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# On group representations and invariant stochastic processes

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*Abstract.* The theory of group representations is explored with regard to statistical applications. Ease of analysis depends rather critically on the amount of symmetry present and this point is examined in detail. 'Stationarity' assumptions are considered for a finite number of variates, with examples of their use in experimental situations.

*Introduction.* Several examples of the use of group representations in the statistical field have been discussed recently by Hannan (1).

The student new to the subject is handicapped by the omission of many of the results of practical importance from the mathematical works such as Murnaghan (7) and Littlewood (6); while the wealth of material presented by more physically inclined writers is apt to be distracting. The present paper is confined to the simplest case, that of finite groups, but aims at a complete account of the relevant theory. Only fundamentals are assumed, most of them implicit in § 1·1: Heine (2) gives a full treatment.

Attention is drawn to an application derived from the last one of Hannan's paper: to each point of a regular structure there corresponds one component of a vector random variable. The covariance matrix is supposed to share the structure's symmetry, which is expressed by invariance under a group. The circulant process is a simple example with the symmetry of a regular polygon. The processes are thus 'stationary', as the term is applied to time-series. These are governed by the additive group, which though Abelian is not compact, let alone finite. The spectral analysis of time-series is analogous to our finite decompositions, the full value of which is attained only if enough symmetry is assumed; enough completely to determine the eigenvectors of the covariance matrix, and so the spectral components of the process. In this, the case of *proper constraint*, the distinct spectral numbers (eigenvalues) possess structural interpretations.

The stage is set mathematically in § 1, with proofs of the less common results. The subject of the next section is the spectral decomposition of a matrix that commutes with a group representation, and the extra conditions that apply when the matrix is real, or real symmetric. Proper constraint occurs, practically speaking, when the representation has distinct components; but this criterion has little intuitive appeal and requires access to the character table. We derive conditions that depend only on transitivity properties: for example, it is necessary (sufficient) that, given any two

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points of the structure, a group element may be found to replace one by the other (each by the other).

The statistical implications are examined generally in § 3 and in particular cases in § 4. Many of the technical points are illustrated by the case of the dodecahedron, while the other applications discussed arise as experimental designs. The paper ends with an actual example.

1. *Mathematical background.* 1·1. *Notation.*

$G$: a finite group, order $g$

$\mathscr{C}_i$ $(i = 1, \ldots, r)$: the classes of $G$, orders $g_i$

$\mathscr{C}_{i'}$: the inverse class of $\mathscr{C}_i$

$\{D_\lambda\}$ $(\lambda = 1, \ldots, r)$: the $r$ inequivalent, irreducible, unitary representations of $G$

$\{D_\lambda^*\}$: the complex conjugate of $\{D_\lambda\}$

$\chi_\lambda(\mathscr{C}_i)$ $(i = 1, \ldots, r)$: the character of $\{D_\lambda\}$

$n_\lambda \equiv \chi_\lambda(I)$: the dimension of $\{D_\lambda\}$

A simple property: $\chi_{\lambda *}(\mathscr{C}_i) = \bar{\chi}_\lambda(\mathscr{C}_i) = \chi_\lambda(\mathscr{C}_{i'})$

$\{D\}$: an $n$-dimensional matrix representation of $G$

$\{D\}$ is equivalent to a direct sum of irreducible representations, possibly repeated

$\chi(\mathscr{C}_i)$ $(i = 1, \ldots, r)$: the character of $\{D\}$

$d_\lambda$: the number of times $\{D_\lambda\}$ appears in $\{D\}$

1·2. *Properties. Orthogonality relations.* $(g_i/g)^{\frac{1}{2}} \chi_\lambda(\mathscr{C}_i)$ are the elements of an $r \times r$ *unitary* matrix.

Hence
$$d_\lambda = \frac{1}{g} \sum_{i=1}^{r} g_i \bar{\chi}_\lambda(\mathscr{C}_i) \chi(\mathscr{C}_i). \tag{1}$$

*Schur's Lemma.* $D_\lambda^{(k)}$ and $D_\mu^{(k)}$ $(k = 1, \ldots, g)$ are the matrices of the (irreducible) representations $\{D_\lambda\}$ and $\{D_\mu\}$. $P$ is a matrix such that

$$P D_\lambda^{(k)} = D_\mu^{(k)} P \quad (k = 1, \ldots, g).$$

Then, if $\{D_\lambda\}$ and $\{D_\mu\}$ are not equivalent, $P = \mathbf{0}$; while if $\{D_\lambda\}$ and $\{D_\mu\}$ are identical, $P = pI$, for some scalar $p$.

*Wigner's Test.* Irreducible representations are classified as follows:

First kind—$\{D_\lambda\}$ is equivalent to a set of real matrices.

Second kind—$\{D_\lambda\}$ is equivalent to the complex conjugate representation $\{D_\lambda^*\}$ but not to a set of real matrices.

Third kind—$\{D_\lambda\}$ and $\{D_\lambda^*\}$ are not equivalent.

It can be shown, although we shall not use it, that

$$\epsilon_\lambda \equiv \frac{1}{g} \sum_{k=1}^{g} \chi_\lambda(a_k^2),$$

where $a_k$ are the elements of $G$, has the values 1, $-1$, or 0, respectively, when $\{D_\lambda\}$ is of the first, second or third kind.

Three more properties will be needed. They are not new, but proofs are given in § 1·4 because in the literature they are hard to find in suitable form. See, however, Johnston ((4), p. 125), for one of them. The results are:

Theorem 1. *The number of irreducible representations of the third kind is equal to the number of classes not self-inverse.*

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THEOREM 2. *If $\{D_\lambda\}$ is equivalent to its conjugate, then $\{D_\lambda^*\} = U^{-1}\{D_\lambda\}U$ for some unitary $U$; and $U$ is symmetric (skew-symmetric) if $\{D_\lambda\}$ is of the first (second) kind.*

THEOREM 3. *$R$ is an invariant subspace (of dimension $n_\lambda$) corresponding to the irreducible component $\{D_\lambda\}$ of a* real *representation $\{D\}$. $R^*$ comprises the complex conjugates of the vectors of $R$. Then $R$ and $R^*$ have zero intersection unless $\{D_\lambda\}$ is of the first kind.*

COROLLARY. *In a real representation each irreducible component of the second kind occurs an even number of times, while those of the third kind occur in complex conjugate pairs.*

1·3. *Permutation representations.* A group of permutations of $n$ symbols $X_1, \ldots, X_n$ is *transitive* if any symbol may be replaced by any other. An intransitive group divides the symbols into transitive sets.

Suppose $\{D\}$ represents $G$ by permutation matrices of dimension $n$. If $\{D\}$ is intransitive, it is nevertheless a direct sum of transitive representations, one for each transitive set. In deriving a simplified form of (1) we lose nothing in considering only transitive representations $\{D\}$.

For $j = 1, \ldots, n$, the elements of $G$ that leave $X_j$ invariant form a subgroup $H_j$. Since $\{D\}$ is transitive these $H_j$ are isomorphic: in fact $H_j \leftrightarrow a^{-1}H_k a$, where $a \in G$ and replaces $X_j$ by $X_k$. The left cosets of $H_1$ in $G$ comprise the elements that replace $X_1$ by $X_1, \ldots, X_n$, respectively. So $\{D\}$ may be regarded as permuting the cosets of $H_1$. The order of $H_1$ is $h \equiv g/n$.

Suppose just $h_i$ elements of $H_1$ belong to class $\mathscr{C}_i$ of $G$. Then just $h_i$ elements of $H_k$ belong to $\mathscr{C}_i$ $(k = 1, \ldots, n)$. Hence

$$g_i \operatorname{tr} D^{(k)} = nh_i \quad \text{if} \quad a_k \in \mathscr{C}_i.$$

But $n$ is the index of $H_1$ in $G$, viz. $g/h$. Therefore

$$\chi(\mathscr{C}_i) = gh_i/hg_i \quad (i = 1, \ldots, r).$$

Thus for a transitive permutation representation (1) becomes

$$d_\lambda = \frac{1}{h}\sum_{i=1}^{r} h_i \overline{\chi}_\lambda(\mathscr{C}_i), \tag{2}$$

where $h_i$ is the number of elements of $H_1$ belonging to $\mathscr{C}_i$, and $h = \sum_{i=1}^{r} h_i$.

The composition of an intransitive representation may be found by applying (2) to each transitive set, and adding the results.

An important special case is the *regular representation*, for which $H_1$ consists of the identity alone. There are really two, corresponding to pre- and post-multiplication.

$$\left.\begin{aligned} \{D\}_{\text{L. Reg.}}\colon\ D_{ij}^{(k)} &= 1 \quad \text{if} \quad a_i a_j^{-1} = a_k \\ &= 0 \quad \text{otherwise} \end{aligned}\right\} \quad (i, j, k = 1, \ldots, g),$$

$$\left.\begin{aligned} \{D\}_{\text{R. Reg.}}\colon\ D_{ij}^{(k)} &= 1 \quad \text{if} \quad a_i^{-1} a_j = a_k \\ &= 0 \quad \text{otherwise} \end{aligned}\right\} \quad (i, j, k = 1, \ldots, g).$$

Note that any matrix from one representation commutes with any matrix from the other. Symbolically $\{D\}_{\text{L. Reg.}} \curlyvee \{D\}_{\text{R. Reg.}}$.



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Left- and right-regular representations are equivalent, because (2) gives for either

$$d_\lambda = \bar{\chi}_\lambda(I) = n_\lambda.$$

1·4. *Proof of Theorem* 1. The orthogonality relations show that no two rows (no two columns) of the character table are alike. It follows easily that

$$\{D_\lambda\} \quad \text{and} \quad \{D_\lambda^*\} \quad \text{are equivalent iff } \chi_\lambda(\mathscr{C}_i) \text{ are real} \quad (i = 1, \ldots, r).$$

$\mathscr{C}_i$ is self-inverse iff $\chi_\lambda(\mathscr{C}_i)$ are real $(\lambda = 1, \ldots, r)$; because $\chi_\lambda(\mathscr{C}_{i'}) \equiv \bar{\chi}_\lambda(\mathscr{C}_i)$.

Now suppose there are just $p$ pairs $\{D_\lambda\}$, $\{D_\lambda^*\}$, of representations of the third kind $(\lambda = 1, \ldots, p)$ and just $q$ pairs $\mathscr{C}_j$, $\mathscr{C}_{j'}$ of classes not self-inverse $(j = 1, \ldots, q)$. Then the $r^2$ entries in the character table are real, except for $4pq$ of them, which form $pq$ blocks.

| | $\mathscr{C}_j$ | $\mathscr{C}_{j'}$ |
|---|---|---|
| $\{D_\lambda\}$ | $\chi$ | $\bar{\chi}$ |
| $\{D_\lambda^*\}$ | $\bar{\chi}$ | $\chi$ |

Let $\boldsymbol{\alpha}_i$ $(i = 1, \ldots, r)$ be the (normal unitary) $r$-vectors with $\lambda$th components

$$(g_i/g)^{\frac{1}{2}} \chi_\lambda(\mathscr{C}_i).$$

$\mathscr{C}_{j'}$ contains *just* the inverses of $\mathscr{C}_j$, so $g_{j'} = g_j$ and $\boldsymbol{\alpha}_{j'} = \bar{\boldsymbol{\alpha}}_j$ $(j = 1, \ldots, q)$. Now the previous remark shows that the vectors $(\boldsymbol{\alpha}_j - \boldsymbol{\alpha}_{j'})/\sqrt{2}$ belong to a certain subspace of dimension $p$. But they are normal unitary, and $q$ in number. Hence $q \leqslant p$. Similarly $p \leqslant q$. So $p = q$.

*Proof of Theorem* 2. Suppose $\{D_\lambda^*\} = T^{-1}\{D_\lambda\}T$. Since this is a unitary representation, we have

$$T^{-1}\{D_\lambda\}T = [T^{-1}\{D_\lambda\}T]^{\dagger-1} = T^\dagger\{D_\lambda\}T^{\dagger-1}.$$

By Schur's Lemma, $TT^\dagger = t.I$; and $t$ is real and positive because $TT^\dagger$ is Hermitian and positive-definite. So $\{D_\lambda^*\} = U^{-1}\{D_\lambda\}U$, where $U \equiv t^{-\frac{1}{2}}T$ is unitary.

Now

$$\{D_\lambda\} = [\overline{U^{-1}\{D_\lambda\}U}] = U'\{D_\lambda^*\}U'^{-1} = U'U^{-1}\{D_\lambda\}UU'^{-1},$$

and so, by Schur's Lemma, $U'U^{-1} = aI$. Hence $U = (U')' = (aU)' = aU' = a^2U$, and so $a^2 = 1$ and $U' = \pm U$. Thus $U$ is either symmetric or skew-symmetric.

Now suppose $\{D_\lambda\}$ is of the first kind and $P^{-1}\{D_\lambda\}P$ is a real (orthogonal) representation. As in the first part of the proof, $P$ may be chosen to be unitary and then

$$P^{-1}\{D_\lambda\}P = [\overline{P^{-1}\{D_\lambda\}P}] = \bar{P}^{-1}U^{-1}\{D_\lambda\}U\bar{P},$$

so

$$\{D_\lambda\}PP'U^{-1} = PP'U^{-1}\{D_\lambda\}.$$

By Schur's Lemma, $PP'U^{-1} = b.I$ and $U = b^{-1}PP'$.

So $U$ is symmetric if $\{D_\lambda\}$ is of the first kind.

Conversely suppose $U$ is symmetric. Then, if $U \equiv e^{iH}$,

$$e^{-iH} = U^{-1} = U^\dagger = \bar{U} = e^{-i\bar{H}}.$$

Hence $U^2 = e^{iH}.e^{i\bar{H}} = e^{i(H+\bar{H})} = e^{2i\Sigma}$, where $\Sigma$ is real symmetric. So $U = e^{iS}$, where $S$ is real symmetric. Now define $P \equiv e^{\frac{1}{2}iS}$. Then $\bar{P} = e^{-\frac{1}{2}iS}$; so $U\bar{P} = e^{\frac{1}{2}iS} = P$.

Hence

$$[\overline{P^{-1}\{D_\lambda\}P}] = \bar{P}^{-1}U^{-1}\{D_\lambda\}U\bar{P} = P^{-1}\{D_\lambda\}P.$$

Thus $P^{-1}\{D_\lambda\}P$ is real and $\{D_\lambda\}$ is of the first kind if $U$ is symmetric.

Hence $U$ is symmetric (skew-symmetric) if $\{D_\lambda\}$ is of the first (second) kind.

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*Proof of Theorem* 3. Since $\{D\}$ is real, $R^*$ also is an invariant subspace. Hence $R \cap R^*$ is an invariant subspace, and so $R$ and $R^*$ either coincide or have only the zero vector in common, for $\{D_\lambda\}$ is irreducible. Suppose $R$ and $R^*$ do coincide, with normal unitary basis $(\mathbf{z}_i)$ $(i = 1, \ldots, n_\lambda)$. Then

$$\bar{\mathbf{z}}_i = \sum_{j=1}^{n_\lambda} u_{ij} \mathbf{z}_j \quad (i = 1, \ldots, n_\lambda), \tag{3}$$

where the $u_{ij}$ form a unitary matrix $U$. But $U$ is symmetric because

$$u_{ij} = \bar{\mathbf{z}}_j' \bar{\mathbf{z}}_i = \bar{\mathbf{z}}_i' \bar{\mathbf{z}}_j = u_{ji}.$$

Now the $\mathbf{z}_i$ transform according to $\{D_\lambda\}$, so that $\bar{\mathbf{z}}_i$ transform according to $\{D_\lambda^*\}$; but by (3) the $\bar{\mathbf{z}}_i$ transform according to $U\{D_\lambda\}U^{-1}$. Hence $\{D_\lambda^*\} = U\{D_\lambda\}U^{-1}$ and, by the previous theorem, $\{D_\lambda\}$ must be of the first kind.

If $\{D_\lambda\}$ is not of the first kind, the only alternative is that $R$ and $R^*$ have zero intersection.

2. *Constrained matrices.* We shall be concerned with matrices $M$ that commute with every member of a permutation representation $\{D\}$ of a finite group $G$. $M$ will be said to be *constrained* by $\{D\}$ and we shall use the notation

$$M \curlyvee \{D\}. \tag{4}$$

If the precise definition of $\{D\}$ is

$$\left.\begin{aligned} D_{ij}^{(k)} = 1 \quad &\text{if} \quad a_k \text{ replaces } X_j \text{ by } X_i \\ = 0 \quad &\text{otherwise} \end{aligned}\right\} \quad (k = 1, \ldots, g;\ i, j = 1, \ldots, n),$$

then the $(i, j)$th and $(p, q)$th elements of $M$ are equal, whenever there is an $a_k \in G$ that replaces $X_i$ by $X_p$ and $X_j$ by $X_q$. But the real significance of (4) becomes plain when $\{D\}$ is decomposed into irreducible representations.

2·1. *Idempotents and canonical bases.* Suppose a reduction of $\{D\}$ is

$$Q^{-1}D^{(k)}Q = \begin{pmatrix} \ddots & & \\ & D_\lambda^{(k)} & \mathbf{0} \\ & & \ddots \\ & \mathbf{0} & \ddots \end{pmatrix} \quad (k = 1, \ldots, g),$$

where, since the permutation matrices $D^{(k)}$ are unitary, $Q$ too is unitary. It is assumed that $Q$ is chosen so that equivalent irreducible representations are identical and are neighbours in the reduced form $Q^{-1}\{D\}Q$.

Now $Q^{-1}MQ$ commutes with each $Q^{-1}D^{(k)}Q$, and application of Schur's Lemma to each submatrix gives, where $I_\lambda$ is the unit matrix of dimension $n_\lambda$,

$$M = Q \left(\begin{array}{c|c|c} \ddots & \mathbf{0} & \\ \hline \mathbf{0} & \begin{matrix} m_\lambda^{11} I_\lambda & \cdots & m_\lambda^{1 d_\lambda} I_\lambda \\ \vdots & & \vdots \\ m_\lambda^{d_\lambda 1} I_\lambda & \cdots & m_\lambda^{d_\lambda d_\lambda} I_\lambda \end{matrix} & \mathbf{0} \\ \hline & \mathbf{0} & \ddots \end{array}\right) Q^{-1}$$

$$\equiv \sum_{\lambda=1}^{r} \sum_{i,j=1}^{d_\lambda} m_\lambda^{ij} E_\lambda^{ij} \tag{5}$$



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by appropriate definition of the matrices $E_\lambda^{ij}$. For each $\lambda$, the arbitrary complex numbers $m_\lambda^{ij}$ form a $d_\lambda \times d_\lambda$ matrix $M_\lambda$, and, if we write

$$M \equiv \sum_{\lambda=1}^{r} [M_\lambda],$$

$$\left(\sum_{\lambda=1}^{r} [M_\lambda]\right)\left(\sum_{\lambda=1}^{r} [N_\lambda]\right) = \sum_{\lambda=1}^{r} [M_\lambda N_\lambda].$$

Since $Q$ is unitary, $M$ is Hermitian iff each $M_\lambda$ is Hermitian.

If $d_\lambda \leqslant 1$ $(\lambda = 1, \ldots, r)$, so that no representation is repeated in $\{D\}$, we have simply

$$M = \sum_{\lambda=1}^{r} m_\lambda E_\lambda, \tag{6}$$

where the $E_\lambda$ are mutually annihilating idempotents, of ranks $n_\lambda$ (for $d_\lambda = 1$). Note $E_\lambda \equiv \mathbf{0}$ for $d_\lambda = 0$.

These idempotents will be expressed in terms of the class matrices

$$C_i \equiv \Sigma_i D^{(k)} \quad (i = 1, \ldots, r),$$

where $\Sigma_i$ means summation over those $k$ for which $a_k \in \mathscr{C}_i$. Since

$$C_i \curlyvee \{D\} \quad (i = 1, \ldots, r),$$

$$C_i = \sum_{\lambda=1}^{r} c_\lambda^{(i)} E_\lambda \quad (i = 1, \ldots, r)$$

and $n_\lambda c_\lambda^{(i)} = \Sigma_i \operatorname{tr} D_\lambda^{(k)} = g_i \chi_\lambda(\mathscr{C}_i)$, necessarily for $d_\lambda = 1$ and by definition if $d_\lambda = 0$. We now appeal to the orthogonality relations and obtain the inversion

$$E_\lambda = \frac{n_\lambda}{g} \sum_{i=}^{r} \bar{\chi}_\lambda(\mathscr{C}_i) C_i \quad (\lambda = 1, \ldots, r). \tag{7}$$

The equation (6) expresses the elements of $M$ as linear functions of its eigenvalues $m_\lambda$. Conversely,

$$m_\lambda E_\lambda = M E_\lambda \quad (d_\lambda = 1).$$

Returning to the general case, where irreducible representations may be repeated any number of times in $\{D\}$, and defining

$$\left.\begin{aligned} E_\lambda = \sum_{i=1}^{d_\lambda} E_\lambda^{ii} \quad & \text{when} \quad d_\lambda \geqslant 1 \\ = \mathbf{0} \quad & \text{when} \quad d_\lambda = 0 \end{aligned}\right\} \quad (\lambda = 1, \ldots, r),$$

we find that (7) holds just as before. Note that

$$m_\lambda^{ij} E_\lambda^{ij} = E_\lambda^{ii} M E_\lambda^{jj} \quad (\lambda = 1, \ldots, r;\ i, j = 1, \ldots, d_\lambda).$$

Before using this or (5) we should have to find the $E_\lambda^{ij}$, which are not yet fully determined. If $\Omega_\lambda$ is the range space of $E_\lambda$, of dimension $n_\lambda d_\lambda$, it can be decomposed in two complementary ways:

$$\Omega_\lambda = R_1 \oplus \ldots \oplus R_{d_\lambda}, \quad \dim R_j = n_\lambda,$$

$$\Omega_\lambda = V_1 \oplus \ldots \oplus V_{n_\lambda}, \quad \dim V_i = d_\lambda,$$

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where each $R_j$ is an invariant subspace for every $D^{(k)}$, and where $V_i$ comprises vectors transforming according to the $i$th column of $\{D_\lambda\}$. The decomposition is far from unique, for a particular representation $\{D_\lambda\}$ must be chosen from among those equivalent to it, to fix the $V_i$. Even then, all unitary bases of $V_1$ serve equally well to fix the $R_j$.

In principle, a *canonical basis* $(\mathbf{x}_{p,q})$ $(p = 1, \ldots, n_\lambda; q = 1, \ldots, d_\lambda)$ may be found, where

$$\mathbf{x}_{p,q} \in V_p \cap R_q.$$

Then

$$E_\lambda^{ij} = \sum_{p=1}^{n_\lambda} \mathbf{x}_{p,i} \mathbf{x}_{p,j}^\dagger. \tag{8}$$

2·2. *Real constrained matrices.* Suppose a constrained matrix $M = \sum_{\lambda=1}^{r} [M_\lambda]$ is real: the form of the conditions on the $m_\lambda^{ij}$ depends on the kind of irreducible representation $\{D_\lambda\}$. Convention: typical representations of the three kinds will be denoted respectively by $\{D_\rho\}$, $\{D_\mu\}$ and $\{D_\nu\}$.

If $d_\lambda \leqslant 1$ $(\lambda = 1, \ldots, r)$ the analysis is simple, for:

$\{D_\rho\}$ of first kind $\Rightarrow$ character $\chi_\rho(\cdot)$ is real $\Rightarrow E_\rho$ is real, by (7). Since $m_\rho E_\rho = M E_\rho$, $m_\rho$ is real.

$\{D_\mu\}$ of second kind $\Rightarrow d_\mu$ even, since the permutation representation $\{D\}$ is real (by corollary to Theorem 3). So $d_\mu = 0$.

$\{D_\nu\}$ of third kind $\Rightarrow \{D_\nu\}$ and $\{D_\nu^*\}$ both appear in $\{D\}$ if either does. Since $E_\nu = \bar{E}_{\nu^*}$, by (7), the following is real,

$$m_\nu E_\nu + m_{\nu^*} E_{\nu^*} \equiv M(E_\nu + E_{\nu^*}).$$

By subtraction of the complex conjugate, we have

$$(m_\nu - \overline{m}_{\nu^*}) E_\nu + (m_{\nu^*} - \overline{m}_\nu) E_{\nu^*} = \mathbf{0},$$

and therefore

$$m_\nu = \overline{m}_{\nu^*}.$$

Now permit $d_\lambda > 1$ and consider the three cases in turn:

*First kind.* Take $\{D_\rho\}$ in real form and let $(\mathbf{z}_{p,q})$ $(p = 1, \ldots, n_\rho;\ q = 1, \ldots, d_\rho)$ be a canonical basis for $\Omega_\rho$. Then the real parts $(\mathbf{x}_{p,q})$ of these vectors have the same transformation properties and form an alternative canonical basis. By (8), the $E_\rho^{ij}$ are now real. Since

$$m_\rho^{ij} E_\rho^{ij} = E_\rho^{ii} M E_\rho^{jj} \quad (i, j = 1, \ldots, d_\rho), \tag{9}$$

the $m_\rho^{ij}$ are real. So $M_\rho$ is real.

*Second kind.* $\{D_\mu\}$ appears an even number of times in $\{D\}$. Suppose $(\mathbf{z}_p)$ $(p = 1, \ldots, n_\mu)$ is a unitary basis of an invariant subspace $R_1$ transforming according to $\{D_\mu\}$. Then the conjugates $\bar{\mathbf{z}}_p$ transform according to $\{D_\mu^*\}$ and span another invariant subspace $R_1^*$ of $\Omega_\mu$. $R_1$ and $R_1^*$ are distinct, by Theorem 3. But equivalent representations are to be identical in the reduced form $Q^{-1}\{D\}Q$, so that a suitable basis for $R_1^*$ is, with the summation convention,

$$\boldsymbol{\zeta}_p = u_{pq} \bar{\mathbf{z}}_q \quad (p = 1, \ldots, n_\mu),$$

where the $u_{pq}$ form a unitary matrix $U$ such that $\{D_\mu^*\} = U^{-1}\{D_\mu\}U$. By Theorem 2, $U$ is skew-symmetric.

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If $R_2$ is chosen orthogonal to $R_1$ and to $R_1^*$, then $R_2^*$ is orthogonal to $R_1^*$ and to $R_1$. $R_2$ and $R_2^*$ are orthogonal by Theorem 3. Eventually we achieve the decomposition

$$\Omega_\mu = R_1 \oplus \ldots \oplus R_{\frac{1}{2}d_\mu} \oplus R_1^* \oplus \ldots \oplus R_{\frac{1}{2}d_\mu}^*,$$

where $R_j^*$ comprises complex conjugates of vectors of $R_j$ $(j = 1, \ldots, \frac{1}{2}d_\mu)$.

Now suppose $(\mathbf{x}_p)$, $(\mathbf{y}_p)$, $(\boldsymbol{\xi}_p)$, $(\boldsymbol{\eta}_p)$ are bases of four spaces $R_i$, $R_j$, $R_i^*$, $R_j^*$ with

$$\boldsymbol{\xi}_p = u_{pq}\bar{\mathbf{x}}_q, \quad \boldsymbol{\eta}_p = u_{pq}\bar{\mathbf{y}}_q \quad (p = 1, \ldots, n_\mu).$$

Using the natural notation we have

$$E_\mu^{ij}\colon\ R_j \to R_i, \qquad E_\mu^{ij^*}\colon\ R_j^* \to R_i,$$

$$E_\mu^{i^*j}\colon\ R_j \to R_i^*, \quad E_\mu^{i^*j^*}\colon\ R_j^* \to R_i^*$$

and then by (8) $$E_\mu^{i^*j^*} = \boldsymbol{\xi}_p \boldsymbol{\eta}_p^\dagger = u_{pq}\bar{u}_{ps}\bar{\mathbf{x}}_q \mathbf{y}_s' = \delta_{qs}\bar{\mathbf{x}}_q \mathbf{y}_s',$$

since $U$ is unitary. Therefore $$E_\mu^{i^*j^*} = \overline{E_\mu^{ij}} \tag{10}$$

and $$E_\mu^{ij^*} = \mathbf{x}_p \boldsymbol{\eta}_p^\dagger = \bar{u}_{pq}\mathbf{x}_p \mathbf{y}_q'$$

while $$E_\mu^{i^*j} = \boldsymbol{\xi}_p \mathbf{y}_p^\dagger = u_{pq}\bar{\mathbf{x}}_q \mathbf{y}_p^\dagger = -u_{pq}\bar{\mathbf{x}}_p \mathbf{y}_q^\dagger,$$

since $U$ is skew. Hence $$E_\mu^{ij^*} = -\overline{E_\mu^{i^*j}}. \tag{11}$$

These results hold whether $i = j$ or not. In particular—dropping the summation convention—$(E_\mu^{kk} + E_\mu^{k^*k^*})$ are real; $k = 1, \ldots, \frac{1}{2}d_\mu$. So if $M$ is real, the following is real

$$(E_\mu^{ii} + E_\mu^{i^*i^*})\, M (E_\mu^{jj} + E_\mu^{j^*j^*}) \equiv m_\mu^{ij} E_\mu^{ij} + m_\mu^{ij^*} E_\mu^{ij^*} + m_\mu^{i^*j^*} E_\mu^{i^*j^*} + m_\mu^{i^*j} E_\mu^{i^*j}.$$

By subtraction of the complex conjugate, and using (10) and (11) we have

$$(m_\mu^{ij} - \overline{m_\mu^{i^*j^*}})\, E_\mu^{ij} + (m_\mu^{i^*j^*} - \overline{m_\mu^{ij}})\, E_\mu^{i^*j^*} + (m_\mu^{ij^*} + \overline{m_\mu^{i^*j}})\, E_\mu^{ij^*} + (m_\mu^{i^*j} + \overline{m_\mu^{ij^*}})\, E_\mu^{i^*j} = \mathbf{0}.$$

Therefore $$m_\mu^{ij} = \overline{m_\mu^{i^*j^*}}; \quad m_\mu^{ij^*} = -\overline{m_\mu^{i^*j}}.$$

In other words, if $M_\mu$ is dissected into four blocks

$$M_\mu \equiv \begin{pmatrix} M_\mu^{00} & M_\mu^{01} \\ M_\mu^{10} & M_\mu^{11} \end{pmatrix},$$

then $$M_\mu^{00} = \overline{M_\mu^{11}}, \quad M_\mu^{01} = -\overline{M_\mu^{10}}. \tag{12}$$

*Third kind.* $\{D_\nu\}$ and $\{D_\nu^*\}$ appear in $\{D\}$ the same number of times (by corollary to Theorem 3). If $(\mathbf{z}_{p,q})$ $(p = 1, \ldots, n_\nu;\ q = 1, \ldots, d_\nu)$; is a canonical basis of $\Omega_\nu$ then $(\bar{\mathbf{z}}_{p,q})$ is a canonical basis of $\Omega_{\nu^*}$. Hence, by (8)

$$E_\nu^{ij} = \overline{E_{\nu^*}^{ij}} \quad (i, j = 1, \ldots, d_\nu).$$

Hence if $M$ is real, so is

$$(E_\nu^{ii} + E_{\nu^*}^{ii})\, M (E_\nu^{jj} + E_{\nu^*}^{jj}) \equiv m_\nu^{ij} E_\nu^{ij} + m_{\nu^*}^{ij} E_{\nu^*}^{ij}.$$

Hence $$(m_\nu^{ij} - \overline{m_{\nu^*}^{ij}})\, E_\nu^{ij} + (m_{\nu^*}^{ij} - \overline{m_\nu^{ij}})\, E_{\nu^*}^{ij} = \mathbf{0},$$

and so $$m_\nu^{ij} = \overline{m_{\nu^*}^{ij}},$$

i.e. $$M_\nu = \bar{M}_{\nu^*}. \tag{13}$$

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These conditions on the parameters $m_\rho^{ij}$, $m_\mu^{ij}$, $m_\nu^{ij}$ are together sufficient, as well as necessary, for $M$ to be real. (5) shows that there are just $\sum_{\lambda=1}^{r} d_\lambda^2$ arbitrary complex constants in the general constrained matrix. (9), (12) and (13) confirm that a *real* constrained matrix contains just

$$f \equiv \sum_{\lambda=1}^{r} d_\lambda^2$$

arbitrary real constants.

2·3. *Real, symmetric, constrained matrices.* These are real Hermitian so that each $M_\lambda$ is Hermitian. We discuss the three cases in turn.

$M_\rho$ is real symmetric and contains $\frac{1}{2}d_\rho(d_\rho+1)$ arbitrary constants.

$M_\mu^{00}$ is Hermitian and contains $(\frac{1}{2}d_\mu)^2$ constants; $M_\mu^{11}$ is automatically Hermitian, by (12). $M$ Hermitian implies $(M_\mu^{01})' = \overline{M_\mu^{10}}$; $M$ real implies $M_\mu^{01} = -\overline{M_\mu^{10}}$. So $M_\mu^{01}$ is (complex) skew-symmetric and contains $\frac{1}{2}d_\mu(\frac{1}{2}d_\mu - 1)$ constants. The total number of constants associated with $\{D_\mu\}$ is $\frac{1}{2}d_\mu(d_\mu-1)$.

$M_\nu$ is Hermitian and contains $d_\nu^2$ constants; but these determine $M_{\nu*}$ also, by (13). The number of constants associated with each of $\{D_\nu\}$, $\{D_\nu^*\}$ is $\frac{1}{2}d_\nu^2$.

So a *real symmetric* constrained matrix contains just

$$f' \equiv \frac{1}{2}\left(\sum_{\lambda=1}^{r} d_\lambda^2 + \sum_{\lambda=1}^{r} \epsilon_\lambda d_\lambda\right)$$

arbitrary real constants, where $\epsilon_\rho \equiv 1$, $\epsilon_\mu \equiv -1$, $\epsilon_\nu \equiv 0$. The reduction in the 'degrees of freedom' of a real constrained matrix, when it is required to be symmetric, is

$$f - f' \equiv \frac{1}{2}\left(\sum_{\lambda=1}^{r} d_\lambda^2 - \sum_{\lambda=1}^{r} \epsilon_\lambda d_\lambda\right). \tag{14}$$

Later, this result will provide a sufficient condition for $d_\lambda \leqslant 1$ $(\lambda = 1, \ldots, r)$. The advantage of that situation lies in the simplicity of (6). If $M$ is real symmetric, however, it is enough if

$$d_\rho, d_\nu \leqslant 1, \quad d_\mu \leqslant 2 \tag{15}$$

for these are the conditions under which the eigenvectors of such a constrained matrix are completely determined and

$$M = \sum_{\lambda=1}^{r} m_\lambda E_\lambda,$$

where (7) gives the $E_\lambda$; the eigenvalues $m_\lambda$ are real and $m_\nu = m_\nu^*$ for representations of the third kind.

If (15) holds, constraint by $\{D\}$ will be called *proper constraint*. In particular, a regular representation imposes proper constraint only if the irreducible representations are one-dimensional, except those of the second kind, which may be two-dimensional.

2·4. *Constraint by both regular representations.* Suppose that $M$ is constrained by both the left- and right-regular representations of a group $G$

$$M \curlyvee \{D\}_{\text{L. Reg}}, \quad M \curlyvee \{D\}_{\text{R. Reg}}.$$

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Let $D_L^{(k)}$ and $D_R^{(k)}$ $(k = 1, \ldots, g)$ be the matrices of these representations; then $M$ commutes with every product

$$D_L^{(k_1)} D_R^{(k_2)} \quad (k_1, k_2 = 1, \ldots, g)$$

and these products form a permutation representation $\{D\}_{\text{L.}\times\text{R.}}$ of the direct product $G \times G$, because $\{D\}_{\text{L. Reg.}} \curlyvee \{D\}_{\text{R. Reg.}}$.

$\{D\}_{\text{L.}\times\text{R.}}$ is transitive and the generating subgroup $H_1$ consists of the elements $(a_k, a_k)$, $k = 1, \ldots, g$. In terms of the characters $\chi_\lambda(\mathscr{C}_i)$ of $G$ the characters of $G \times G$ are

$$\chi_{\lambda\times\mu}(\mathscr{C}_i \times \mathscr{C}_j) = \chi_\lambda(\mathscr{C}_i)\, \chi_\mu(\mathscr{C}_j).$$

Applying (2), the composition of $\{D\}_{\text{L.}\times\text{R.}}$ is

$$d_{\lambda\times\mu} = \frac{1}{g} \sum_{i=1}^{r} g_i \overline{\chi}_\lambda(\mathscr{C}_i)\, \overline{\chi}_\mu(\mathscr{C}_i) = \frac{1}{g} \sum_{i=1}^{r} g_i \chi_{\lambda^*}(\mathscr{C}_i)\, \overline{\chi}_\mu(\mathscr{C}_i)$$
$$= \delta_{\lambda^*\mu}, \text{ by the orthogonality relations.}$$

So no irreducible representation of $G \times G$ is repeated in $\{D\}_{\text{L.}\times\text{R.}}$, and

$$M = \sum_{\lambda=1}^{r} m_\lambda E_{\lambda\times\lambda^*}.$$

Applying (7)
$$E_{\lambda\times\lambda^*} = \frac{n_\lambda^2}{g^2} \sum_{i,j=1}^{r} \overline{\chi}_\lambda(\mathscr{C}_i)\, \overline{\chi}_{\lambda^*}(\mathscr{C}_j)\, C_i^L C_j^R,$$

where $C_i^L$, $C_j^R$ $(i, j = 1, \ldots, r)$ are the class matrices for $\{D\}_{\text{L. Reg.}}$ and $\{D\}_{\text{R. Reg.}}$ respectively. But because (for any $p, q = 1, \ldots, g$) $a_p a_q^{-1}$ and $a_p^{-1} a_q$ belong to inverse classes

$$C_j^L = C_j^R \quad (j = 1, \ldots, r).$$

Hence
$$E_{\lambda\times\lambda^*} = \left( \frac{n_\lambda}{g} \sum_{i=1}^{r} \overline{\chi}_\lambda(\mathscr{C}_i)\, C_i^L \right)^2 = (E_\lambda^L)^2 = E_\lambda^L,$$

and so
$$M = \sum_{\lambda=1}^{r} m_\lambda E_\lambda^L.$$

This appears to favour $\{D\}_{\text{L. Reg.}}$ at the expense of $\{D\}_{\text{R. Reg.}}$, but $E_\lambda^L \equiv E_{\lambda^*}^R$, so impartiality is preserved. In fact the $(p, 1)$th element of $M$ is, if $a_p \in \mathscr{C}_i$,

$$M_i = \frac{1}{g} \sum_{\lambda=1}^{r} n_\lambda \overline{\chi}_\lambda(\mathscr{C}_i)\, m_\lambda \quad (i = 1, \ldots, r), \tag{16}$$

and, conversely,
$$m_\lambda = \frac{1}{n_\lambda} \sum_{i=1}^{r} g_i \chi_\lambda(\mathscr{C}_i)\, M_i \quad (\lambda = 1, \ldots, r). \tag{17}$$

If $M$ is real symmetric, then the $m_\lambda$ are real, and $m_\nu = m_{\nu^*}$ for representations of the third kind.

2·5. *Conditions for proper constraint.* (2) shows that the identical representation $\{D_1\}$, which is one-dimensional and represents every element of $G$ by unity, appears just once in a transitive permutation representation $\{D\}$. In general it appears just as many times as there are transitive sets. So a necessary condition for proper constraint is that $\{D\}$ be transitive.

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The subgroup $H_1$ of those $a_k \in G$ that leave $X_1$ invariant divides the symbols $X_1, \ldots, X_n$ into transitive sets (*orbits*). However, the remark following (4) shows that, for transitive $\{D\}$, the same elements are permuted in every row of a constrained matrix $M$. So the number of orbits is the number of arbitrary constants in the general constrained matrix. We pointed out at the end of § 2·2 that this number was

$$f \equiv \sum_{\lambda=1}^{r} d_\lambda^2.$$

If we ask that $M$ be symmetric, additional equalities may be imposed on its elements. Because $\{D\}$ is transitive, it is enough to consider the additional constraints on the elements of the first row and first column. In fact the reduction in the number of arbitrary constants, when $M$ is required to be symmetric, is just *half the number of orbits whose symbols cannot be exchanged with* $X_1$. If there are $K$ such orbits, then by (14)

$$K = 2(f - f') = \sum_{\lambda=1}^{r} d_\lambda^2 - \sum_{\lambda=1}^{r} \epsilon_\lambda d_\lambda. \tag{18}$$

Now suppose any pair of symbols $X_i$, $X_j$ may be exchanged, by some $a_k \in G$. Then $K = 0$ and by (18)

$$d_\lambda = 0 \quad \text{or} \quad d_\lambda = \epsilon_\lambda = 1 \quad (\lambda = 1, \ldots, r).$$

So every irreducible representation present in $\{D\}$ is of the first kind and is not repeated. We have found a sufficient condition for $d_\lambda \leqslant 1$ $(\lambda = 1, \ldots, r)$ and for proper constraint. $K = 0$ is not a necessary condition; but (18) and the use of Theorem 1 to establish the number of representations of the third kind, will often settle the matter in practice. In particular, if every class of $G$ is self-inverse, then $K = 0$ is necessary for $d_\lambda \leqslant 1$ $(\lambda = 1, \ldots, r)$ and $K \equiv 0 \pmod 6$ is necessary for proper constraint.

Note: (18) can be derived without appeal to (14), by using Wigner's Test to show that the number of orbits whose symbols *can* be exchanged with $X_1$ is $\sum_{\lambda=1}^{r} \epsilon_\lambda d_\lambda$.

3. *Invariant processes.* 3·1. *Covariance structure and linear models.* $\mathbf{X}$ and $\mathbf{Y}$ are random vectors of $n$ complex components, with zero expectation. $\mathbf{Y} = B\mathbf{X}$ is a linear model, where $B$ is a non-singular matrix of unknown constants. We suppose that there is latitude in the choice of coordinate system so that the processes are stochastically invariant under a group $G$ of permutations of the components. Thus $G$ has a representation $\{D\}$ defined by

$$\left.\begin{aligned} D_{ij}^{(k)} &= 1 \quad \text{if} \quad a_k \text{ replaces } X_j \text{ by } X_i \\ &= 0 \quad \text{otherwise} \end{aligned}\right\} \quad (k = 1, \ldots, g;\ i, j = 1, \ldots, n).$$

In particular the covariance matrices $\Gamma_X \equiv \mathscr{E}(\mathbf{X}\mathbf{X}^\dagger)$, $\Gamma_Y \equiv \mathscr{E}(\mathbf{Y}\mathbf{Y}^\dagger)$, and the matrix $B$ of any linear model, are assumed to depend on structural properties invariant under the transformations

$$\mathbf{X}^* = D^{(k)}\mathbf{X}, \quad \mathbf{Y}^* = D^{(k)}\mathbf{Y} \quad (k = 1, \ldots, g).$$

Thus $$\Gamma_{X^*} = \Gamma_X, \quad \Gamma_{Y^*} = \Gamma_Y, \quad \mathbf{Y}^* = B\mathbf{X}^*,$$

and so $$D^{(k)}\Gamma_X D^{(k)\dagger} = \Gamma_X, \quad D^{(k)}\Gamma_Y D^{(k)\dagger} = \Gamma_Y, \quad [D^{(k)}]^{-1} B D^{(k)} = B.$$

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Since permutation matrices are unitary,

$$\Gamma_X \curlyvee \{D\}, \quad \Gamma_Y \curlyvee \{D\}, \quad B \curlyvee \{D\}.$$

Hence our interest in constrained matrices. With proper constraint the spectral numbers $\gamma_\lambda$ can be given structural interpretations; otherwise the parameters $\gamma_\lambda^{ij}$ depend on the (arbitrary) choice of canonical basis. In the notation of (5),

$$\Gamma_X = \sum_{\lambda=1}^{r} [\Gamma_{X\lambda}], \quad \Gamma_Y = \sum_{\lambda=1}^{r} [\Gamma_{Y\lambda}], \quad B = \sum_{\lambda=1}^{r} [B_\lambda].$$

$\Gamma_X, \Gamma_Y$ may be constrained more heavily than $B$, for they are Hermitian and positive-definite. But

$$\sum_{\lambda=1}^{r} [\Gamma_{Y\lambda}] = \Gamma_Y = B\Gamma_X B^\dagger = \sum_{\lambda=1}^{r} [B_\lambda] \sum_{\lambda=1}^{r} [\Gamma_{X\lambda}] \sum_{\lambda=1}^{r} [B_\lambda^\dagger]$$

$$= \sum_{\lambda=1}^{r} [B_\lambda \Gamma_{X\lambda} B_\lambda^\dagger],$$

and so

$$\Gamma_{Y\lambda} = B_\lambda \Gamma_{X\lambda} B_\lambda^\dagger \quad (\lambda = 1, \ldots, r).$$

An interesting point, to which it is hoped to return on another occasion, is that $B$ is not fully determined by $\Gamma_X$ and $\Gamma_Y$.

3·2. *Estimation of the spectral numbers of a real, properly invariant process.* Suppose now that $\mathbf{X}$ is real and that $\{D\}$ imposes proper constraint. Then

$$\Gamma = \sum_{\lambda=1}^{r} \gamma_\lambda E_\lambda,$$

where $\gamma_\nu = \gamma_{\nu^*}$ for representation of the third kind. A simple argument shows that the only quadratic forms invariant under $G$ which estimate the spectral numbers $\gamma_\lambda$ are

$$\hat{\gamma}_\rho = \frac{1}{n_\rho} \mathbf{X}' E_\rho \mathbf{X}, \quad \hat{\gamma}_\mu = \frac{1}{2n_\mu} \mathbf{X}' E_\mu \mathbf{X}, \quad \hat{\gamma}_\nu = \frac{1}{2n_\nu} \mathbf{X}' F_\nu \mathbf{X},$$

where $F_\nu \equiv E_\nu + E_{\nu^*}$. If $\mathbf{X}$ is multivariate normal these estimates are distributed as multiples of $\chi^2$ on $n_\rho$, $2n_\mu$, $2n_\nu$ degrees of freedom.

3·3. *Test for equality of the components of the mean of a real, properly invariant process.* Suppose a sample of $N$ vectors from an $n$-dimensional multivariate normal distribution with properly constrained, but unknown, covariance matrix is given

$$\mathbf{z}_\alpha \in N(\boldsymbol{\mu}, \Gamma) \quad (\alpha = 1, \ldots, N).$$

Nothing is assumed about $\boldsymbol{\mu}$ and we want to test the hypothesis that its components are all equal. Arbitrary real orthogonal bases may be chosen for the subspaces $\Omega_\rho$, $\Omega_\mu$, $\Omega_\nu \oplus \Omega_{\nu^*}$ associated with the irreducible representations present in $\{D\}$. The spaces are spanned by the columns of $E_\rho$, $E_\mu$, $F_\nu$. The scalar products with $\mathbf{z}$ of the vectors of the orthogonal bases are spectral components $\xi_1, \ldots, \xi_n$: they are statistically independent, with variances $\gamma_\lambda$.

The component associated with the identical representation is always

$$\xi_1 \equiv \frac{1}{\sqrt{n}} \sum_{i=1}^{n} z_i.$$

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The other components are linear *contrasts* $\Sigma c_i z_i$, with $\Sigma c_i = 0$. If $\mathscr{E}(\boldsymbol{\xi}) = \boldsymbol{\nu}$, the null hypothesis is

$$H\colon\ \nu_2 = \ldots = \nu_n = 0,$$

and it is natural to discard $\xi_1$ as irrelevant.

An $F$-test should be employed within each of the subspaces $\Omega_\rho$, $\Omega_\mu$, $\Omega_\nu \oplus \Omega_{\nu^*}$, for the usual reasons of sufficiency, and of invariance with respect to orthogonal transformations and changes of scale. In fact

$$\frac{N(N-1)\,\mathbf{z}_\cdot' F_\lambda \mathbf{z}_\cdot}{\sum_{\alpha=1}^{N} \mathbf{z}_\alpha' F_\lambda \mathbf{z}_\alpha - N\,\mathbf{z}_\cdot' F_\lambda \mathbf{z}_\cdot} \equiv W_\lambda$$

should be referred to the $F$ distribution on $n_\lambda'$, $n_\lambda'(N-1)$ degrees of freedom, where

$$\mathbf{z}_\cdot = \frac{1}{N}\sum_{\alpha=1}^{N} \mathbf{z}_\alpha, \quad F_\rho = E_\rho, \quad F_\mu = E_\mu, \quad F_\nu = E_\nu + E_{\nu^*};$$

$$n_\rho' = n_\rho, \quad n_\mu' = 2n_\mu, \quad n_\nu' = 2n_\nu.$$

It is far from clear how these tests should be combined. The likelihood ratio criterion, asymptotically $\chi^2_{n-1}$, is

$$N\Sigma n_\lambda' \log\left(1 + \frac{W_\lambda}{N-1}\right);$$

but perhaps a more practical course is to reformulate the problem as one of multiple decision and ask *which* of the subspaces contain a non-zero component of $\boldsymbol{\mu}$. $H$ would then be rejected if $W_\lambda > c(n_\lambda')$ for *any* $\lambda$, where the constants $c(n_\lambda')$ have yet to be fixed. If they are chosen so that the component tests are the same size $\alpha$, and the number of subspaces is $s$, then the composite test has size $1-(1-\alpha)^s$.

Finally, we consider the extension to cases where $\{D\}$ does not impose proper constraint. Provided $\{D\}$ is transitive, the 'overall' component $\xi_1$ is independent of all the contrasts, as before, but the multivariate character of the problem can no longer be evaded.

Within $\Omega_\rho$ we have to test for the vanishing of the mean vectors of $d_\rho$-variate populations, $n_\rho$ in number, with common covariance matrix $\Gamma_\rho$:

$$\boldsymbol{\xi}_\alpha^{(j)} \in N(\boldsymbol{\nu}^{(j)}, \Gamma_\rho) \quad (j = 1, \ldots, n_\rho;\ \alpha = 1, \ldots, N),$$

$$H_\rho\colon\ \boldsymbol{\nu}^{(j)} = \mathbf{0} \quad (j = 1, \ldots, n_\rho).$$

Lehmann (5) shows (§ 10 of Chapter 7) how to reduce this problem by sufficiency and invariance to the roots of $G_\rho^{-1} M_\rho$, where

$$G_\rho \equiv \sum_{j=1}^{n_\rho}\sum_{\alpha=1}^{N} (\boldsymbol{\xi}_\alpha^{(j)} - \boldsymbol{\xi}_\cdot^{(j)})(\boldsymbol{\xi}_\alpha^{(j)} - \boldsymbol{\xi}_\cdot^{(j)})',$$

$$M_\rho \equiv \sum_{j=1}^{n_\rho} \boldsymbol{\xi}_\cdot^{(j)} \boldsymbol{\xi}_\cdot^{(j)\prime}, \quad \boldsymbol{\xi}_\cdot^{(j)} \equiv \frac{1}{N}\sum_{\alpha=1}^{N} \boldsymbol{\xi}_\alpha^{(j)}.$$

$G_\rho^{-1}$ exists almost surely, provided $(N-1)n_\rho \geqslant d_\rho$; and this is so because (2) implies for transitive $\{D\}$, $d_\lambda \leqslant n_\lambda$ $(\lambda = 1, \ldots, r)$.

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Within $\Omega_\nu \oplus \Omega_{\nu^*}$ the situation is similar but the common covariance matrix is $2d_\nu$-dimensional and has the form

$$\begin{pmatrix} S & A \\ -A & S \end{pmatrix} \equiv V \begin{pmatrix} \Gamma_\nu & \\ & \overline{\Gamma_\nu} \end{pmatrix} V^\dagger,$$

where $$\Gamma_\nu \equiv S + iA \quad \text{and} \quad V = \frac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ iI & -iI \end{pmatrix}.$$

The complex analogues of Lehmann's arguments show that the test should depend on the roots of a certain $d_\nu$-dimensional matrix.

Within $\Omega_\mu$ we cannot escape so easily into real coordinates and the $d_\mu$-dimensional common covariance matrix has the form

$$\begin{pmatrix} H & K \\ -\overline{K} & \overline{H} \end{pmatrix},$$

where $H = \Gamma_\mu^{00}$ is Hermitian and $K = \Gamma_\mu^{01}$ is (complex) skew-symmetric.

Representations of the second kind occur in the quaternion groups, but otherwise appear to be rare. The distinction between $d_\lambda \leqslant 1$ $(\lambda = 1, \ldots, r)$ and proper constraint is mainly of theoretical interest.

4. *Applications.* In the following examples of invariant processes $\mathbf{X} = \mathbf{z} - \boldsymbol{\mu}$, where $\mathbf{z}_\alpha \in N(\boldsymbol{\mu}, \Gamma)$; $\alpha = 1, \ldots, N$ and $\Gamma \curlyvee \{D\}$.

4·1. *Circulant process.* This is asymptotically equivalent to the stationary time-series/line-transect. $G$ is the cyclic group $C_n$ and $\{D\}$ is the regular representation (left- and right-coincide because the group is Abelian). Every element forms a class and each irreducible representation appears just once. If $\alpha \equiv e^{2\pi i/n}$ the character table is

$$\chi_\lambda(\mathscr{C}_j) = \alpha^{\lambda j} \quad (\lambda, j = 0, 1, \ldots, n-1).$$

By (16) and (17) the elements and eigenvalues of the constrained matrix

$$\begin{pmatrix} M_0 & M_{n-1} & \cdot & \cdot & \cdot & M_1 \\ M_1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & M_{n-1} \\ M_{n-1} & \cdot & \cdot & \cdot & M_1 & M_0 \end{pmatrix}$$

are related by

$$M_j = \frac{1}{n} \sum_{\lambda=0}^{n-1} \alpha^{-\lambda j} m_\lambda \quad (j = 0, 1, \ldots, n-1),$$

$$m_\lambda = \sum_{j=0}^{n-1} \alpha^{\lambda j} M_j \quad (\lambda = 0, 1, \ldots, n-1).$$

If $n$ is odd, all the $\{D_\lambda\}$ except $\{D_0\}$ are of the third kind; so a real, symmetric $M$ has $\frac{1}{2}(n-1)$ pairs of repeated eigenvalues

$$\gamma_\lambda = \Gamma_0 + 2 \sum_{j=1}^{\frac{1}{2}(n-1)} \Gamma_j \cos j\lambda\theta \quad (\lambda = 0, 1, \ldots, \tfrac{1}{2}(n-1)),$$

where $\theta \equiv 2\pi/n$. If $n$ is even, $\{D_{\frac{1}{2}n}\}$ is real and $\Gamma$ has $\frac{1}{2}(n-2)$ pairs of repeated eigenvalues and two simple ones

$$\gamma_\lambda = \Gamma_0 + (-1)^\lambda \Gamma_{\frac{1}{2}n} + 2 \sum_{j=1}^{\frac{1}{2}(n-2)} \Gamma_j \cos j\lambda\theta \quad (\lambda = 0, 1, \ldots, \tfrac{1}{2}n).$$

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A closely related process is that invariant under the dihedral group $D_n$. The only difference is that $M_j = M_{n-j}$ $(j = 1, \ldots, n-1)$; hence $M$ is just a symmetric circulant matrix. The covariance structure of a real process is the same whether circulant or dihedral invariance is assumed; but the latter is more stringent for linear models. The distinction is not entirely academic, as we shall see when cyclic factorial models are discussed.

4·2. *Dodecahedron.* A random variable is associated with each of the twenty vertices of a dodecahedron. One hemisphere is sketched below (Fig. 1), the points being labelled according to their distance from an arbitrarily chosen pole $x$. The remaining points are antipodal to those shown, and are labelled by the corresponding Greek letters $\xi$, $\eta_1$, $\eta_2$, $\eta_3$, $\zeta_1$, …, $\zeta_6$. The symmetry of the figure is reflected in the extent to which such a choice of coordinate system is arbitrary. If it is meaningful, in the physical setting of the problem, to distinguish between clockwise and anticlockwise, there are just $20 \times 3$ ways of so labelling the points. They correspond to the 60 proper rotations of the figure, which generate the icosahedral group $\mathfrak{G}_5$.

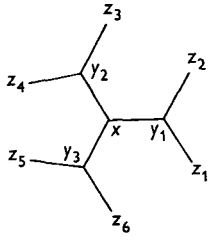



Fig. 1

The classes of the group comprise abstractly identical elements and are recognized to be the following.

| Points on axis of rotation | Angle of rotation | Order of class |
|---|---|---|
| The identity | 0 | 1 |
| Mid-points of opposite edges | $\pi$ | 15 |
| Opposite vertices | $\pm\frac{2}{3}\pi$ | 20 |
| Centres of opposite faces | $\pm\frac{2}{5}\pi$ | 12 |
| Centres of opposite faces | $\pm\frac{4}{5}\pi$ | 12 |

Every class is self-inverse so there are no irreducible representations of the third kind. Hence, if the statistical parameters do not depend on the particular coordinate system chosen, a necessary condition for proper constraint is $K \equiv 0 \pmod 6$.

The points which may not be exchanged with $x$ are the $z_i$ $(i = 1, \ldots, 6)$; these lie on two orbits with respect to the rotations about the axis through $x$. So $K = 2$ and there is not enough symmetry for proper constraint. There are eight orbits altogether so $\sum_{\lambda=1}^{5} d_\lambda^2 = 8$. Since $d_1 = 1$, by transitivity, the $d_\lambda$ are 1, 1, 1, 1, 2. Now (18) shows, incidentally, that all five representations of $\mathfrak{G}_5$ are real.

To obtain proper constraint we must assume invariance under the improper rotations. These are generated by inclusion of the antipodal reflexion, so that $G$ is now $\mathfrak{G}_5 \times C_2$. Any of the $z_i$ may be exchanged with $x$, so $K = 0$ and no representation is

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repeated. The relationship between two points now depends only on the distance between them, and there are six orbits. So the $d_\lambda$ are 1 (six times) and 0 (four times).

It is quite simple to find the class matrices and, by looking up the table of characters, determine the idempotents $E_\lambda$. The covariances $\Gamma_i$, of variates $i$ edges apart, are related to the variances $\gamma_\lambda$ of the spectral components by

$$30\begin{pmatrix}\Gamma_0+\Gamma_5\\ \Gamma_1+\Gamma_4\\ \Gamma_2+\Gamma_3\end{pmatrix}=\begin{pmatrix}3 & 15 & 12\\ 3 & 5 & -8\\ 3 & -5 & 2\end{pmatrix}\begin{pmatrix}\gamma_0\\ \gamma_2\\ \gamma_4\end{pmatrix},$$

$$10\begin{pmatrix}\Gamma_0-\Gamma_5\\ \Gamma_1-\Gamma_4\\ \Gamma_2-\Gamma_3\end{pmatrix}=\begin{pmatrix}3 & 4 & 3\\ \sqrt{5} & 0 & -\sqrt{5}\\ 1 & -2 & 1\end{pmatrix}\begin{pmatrix}\gamma_1\\ \gamma_3\\ \gamma_5\end{pmatrix}.$$

To interpret the spectral numbers, we confine error to each subspace of spectral components in turn. For each $\lambda$, let $\mathbf{S}_\lambda$ denote the set of autocorrelations $\rho_i \equiv \Gamma_i/\Gamma_0$ $(i = 0, \ldots, 5)$ when only $\gamma_\lambda$ is non-zero. The $\mathbf{S}_\lambda$ are tabulated below, with the dimension $n_\lambda$ of the corresponding subspace.

| $n_\lambda$ | | $\rho_0$ | $\rho_1$ | $\rho_2$ | $\rho_3$ | $\rho_4$ | $\rho_5$ |
|---|---|---|---|---|---|---|---|
| 1 | $\mathbf{S}_0$ | 1 | 1 | 1 | 1 | 1 | 1 |
| 3 | $\mathbf{S}_1$ | 1 | $\frac{1}{3}\sqrt{5}$ | $\frac{1}{3}$ | $-\frac{1}{3}$ | $-\frac{1}{3}\sqrt{5}$ | $-1$ |
| 5 | $\mathbf{S}_2$ | 1 | $\frac{1}{3}$ | $-\frac{1}{3}$ | $-\frac{1}{3}$ | $\frac{1}{3}$ | 1 |
| 4 | $\mathbf{S}_3$ | 1 | 0 | $-\frac{1}{2}$ | $\frac{1}{2}$ | 0 | $-1$ |
| 4 | $\mathbf{S}_4$ | 1 | $-\frac{2}{3}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $-\frac{2}{3}$ | 1 |
| 3 | $\mathbf{S}_5$ | 1 | $-\frac{1}{3}\sqrt{5}$ | $\frac{1}{3}$ | $-\frac{1}{3}$ | $\frac{1}{3}\sqrt{5}$ | $-1$ |

The separation of two points $i$ edges apart on the dodecahedron is naturally expressed by the angle $\theta_i$ subtended at the centre of the circumscribing sphere. The angles are $(i = 0, \ldots, 5)$:

$$0, \quad \cos^{-1}\tfrac{1}{3}\sqrt{5}, \quad \cos^{-1}\tfrac{1}{3}, \quad \pi-\cos^{-1}\tfrac{1}{3}, \quad \pi-\cos^{-1}\tfrac{1}{3}\sqrt{5}, \quad \pi.$$

Each $\mathbf{S}_\lambda$ now suggests the behaviour of a Legendre polynomial; in fact if $\mathbf{P}_\lambda$ denotes the set of values $P_\lambda(\cos\theta_i)$ of the $\lambda$th Legendre polynomial, then

$$\left.\begin{aligned}\mathbf{S}_0 = \mathbf{P}_0, \quad \mathbf{S}_1 = \mathbf{P}_1, \quad \mathbf{S}_2 = \mathbf{P}_2,\\ 56\mathbf{S}_3 = -9\mathbf{P}_5+63\mathbf{P}_3+2\mathbf{P}_1,\\ 56\mathbf{S}_4 = 81\mathbf{P}_4-25\mathbf{P}_2, \quad 7\mathbf{S}_5 = 9\mathbf{P}_5-2\mathbf{P}_1.\end{aligned}\right\} \tag{19}$$

So the labelling of the component representations $\{D_0\}, \ldots, \{D_5\}$ was not without its point.

The spherical harmonics of degree $m$ form a $(2m+1)$-dimensional invariant subspace of the rotation group of the sphere. They are invariant also for the antipodal reflexion. Hence they generate a representation $\{D\}_m$, specified below for $m = 0, \ldots, 5$, of $\mathfrak{S}_5 \times C_2$:

| $m$: | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $\{D\}_m$: | $\{D_0\}$ | $\{D_1\}$ | $\{D_2\}$ | $\{D_3\}+\{D_5\}$ | $\{D_2\}+\{D_4\}$ | $\{D_1\}+\{D_5\}+\{D_a\}$. |

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where $\{D_a\}$ does not appear in the dodecahedral representation $\{D\}$. Now restrict the domain of the spherical harmonics to the twenty vertices, so that they become 20-vectors. Their independence may be lost, but not their transformation properties. Vectors transforming according to $\{D_\lambda\}$ form the image-space $\Omega_\lambda$ of the idempotent $E_\lambda$, so

$$\mathbf{P}_0 \in \Omega_0, \quad \mathbf{P}_1 \in \Omega_1, \quad \mathbf{P}_2 \in \Omega_2, \quad \mathbf{P}_3 \in \Omega_3 \oplus \Omega_5, \quad \mathbf{P}_4 \in \Omega_2 \oplus \Omega_4, \quad \mathbf{P}_5 \in \Omega_1 \oplus \Omega_5.$$

Now the $\mathbf{P}_\lambda$ have axial symmetry, and in each $\Omega_\lambda$ only $\mathbf{S}_\lambda$ has axial symmetry. This explains the rather uneven result (19).

Random variables could be associated with the faces of the dodecahedron. Any two of these may be exchanged by a *proper* rotation so that invariance under $\mathfrak{G}_5$ is all we need ask. By contrast, if the edges were the subjects even $\mathfrak{G}_5 \times C_2$ would not be enough for proper constraint; for edges such as $(xy_1)$ and $(z_4 z_5)$ cannot be exchanged —and $K = 0$ is necessary, since all $\{D_\lambda\}$ are real.

4·3. *Additive block effect.* In a controlled experiment the independent realizations of the process $\mathbf{z}$ may comprise observations from $N$ 'blocks'. Block effects and interactions are expressed by departure of the covariance matrix $\Gamma$ from diagonal form. So an invariant process is one where certain kinds of interaction between blocks and the systematic factors are excluded.

In applications one may want to consider the possibility that there is no interaction at all, but only block main effects. The model then reduces to

$$z_{i\alpha} = \mu_i + b_\alpha + \epsilon_{i\alpha} \quad (i = 1, \ldots, n;\ \alpha = 1, \ldots, N), \tag{20}$$

where the $\mu_i$ are constants: $\mathscr{E}(b_\alpha) = \mathscr{E}(\epsilon_{i\alpha}) = 0$; $\mathscr{E}(b_\alpha^2) = \sigma_b^2$; $\mathscr{E}(\epsilon_{i\alpha}^2) = \sigma^2$; and all random variables are independent.

In vector notation, we have

$$\mathbf{z}_\alpha = \boldsymbol{\mu} + b_\alpha \mathbf{1} + \boldsymbol{\epsilon}_\alpha \quad (\alpha = 1, \ldots, N).$$

The vector $\mathbf{1}$ has unit components, so that $\mathbf{1}\mathbf{1}' = nE_1$, where $E_1$ is the first idempotent of *any* transitive permutation representation $\{D\}$. Hence

$$\begin{aligned} \Gamma &= n\sigma_b^2 E_1 + \sigma^2 I \\ &= (n\sigma_b^2 + \sigma^2) E_1 + \sigma^2 \sum_{\lambda=2}^{r} E_\lambda, \end{aligned}$$

and the spectral numbers of the invariant process are

$$\gamma_1 = n\sigma_b^2 + \sigma^2, \quad \gamma_2 = \ldots = \gamma_r = \sigma^2.$$

Note that (20) is consistent with invariance under any group.

4·4. *Regular representations.* The two invariant processes discussed so far arise naturally from geometrical figures—the regular polygon and the dodecahedron—but the intuitive framework may have quite a different basis. Suppose the $n$ components can be identified with the elements of a group $G$

$$z_k \leftrightarrow a_k \quad (k = 1, \ldots, g).$$

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It may be reasonable to make one of the following assumptions:

(I) $\operatorname{cov}(z_i, z_j)$ depends only on $a_j^{-1}a_i$; i.e. $\Gamma \curlyvee \{D\}_{\text{L. Reg.}}$.
(II) $\operatorname{cov}(z_i, z_j)$ depends only on $a_i a_j^{-1}$; i.e. $\Gamma \curlyvee \{D\}_{\text{R. Reg.}}$.
(III) $\operatorname{cov}(z_i, z_j)$ depends only on the class to which both $a_j^{-1}a_i$ and $a_i a_j^{-1}$ belong.

The practical justification of III is that classes comprise abstractly identical elements of a group.

III implies I and II, obviously. I and II together imply III, for, if $a_j^{-1}a_i$ and $a_q^{-1}a_p$ belong to the same class,

$$a_j^{-1}a_i = a^{-1}a_q^{-1}a_p a, \quad \text{say}$$
$$= (a_q a)^{-1}a_p a.$$

So if $a_s \equiv a_p a$; $a_t \equiv a_q a$; then

$$\operatorname{cov}(z_i, z_j) = \operatorname{cov}(z_s, z_t) \quad \text{by I.}$$

But $a_s a_t^{-1} = a_p a a^{-1} a_q^{-1} = a_p a_q^{-1}$; so

$$\operatorname{cov}(z_s, z_t) = \operatorname{cov}(z_p, z_q) \quad \text{by II.}$$

Therefore $\operatorname{cov}(z_i, z_j) = \operatorname{cov}(z_p, z_q)$.

So we may rewrite:

(III) $\Gamma \curlyvee \{D\}_{\text{L. Reg.}}, \quad \Gamma \curlyvee \{D\}_{\text{R Reg.}}$.

We saw in §2·4 that the spectral components could then be identified, while (16) and (17) related explicitly covariances and spectral numbers. Some examples follow: except in the first one $G$ is Abelian and assumptions I, II and III coincide.

4·41. *Permutation experiment.* Suppose an industrial product undergoes $m$ different operations $A, B, C, \ldots$ and one wants to study the effect of changing the order of their application. Raw material is available in $N$ independent batches. Each is split into $m!$ parts and the operations carried out in a different order on each part. Batch interactions may be present if the responses depend rather subtly on the exact composition of the raw material. $G$ is the symmetric group of degree $m$ and assumptions I or II impose 'operation-wise' or 'position-wise' metrics among the treatments. There are no representations of the third kind (Theorem 1) and, under assumption III, $\Gamma$ has $r$ distinct elements.

Take the case $m = 3$ and assume III, i.e. that $\operatorname{cov}(z_{i\alpha}, z_{j\alpha})$ depends only on whether treatments $a_i$ and $a_j$ differ by the identity, by a cyclic permutation, or by a transposition. The spectral components

$$\xi_1 = (z_{ABC} + z_{BCA} + z_{CAB} + z_{ACB} + z_{CBA} + z_{BAC})/\sqrt{6},$$
$$\xi_2 = (z_{ABC} + z_{BCA} + z_{CAB} - z_{ACB} - z_{CBA} - z_{BAC})/\sqrt{6}$$

have variances $\gamma_1$, $\gamma_2$. The remaining orthogonal contrasts $\xi_3, \ldots, \xi_6$ each have variance $\gamma_3$.

4·42. $2^p$ *factorial.* A factorial experiment in which each of $p$ factors is at two levels is completely replicated in $N$ blocks. $G$ is the $p$-fold direct product $C_2 \times \ldots \times C_2$. The invariance assumption is that $\operatorname{cov}(z_{i\alpha}, z_{j\alpha})$ depends only on *which* factors are at different levels in treatments $a_i$ and $a_j$. The table of characters has a familiar look, each

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entry being $\pm 1$. The spectral components are the usual $2^p$ orthogonal contrasts for treatment main effects and mutual interactions:

$$\xi_1 = (z_{(1)} + z_A + z_B + z_{AB} + \ldots)/2^{\frac{1}{2}p}$$
$$\xi_2 = (z_{(1)} - z_A + z_B - z_{AB} + \ldots)/2^{\frac{1}{2}p}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Each has a different variance and the hypothesis-testing problem of § 3·3 reduces to $(2^p - 1)$ $t$-statistics.

Conversely:
$$z_{(1)} = (\xi_1 + \xi_2 + \xi_3 + \xi_4 + \ldots)/2^{\frac{1}{2}p}$$
$$z_A = (\xi_1 - \xi_2 + \xi_3 - \xi_4 + \ldots)/2^{\frac{1}{2}p}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

which is an alternative statement of the invariance assumption.

4·43. *Cyclic factorial.* Extension to more than two levels is reasonable only if the factors are periodic. Examples are the seasons, longitudinal position, and sunspot influences. Suppose $(i = 1, \ldots, p)$ the $i$th factor is at $\nu_i$ levels. The factors may be circulant or dihedral. The weakest assumption is that all are circulant and $\{D\}$ is the regular representation of $C_{\nu_1} \times C_{\nu_2} \times \ldots \times C_{\nu_p}$. With this model the following need not, in general, be the same:

$$\operatorname{cov}[A^0 B^0 \ldots, A^r B^s \ldots]; \quad \operatorname{cov}[A^0 B^0 \ldots, A^{-r} B^s \ldots].$$

The number of subspaces of spectral components is

$$\frac{1}{2}\left(\prod_{i=1}^{p} \nu_i + 2^{p'}\right),$$

where $p'$ is the number of *even* $\nu_i$.

The opposite extreme is to assume all factors dihedral, so that $G$ is

$$D_{\nu_1} \times D_{\nu_2} \times \ldots \times D_{\nu_p}.$$

The cycles lose their sense of direction and there are only

$$(\tfrac{1}{2})^p \prod_{\nu_i \text{ even}} (\nu_i + 2) \prod_{\nu_j \text{ odd}} (\nu_j + 1)$$

subspaces of spectral components.

*Discussion.* We have distinguished structural relationships from those imposed by an arbitrary choice of coordinate system. This choice may consist simply of deciding which of two levels in a factorial experiment is to be regarded as the control. A sufficient condition for proper constraint is that the structural relationships are all reflexive (symmetric); for then $K = 0$. James (3), who is responsible for many of these ideas, has noted the importance of symmetric relationships.

A problem may have a suitable structure, although the parameters do not conform to it. Nevertheless, tests based on § 3·3 have just the required invariance; they emphasize the natural alternative hypotheses and concentrate power against them. An example of this kind is due to Rao (8): borings are taken from four sides of 28 cork-trees, to determine whether growth is uniform round the trunks. Making no assumption about the covariances, Rao obtains a value of Hotelling's $T^2$ significant at 1 %.



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A circulant model is plausible—indeed prior information has already singled out one spectral component—but the data reveal definite correlation between $\xi_1$ and $\xi_2$. The sample means and variances of the spectral components are:

$$\begin{aligned}
\xi_1 &= \tfrac{1}{2}(N+E+S+W), & \hat{\nu}_1 &= 95{\cdot}79, & \hat{\gamma}_1 &= 970{\cdot}81,\\
\xi_2 &= \tfrac{1}{2}(N-E+S-W), & \hat{\nu}_2 &= 4{\cdot}43, & \hat{\gamma}_2 &= 32{\cdot}18,\\
\xi_3 &= (E-W)/\sqrt{2}, & \hat{\nu}_3 &= 0{\cdot}71, & \hat{\gamma}_3 &= 41{\cdot}675.\\
\xi_4 &= (N-S)/\sqrt{2}, & \hat{\nu}_4 &= 0{\cdot}61, &
\end{aligned}$$

The test for $\nu_2 = 0$ gives $F_{1,27} = 17{\cdot}08$, significant at 0·1 %; that for $\nu_3 = \nu_4 = 0$ gives $F_{2,54} = 0{\cdot}29$.

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