Rounding versus truncation estimates in difference calculations

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Two common methods of approximating difference problems are by rounding or truncating the minuend and the subtrahend before calculating the difference. Since rounding generally gives a better approximation of a number than truncation, one might expect that rounding also generally gives a better approximation in difference problems. We show that intuition to be false.

Introduction

Here is a typical fourth grade homework problem:

1. Estimate 574 – 429 by rounding to the nearest hundreds place.
2. Estimate 574 – 429 by truncating to the hundreds place.
3. Which is closer to the actual answer?
4. Repeat 1 to 3 for 509 – 474.

The answers for 574 – 429 are:

1. Rounding yields 600 – 400 = 200.
2. Truncation yields 500 – 400 = 100.
3. Since the actual answer is 145, truncation is more accurate.

On the other hand, the answers for 509 – 474 are:

1. Rounding yields 500 – 500 = 0.
2. Truncation yields 500 – 400 = 100
3. Since the actual answer is 35, rounding is more accurate.

Some readers may have been assigned problems of this type and wondered about the answer to the question that begs to be asked: which method, rounding or truncation, is better *in general*? Since rounding a single number is more accurate than truncating – with two possibilities of estimation rather than one – it would seem that rounding would be the more accurate method for difference problems as well. The purpose of this paper is to show this intuition is false.

Standard rounding versus truncation

In this section, we use the common ‘half round up’ method of rounding. For example, when rounding to the nearest tens place, the numbers 80-84 round to 80 while the numbers 85-89 round to 90.

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Claim: For difference problems, estimates by rounding and truncating have an equal probability of being more accurate.

We will first illustrate this claim by estimating 'ninety-something minus twenty-something' by both truncating and rounding to the nearest tens digit. If y represents 'ninety-something' and x represents 'twenty-something' then the square of lattice points in Figure 1 represents the 10×10 possible ordered pairs (x, y).

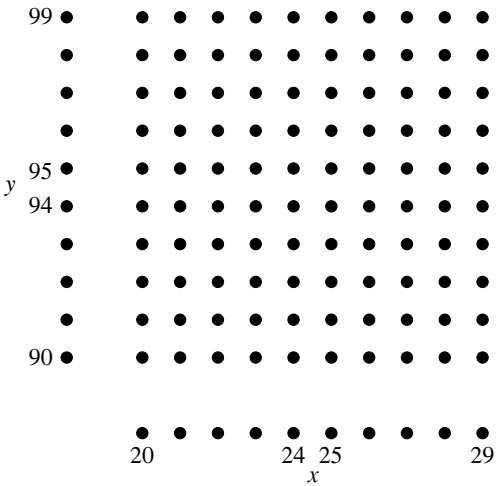


FIGURE 1: The ordered pairs (20, 90) to (29, 99)

Computing the value of the difference $y - x$ for each ordered pair (x, y) results in the 19 possible differences 61 to 79. Ordered pairs with common differences lie on the same line with slope 1, as illustrated in Figure 2.

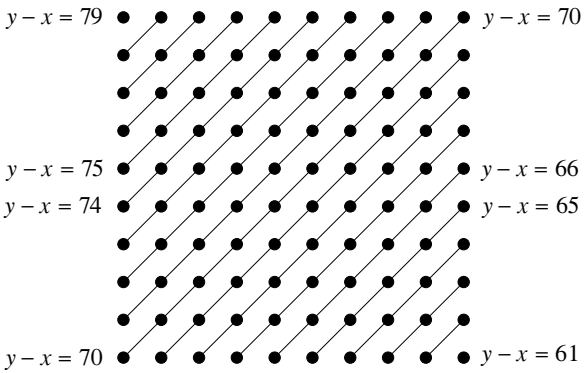


FIGURE 2: Ordered pairs with common differences, from 61 to 79

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When using truncation to estimate any of these differences, y is estimated by $y_T = 90$, while x is estimated by $x_T = 20$. So the truncated difference is $d_T = y_T - x_T = 70$ for each of these 100 values of (x, y).

By contrast, rounding yields four cases, since y is estimated by either 90 or 100, while x is estimated by either 20 or 30. Two of these have identical differences. These differences $d_R = y_R - x_R$ are illustrated in Figure 3.

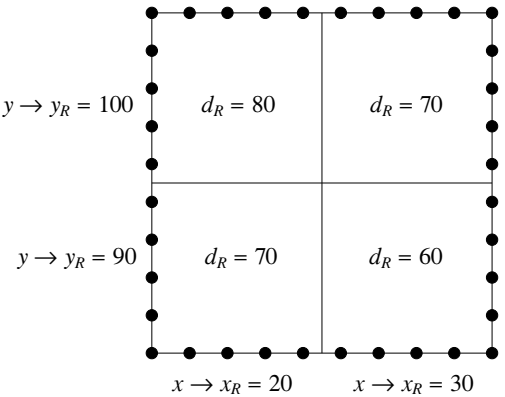


FIGURE 3: The four possible differences obtained by rounding

Now, for each of the possible ordered pairs (x, y), we will compare the actual difference $d_T = y_T - x_T$ with the truncated approximation $d_R = y_R - x_R$ and the rounded approximation $d_R = y_R - x_R$. Notice for the bottom left and top right quadrants of Figure 3, $d_R = d_T$, so the errors of approximation are equal for those 50 ordered pairs. Thus it only remains to study the bottom right quadrant (where $d_T = 70$ and $d_R = 60$) and top left quadrant (where $d_T = 70$ and $d_R = 80$). But comparing the actual differences in Figure 2 with these approximated differences, we see that truncation and rounding yield equal errors for points on the lines $y - x = 65$ and $y - x = 75$. Of the 25 remaining points in each of these two quadrants, exactly half of the actual differences are closer to d_R, while the other half are closer to d_T. This is illustrated in Figure 4.

Consequently, given the problem of finding an arbitrary difference of the form 'ninety-something minus twenty-something', there is an equal likelihood that truncation or rounding is a better approximation than the other.

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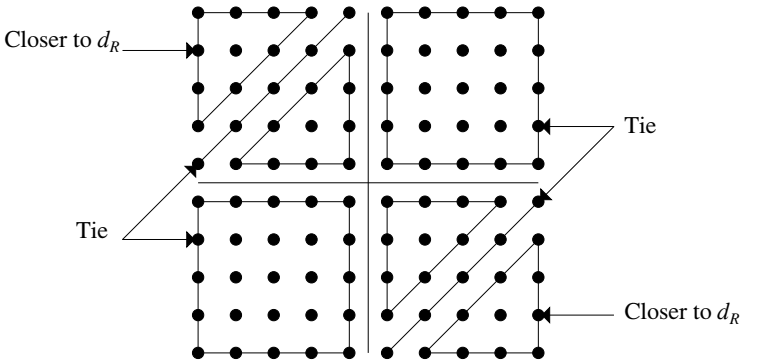


FIGURE 4: It's a tie! For points in the two inner triangles, d_R is closer to d_T.

The general case

Given a difference of two whole numbers (this also works for decimals), the comparison of a rounding approximation and a truncation approximation follows the same form as the above example. In general, we will round and truncate two numbers to the nearest 10^k (in our previous example, $k = 1$). Plotting the $10^k \times 10^k$ lattice points (x, y), points with common difference $y - x$ lie on lines with slope 1 as before. And, as before, there are four quadrants, one for each of the possible rounding estimates of x and y. The bottom left and top right quadrants again satisfy $d_T = d_R$. To determine which estimate is closer to the actual answer for the other two quadrants, consider the distances between the lattice points (x, y) and each of the lines $y - x = d_T + 10^k$, $y - x = d_T$ and $y - x = d_T - 10^k$. This is illustrated in Figure 5.

Those points (x, y) closer to the line $y - x = d_T$ will have a difference $y - x$ better estimated by truncation; those points closer to either $y - x = d_T + 10^k$ or $y - x = d_T - 10^k$ will have a difference better estimated by rounding. Since the number of lattice points in each case is equal, there is an equal probability that each method is closer to the actual answer.

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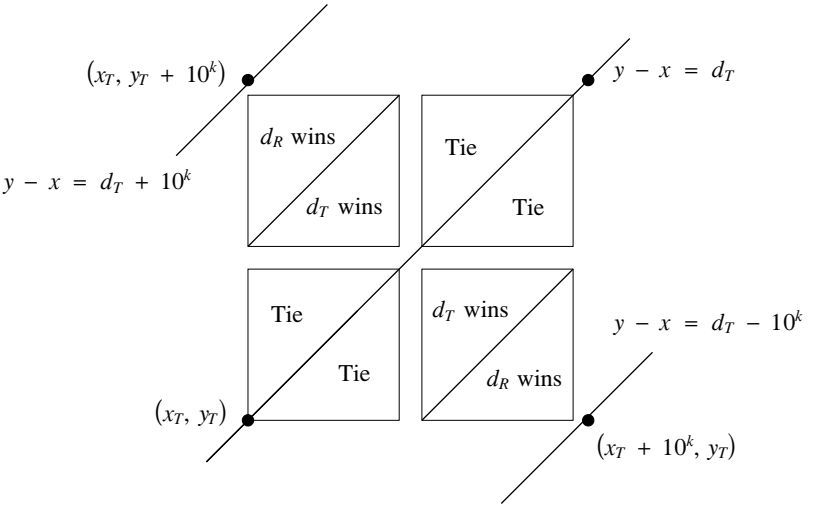


FIGURE 5: Realising the winner via distances to 'difference' lines

Banker's rounding versus truncation

In this section, we use another method of rounding, sometimes called 'banker's rounding', where numbers ending in 5 round to the nearest even digit, instead of always rounding up. For example, when rounding to the nearest tens place, the numbers 80 to 85 round to 80 while the numbers 86 to 89 round to 90.

One might expect the approximation of difference problems using banker's rounding to be better than that of usual rounding, or at least no worse. However, once again, the results are counterintuitive. In fact, banker's rounding approximations of difference problems actually fare worse than those using truncation.

Here we will let d_B denote the estimated difference of the two numbers using banker's rounding; as before, d_T will denote the estimation using truncation.

Claim: For difference problems, estimates by truncating have a greater probability of being more accurate than estimates by banker's rounding.

As before, we will illustrate this using the difference of 2 two-digit numbers. Obviously, banker's rounding estimates only differ from usual rounding estimates for numbers with a units digit of 5. So focusing on the changes in the horizontal and vertical dividing lines in the grid of possible differences is the key. We will also focus on the top left and bottom right quadrants, where rounding and truncating differ.

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There are four cases to consider:

Case I: Both tens digits are odd.

In this case, banker's rounding and usual rounding coincide, so there is no difference from the previous case.

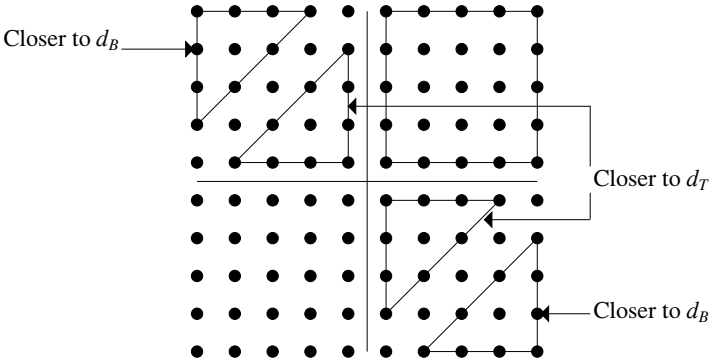


FIGURE 6: Odd – Odd. No change

Case II: Both tens digits are even.

In this case, both of the dividing lines shift. Figure 7 illustrates the corresponding shifting of the triangles containing points for which truncation is more accurate. Since these triangles contain the same number of points as before, there is no change in probability.

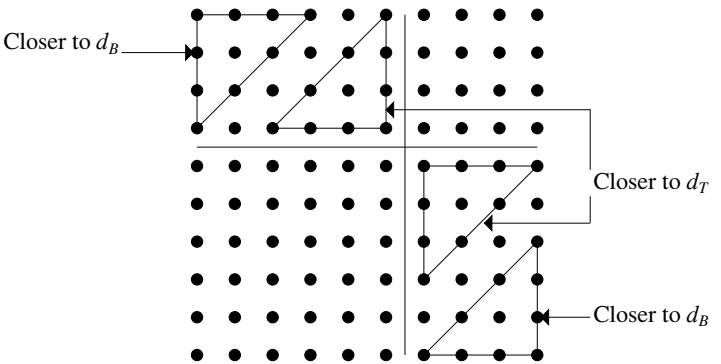


FIGURE 7: Even – Even. No change

Case III: Even tens digit minus odd tens digit.

In this case, the horizontal dividing line is shifted up. The net result is an advantage of truncation over banker's rounding. As Figure 8 illustrates, the points in the triangles for which banker's rounding holds an advantage are unchanged, while the total number of points in the triangles for which truncation holds an advantage increase by 1, from $10 + 10$ to $6 + 15$.

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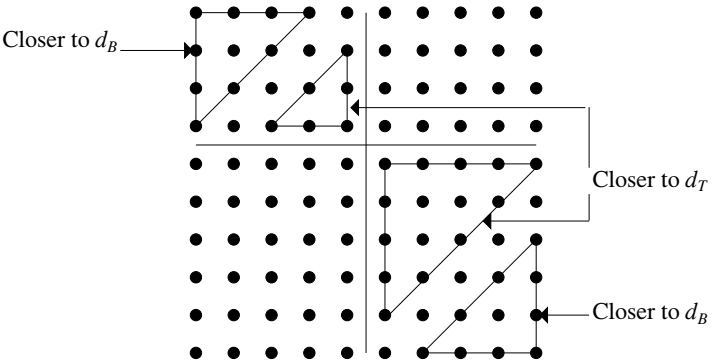


FIGURE 8: Even – Odd. Advantage truncation

Case IV: Odd tens digit minus evens tens digit. In this case, the vertical dividing line is shifted right. As with Case III, the net result is an advantage of truncation over banker's rounding, with truncation gaining an additional point.

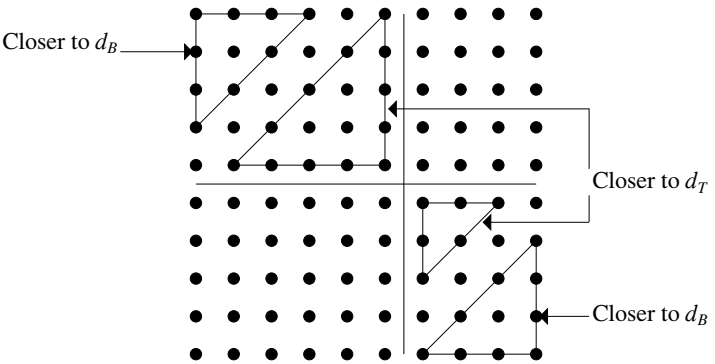


FIGURE 9: Odd – Even. Advantage truncation

Notice that in Cases III and IV, the triangles containing points of advantage for truncation consist of the sum of the triangular numbers 6 and 15, while the points of advantage for banker's rounding consist of the sum of the triangular number 10 with itself. If $t_n = 1 + 2 + \ldots + n$ denotes the nth triangular number, then it is easy to verify that $1 + 2t_n = t_{n-1} + t_{n+1}$. Since the set of points of advantage for truncation always forms two triangles, and shifting one of the horizontal or vertical lines (as in Cases III and IV) results in replacing two triangles each containing t_n points with one containing t_{n-1} points and one containing t_{n+1} points, truncation will always gain exactly one extra point of advantage.

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Summary

Both rounding and truncation estimation problems, such as that at the beginning of this paper, are regularly assigned to young students. It is quite possible that many who assign these problems have not had time to consider whether there is a general advantage of one estimation method over another. If they have had the time, intuition might indicate that rounding would be the more accurate method of estimation. As we have seen, this intuition is, somewhat surprisingly, incorrect.

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