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GROUPS WITH DENSE TRANSITIVELY NORMAL SUBGROUPS

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Abstract

A subgroup X of a group G is said to be *transitively normal* if X is normal in any subgroup Y of G such that $X \leq Y$ and X is subnormal in Y. We investigate the structure of generalised soluble groups with *dense transitively normal subgroups*, that is, groups in which every nonempty open interval in their subgroup lattice contains a transitively normal subgroup. In particular, it will be proved that nonperiodic generalised soluble groups with dense transitively normal subgroups are abelian.

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1. Introduction

Let $\mathfrak{L}$ be a set of subgroups of a group G. We shall say that G has *dense* $\mathfrak{L}$*-subgroups* if for each pair (H, K) of subgroups of G such that $H < K$ and H is not maximal in K, there exists an $\mathfrak{L}$-subgroup X such that $H < X < K$. Clearly, every group in which all subgroups belong to the set $\mathfrak{L}$ has dense $\mathfrak{L}$-subgroups. Several authors have investigated the structure of groups with dense $\mathfrak{L}$-subgroups for many different choices of the set $\mathfrak{L}$ (see for instance [1, 5, 9, 15]). In particular, Mann [9] considered groups with dense normal or subnormal subgroups, proving, among other results, that any infinite group with dense normal subgroups is nilpotent (see also [1, Corollary 3.2]). The consideration of the symmetric group of degree 3 shows that there exist finite nonnilpotent groups with dense normal subgroups. Furthermore, Vincenzi [15] investigated locally soluble groups with dense pronormal subgroups. Recall that a subgroup X of a group G is said to be *pronormal* if the subgroups X and X^g are conjugate in $\langle X, X^g \rangle$ for all $g \in G$. Obvious examples of pronormal subgroups are normal subgroups and maximal subgroups of arbitrary groups; moreover, Sylow subgroups of finite groups and Hall subgroups of finite soluble groups are always pronormal. We point out that there exist infinite groups containing Sylow subgroups which are not pronormal. Let X be a pronormal subgroup of a group G and consider a subgroup H of G such that $X \leq H \leq N_G(X)$. If g is an element of $N_G(H)$, there exists $y \in \langle X, X^g \rangle$ such that $X^g = X^y$ and, hence, $gy^{-1} \in N_G(X)$. However, $\langle X, X^g \rangle \leq H \leq N_G(X)$

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so that $g = (gy^{-1})y$ lies to $N_G(X)$. Therefore, X is normal in $N_G(H)$. This latter argument shows that X is a transitively normal subgroup of G.

Following [6], a subgroup X of a group G is said to be *transitively normal* if X is normal in any subgroup Y of G such that $X \leq Y$ and X is subnormal in Y. Clearly, self-normalising and Sylow subgroups of arbitrary groups are transitively normal. Moreover, all subgroups of a group G are transitively normal if and only if G is a $\overline{T}$-group. Recall here that a group G is said to be a *T-group* if normality in G is a transitive relation, that is, if every subnormal subgroup of G is normal. The structure of finite soluble T-groups has been described by Gaschütz [3], while Robinson [11] investigated infinite soluble groups with the property T. In particular, it turns out that soluble T-groups are metabelian, locally supersoluble and that any finitely generated soluble T-group is either finite or abelian. Clearly, a simple group has the property T so that T is not subgroup closed. A group G is called a $\overline{T}$*-group* if all its subgroups are T-groups. It can be proved that every finite soluble T-group satisfies the property $\overline{T}$ and that the finite $\overline{T}$-groups are soluble.

In this article, the structure of groups with dense transitively normal subgroups will be investigated. Of course, all subgroups of *Tarski groups* (that is, infinite simple groups whose proper nontrivial subgroups have prime order) are pronormal. To avoid Tarski groups and other similar pathologies, our results will be proved in the universe of locally graded groups. A group G is said to be *locally graded* if every finitely generated nontrivial subgroup of G has a proper subgroup of finite index. Locally graded groups form a large class of generalised soluble groups, containing, in particular, all locally (soluble-by-finite) groups.

Most of our notation is standard and can be found in [12].

2. Results

Clearly, the class of groups with dense transitively normal subgroups is closed with respect to subgroups and homomorphic images. Our first result shows in particular that a locally nilpotent group with dense transitively normal subgroups has dense normal subgroups.

LEMMA 2.1. *Let G be a locally nilpotent group with dense transitively normal subgroups. Then, G is nilpotent with class at most* 3.

PROOF. We may suppose that G is finitely generated. Let H be a proper subgroup of G and assume that H is not maximal in its normal closure H^G. By hypothesis, there exists a transitively normal subgroup X of G such that $H < X < H^G$. Since G is nilpotent, then X is normal in G. It follows that $H^G \leq X$, which is a contradiction. Therefore, H is maximal in its normal closure and, hence, it is subnormal with defect at most 2. It follows that G has nilpotent class at most 3 by a result of Madavianary (see [8, Theorem 1]). □

The structure of locally graded groups with dense transitively normal subgroups will be investigated separately in the periodic and in the nonperiodic case.

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Let G be a periodic locally graded $\overline{T}$-group. If G is not metabelian, there exists a finitely generated subgroup E of G such that $E^{(2)} \neq \{1\}$. Clearly, $E^{(2)}$ is finitely generated and, hence, it contains a proper subgroup H_E such that E/H_E is a finite $\overline{T}$-group. Therefore, $E^{(2)} \leq H_E$, which is a contradiction. Thus, G is metabelian and so even locally supersoluble.

THEOREM 2.2. *Let G be a periodic locally graded group with dense transitively normal subgroups. Then, G is hypercyclic (and so even locally supersoluble) and its commutator subgroup G' is nilpotent with class at most* 3.

PROOF. First, assume that G is finite. By Lemma 2.1, it is enough to prove that G is supersoluble. Let G be a counterexample of minimum order. As the maximal subgroups of G are supersoluble, then G is soluble by a celebrated theorem of Huppert. Let N be a minimal normal subgroup of G. Clearly, N is abelian with exponent a prime p and G/N is supersoluble. If $|N| > p$, by hypothesis, N contains a proper nontrivial subgroup K which is transitively normal in G and so even normal in G, which is a contradiction. It follows that N has order p and, hence, G is supersoluble. This last contradiction shows the statement when G is finite.

In the general case, let H be a finitely generated subgroup of G such that $H^{(3)} \neq \{1\}$. By hypothesis, $H/H^{(3)}$ is finite and, hence, $H^{(3)}$ is finitely generated. It follows that $H^{(3)}$ contains a proper normal subgroup K such that the factor group $H^{(3)}/K$ is finite, and the first part of the proof yields the contradiction $H^{(3)} \leq K_H$. It follows that G is a soluble group (with derived length at most 3) and, in particular, it is locally finite. Let $\mathfrak{F}$ be the set of all finite subgroups of G. As

$$G' = \bigcup_{L \in \mathfrak{F}} L',$$

it follows that G' is nilpotent with class at most 3.

Finally, let A be the last nontrivial term of the derived series of G. If A is not cyclic, it contains a subgroup B of order pq, where p and q are not necessarily distinct prime numbers. By hypothesis, there exists a transitively normal subgroup X of B of order p or q. As X is subnormal in G, it is also normal. It follows that G contains a nontrivial cyclic normal subgroup and, hence, it is hypercyclic since the hypotheses are inherited by homomorphic images. The statement is proved. □

To examine the nonperiodic case, we first observe that the infinite dihedral group $D_\infty = \langle x \rangle \ltimes \langle a \rangle$, where a has infinite order, $x^2 = 1$ and $xax = a^{-1}$, has nondense transitively normal subgroups. In fact, suppose for a contradiction that D_∞ contains a transitively normal subgroup K such that

$$\langle x, a^{16} \rangle < K < \langle x, a^4 \rangle.$$

As $D_\infty/\langle a^{16} \rangle$ is nilpotent, then K is normal in D_∞. However,

$$x^a = a^{-1}xa = xaxxa = xa^2$$

does not belong to $\langle x, a^4 \rangle$ and, hence, $x^a \notin K$, which is a contradiction.

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Note that if X is an ascendant transitively normal subgroup of a group G, then arguing by transfinite induction, it easy to prove that X is normal in G.

LEMMA 2.3. *Let G be a group with dense transitively normal subgroups. If a is an element of infinite order in G such that $\langle a\rangle$ is an ascendant subgroup of G, then $a \in Z(G)$.*

PROOF. Let p and q be distinct prime numbers. As $\langle a^{p^2}\rangle$ and $\langle a^{q^2}\rangle$ are not maximal in $\langle a\rangle$, then $\langle a^p\rangle$ and $\langle a^q\rangle$ are transitively normal in G, and so even normal in G. It follows that $\langle a\rangle$ itself is a normal subgroup of G and, hence, $a \in Z(G)$, since G has no infinite dihedral sections. □

As an immediate consequence of Lemmas 2.1 and 2.3, a nonperiodic locally nilpotent group with dense transitively normal subgroups is abelian.

THEOREM 2.4. *Let G be a nonperiodic locally graded group all of whose sections have abelian chief factors. If G has dense transitively normal subgroups, then G is abelian.*

PROOF. Clearly, we may suppose that G is finitely generated. First, we prove the statement when G is soluble. If G' is periodic, then factorising by the last nontrivial term of the derived series of G, it can be assumed that G' is abelian nontrivial. By Hall's lemma (see [12, Lemma 14.1.3]), G' is the normal closure of a finite subset, and an application of Zorn's lemma yields the existence of a maximal proper G-invariant subgroup M of G'. Factorising by M, it can be assumed that G' is a minimal normal subgroup of G and hence it has exponent a prime p. It follows that the order of G' must be exactly p since G has dense transitively normal subgroups. Let $G' = \langle x\rangle$ and consider an element g of infinite order of G. As $G/C_G(G')$ has order dividing $p-1$, then $g^{p-1} \in C_G(G')$ and, hence, $[g^{p-1}, x] = 1$. It follows that $\langle g^{p-1}\rangle G'$ is an abelian normal subgroup of $\langle g\rangle G'$, so that by Lemma 2.3, $\langle g^{p-1}\rangle G' \leq Z(\langle g\rangle G')$. In particular, $[g, x] = 1$. Thus, $\langle g\rangle$ is subnormal in G and, hence, $g \in Z(G)$ again by Lemma 2.3. Therefore, G is abelian because it is generated by elements of infinite order. Now, let G' be not periodic. Clearly, arguing by induction on the derived length of G, we can assume that G is metabelian. It follows from Lemma 2.3 that $G' \leq Z(G)$ and, hence, G is abelian. This last contradiction proves the theorem when G is soluble.

In the general case, assume first that the factor group $G/G^{(3)}$ is periodic. As G is locally graded finitely generated, there exists a proper normal subgroup N of $G^{(3)}$ such that G/N is finite. This is a contradiction by Theorem 2.2 and, hence, the first part of the proof yields $G' = G^{(3)}$. Assume for a contradiction that $G' \neq \{1\}$. By Hall's lemma, G' is the normal closure of a finite set and so it contains a maximal proper G-invariant subgroup L. It follows that G'/L is a minimal normal subgroup of G/L and, hence, it is abelian by hypothesis. This last contradiction completes the proof of the theorem. □

Let G be any group. Recall that G is said to be *radical* if it has an ascending (normal) series all of whose factors are locally nilpotent; moreover, G is called *hypoabelian* if

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it has a descending series with abelian factors, that is, if it does not contain perfect nontrivial subgroups. Clearly, both radical and hypoabelian groups are locally graded.

COROLLARY 2.5. *Let G be a nonperiodic group with dense transitively normal subgroups.*

(1) *If G is hypoabelian, then G is abelian.*
(2) *If G is locally soluble, then G is abelian.*
(3) *If G is radical, then G is abelian.*

PROOF. Clearly, if G is hypoabelian, then the chief factors of all its sections are abelian and, hence, G is abelian by Theorem 2.4. Moreover, as all chief factors of a locally soluble group are abelian (see [12, Theorem 12.5.1]), then condition (2) follows again from Theorem 2.4. Finally, assume that G is radical and let N be a minimal normal subgroup of G. If H is the Hirsch–Plotkin radical of G, then $N \leq H$ (see [12, Exercise 12.1.3]). However, H is nilpotent by Lemma 2.1, so that $N' < N$. Therefore, N is abelian and, hence, by Theorem 2.4, G itself is abelian. □

COROLLARY 2.6. *Let G be a nonperiodic locally graded group with dense transitively normal subgroups. Then, the set T of periodic elements of G is a central subgroup of G.*

PROOF. Let x and y be elements of finite order of G. Assume that the subgroup $H = \langle x, y\rangle$ is infinite. As G is locally graded, we can construct a strictly descending sequence $(H_n)_{n\in\mathbb{N}}$ of normal subgroups,

$$H = H_1 > H_2 > \cdots > H_n \ldots,$$

of H such that H/H_n is finite for all n. Put

$$X = \bigcap_{n\in\mathbb{N}} H_n.$$

By Theorem 2.2, the factor group H/X is an infinite finitely generated soluble group. Therefore, H/X is not periodic and, hence, it is abelian by Theorem 2.4. In particular, H/H' is infinite, which is a contradiction. It follows that H is finite and, hence, T is a subgroup of G.

Finally, let $x \in T$ and $y \in G \setminus T$. Clearly, $K = \langle x, y\rangle = \langle y\rangle \ltimes T_1$, where T_1 is the torsion subgroup of H. By Theorem 2.2, K is a nonperiodic soluble group with dense transitively normal subgroups. It follows from Theorem 2.4 that K is abelian. Thus, x commutes with all elements of $G \setminus T$ and, hence, $x \in Z(G)$. □

Note that it is an *open question* (see [4, Question 14.36 (Francesco de Giovanni)]) whether a nonperiodic locally graded $\overline{T}$-group is abelian. It is easy to show that Corollary 2.6 reduces this latter question to the torsion-free case. In fact, let G be a nonperiodic locally graded $\overline{T}$-group. Clearly, G has dense transitively normal subgroups and, hence, by Corollary 2.6, the set T of its elements of finite order is an abelian normal subgroup of G. It follows by [7] that G/T is locally graded. Thus, if

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we assume that the above conjecture is true for torsion-free groups, then $G' \leq Z(G)$. Therefore, G is nilpotent and so even abelian.

Following Müller [10], a subgroup H of a group G is called *weakly normal* if $H^g = H$ whenever g is an element of G such that $H^g \leq N_G(H)$. It is easy to prove that every pronormal subgroup of a group G is weakly normal, while a weakly normal subgroup of G is always transitively normal. In [14], it has been proved that a nonperiodic locally graded group, all of whose subgroups are weakly normal, is abelian.

We conclude the article by proving a slight generalisation of this result. First, recall that a group is said to be *HNN-free* if it contains no subgroups which are proper HNN-extensions (see [13]). It is easy to show that a group G is HNN-free if and only if $H^g = H$ whenever H is a subgroup and g is an element of G such that $H^g \leq H$. We say that a subgroup H of a group G is *almost weakly normal* if there exists a subgroup K of G containing H such that H is weakly normal in K and the index $|G : K|$ is finite.

PROPOSITION 2.7. *Let G be a nonperiodic locally graded $\overline{T}$-group in which every subgroup is almost weakly normal. Then, G is abelian.*

PROOF. Let H be a subgroup of G such that $H^g \leq H$ for some element g of G. By hypothesis, there exists a subgroup K of finite index of G such that H is weakly normal in K. Then, $g^n \in K$ for some positive integer n, and so $H^{g^n} \leq N_K(H)$. It follows that $H^{g^n} = H$ and, hence, $H^g = H$. Thus, G is HNN-free.

We may assume that G is finitely generated. Then, the factor group $G/G^{(2)}$ is either abelian or finite since G is a $\overline{T}$-group. In particular, $G/G^{(2)}$ has finite rank. Now, let G/N be a finite quotient of G. As G/N is a $\overline{T}$-group, then $G^{(2)} \leq N$. It follows that there is an upper bound for the (Prüfer) ranks of the finite quotients of G. We have already proved that G is HNN-free so that it is polycyclic by [13, Theorem B]. Therefore, G is abelian since it is not periodic. □

We point out that in the last result, the hypothesis that G is a $\overline{T}$-group cannot be dropped since there exists a metabelian nonperiodic group G with trivial centre such that all subgroups of G are almost weakly normal (see [2, Example 4.5])

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