A Structural Justification of Probabilism: From Partition Invariance to Subjective Probability

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A new justification of probabilism is developed that pays close attention to the structure of the underlying space of possibilities. Its central assumption is that rational numerical degrees of belief ought to be partition invariant. By means of a representation theorem, one can prove that if graded belief satisfies the resulting set of postulates, rational degrees of belief may be identified with probabilities.

1. Introduction: Two Plausible Constraints on Rational Belief. Rational belief is naturally taken to be an attitude toward possibilities or possible cases, such as the 100 possibilities of one ticket winning in a lottery of a hundred tickets, the possible ways the weather might be like at a certain time and location, and so on. Indeed, standard possible worlds semantics for rational all-or-nothing belief involves a distinction between *live* (doxastically accessible) possibilities and the rest, such that a proposition X is rationally believed by an agent at a time if and only if (iff) X is true at all the possibilities that are live for the agent at the time. If propositions are themselves identified with sets of possibilities, one may say equivalently: X is rationally believed just in case it includes all live possibilities as members. For instance, the proposition expressed by 'Either ticket 1 wins or ticket 2 wins' is rationally

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believed iff every live possibility is included in that proposition; that is, every live possible case is such that the proposition is true in that case (ticket 1 wins or ticket 2 wins). From this it follows that whether an agent rationally believes a potential belief content X only depends on the joint doxastic status of the possibilities in that content X taken together—whether they include all possibilities that are live for the agent or not. Call this the *aiming-at-the-content constraint* on rational all-or-nothing belief.

Accordingly, for *graded* belief, the numerical degree of belief that a rational agent assigns to a proposition X should be a function F of the degrees of belief the agent assigns to the possibilities included in X. *Probabilism* (Jeffrey 1992) requires that function F to correspond to the addition function of real numbers, as becomes transparent from the finite additivity axiom

$$\text{if } Y \cap Z = \varnothing, \text{ then } P(Y \cup Z) = P(Y) + P(Z),$$

which constitutes the core tenet of the theory.[1] For instance, the subjective probability of the proposition expressed by 'Either ticket 1 wins or ticket 2 wins' is the sum of the probabilities of the mutually exclusive propositions that are expressed by the respective disjuncts. If logical space is partitioned into finitely many possible cases, and if every way of aggregating these possible cases counts as a proposition, finite additivity entails that the subjective probability of a proposition X is just the sum of the subjective probabilities of the (singleton sets of) possibilities included in X, that is, of the possible cases in which X is true. Analogously to the case of all-or-nothing belief, an agent's rational degree of belief in a potential belief content X only depends on the joint doxastic status of the possibilities in the content X taken together: on what their rational degrees of belief are like. Call this the *aiming-at-the-content constraint* on rational graded belief.[2]

In what follows, I want to give a new kind of justification of probabilism, that is, of the view that *rational belief comes in numerical degrees* and that *these degrees of belief conform to the axioms of subjective probability*. Probabilism in that sense just concerns synchronic norms on graded belief and does not involve any diachronic postulates, such as for the rational update of graded belief by conditionalization (e.g., in the same sense of the term, Joyce [1998] aimed to justify probabilism without extending his justification to diachronic postulates). I will leave the topic of diachronic norms to follow-up work.

1. We take subjective probability theory to be committed to finite additivity while leaving open the status of countably infinite additivity or σ-additivity, one of Kolmogorov's axioms of probability theory. See, e.g., Howson (2008) for more on this.

2. One might argue that these aiming-at-the-content constraints derive from the familiar idea that (all-or-nothing/graded) belief in X aims at the truth of X. But I will not pursue this thought here in more detail.

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One of the premises of my justification of probabilism will express the aiming-at-the-content constraint on rational graded belief: the rational degree of belief in a proposition *X* is a function of the degrees of belief assigned to the possibilities in *X*. Instead of presupposing that function to be the addition of real numbers, I am going to justify this on the basis of further postulates. The approach is going to differ from existing justifications of probabilism (see Hájek 2008a; Paris 1994, chap. 3, for surveys), such as epistemic/accuracy-theoretic ones (see Joyce 1998; Pettigrew 2016), logical/plausibility-based ones (such as Cox 1946, 1961), or pragmatic/decision-theoretic ones (see Hájek 2008b), by the following feature: it will combine the aiming-at-the-content constraint on graded belief with postulates that concern the *structure of the underlying space of possibilities*. It is for that reason that I am going to calling the justification a *structural* one.

For what are possibilities or possible cases anyway? As the lottery and the weather example from above demonstrate, possibilities can be individuated differently: they may be "cut" out of logical space in different "sizes" and "shapes." For instance, the proposition expressed by 'Ticket 1 wins' may be reconstructed as the set {Ticket-1-wins}, which is a subset of the 101-member space {Ticket-1-wins, . . . , Ticket-100-wins, No-lottery-takes-place} of lottery possibilities. (No-lottery-takes-place is meant to cover also incorrect implementations of the lottery and coincides with the negation of the disjunction of the 100 ticket propositions.) Alternatively, what is expressed by 'Ticket 1 wins' may be understood to be the set {Ticket-1-wins-&-Sunny-tomorrow, Ticket-1-wins-&-Not-sunny-tomorrow} and hence as a subset of the 202-member space {Ticket-1-wins-&-Sunny-tomorrow, Ticket-1-wins-&-Not-sunny-tomorrow, . . . , Ticket-100-wins-&-Sunny-tomorrow, Ticket-100-wins-&-Not-sunny-tomorrow, No-lottery-takes-place-&-Sunny-tomorrow, No-lottery-takes-place-&-Not-sunny-tomorrow} of combined lottery-weather possibilities, and so forth. Clearly, at least in principle, there are infinitely many ways of reconstructing the content of a declarative sentence as a set of possible cases.

At the same time, {Ticket-1-wins} and {Ticket-1-wins-&-Sunny-tomorrow, Ticket-1-wins-&-Not-sunny-tomorrow} still cover the same "region" in logical space (see the rectangle marked 'X' in fig. 1*a* and the two corresponding rectangles marked 'X' in fig. 1*b*): they have the same truth condition, impose the same constraint on what the world is like—the constraint that ticket 1 wins the lottery. For that reason, an ideally rational agent should assign the same degree of belief to them. In other words: rational degrees of belief should be invariant under (re)partitioning. Call this the *partition-invariance constraint* on rational graded belief.

Partition invariance seems plausible whatever additional interpretation or justification one intends to ascribe to degrees of belief. From an epistemic point of view, the inaccuracy or "distance from the truth" of an agent's degree of belief in {Ticket-1-wins} should equal the inaccuracy

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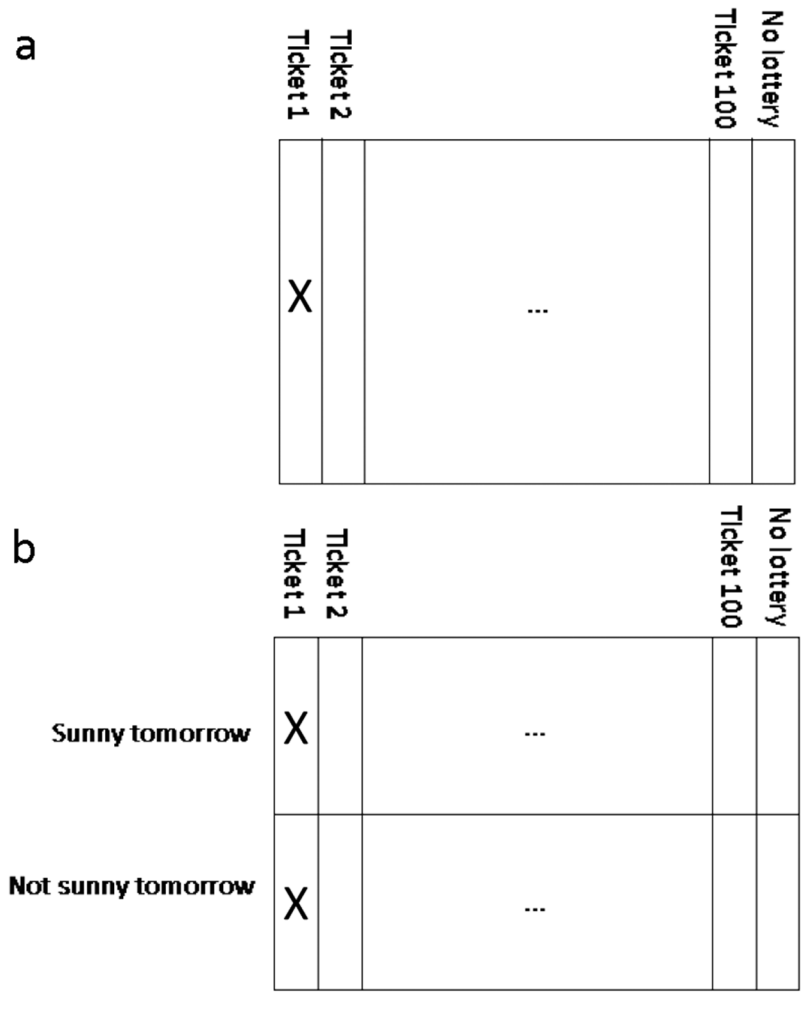


Figure 1. Two partitionings of logical space.

or "distance from the truth" of the same agent's degree of belief in {Ticket-1-wins-&-Sunny-tomorrow, Ticket-1-wins-&-Not-sunny-tomorrow}. After all, whatever the actual world is like, if ticket 1 wins in it, then either ticket 1 wins in it and it is sunny there tomorrow (at a given time and location) or ticket 1 wins in it and it is not sunny there tomorrow and vice versa. Therefore, if the aim of an agent's degrees of belief is to get as close as possible to the truth, her degrees of belief in {Ticket-1-wins} and {Ticket-1-wins-&-Sunny-tomorrow, Ticket-1-wins-&-Not-sunny-tomorrow} should be on an equal

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footing. Similarly, any kind of logical understanding of rational degrees of belief should leave {Ticket-1-wins} and {Ticket-1-wins-&-Sunny-tomorrow, Ticket-1-wins-&-Not-sunny-tomorrow} with the same degree, for the same reason for which the two sentences 'Ticket 1 wins' and 'Ticket 1 wins and it is sunny tomorrow, or Ticket 1 wins and it is not sunny tomorrow' are logically equivalent and hence bear the same logical relationships to sentences that might express an agent's evidence.

Analogous considerations apply if one focuses instead on the pragmatic aspects of rational degrees of belief. Indeed, in decision theory, the corresponding partition-invariance desideratum for the value of actions provided one of the reasons for which Jeffrey (1983) abandoned Leonard Savage's partition-sensitive theory and developed his own partition-invariant "Logic of Decision" (see Joyce 1999, secs. 2.6, 3.4, 4.2, for a survey of the debate): rational decision-making in a "small world," in which not all potentially available distinctions have been made, should remain unaffected when the underlying set of states or worlds is refined. In order to decide rationally, we should not be forced to enter the "grand world," in which every conceivable distinction has been drawn already. As Huttegger (2017, 81–82) summarizes Jeffrey's achievement: "Partition invariance shows that any [small world] model using Jeffrey's decision theory is . . . consistently embeddable in the large world." The same desideratum of partition invariance may also be applied to rational degrees of belief, as the following sections will show, in which I am going to count partition invariance among the characterizing properties of rational graded belief.

In that respect, rational graded belief might actually differ from its all-or-nothing counterpart: for some recent formal theories of rational belief, which are dealing jointly with all-or-nothing belief and degrees of belief, argue that rational all-or-nothing belief is sensitive to how the underlying set of possibilities is partitioned (see Lin and Kelly 2012a, 2012b; Leitgeb 2014, 2017).[3] (Yalcin [2018] argues for the same conclusion on grounds that are independent of considerations on graded belief.) Lin and Kelly (2012a, sec. 13) even prove that, given some fairly plausible premises—including the logical closure of rational belief—rational all-or-nothing belief could not be partition invariant: it is impossible for a rational agent to always maintain, in each and every case of repartitioning, her classification of propositions into believed ones, disbelieved ones, and ones that are neither. (The results of

3. Assigning the same rational degree of belief to any two belief contents with the same truth conditions might be understood as graded belief aiming at the truth and nothing more. In contrast, rational all-or-nothing belief might aim at more than just the truth (e.g., it might aim also at a partitioning of possible cases that is contextually suitable for matters of acceptance or assertion).

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sec. 6 of Schurz [2019] may also be interpreted along these lines.) At the same time, all these theories still assume rational graded belief to be partition invariant. Hence, if these theories are right, rational all-or-nothing belief might depend on additional pragmatic-contextual features, including the relevant partitioning of cases, which rational graded belief manages to steer clear of. In the terminology of Lewis's (1988) account of *subject matter* as partition, rational all-or-nothing belief might be sensitive to subject matter (winning ticket vs. winning ticket & weather), while rational graded belief is not. In the terminology of Groenendijk and Stokhof's (1984) theory of *questions* as partitions, rational all-or-nothing belief might be sensitive to the question relative to which the agent is operating in a context (which ticket will win? vs. which ticket will win and what will the weather be like?), while rational graded belief is not. And if partitions of possibilities are taken to be correspond to an agent's underlying linguistic framework, then rational all-or-nothing belief might be language sensitive, whereas rational graded belief is language invariant. (All of the references stated above interpret partitions in at least some of these ways, but more interpretations would be available: e.g., Suppes [2016] interprets partitions as *experiments*.) This does not mean that, prima facie, rational all-or-nothing belief should not be partition invariant, too—it is just that this desideratum might be overruled by further desiderata, such as the logical closure of belief or assumptions on how rational all-or-nothing and graded belief ought to interact.[4]

In any case: let us put the question whether rational all-or-nothing belief is partition invariant to one side now. For the rest of the article, we are going to concentrate solely on rational graded belief, for which partition invariance does seem like a powerful and satisfiable desideratum. (That holds at least as far as perfectly rational agents are concerned—actual degrees of belief of actual human agents might be a different matter.) For instance, where the aiming-at-the-content constraint will require the rational degree of belief $B(\{X, Y, Z\})$ in the proposition $\{X, Y, Z\}$ to be a function of the rational degrees of belief in the three possible cases X, Y, Z—that is, $B(\{X, Y, Z\}) = F(B(\{X\}), B(\{Y\}), B(\{Z\}))$—the partition-invariance constraint is going to add to this, for example, that $B(\{X, Y, Z\}) = B(\{X,$

4. As far as people's actual beliefs are concerned, empirical work (Windschitl and Well 1998) seems to support a similar conclusion: if presented with two stories in both of which the subject owns the same number of tickets in a raffle but where the two stories claim the remaining tickets to be distributed differently over other people (corresponding to different partitions of the set of tickets), the *numerical* probability estimates by test persons concerning focal outcomes were found to remain invariant, while their *verbal nonnumerical* certainty estimates proved to be sensitive to the partitionings. Windschitl and Well (1998) refer to this as the "alternative outcomes effect." (See Leitgeb [2017, 141], for a summary and an application to a theory of rational belief.)

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$Y \cup Z\}) = B(\{X \cup Y, Z\})$, since both $\{X, Y \cup Z\}$ and $\{X \cup Y, Z\}$ result from repartitioning $\{X, Y, Z\}$. With only the aiming-at-the-content constraint in place, anything could happen to an agent's degrees of belief by repartitioning the logical space of possibilities; not so with partition invariance added.

This is thus the plan of the article: section 2 will start by introducing partitions as contents of belief, after which it will enumerate some plausible postulates for rational degrees of belief, including formalizations of the aiming-at-the-content constraint and the partition-invariance constraint. The other postulates include a monotonicity requirement for rational degrees of belief and some assumptions on how logical space can be (re)partitioned. Section 3 will derive from these postulates—subject to one conventional choice and relative to a given algebra—the existence of a uniquely determined isomorphism between rational graded belief and a subjective probability measure. From this representation theorem it follows that, at least for all formal purposes, rational degrees of belief can be identified with probabilities. In that sense, probabilism will be supported by the theory. Section 4 will summarize what has been achieved and suggest future extensions of the theory.

2. Partitions and Postulates. In order to rationally reconstruct the relevant notion of partition invariance in a formal setting, we are going to introduce a framework in which the partition invariance of rational graded belief can be formulated easily, partition invariance could fail in principle, and postulating partition invariance will thus amount to a substantial constraint on rational degrees of belief. (In the next section, this "substance" will show up in the proof of an important associativity law and, ultimately, in the definability of probabilities as uniquely determined representations of rational degrees of belief.)

There is certainly more than one way of setting up a formal framework like that, but what we opt for is likely to be the simplest possible choice: we are going to presuppose that degrees of belief are assigned to partitions, where partitions are defined *set-theoretically*, just as it is usually done in mathematical contexts. Partitions in that sense have already been the objects of philosophical investigation for various other purposes (see Groenendijk and Stokhof [1984], Lewis [1988], and other references from the last section, and see Jeffrey [1983, 1992] and Joyce [1999] for theories of learning and decision-making based on partitions as sets). For the same reason, the set-theoretic partitions framework to which I will now turn is motivated independently.

For a start, let us pretend to be given "the" nonempty set W of "maximally fine-grained" possibilities, that is, possibilities that take everything into account: the lottery, the weather tomorrow, and whatever else. Some may take W very seriously by regarding it to be "the real" set of "ultimate"

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possibilities.[5] Others may just view it as one of the useful fictions that rational reconstructions sometimes involve (see Carnap 1967, sec. 99). Either way, in what follows, W will be regarded as the logical space of "maximally fine-grained possibilities," by which we may refer to the members of W or, alternatively, to their singleton sets. As we will see, one of the postulates below will entail W to be infinite. The more "realistic" logical spaces of possibilities that one encounters in actual instances of uncertain reasoning are going to coincide with some more or less "coarse-grained" partitionings of W, including partitions with only finitely many partition cells. When rational degrees of belief will be represented by subjective probabilities at the end of section 3, some of these more coarse-grained partitions may be used as the sample space of the resulting probability measure.

Let us turn to partitions on W now in more detail: when X is a subset of W, a (generalized) *partition* π of X is a set of nonempty pairwise disjoint subsets of X, such that the union of the members (partition cells) of π, that is, $\cup\pi$, is X. Note that, in contrast with standard usage, we allow for the empty set to count as a partition in the present sense of the term. A partition π of the *full set* W corresponds to a "logical space" of more or less coarse-grained "logically possible worlds," that is, partition cells. We are going to refer to not just W but also to $\{W\}$ and any other partition of W as 'logical space'. Call π a *partition* (simpliciter, and relative to W) just in case there is a subset X of W, such that π is a partition of X. Hence, a partition (simpliciter) does not necessarily partition the complete logical space W but might well partition just a proper subset thereof. We will assume to be given a set of such partitions π (on W), which we will denote by Π. Note that Π does not need to coincide with the set of all partitions on W; instead, postulate P5 below will merely enumerate some closure conditions on Π. The given set Π of partitions is not assumed to be a (Boolean) algebra of sets either; for instance, there is no postulate that would guarantee the existence of a uniquely determined Boolean "negation" or complement of a partition π in Π.

If π and π' are partitions, such that $\cup\pi = \cup\pi'$, that is, the union of the partition cells of π coincides with the union of the partition cells of π', then π and π' are both partitions of $\cup\pi$, or equivalently $\cup\pi'$: they occupy the same region in logical space. If, additionally, every partition cell of π is the union of partition cells in π', then π' is *at least as fine-grained as* π, and if furthermore some partition cell in π' is not the union of partition cells in π,

5. One might want to argue that W should be a *proper class* rather than a set: in that case, one should substitute 'class' for 'set' throughout the article, and one should think of all subsequent considerations to be carried out in a theory of classes rather than a theory of sets. But I am going to ignore this for the sake of simplicity.

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then π' is *strictly more fine-grained* than π. For instance, both {Ticket-1-wins} and {Ticket-1-wins-&-Sunny-tomorrow, Ticket-1-wins-&-Not-sunny-tomorrow} might be members of Π, where Ticket-1-wins, Ticket-1-wins-&-Sunny-tomorrow, and Ticket-1-wins-&-Not-sunny-tomorrow are nonempty subsets of W, such that Ticket-1-wins is the union of Ticket-1-wins-&-Sunny-tomorrow and Ticket-1-wins-&-Not-sunny-tomorrow. Hence, it holds that $\cup${Ticket-1-wins-&-Sunny-tomorrow, Ticket-1-wins-&-Not-sunny-tomorrow} = $\cup$ {Ticket-1-wins}, which means the two partitions cover the same region in logical space. At the same time, {Ticket-1-wins-&-Sunny-tomorrow, Ticket-1-wins-&-Not-sunny-tomorrow} is strictly more fine-grained than {Ticket-1-wins}. In what follows, $\varnothing$ will serve as the logically impossible partition that rules out every logical possibility whatsoever. And $\{W\}$ is the most coarse-grained partition of W that is compatible with every logical possibility whatsoever; we will regard it, and its fine-grainings, as logically necessary partitions. It is partitions in that sense to which rational degrees of belief will be assigned in our formal reconstruction (so long as they are members of the given set Π of partitions).[6]

Now consider an ideally rational agent at some point in time: we are going to denote her numerical degree-of-belief (graded belief, strength of belief) assignment at the time by 'B', and we will denote the function by which her degree of belief in π will be determined from the degrees of belief in the possibilities at which π is true by 'F'. As we are going to see, plausible assumptions for B and F will entail ultimately that B may be identified with a subjective probability measure P, and F may be identified with the addition function for real numbers in probabilism's finite additivity axiom.

These are our postulates for the set Π of partitions and the B and F functions. We have already mentioned:

(P1) Domain: The domain of B is Π.

As before, propositions may be identified with sets of possibilities, but now the possibilities in question may result from different partitionings: for example, both {Ticket-1-wins} and {Ticket-1-wins-&-Sunny-Tomorrow, Ticket-1-wins-&-Not-Sunny-Tomorrow} might count as propositions in that refined sense. The only difference to the standard reconstruction of propositions in possible worlds semantics is that the members π of Π "wear" the granularity of their underlying possibilities "on their sleeves": the possibilities over which any such π ranges are just the members of π, and π may be more or less fine-grained.

6. Alternatively, one might start from an axiomatic treatment of logical space in which partitions, possibilities, and their relations are considered as *primitives* rather than defined in set-theoretic terms. For the sake of simplicity, we will not do so here.

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By assuming that our ideally rational agent assigns degrees of beliefs to all partitions in Π, we do not want to rule out that, at a time and for a certain purpose, the agent might only be interested in (focus on, pay attention to) partitions that are subsets of one particular partition σ of logical space. For instance, at a time, the agent might concentrate on which ticket will win in a lottery, which is why she might solely dedicate herself to subsets of the partition {Ticket-1-wins, . . . , Ticket-100-wins, No-lottery-takes-place}. Indeed, as mentioned above, when we are going to represent rational degrees of belief as probabilities at the end of section 3, we will do so relative to such a "fixed" partition σ of logical space, such that σ may be interpreted as formalizing the distinctions the agent is interested to make in a particular context. For example, if her interests changed in the way that she would start focusing on subsets of the refined partition $\sigma' =$ {Ticket-1-wins-&-Sunny-tomorrow, Ticket-1-wins-&-Not-sunny-tomorrow, . . . , Ticket-100-wins-&-Sunny-tomorrow, Ticket-100-wins-&-Not-sunny-tomorrow, No-lottery-takes-place-&-Sunny-tomorrow, No-lottery-takes-place-&-Not-sunny-tomorrow}, then the results from section 3 will show that the agent's degrees of belief may also be represented by probabilities relative to that refined partition of possibilities. In terms of an analogy with the more traditional literature on belief, B might be understood as the agent's rational degrees-of-*dispositional*-belief assignment on Π, while the agent's degrees of *occurrent* belief in a context might be restricted to partitions in Π that are subsets of one particular partition (σ or σ' or . . .) of logical space that the agent attends to in the relevant context. If interpreted along these lines, the probabilistic representation of degrees of belief in section 3 may be understood as the representation of degrees of dispositional belief (B) becoming occurrent in a particular context, where such a context involves σ and some algebra of subsets of σ that are at the same time members of Π.

Second, as we are dealing with graded belief, we postulate:

(P2) Rational degrees of belief:

i) $B(\varnothing)$ and $B(\{W\})$ are defined (thus $\varnothing$, $\{W\} \in \Pi$), and the codomain of B is $[B(\varnothing), B(\{W\})]$, where $B(\varnothing) < B(\{W\})$.
ii) If $\pi \subseteq \pi'$ and there exists a ρ in Π, such that $\rho \subseteq \pi'$, $\rho \cap \pi = \varnothing$, and $B(\rho) > B(\varnothing)$, then $B(\pi) < B(\pi')$.

Here, $[B(\varnothing), B(\{W\})]$ is the closed interval of real numbers (not strictly) between $B(\varnothing)$ and $B(\{W\})$: rational degrees of belief ought to be bounded by the rational degrees of belief in the logically impossible partition $\varnothing$ and in the logically necessary partition $\{W\}$. One might wonder why the right boundary of the interval should be given by the B-image of $\{W\}$ rather than of some other logically necessary partition, but P4 below (partition invariance)

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will prove that to be irrelevant. As far as the assumption $B(\varnothing) < B(\{W\})$ is concerned, one could easily derive a variant of the result below in which (mutatis mutandis) $B(\{W\}) < B(\varnothing)$ rather than the converse would be assumed. What is important about P2.i is merely the claim that rational degrees of belief are real numbers that are bounded by the real numbers $B(\varnothing)$ and $B(\{W\})$ and that $B(\varnothing)$ is numerically distinct from $B(\{W\})$.

Postulate P2.ii adds to this a special type of strict monotonicity requirement for B: in contrast with plausibility measures (Halpern 2003, sec. 2.8), which generalize almost all kinds of degree-of-belief functions that one can find in the existing literature and which satisfy the monotonicity axiom 'if $\pi \subseteq \pi'$ then $B(\pi) \leq B(\pi')$' (stated in present terminology), P2.ii involves an additional antecedent condition ('there exists a ρ . . .') by which even the strict inequality '$B(\pi) < B(\pi')$' can be derived. Roughly: the strictly larger the set of "possible cases" from the viewpoint of an agent, the strictly greater the agent's rational degree of belief in the set of "possible cases" should be. (But one can show that plain monotonicity also follows from the full set of postulates P1–P5.) A further difference to plausibility measures is that P2.i identifies degrees of belief with real numbers, which are of course totally ordered by the $\leq$ relation for real numbers, whereas plausibility measures are merely taken to map propositions to some partially ordered set of plausibility "values."

Over and above postulates P1–P5 from this section, we will not presuppose any particular interpretation or justification of the degrees of belief assigned by B except that any such number is going to measure quantitatively the "plausibility" that a given ideally rational agent attaches to a partition (proposition). Thus, for instance, we will leave open whether any such degree of belief corresponds epistemically to an expected distance from the truth or logically to a logical degree of credibility given one's total evidence or pragmatically to a fair betting quotient or the like.

Third, as highlighted at the beginning of this article, rational belief in a content is an attitude toward the possibilities in which that content is true. In the current context, such possibilities may be given by any kind of partition π, for example, $\{X_1, \ldots, X_n\}$. The agent's rational degree of belief in π should therefore be a function, F, of her rational degrees of belief in the (singleton partitions of) the partition cells of π—here, $\{X_1\}, \ldots, \{X_n\}$—as these are the possibilities at which π is true. It follows that F should really be a variably polyadic function that is applicable to any finite number n of degrees of belief whatsoever. (I will put applications of F to infinitely many arguments to one side.) Fortunately, with the help of the partition-invariance principle P4 from below, it will suffice to focus solely on applications of F to two degrees of belief at a time, which is going to simplify the formal treatment of F. Accordingly, we formulate the aiming-at-the-content constraint on rational graded belief just for $n = 2$.

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(P3) Aiming at the content: F is a partial mapping from $[B(\varnothing), B(\{W\})]^2$ to $[B(\varnothing), B(\{W\})]$, such that:

i) $F(x, y)$ is defined iff there are partitions $\{X, Y\}$ (where $X \cap Y = \varnothing$), $\{X\}$, $\{Y\}$ in Π, such that $x = B(\{X\})$ and $y = B(\{Y\})$;
ii) for all partitions $\{X, Y\}$ (where $X \cap Y = \varnothing$), $\{X\}$, $\{Y\}$ in Π (thus $F(B(\{X\}), B(\{Y\}))$ being defined by i and P1): $B(\{X, Y\}) = F(B(\{X\}), B(\{Y\}))$;
iii) in addition, we postulate $B(\{X\}) = F(B(\{X\}), B(\varnothing)) = F(B(\varnothing), B(\{X\}))$. (The set $[B(\varnothing), B(\{W\})]^2$ is the set $[B(\varnothing), B(\{W\})] \times [\mathrm{B}(\varnothing), B(\{W\})]$ of pairs of numbers in the closed interval $[B(\varnothing), B(\{W\})]$.)

Postulate P3.ii formalizes the aiming-at-the-content constraint for binary partitions, that is, partitions that include precisely two possibilities. Postulate P3.i states that F is defined when applied to the rational degrees of belief of the singletons of the partition cells of such binary partitions, and it is defined only when it can be applied to rational degrees of belief of such a kind. For instance, if $F(x, y)$ is defined, P3.i implies that there are partitions $\{X, Y\}$, $\{X\}$, $\{Y\}$ in Π, such that $x = B(\{X\})$ and $y = B(\{Y\})$, and P3.ii implies that $B(\{X, Y\}) = F(B(\{X\}), B(\{Y\}))$. If additionally x and y are greater than $B(\varnothing)$, then P2.ii from before, using $\{X\} \subseteq \{X, Y\}$, $\{Y\} \subseteq \{X, Y\}$, $\{X\} \cap \{Y\} = \varnothing$, and $B(\{Y\}) > B(\varnothing)$, yields that $x = B(\{X\}) < B(\{X, Y\}) = F(B(\{X\}), B(\{Y\})) = F(x, y)$ (and the same holds for y), and hence it follows that $F(x, y)$ is greater than $B(\varnothing)$, too. In a similar way, one can use P3 and P2 to show that if $B(\{X\}) = B(\{W\})$ and $B(\{Y\}) > B(\varnothing)$, then $F(B(\{X\}), B(\{Y\}))$ must be undefined; that is, $\{X, Y\}$ cannot be a member of Π.

Postulate P3.iii is justified by the thought that $B(\{X\})$ should depend functionally, as realized by the (variably polyadic) function F, on all the possibilities that compose $\{X\}$, but since the partition $\{X\}$ may always be regarded as consisting of the possibilities in $\{X\}$ taken together with no further possibilities at all (as "collected" by $\varnothing$), $B(\{X\})$ should equal $F(B(\{X\}), B(\varnothing))$ and $F(B(\varnothing), B(\{X\}))$.

Now we are in the position to make the intended partition-invariance constraint on rational belief more precise:

(P4) Partition invariance: For all partitions π, π' in Π, if $\cup\pi = \cup\pi'$, then $B(\pi) = B(\pi')$.

As explained before, if $\cup\pi = \cup\pi'$, then π and π' are partitions of one and the same subset of W (the set $\cup\pi$ or $\cup\pi'$), in which case $B(\pi)$ should equal $B(\pi')$, as explained in section 1. Once again, P4 does not presuppose that

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our ideally rational agent is paying attention simultaneously to all partitions of possibilities whatsoever. But, for example, if she first concentrates on the lottery, assigning a certain degree of belief to ticket 1 winning (corresponding to the partition $\pi = \{\text{Ticket-1-wins}\}$), and then someone brings up the question whether the winner of the lottery will be able to enjoy the weather tomorrow, she should be able to shift her attention to the combined lottery-and-weather possibilities and yet maintain her degree of belief in ticket 1 winning (which will now correspond to the refinement $\pi' = \{\text{Ticket-1-wins-\&-Sunny-tomorrow, Ticket-1-wins-\&-Not-sunny-tomorrow}\}$ of π). Similarly, if the agent were introduced to a new concept, say, *dark matter*, she should be capable of grasping the refined partition and assigning the same degree of belief to $\pi'' = \{\text{Ticket-1-wins-\&-Sunny-tomorrow-\&-Dark-matter-exists, Ticket-1-wins-\&-Sunny-tomorrow-\&-Not-dark-matter-exists, Ticket-1-wins-\&-Not-sunny-tomorrow-\&-Dark-matter-exists, Ticket-1-wins-\&-Not-sunny-tomorrow-\&-Not-dark-matter-exists}\}$ as she had done to π and π'. And so forth. In particular, if π and $\{\cup\pi\}$ are in Π, P4 yields that $B(\{\cup\pi\}) = B(\pi)$, where $\cup\pi$ is again the union of all partitions cells in π.

In line with the informal discussion from the introduction, P3 and P4 are going to constitute the core principles of our theory. Finally, we add the following assumptions on Π, which, collectively, maintain that the agent's space of possible (re)partitionings is sufficiently "rich":

(P5) (Re)partitioning:

i) For every singleton partition $\{X\}$ in Π with $B(\{X\}) > B(\emptyset)$, and for every real number y in $(B(\emptyset), B(\{X\}))$, there is a partition $\pi = \{Y, Z\}$ of X in Π, such that $\{Y\}, \{Z\} \in \Pi$, and $B(\{Y\}) = y$ and $B(\{Z\}) \in (B(\emptyset), B(\{X\}))$.

ii) For every singleton partition $\{X\}$ in Π with $B(\{X\}) > B(\emptyset)$, and for all real numbers y and z in $(B(\emptyset), B(\{X\}))$, such that $F(y, z)$ is defined and $B(\{X\}) = F(y, z)$, there is a partition $\pi = \{Y, Z\}$ of X in Π, such that $\{Y\}, \{Z\} \in \Pi$, and $B(\{Y\}) = y$ and $B(\{Z\}) = z$.

iii) For all partitions $\{X_1, \ldots, X_m\}$ (with $m \geq 1$) and $\{Y_1, \ldots, Y_n\}$ (with $n \geq 1$) in Π, such that for all i, j it holds that $X_i \cap Y_j = \emptyset$, it is the case that $\{X_1, \ldots, X_m, Y_1, \ldots, Y_n\} \in \Pi$.

iv) For every partition π in Π: $\{\cup\pi\} \in \Pi$.

v) There is no x in $(B(\emptyset), B(\{W\}))$, such that for all i there is a partition $\{X_1, \ldots, X_i\}$ in Π with i partition cells $X_1, \ldots, X_i$ for which $\{X_1\}, \ldots, \{X_i\} \in \Pi$ and $B(\{X_1\}) = \ldots = B(\{X_i\}) = x$.

Postulates P5.i–v express structural closure conditions on our ideally rational agent's set Π of partitions. Postulates P5.i–ii are refinement postulates:

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they state that any singleton partition $\{X\}$ in Π that is assigned a degree of belief greater than $B(\varnothing)$ can be refined to a particular kind of binary partition $\{Y, Z\}$ that is also a member of Π. Postulate P5.iii expresses a different kind of closure condition on Π: Π is closed under taking unions of partitions of a certain kind. Postulate P5.iv is a coarse-graining postulate: any partition in Π can be coarse-grained to a singleton partition in Π that covers the same region of logical space. Postulate P5.v serves as a kind of "Archimedean property" for Π, B, and F in section 3: for every $x > B(\varnothing)$ there is a finite upper boundary to the cardinality of any partition all the (singletons of) partition cells of which have degree of belief of x.

Axioms in a similar ballpark as P5.i–v are known from representation theorems for qualitative/comparative probability (for surveys, see chap. 5 of Krantz et al. [1971] and sec. 5.7 of Suppes [2002]). For example, axioms 5, 5′, and 5″ in section 5.2.3 of Krantz et al. (1971) postulate the existence of arbitrarily fine-grained finite partitions, much as P5.i–ii do; qualitative probability structures are defined on algebras of sets (see Krantz et al. 1971, 204; Suppes 2002, 227), which implies that these structures obey a closure condition similar to P5.iii; and Archimedean qualitative probability structures (see Krantz et al. 1971, 205) are defined to satisfy a property of a type similar to that expressed by P5.v.

Let me now explain P5.i–v in a bit more detail. In P5.i, a partition $\{Y, Z\}$ is claimed to exist, such that $B(\{Y\})$ is any given real number y in the open interval $(B(\varnothing), B(\{X\}))$, that is, for which $B(\varnothing) < y < B(\{X\})$ holds. For instance, if y is, say, $(B(\varnothing) + B(\{W\}))/2$, there should be some way for our ideally rational agent to divide logical space $\{W\}$ $(=\{X\})$ into a binary partition $\{Y, Z\}$, such that $B(Y) = (B(\varnothing) + B(\{W\}))/2$. As mentioned before, this does not mean that our ideally rational agent needs to be interested in any such partition $\{Y, Z\}$ in every context and for every purpose; $\{Y, Z\}$ is merely meant to be available to her if she were to attend to it. If F had been postulated to be a continuous function in P3, then P5.i could in fact be weakened to postulate the existence of $\{Y, Z\}$, such that $B(\{Y\})$ would be any given rational number y with $B(\varnothing) < y < B(\{X\})$, and yet the same results would have gone through as below. In any case, P5.i entails that W and Π are infinite—thus, infinitely fine-grained distinctions are potentially or dispositionally available to the agent—and the degree-of-belief mapping B is onto (by letting $\{X\} = \{W\}$ again). The claim is not that this is a psychologically realistic dispositional feature of human agents but rather a plausible dispositional feature of ideally rational agents. (P5.i and ii also bear some resemblance with continuity and nonatomicity axioms in decision theory.)

In P5.ii, $B(\{Y\})$ and $B(\{Z\})$ are any numbers y, z, respectively, such that $B(\{X\}) = F(y, z)$. While P3.i–ii would suffice to prove that, if $F(y, z)$ is defined, then there are partitions $\{Y, Z\}$, $\{Y\}$, $\{Z\}$ in Π, such that $y = B(\{Y\})$ and $z = B(\{Z\})$, P5.i shows that for any given singleton partition

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$\{X\}$ in Π with $B(\{X\}) > B(\varnothing)$ and $B(\{X\}) = F(y, z)$, there is a partition $\{Y, Z\}$ of the given singleton partition X in Π, such that $\{Y\}$, $\{Z\}$ are partitions in Π, and $y = B(\{Y\})$ and $z = B(\{Z\})$. For instance, with appropriate Π, B, and F in place, $\{X\}$ might be {Ticket-1-wins}, B({Ticket-1-wins}) might equal $F(y, z) > 0$ for suitable numbers y and z, and the required partition $\{Y, Z\}$ might happen to be {Ticket-1-wins-&-Sunny-tomorrow, Ticket-1-wins-&-Not-sunny-tomorrow}, such that {Ticket-1-wins-&-Sunny-tomorrow}, {Ticket-1-wins-&-Not-sunny-tomorrow} are members of Π, and B({Ticket-1-wins-&-Sunny-tomorrow}) $= y$ and B({Ticket-1-wins-&-Not-sunny-tomorrow}) $= z$.

Postulate P5.iii expresses that Π is closed under finite unions as long as these unions yield partitions again, while P5.iv says that Π is also closed under taking the union of all partition cells of a partition and turning the result into a singleton partition. For instance, it follows from P5.iv that for every partition $\{X, Y\}$ in Π, the singleton partition $\{X \cup Y\}$ is a member of Π, too. Or more concretely, if {Ticket-1-wins-&-Sunny-tomorrow, Ticket-1-wins-&-Not-sunny-tomorrow} is in Π, then {Ticket-1-wins-&-Sunny-tomorrow $\cup$ Ticket-1-wins-&-Not-sunny-tomorrow}, that is, {Ticket-1-wins}, is a member of Π, too.

Finally, P5.v puts on record that for any given x in the semiopen interval $(B(\varnothing), B(\{W\})]$, that is, where $B(\varnothing) < x \leq B(\{W\})$, there cannot be arbitrarily fine-grained finite partitions π in Π, such that the singleton set of each partition cell of π is always assigned the same given number x as a rational degree of belief. Once the probabilistic representation theorem of the next section is in place, this will correspond to the Archimedean fact that, for $x > P(\varnothing)$, there cannot be arbitrarily large finite sums $x + \ldots + x$ that coincide with probabilities of propositions. This concludes the list of assumptions.

3. From Postulates to Probabilistic Representation. Our strategy will now be to derive some consequences from the postulates of the last section, including, as we will see shortly, the consequence that the function F mentioned in P3 and P5 must be associative; for that purpose, the partition-invariance postulate P4 will prove to be crucial. Afterward we are going to invoke an existing mathematical result to characterize the very functions B and F that satisfy the emerging constraints. The relevant mathematical result is theorem 4.2 in Narens (2007, 33), which itself is a descendant of a central theorem in representational extensive measurement theory that is due to Hölder and that concerns the real-valued representation of special associative concatenation functions on total orders (see sec. 2.2 of Krantz et al. 1971). Hölder's theorem and variants thereof have been applied extensively, for example, in the representation of qualitative/comparative probability structures (see Krantz et al. 1971, chaps. 3 and 5). In the current context, our goal

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is to develop a new application of a variant of Hölder's theorem in support of the structural justification of probabilism that was motivated in section 1 and that is based on the postulates in section 2.

We begin with the following definition (see definition 4.4 in Narens 2007, 32):

> Definition. $\langle X_{\text{ext}}, \preccurlyeq, \oplus \rangle$ is an *extensive structure with maximal element m* iff $\preccurlyeq$ is a binary relation on X_{ext}, $\oplus$ is a partial binary function on X_{ext}, and m is a member of X_{ext}, such that the following axioms are satisfied for all x, y, z, u, v in X_{ext}:
>
> 1. Relation $\preccurlyeq$ is a weak ordering on X_{ext}, that is, $\preccurlyeq$ is transitive and connected on X_{ext}, and furthermore $x \preceq m$. (In what follows, let $x \prec y$ just in case $x \preceq y$ and not $y \preceq x$.)
> 2. If $x \oplus y$ is defined and $u \preceq x$ and $v \preceq y$, then $u \oplus v$ is defined.
> 3. If $x \oplus y$ is defined, then $x \prec x \oplus y$.
> 4. If $x \oplus (y \oplus z)$ is defined, then $(x \oplus y) \oplus z$ is defined and $x \oplus (y \oplus z) \sim (x \oplus y) \oplus z$. (Note that $\sim$ is defined as $\preccurlyeq$ in both directions. Definedness of $x \oplus (y \oplus z)$ also means that $y \oplus z$ is defined, and analogously for $(x \oplus y) \oplus z$.)
> 5. If $x \prec y$, then there exists a w in X_{ext}, such that $x \oplus w \preceq y$.
> 6. There exist u and v in X_{ext}, such that $u \oplus v = m$.
> 7. (Archimedean Axiom) There does not exist an infinite sequence $x_1, \ldots, x_i, \ldots$ of members of X_{ext}, such that for each i: $x_{i+1} = x_i \oplus x_1$.

The following theorem about extensive structures with maximal elements can be found in Narens (2007, 33):

> Theorem 1. Let $\langle X_{\text{ext}}, \preccurlyeq, \oplus \rangle$ be an extensive structure with maximal element m. Then it holds:
>
> (Existence) There exists a function φ from X_{ext} to $\mathbb{R}^+$, such that for all x and y in X_{ext},
>
> $$x \preceq y \text{ iff } \varphi(x) \leq \varphi(y), \tag{1}$$
>
> and, where $x \oplus y$ is defined,
>
> $$\varphi(x \oplus y) = \varphi(x) + \varphi(y). \tag{2}$$
>
> (Uniqueness) If φ and ψ satisfy all instances of (1) and (2), then (i) there exists an r in $\mathbb{R}^+$, such that $\varphi = r\psi$, and (ii) for each s in $\mathbb{R}^+$, $s\varphi$ satisfies all instances of (1) and (2).

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What we will prove now is:

> Theorem 2. Given postulates P1–P5 from section 2, it follows that $\langle (B(\varnothing), B(\{W\})], \leq, F \restriction_{(B(\varnothing),\ B(\{W\})]} \rangle$ is an *extensive structure with maximal element* $B(\{W\})$.

($\mathbb{R}^+$ in theorem 1 is the set of positive real numbers; $F \restriction_{(B(\varnothing),\ B(\{W\})]}$ in theorem 2 is the restriction of F to $(B(\varnothing), B(\{W\})]^2$.)

Ultimately, given just one conventional choice concerning the representation of $B(\{W\})$, theorem 2 will allow us to use theorem 1 to support the identification of the rational degree-of-belief function B from section 2 with a (finitely additive) probability measure and of the function F from section 2 with the addition function $+$ for real numbers restricted to [0, 1].

For the proof of theorem 2, let X_{ext} be $(B(\varnothing), B(\{W\})]$, let $\preccurlyeq$ be the $\leq$ relation for real numbers restricted to X_{ext}, and let $m = B(\{W\})$. Furthermore, let $\oplus$ be partially defined on X_{ext} like this: $x \oplus y = z$ iff $F(x, y) = z$. Recall that, by P3 from section 2, F is itself a partial mapping from $[B(\varnothing), B(\{W\})]$ to $[B(\varnothing), B(\{W\})]$. And $\oplus$ is still a mapping to the half-open interval $(B(\varnothing), B(\{W\})]$, since, as shown when discussing P3 in section 2, if x and y are greater than $B(\varnothing)$, P3.i–ii and P2.ii imply $F(x, y)$ is greater than $B(\varnothing)$, too.

> *Proof of theorem 2*. (1) Clearly, $\preccurlyeq$ (i.e., $\leq$) is a weak ordering on X_{ext}, and for all x in X_{ext} $(=(B(\varnothing), B(\{W\})]) : x \preceq m$ $(=B(\{W\}))$.
>
> (2) Assume $x \oplus y$ to be defined and $u \preceq x$ and $v \preceq y$. Since $x \oplus y$ is defined, it follows that $F(x, y)$ is defined, and hence by P3.i–ii there are partitions $\{X, Y\}$ (with $X \cap Y = \varnothing$), $\{X\}$, $\{Y\}$ in Π, such that $x = B(\{X\})$, $y = B(\{Y\})$, and $x \oplus y = F(x, y) = B(\{X, Y\})$. By $u \leq x$ and $v \leq y$, $B(\varnothing) < x$, $B(\varnothing) < y$, and P5.i, there is a partition $\{X', X''\}$ of X in Π (with $X' = X'' = X$ if $u = x$), and there is partition $\{Y', Y''\}$ of Y in Π (with $Y' = Y'' = Y$ if $v = y$), such that $\{X'\}, \{X''\} \in \Pi$, $\{Y'\}, \{Y''\} \in \Pi$, $u = B(\{X'\})$, and $v = B(\{Y'\})$. Since $X' \subseteq X$ and $Y' \subseteq Y$, it holds that X' and Y' are disjoint, and therefore P5.iii with $\{X'\} \in \Pi$ and $\{Y'\} \in \Pi$ implies that $\{X', Y'\} \in \Pi$. Thus, P3.i entails that $F(u, v)$ is defined, and hence (as u and v are members of $X_{\text{ext}} = (B(\varnothing), B(\{W\})]$) $u \oplus v$ is also defined.
>
> (3) Assume that $x \oplus y$, that is, $F(x, y)$, is defined. By P3.i–ii, there are partitions $\{X, Y\}$ (with $X \cap Y = \varnothing$), $\{X\}$, $\{Y\}$ in Π, such that $x = B(\{X\})$, $y = B(\{Y\})$, and $B(\{X, Y\}) = F(B(\{X\}), B(\{Y\}))$. Since $\{X\} \subseteq \{X, Y\}$ and furthermore $\{Y\} \subseteq \{X, Y\}$, $\{Y\} \cap \{X\} = \varnothing$, and $B(\{Y\}) = y \in X_{\text{ext}} = (B(\varnothing), B(\{W\})]$, it follows with P2.ii that $x = B(\{X\}) < B(\{X, Y\}) = F(B(\{X\}), B(\{Y\})) = F(x, y)$; that is, $x \prec x \oplus y$.
>
> (4) Assume $x \oplus (y \oplus z)$ to be defined. It follows that $F(x, F(y, z))$ is defined, too. We will show first the definedness claims and the existence claims for the relevant partitions in Π that are required for the application

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of P3.ii and P4; afterward, we will use P3.ii and P4 to prove that $F(x, F(y, z)) = F(F(x, y), z)$ and hence that $x \oplus (y \oplus z) = (x \oplus y) \oplus z$.

By $F(x, F(y, z))$ being defined and P3.i, there are partitions $\{X, A\}$ (with $X \cap A = \varnothing$), $\{X\}$, $\{A\}$ in Π, such that $B(\{X\}) = x$, $B(\{A\}) = F(y, z)$. As mentioned before, since y, z are in X_{ext} and hence greater than $B(\varnothing)$, P3.i–ii and P2.ii imply that $B(\{A\}) = F(y, z) > B(\varnothing)$. Moreover, as follows from the discussion of P3.ii in section 2 again, both y and z must be strictly less than $B(\{A\})$. By P5.ii, there is therefore a partition $\{Y, Z\}$ of A in Π, such that $\{Y\}, \{Z\} \in \Pi$, and $B(\{Y\}) = y$ and $B(\{Z\}) = z$. Since $Y \subseteq A$, it holds that $X \cap Y = \varnothing$, which is why P5.iii with $\{X\} \in \Pi$ and $\{Y\} \in \Pi$ yields that $\{X, Y\} \in \Pi$; thus, P3.i implies that $F(B(\{X\}), B(\{Y\}))$ (and so $F(x, y)$) is defined, and P3.ii implies that $B(\{X, Y\}) = F(B(\{X\}), B(\{Y\}))$. As $\{X, Y\} \in \Pi$, P5.iv has it that $\{X \cup Y\}$ is a member of Π, too. Since Z is disjoint from both X and Y, it follows from P5.iii again that $\{X \cup Y, Z\} \in \Pi$, and thus P3.i implies that $F(B(\{X \cup Y\}), B(\{Z\}))$ is defined. P4 entails that $B(\{X \cup Y\}) = B(\{X, Y\})$, which is why $F(F(B(\{X\}), B(\{Y\})), B(\{Z\}))$ (and hence $F(F(x, y), z)$) is defined. Finally, since $\{X, A\} \in \Pi$ and $A = Y \cup Z$, it also holds that $\{X, Y \cup Z\}$ is a member of Π, and because $\{Y, Z\}$ is in Π (as shown before), it follows with P5.iv that $\{Y \cup Z\}$ is a member of Π, too.

Relying on this, we can conclude that $F(x, F(y, z)) = F(B(\{X\}), F(B(\{Y\}), B(\{Z\}))) =$, by P3.ii, $= F(B(\{X\}), B(\{Y, Z\})) =$, by P4, $= F(B(\{X\}), B(\{Y \cup Z\})) =$, by P3.ii, $= B(\{X, Y \cup Z\}) =$, by P4, $= B(\{X \cup Y, Z\}) =$, by P3.ii, $= F(B(\{X \cup Y\}), B(\{Z\})) =$, by P4, $= F(B(\{X, Y\}), B(\{Z\})) =$, by P3.ii, $= F(F(B(\{X\}), B(\{Y\})), B(\{Z\})) = F(F(x, y), z)$. So, as intended, we have proved that $(x \oplus y) \oplus z$ is defined and $x \oplus (y \oplus z) = (x \oplus y) \oplus z$; that is, $x \oplus (y \oplus z) \sim (x \oplus y) \oplus z$.

(5) Suppose $x \prec y$, that is, $x < y$ (where $x, y \in X_{\text{ext}} = (B(\varnothing), B(\{W\})]$). By P2.i, $B(\{W\})$ is defined and $B(\varnothing) < B(\{W\})$. Let y' be such that $x < y' < y$. Postulate P5.i yields the existence of a partition $\{Y', Z\}$ of W in Π, such that $\{Y'\}, \{Z\} \in \Pi$, $B(\{Y'\}) = y'$ and $B(\{Z\}) \in (B(\varnothing), B(\{W\}))$. Another application of P5.i (using $B(\varnothing) < x < y'$) proves the existence of a partition $\{X, Z'\}$ of Y' in Π, such that $\{X\}, \{Z'\} \in \Pi$, $B(\{X\}) = x$ and $B(\{Z'\}) \in (B(\varnothing), B(\{Y'\})) \subseteq (B(\varnothing), B(\{W\})] = X_{\text{ext}}$. By P3.ii, $B(\{X, Z'\}) = F(B(\{X\}), B(\{Z'\}))$, and P4 implies with $\cup\{X, Z'\} = \cup\{Y'\}$ that $B(\{X, Z'\}) = B(\{Y'\})$. So if we let $w = B(\{Z'\})$, we have overall that $F(x, w) = F(B(\{X\}), B(\{Z'\})) = B(\{Y'\}) = y' < y$, and thus also $x \oplus w \preceq y$.

(6) Using P2.i, P5.i, P3.i–ii, and P4, it follows immediately that there exist u and v in X_{ext}, such that $f(u, v) = u \oplus v = m$ $(= B(\{W\}))$.

(7) Suppose for contradiction that there is an infinite sequence $x_1, \ldots,$ $x_{i, \ldots}$ of members of X_{ext} (thus, in particular, $x_1 \in (B(\varnothing), B(\{W\})]$), such that, for each i, $x_{i+1} = x_i \oplus x_1$ $(= F(x_i, x_1))$. In other words, $x_2 = x_1 \oplus x_1$, $x_3 =$

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$x_2 \oplus x_1 = (x_1 \oplus x_1) \oplus x_1, x_4 = x_3 \oplus x_1 = ((x_1 \oplus x_1) \oplus x_1) \oplus x_1, \ldots$ Analogously to what was shown after P3 in section 2, since $x_1 > B(\varnothing)$, P3.i–ii and P2.ii imply that $x_i > B(\varnothing)$ for all i, and it holds that $B(W) > x_i$ for all i.

Given any $i + 1 \geq 2$, we first construct recursively a sequence of sets $X_{i+1,1}, \ldots, X_{i+1,i+1}$ (and auxiliary sets $A_{i+1,1}, \ldots, A_{i+1,i}$) as follows: by $x_{i+1} = x_i \oplus x_1 = F(x_i, x_1)$ and P3.i, it is possible to find partitions $\{A_{i+1,i}, X_{i+1,1}\}$ (and hence $A_{i+1,i} \cap X_{i+1,1} = \varnothing$), $\{A_{i+1,i}\}$, $\{X_{i+1,1}\}$ in Π, such that $B(\{A_{i+1,i}\}) = x_i, B(\{X_{i+1,1}\}) = x_1$. With $A_{i+1,i+1-j}, X_{i+1,j}$ being defined for $1 \leq j \leq i - 1$, using $x_{i+1-j} = x_{i-j} \oplus x_1 = F(x_{i-j}, x_1)$ and P5.ii, it is possible to find a partition $\{A_{i+1,i-j}, X_{i+1,j+1}\}$ of $A_{i+1,i+1-j}$ in Π (thus $A_{i+1,i-j}$, $X_{i+1,j+1} \subseteq A_{i+1,i+1-j}$), such that $\{A_{i+1,i-j}\}, \{X_{i+1,j+1}\} \in \Pi$, and $B(\{A_{i+1,i-j}\}) = x_{i-j}, B(\{X_{i+1,j+1}\}) = x_1$. And with $A_{i+1,1}, X_{i+1,i}$ being defined (so $j = i$), let $X_{i+1,i+1} = A_{i+1,1}$. Hence, by the construction, $B(\{X_{i+1,i+1}\}) = B(\{A_{i+1,1}\}) = x_1$.

Second, one proves inductively, using P5.iii and the properties of the construction from before, that $\{X_{i+1,i+1}, X_{i+1,i}, \ldots, X_{i+1,1}\}$ is a member of Π. This shows that for the given number $x_1 \in (B(\varnothing), B(\{W\})]$, and for every $i \geq 1$ there is a partition $\{X_{i+1,i+1}, X_{i+1,i}, \ldots, X_{i+1,1}\}$ in Π, such that $B(\{X_{i+1,i+1}\}) = \ldots = B(\{X_{i+1,1}\}) = x_1$. (For $i = 0$ the same follows from P2.i and P5.i.) This contradicts P5.v.

By combining theorems 1 and 2, and by assuming additionally that $\varphi(m) = \varphi(B(\{W\})) = 1$ holds by convention—so $B(\{W\})$ is required to be represented by the real number 1, which one may do consistently by the uniqueness part of theorem 1—it follows that there is one and only one representation function φ for $(B(\varnothing), B(\{W\})]$, $\leq$, and $F \restriction_{(B(\varnothing),\ B(\{W\})]}$, such that (1) and (2) in theorem 1 are satisfied.

The next step is to extend φ to the full closed interval $[B(\varnothing), B(\{W\})]$: with φ already being determined uniquely, let φ^+ be the uniquely determined mapping from $[B(\varnothing), B(\{W\})]$ to $[0, 1]$ that is given by $\varphi^+(x) = \varphi(x)$ for x in $(B(\varnothing), B(\{W\})]$, and $\varphi^+(B(\varnothing)) = 0$. By P2.i, φ^+ satisfies (1) in theorem 1 but now extended to $[B(\varnothing), B(\{W\})]$, and since $B(\{X\}) = F(B(\{X\}), B(\varnothing)) = F(B(\varnothing), B(\{X\}))$ by P3.iii, the same holds when $\oplus$ in (2) in theorem 1 is extended to the partial function F on $[B(\varnothing), B(\{W\})]^2$. And choosing $\varphi^+(B(\varnothing)) = 0$ is in fact the only possible way of extending φ to $[B(\varnothing), B(\{W\})]$, such that (1) and (2) hold on all of $[B(\varnothing), B(\{W\})]$, as follows with the help of P3.iii.

Summing up, we have:

Corollary 1 (Unique representation of Π, B, F). Postulates P1–P5 for Π, B, and F (see sec. 2) entail that there exists one and only one function φ^+ from $[B(\varnothing), B(\{W\})]$ to $[0, 1]$, such that $\varphi^+(B(\{W\})) = 1$, $\varphi^+(B(\{\varnothing\})) = 0$, and for all x and y in $[B(\varnothing), B(\{W\})]$,

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$$x \leq y \text{ iff } \varphi^+(x) \leq \varphi^+(y), \tag{1'}$$

and, where $F(x, y)$ is defined,

$$\varphi^+(F(x, y)) = \varphi^+(x) + \varphi^+(y). \tag{2'}$$

Corollary 1 shows that, if one makes a couple of plausible assumptions, then additivity is forced on the numerical representation of degrees of belief.[7]

The function φ^+ in corollary 1 is not just uniquely determined, it can also be proved to be one to one:

Corollary 2. φ^+ in corollary 1 is one to one.

(I omit the proof, which uses postulates P2.i, P3.ii, P4, and P5.i.)

The ultimate step is to show that the composition of φ^+ and B, $\varphi^+ \circ B$, is a finitely additive probability measure if restricted to a suitable algebra of sets. For that purpose, fix any partition σ of W in Π (so σ is logically necessary): σ is going to serve as the "sample space of possible worlds," where the "worlds" in question are really the partition cells of the partition σ of W. As emphasized before, one may think of σ as formalizing the distinctions that the agent is willing and interested to make in a particular context. Let $\mathcal{A}$ be an arbitrary set of partitions in Π that are also subsets of σ, such that $\mathcal{A}$ is a set algebra on σ: for all π, π', (i) if $\pi \in \mathcal{A}$, then $\sigma \backslash \pi \in \mathcal{A}$ (where $\sigma \backslash \pi$ is the set-theoretic complement of π with respect to σ); (ii) if π, $\pi' \in \mathcal{A}$, then $\pi \cup \pi' \in \mathcal{A}$; and (iii) σ, $\varnothing \in \mathcal{A}$. Such partitions σ and algebras $\mathcal{A}$ exist: for example, $\sigma = \{W\}$ is a member of Π by P2.i, $\varnothing$ is a member of Π by P2.i and also a subset of $\{W\}$, and $\{\{W\}, \varnothing\}$ is indeed a set algebra on the partition $\{W\}$ of W.

Now, finally, let P: $\mathcal{A} \rightarrow [0, 1]$, such that $P(\pi) = \varphi^+(B(\pi))$. Since $\varphi^+(B(\{W\})) = 1$ and $\varphi^+(B(\{\varnothing\})) = 0$, and because $B(\sigma) = B(\{W\})$ by P4, it follows that $\varphi^+(B(\sigma)) = 1$ and $\varphi^+(B(\varnothing)) = 0$, or in other words, $P(\sigma) = 1$ and $P(\varnothing) = 0$.

Furthermore, for arbitrary disjoint π and π' in $\mathcal{A}$ (and hence in Π), P5.iv entails that $\{\cup\pi\}$, $\{\cup\pi'\}$ are in Π, and with P5.iii it follows that $\{\cup\pi, \cup\pi'\}$ is a member of Π, too; moreover, $\pi \cup \pi'$ is in $\mathcal{A}$ (and thus also in Π) by $\mathcal{A}$ being a set algebra. So we can conclude that $P(\pi \cup \pi') = \varphi^+(B(\pi \cup \pi')) =$, by P4, $= \varphi^+(B(\{\cup\pi, \cup\pi'\})) =$, by P3.ii, $= \varphi^+(F(B(\{\cup\pi\}), B(\{\cup\pi'\}))) =$, by corollary 1, $= \varphi^+(B(\{\cup\pi\})) + \varphi^+(B(\{\cup\pi'\})) =$, by P4, $= \varphi^+(B(\pi)) + \varphi^+(B(\pi')) = P(\pi) + P(\pi')$. In short, finite additivity holds; that is, for arbitrary disjoint π and π' in $\mathcal{A}$, $P(\pi \cup \pi') = P(\pi) + P(\pi')$.

7. I owe this formulation to an anonymous reviewer.

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Overall, we get:

Corollary 3 (Unique probabilistic representation of Π, B, F relative to σ, $\mathcal{A}$). Given a set algebra $\mathcal{A} \subseteq \Pi$ over a partition σ of W in Π, our postulates P1–P5 for Π, B, and F (see sec. 2) entail that:

i) $\langle \mathcal{A}, B, \leq_{[B(\{\varnothing\}),B(\{W\})]}, F\rangle$ is isomorphic, modulo a monotonic and one-to-one function φ^+ for which $\varphi^+(B(\{W\})) = 1$ and $\varphi^+(B(\{\varnothing\})) = 0$, to $\langle \mathcal{A}, P, \leq_{[0,1]}, +_{[0,1]}\rangle$, where $\leq_{[B(\{\varnothing\}),B(\{W\})]}$ is the restriction of the less-than-equals relation of real numbers to $[B(\{\varnothing\}), B(\{W\})]$, P is a *probability measure* on $\mathcal{A}$ (such that $P(\pi) = \varphi^+(B(\pi))$), $\leq_{[0,1]}$ is the restriction of the less-than-equals relation of real numbers to [0, 1], and where $+$ is the *addition function* of real numbers restricted to [0, 1] (as applied to the probabilities of disjoints members of $\mathcal{A}$ in the finite additivity axiom),
ii) where φ^+ is uniquely determined having the properties in i.[8]

Speaking of φ^+ as a uniquely determined *isomorphism* between $\langle \mathcal{A}, B, \leq_{[B(\{\varnothing\}),B(\{W\})]}, F\rangle$ and $\langle \mathcal{A}, P, \leq_{[0,1]}, +_{[0,1]}\rangle$ via φ^+ is justified for the following reasons: φ^+ is defined on all rational degrees of belief $B(\pi)$ in partitions π in $\mathcal{A}$, it is one-to-one on these degrees of belief (by corollary 2), it is onto with respect to all probabilities $P(\pi)$ for π in $\mathcal{A}$ (by $P(\pi) = \varphi^+(B(\pi))$), and it maps the structure given by $\leq_{[B(\{\varnothing\}),B(\{W\})]}$ and F to that given by $\leq_{[0,1]}$ and $+_{[0,1]}$ (by corollary 1). This holds independently of what the logically necessary "sample space" σ in Π and the set algebra $\mathcal{A}$ of subsets of σ that are partitions in Π are like.

At least for all those purposes for which only the formal structure of B, $[B(\{\varnothing\}), B(\{W\})]$, $\leq_{[B(\{\varnothing\}),B(\{W\})]}$, and F restricted to the members of $\mathcal{A}$ is relevant, it follows that rational degrees of belief may be identified (along φ^+) with probabilities in [0, 1] on $\mathcal{A}$ that are ordered by $\leq_{[0,1]}$ and that can be "composed" by means of $+_{[0,1]}$. This holds in the same sense in which, for example, the natural numbers 0, 1, 2, . . . with their less-than-equals relation and successor function may be identified with the isomorphic structure of the set-theoretic von Neumann ordinals $\varnothing$, $\{\varnothing\}$, $\{\varnothing, \{\varnothing\}\}$, . . . and their subset relation and successor-defined-by-union-and-singleton-function: indeed, for all mathematical purposes, 0 may be defined to be $\varnothing$, 1 may be defined as $\{\varnothing\}$, . . . , $\leq$ may be defined to be $\subseteq$, and $+$ may be defined by the union and singleton functions. Accordingly, for all formal purposes, $B(\pi)$

8. At the same time, it is also easy to construct a model for P1–P5 from sec. 2 by defining B from, e.g., the Lebesgue measure P over [0, 1] and by defining $F(x, y) = x + y$ when $x + y \in [0, 1]$ (otherwise let F be undefined). Hence, P1–P5 are of course jointly satisfiable. It is even possible to show that every (finitely additive) probability measure whatsoever can be viewed as a "coarse-graining" of some function B that satisfies P1–P5 for a suitable W, but I will not deal with this here in more detail.

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may be identified with $P(\pi)$, $[B(\{\varnothing\}), B(\{W\})]$ with $[0, 1]$, $\leq_{[B(\{\varnothing\}),B(\{W\})]}$ with $\leq_{[0,1]}$, and F with $+_{[0,1]}$. It is in that sense that probabilism has been justified by the results from above.

In a stricter sense of the term, however, one might argue that probabilism has not been justified: after all, demonstrating that rational degrees of belief can be identified with probabilities for formal-structural purposes falls short of showing that rational degrees of belief are probabilities. Here I defer to the weakly realist kind of position developed at the end of Zynda (2000), who deals with the same topic in the context of probabilistic representation theorems in general. I understand the postulates from section 2 to capture the "realist" part of my proposal—numerical degrees of belief should satisfy P1–P5, and hence, by the results from the current section, they should be representable as subjective probabilities. At the same time, we might be only pragmatically justified in choosing and accepting the framework of subjective probabilities as the codomain of our representation mapping φ^+, because of the usefulness, simplicity, elegance, . . . of probabilities and because the resulting representation as subjective probabilities is uniquely determined (as shown above). However, this does not rule out that, for certain purposes, there might be other salient ways of representing rational degrees of belief. And the decision to represent rational degrees of belief as subjective probabilities "is not wholly justified by the representation theorems alone, but in part on pragmatic or conventional grounds" (Zynda 2000, 68), such as the conventional grounds on which we have chosen to represent $B(\{W\})$ as the real number 1 ($\varphi^+(B(\{W\})) = 1$). In this weakly realist sense, one may take the argument from this section to make a case that rational degrees of belief "are" probabilities, but only in that sense—I do not want to claim the argument above shows more than just that.

4. Conclusion. Whether one aims to emphasize the epistemic, logical, or pragmatic role of rational degrees of belief, so long as the postulates from section 2 are satisfied, the results from section 3 demonstrate that—up to one conventional choice ($\varphi^+(B(\{W\})) = 1$) and restricted to an algebra of partitions—there is a uniquely determined isomorphism φ^+ between the structure of rational degrees of belief $B(\pi)$ and the well-known structure of probabilities $P(\pi)$. For the same reason, at least to the extent to which just these structures are concerned, there should be no harm in identifying them, and any remaining objection to their identification would have to be directed against the structure of rational degrees of belief that is captured by postulates P1–P5 from section 2.

Some features of subjective probability theory appear in new light, once subjective probabilities $P(\pi)$ are understood to correspond via φ^+ (or rather its inverse) to rational degrees of belief $B(\pi)$ in the sense of section 3. In

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particular, according to the current theory, there is nothing special about using 1 as the respective probability of a logical necessity σ (such that σ is a partition of the logical space W): the real number 1 simply serves as a particularly simple conventional choice for how to represent the rational degree of belief $B(\sigma)$ $(=B(\{W\}))$ that may itself well differ from 1. It remains to be seen in future work whether an extension of the framework and postulates from section 2 to conditional rational degrees of belief (say, $B(\pi|\pi')$) might allow one to eliminate that remaining conventional component, and the same extension might be necessary anyway to give a structural justification of diachronic norms on belief update by means of probabilistic conditionalization.

Another noteworthy feature of the results of section 3 is that, while they show that rational degrees of belief may be identified with probabilities on a suitable algebra $\mathcal{A}$ over a suitable partition σ of logical space, they do not entail the existence of a "most inclusive" such algebra $\mathcal{A}$ on a "maximally fine-grained" partition σ in Π. Hence, rational degrees of belief should really be thought of as corresponding to many probability measures, algebras, and partitions of logical space simultaneously—the B function corresponds to a probability measure on each such algebra, but there may be many such algebras that are subsets of the space Π of partitions. From the viewpoint of the agent, any of these partitions σ and algebras $\mathcal{A}$ might become "occurrent" in a particular context. At the same time, since the uniquely determined function φ^+ translates the values of B into probabilities for each such $\mathcal{A}$ and σ, the respective probabilities will be preserved across the different algebras and partitions. For instance, if in the lottery context an agent's subjective probability measure P is given relative to an algebra $\mathcal{A}$ of ticket propositions over the partition σ = {Ticket-1-wins, . . . , Ticket-100-wins, No-lottery-takes-place}, and the same agent's subjective probability measure P' gets determined in a combined lottery-weather context relative to another algebra $\mathcal{A}'$ of combined ticket-weather propositions over the refined partition σ' = {Ticket-1-wins-&-Sunny-tomorrow, Ticket-1-wins-&-Not-sunny-tomorrow, . . . , Ticket-100-wins-&-Sunny-tomorrow, Ticket-100-wins-&-Not-sunny-tomorrow, No-lottery-takes-place-&-Sunny-tomorrow, No-lottery-takes-place-&-Not-sunny-tomorrow}, it follows that $P(\{\text{Ticket-1-wins}\}) = \varphi^+(B(\{\text{Ticket-1-wins}\})) =$, by P4 (partition invariance), $= \varphi^+(B(\{\text{Ticket-1-wins-\&-Sunny-tomorrow, Ticket-1-wins-\&-Not-sunny-tomorrow}\})) = P'(\{\text{Ticket-1-wins-\&-Sunny-tomorrow, Ticket-1-wins-\&-Not-sunny-tomorrow}\})$. Hence, partition invariance guarantees that the multiple probability spaces that correspond to rational degrees of belief cohere with each other. If the different algebras and partitions are interpreted as what the agent might be interested in (focus on, attend to) at a point in time—as suggested in previous sections—partition invariance expresses that the same agent's subjective probability values will not themselves be affected by the agent's interests.

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Future work should throw more light on how exactly a particular algebra $\mathcal{A}$ and partition σ are determined for the purpose of probabilistic representation, but the framework and postulates from section 2 may at least serve as a plausible starting point for that kind of endeavor.

Similarly, it remains to be investigated whether the formal structure of the framework and postulates from section 2 might be reinterpreted in more objective terms, such that ultimately a representation theorem for objective probabilities (e.g., single case objective chances) could be obtained in which $\mathcal{A}$ and σ would reflect "natural kinds" or correspond to "natural hypotheses" about causal structure, counterfactual structure, or the decision-theoretic structure of decision problems or games. (See Skyrms [1977, 1980] and Skyrms, Bell, and Woodruff [1999] for such interpretations of partitions as underlying objective chances.)

Summing up: P3 (the aiming-at-the-content constraint) and P4 (the partition-invariance constraint) may be ascribed crucial roles in the justification of probabilism. While the aiming-at-the-content constraint applies to rational belief more generally, partition invariance turns out to be a powerful invariance property of rational graded belief in particular: it was of major importance for the representation of rational degrees of belief by subjective probabilities that was described in section 3, and it might distinguish rational graded belief from its all-or-nothing counterpart, as explained in section 1. In a formal setting in which the contents of belief are reconstructed as partitions, and in which rational graded belief in any such content derives from rational graded belief in the possibilities at which that content is true, the axiomatic formulation of partition invariance ought to be considered constitutive of rational graded belief. If so, to the extent to which its formal structure is concerned, rational graded and hence partition-invariant belief can be identified with subjective probability.

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