The following example illustrates this.

Example 2: For experiments, we consider the situation as introduced in Example 1. In our experiment, we took $m \times 10^4$ independent replications for the mth iteration. For the step sizes, we used the accelerated harmonic series; more specifically, we set $a_0 = 100$ and $a_{n+1} = a_n$ if the sign of the derivative estimator has not changed from the nth to $(n+1)$st iteration, otherwise $a_{n+1} = a_n/(a_n + 1)$. The algorithm terminated at a level of precision of $\epsilon = 10^{-4}$. We estimated the gradient $(dE[J_T]/ds_1, dE[J_T]/ds_2)$ with the estimator in Theorem 2 (respectively, the gradient estimation algorithm 1). The stochastic approximation terminated after 24 iterations and yielded 5.490 as the price of the option. The threshold prices leading to the option price were $s_1 = 67.27$ and $s_2 = 61.80$. To compare our results with the true option price, we carried out a series of intensive simulation experiments and obtained 5.489 as the price of the option (with $s_1 = 67.18$ and $s_2 = 60.90$).

4. THE MULTIVARIATIONAL CASE

In this section, we assume that all threshold values depend simultaneously on a parameter $\theta \in \Theta \subset \mathbb{R}$.

Assumption 1 (A1): *The threshold prices s_i are differentiable functions in θ on $\Theta \subset \mathbb{R}$, and we assume* without loss of generality *$s_i(\theta) > d_i$ for $\theta \in \Theta$ and $1 \leq i \leq \eta(T)$.*

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Under (A1), the weak derivative of the transition kernel Q_i, as given in Theorem 1, reads

$$\left(\phi(h_{i,x}^{-1}(s_i(\theta))) \frac{\dfrac{d}{d\theta} s_i(\theta)}{\sigma\sqrt{t_i - t_{i-1}}(s_i(\theta) - d_i)}, \delta_{(s_i(\theta),0)}(\cdot), \delta_{(s_i(\theta),i)}(\cdot) \right),$$

for all $(x,0) \in E_i$ and $1 \le i \le \eta(T)$; whereas for all $(x,v) \in E_i$ with $v \neq 0$, the weak derivative reads $(0, Q_i(\cdot;x,v), Q_i(\cdot;x,v))$. Observe that $s_i(\theta) = \theta$ and $s_j(\theta) = s_j$ for $j \neq i$; this recovers the weak derivative in Theorem 2.

Once the weak derivatives of the transition kernels have been derived we may estimate the derivative of $E[J_T]$ with respect to θ as follows. Let

$$\gamma = \max\{i \le \eta(T) : V(i) = 0\} + 1$$

denote the transition at which we exercise the option; that is, for $\gamma \le \eta(T)$, we exercise the option prematurely, and for $\gamma = \eta(T) + 1$, we exercise it at the expiration date. As for the single-threshold case, we construct two variants of the nominal process X. The main difference between the single-threshold case and the multivariational case is that we pick the ex-dividend date at which we split the sample path randomly. More precisely, we chose this particular ex-dividend date according to a uniform distribution over $\{1,\dots,\gamma\}$.

Algorithm 3

Let σ be uniformly distributed on $\{1,\dots,\gamma\}$ independent from everything else.

1. We start X with $X(0) = (S(0),0)$ and generate the first $\sigma - 1$ transitions of X according to the transition kernel of Q_i.
2. We perform the σth transition according to Q_σ^+: We set $S(\sigma) = s_\sigma(\theta)$ and $V(\sigma) = 0$ (i.e., we do not exercise the option) and note the normalization constant by $c_\sigma(X_{\sigma-1})$.
3. After σ, we again generate the transitions of X according to the nominal transition kernel.

Then it holds true that

$$\frac{d}{d\theta} E[J_T] = E[\gamma c_\sigma(X_{\sigma-1})(E[G_T | X(\sigma) = (s_\sigma,0)] - (s_\sigma - K)e^{r(T-t_\sigma)})];$$

see Pflug [5].

The algorithm is not difficult to implement. We elaborate on the fact that pseudorandom numbers on a computer are reproducible. After generating β_i transitions of X, we choose σ. Then, the random number stream is reset and the process is simulated again, but this time, it is stopped after the $(\sigma - 1)$st transition.

Making use of (A1), the optimization is then carried out to maximize the expected return with respect to θ. Put another way, we consider a one-dimensional

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variant of the optimization problem in Example 2. The stochastic approximation algorithm reads

$$\theta^{m+1} = \Pi_\Theta\left(\theta^m + a_m \frac{d}{d\theta}\tilde{J}_T\right),$$

where θ^m is the parameter value at the beginning of iteration m, a_m is a (positive) sequence of step sizes (see Example 2), $d\tilde{J}_T/d\theta$ is an unbiased estimator for $dE[J_T]/d\theta$, and Π_Θ is a projection onto Θ.

Example 3: We consider the optimization problem in Example 2 again, but this time, we assume that $s_1(\theta) = s_2(\theta) = \theta$ for $\theta \in \Theta$. For our experiment, we took $\theta_1 = 50$, and all other values as in Example 2. The stochastic approximation stopped after five iterations and yielded 5.494 as the option price. The corresponding thresholds were $s_1 = s_2 = 63.133$.

Comparing these results with the ones in Example 2, we see that both approaches yielded an option price of 5.49. However, the two-dimensional optimization approach required 20 times more observations for the algorithm. This indicates that a considerable amount of computation time is saved when a parameterized model is used. Surprisingly enough, the threshold values obtained above and the ones obtained in Example 2 are quite different. This indicates a flexibility in the choice of the thresholds.

In a second experiment, we assumed a linear dependence of $s_1(\theta)$ and $s_2(\theta)$ on θ:

$$s_1(\theta) = -\theta(T - t_1) + K, \qquad s_2(\theta) = -\theta(T - t_2) + K.$$

For this parameterization, the stochastic approximation stopped after five iterations at an option price of 5.493 with $s_1 = 71.061$ and $s_2 = 60.530$. The computational effort is again smaller by a factor of 20 than that of the two-dimensional approach.

Comparing the results in Examples 2 and 3, we observe that although the option prices are (practically) equal, the threshold values are quite different. Furthermore, a comparison of the computational effort indicates that the option can be priced very efficiently when a parameterized model is used. Surprisingly enough, there seems to be a lot of freedom with respect to the choice of the type of the parameterization. We think this insensitivity is as an interesting topic for further research.

Acknowledgments

This research is supported by Deutsche Forschungsgemeinschaft under grant He3139/1-1. Part of this work was done while the author was with the Faculty of Information Technology and Systems, Technical University Delft, The Netherlands, and supported by the EC-TMR project ALAPEDES under grant ERBFMRXCT960074. The author would like to thank Gül Gürkan of Tilburg University, for her valuable insight, knowledge, and comments contributed to this article.

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APPENDIX

PROOF OF THEOREM 1: Let $f: E_i \to \mathbb{R}$ be continuous and bounded. We have to calculate

$$\frac{d}{ds_i}\int_{E_i} f(\gamma)Q_i(d\gamma;x,0) = \frac{d}{ds_i}\int_{[d_i,\infty)\times\{0,i\}} f(\gamma)Q_i(d\gamma;x,0).$$

First, note that the mapping $h_{i,x}^{-1}(r)$ is differentiable on (d_i,∞) with

$$\frac{d}{dr}h_{i,x}^{-1}(r) = (\sigma\sqrt{t_i - t_{i-1}}(r-d_i))^{-1}$$

for $0 < i \le \eta(T)+1$. Furthermore, given $X(i-1) = (x,0)$, the event $V(i) = 0$ is equivalent to

$$\{V(i)=0\} = \{S(i) < s_i\} \overset{(7)}{=} \{Z_i \le h_{i,x}^{-1}(s_i)\}. \tag{12}$$

Hence,

$$\begin{aligned}&\frac{d}{ds_i}\int_{[d_i,\infty)\times\{0,i\}} f(\gamma)Q_j(d\gamma;x,0)\\ &\quad\overset{(12)}{=} \frac{d}{ds_i}\left(\int_{d_i}^{s_i} f(\sigma,0)Q_j(d\sigma,0;x,0) + \int_{s_i}^{\infty} f(\sigma,i)Q_i(d\sigma,i;x,0)\right).\end{aligned}$$

Let $\phi(\cdot)$ denote the density of the standard normal distribution. If we apply the transformation rule for measures, the above expression is equal to

$$\begin{aligned}&\overset{(6)}{=} \frac{d}{ds_i}\left(\int_{d_i}^{s_i} f(\sigma,0)P^{h_i(Z_i;x)}(d\sigma) + \int_{s_i}^{\infty} f(\sigma,i)P^{h_i(Z_i;x)}(d\sigma)\right)\\ &= \frac{d}{ds_i}\left(\int_{d_i}^{h_{i,x}^{-1}(s_i)} f(h_i(y;x),0)\phi(y)(dy) + \int_{h_{i,x}^{-1}(s_i)}^{\infty} f(h_i(y;x),i)\phi(y)(dy)\right)\\ &= \phi(h_{i,x}^{-1}(s_i))\frac{d}{ds_i}h_{i,x}^{-1}(s_i;x)\left(\int_{E_i} f(\gamma)\delta_{(s_i,0)}(d\gamma) - \int_{E_i} f(\gamma)\delta_{(s_i,i)}(d\gamma)\right).\end{aligned}$$

In all other cases, the transition kernel does not depend on s, which concludes the proof. ■

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PROOF OF THEOREM 2: We apply the theory of weak differentiation (see [5, Sect. 4.3.2]) and obtain $dE[J_T]/ds_i$ as follows. We generate two variants of X, denoted by X^+ and X^-, such that $dE[J_T]/ds_i$ can be expressed by the difference between J_T evaluated over X^+ and X^-.

We generate X^+ as follows:

1. We start X^+ with $X^+(0) = (S(0),0)$ and generate the first $i-1$ transitions of X^+ according to the transition kernel of X.
2. The ith transition we perform according to Q_i^+ (see Theorem 1): Provided that $X(i-1) \neq *$, we set $S(i) = s_i$ and $V(i) = 0$; that is, we force the stock price to jump to s_i and do not exercise the option with probability 1.
3. After i, we generate the transitions of X^+ according to the nominal transition kernel again. We do this until X^+ hits the absorbing state or terminates [i.e., until either $V^+(j) = j$ or $j = \eta(T)+1$].

We generate X^- in exactly the same way, except that we generate the transition in the second step according to Q_i^-. This means that we set $S(i) = s_i$ and exercise the option with probability 1 [i.e., $V(i) = i$]. Set

$$G_T^+ = \sum_{j=1}^{\eta(T)+1} g(X^+(j)) \quad \text{and} \quad G_T^- = \sum_{j=1}^{\eta(T)+1} g(X^-(j)).$$

The weak derivative estimator {see (4.42) in Pflug [5, p. 256]} is then given by

$$\frac{d}{ds_i} E[J_T] = E[c_i(X(i-1))(G_T^+ - G_T^-)|X^+(0) = X^-(0) = (S(0),0)].$$

If the nominal observation terminates before i, then the variants X^+ and X^- reach the absorbing state $*$ before transition i. Put another way, it holds that

$$\beta_i = 1 \Rightarrow G_T^+ - G_T^- = 0. \tag{13}$$

Furthermore, the construction of X^+ and X^- implies that the first $i-1$ states of X^+ and X^- are equal; that is,

$$X^+(j) = X^-(j), \qquad 0 \leq j < i. \tag{14}$$

Finally, we note that the ith states of X^+ and X^- are deterministic. In particular, provided that $\beta_i = 0$, we have

$$X^+(i) = (s_i,0) \quad \text{and} \quad X^-(i) = (s_i,i). \tag{15}$$

Let

$$G(i) := \sum_{j=i}^{\eta(T)+1} g(X^+(j)) - \sum_{j=i}^{\eta(T)+1} g(X^-(j)).$$

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Combining the above arguments yields

$$
\begin{aligned}
&E[c_i(X(i-1))(G_T^+ - G_T^-)|X^+(0) = X^-(0) = (S(0),0)] \\
&\quad \overset{(13)}{=} E[c_i(X(i-1))1_{\beta_i=0}G(0)|X^+(0) = X^-(0) = (S(0),0)] \\
&\quad \overset{(14)}{=} E[c_i(X(i-1))1_{\beta_i=0}G(i)|X^+(0) = X^-(0) = (S(0),0)] \\
&\quad \overset{(15)}{=} E[c_i(X(i-1))1_{\beta_i=0}G(i)|X^+(i) = (s_i,0), X^-(i) = (s_i,i)] \\
&\quad \overset{(8)}{=} E\left[c_i(X(i-1))1_{\beta_i=0}\sum_{j=i}^{\eta(T)+1} g(X^+(j))|X^+(i) = (s_i,0)\right] \\
&\qquad - E[c_i(X(i-1))1_{\beta_i=0}(s_i - K)e^{r(T-t_i)}] \\
&\quad = E\left[c_i(X(i-1))1_{\beta_i=0}\left(E\left[\sum_{j=i}^{\eta(T)+1} g(X^+(j))|X(i) = (s_i,0)\right] - (s_i - K)e^{r(T-t_i)}\right)\right],
\end{aligned}
$$

where the last equality follows from the Markov property of X. This proves the theorem. ■

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