10 Correlation Functions

Correlation observables constitute an essential component of the toolset used by researchers in the study of the dynamics of elementary and nuclear collisions, and for measurements of the properties of the matter produced in these collisions. As such, correlation functions span a wide variety of forms and provide sensitivity to a broad range of phenomena and nuclear matter properties. Although they seemingly take many different forms, correlation functions all share a core definition and have common properties.

We introduce and motivate the notion of correlation function as an extension of the concept of covariance between two random variables in §10.1. The concept of correlation function is, however, not limited to two variables (or fields) and can be readily developed for an arbitrary number of random variables in the form of cumulant functions, discussed in §10.2. Cumulants play a very important role, in particular, in studies of collective properties, such as flow, of the medium produced in high-energy collisions. They are also essential for proper studies of multiparticle correlations.

Correlation functions can be broadly divided into differential and integral correlation functions. Differential correlations can be studied as a function of selected kinematical variables of two or more particles. They may be averaged over all interactions measured for a given type of collision or studied as a function of global event observables, such as the total transverse energy or the charged particle multiplicity measured in a specific kinematic range. The basic definition of inclusive correlation functions, that is, correlation averaged over all events, is introduced in §10.2. Semi-inclusive correlations, measured as a function of global observables, are discussed in §10.3. (Specific examples involving correlation measurements as a function of the relative emission angle and rapidity are discussed in §11.1.1.) The concept of differential correlation functions is extended, in §10.8, to include correlation functions weighted by particle properties. An example of such weighted correlation functions involving transverse momentum deviates is presented in §11.1.5.

Integral correlation functions are introduced in §10.4 in the form of factorial cumulants. They are useful, in particular, for the study of fluctuations of particle multiplicity, net charge, transverse momentum, or the relative yield of production of different particle species, discussed in Chapter 11. Measurements of correlation functions also play a central role in studies of collective flow, particularly in heavy-ion collisions. Flow constitutes a subfield of study of its own that encompasses a wide variety of techniques, introduced here in §11.4.

Correlation functions are subject, like any other observables, to instrumental effects that must be properly accounted for to extract meaningful physics results. Techniques to

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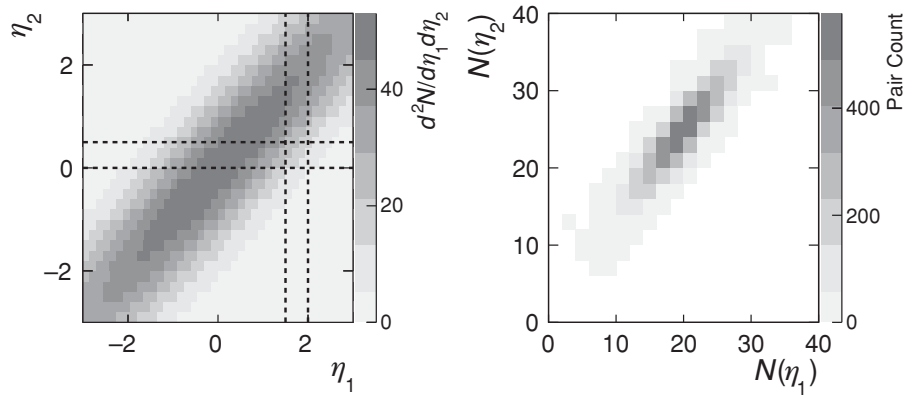


Fig. 10.1 Illustration of the notion of covariance between the particle production yields measured at two points (η_1 and η_2) in momentum space with a simple Gaussian correlation model. The left-hand plot shows the pair yield vs. the pseudorapidities of the particles, average over a large number of events. The right-hand plot displays the number of particles in the wide rapidity bins shown on the left with dashed lines. It reveals that the particle yields in bins η_1 and η_2 are tightly correlated.

account for the finite acceptance of measurements, particle detection efficiency, and various other instrumental effects are discussed in §12.4.

10.1 Extension of the Notion of Covariance

We introduce the notion of **correlation function** on the basis of the covariance of the number of particles detected event-by-event at two distinct points $\vec{p}_1$ and $\vec{p}_2$ in momentum space, as illustrated in Figure 10.1.

Let N_i represent the number of particles produced in the volumes Ω_i, $i = 1, 2$, defined by the ranges $p_{T,i}^{\min} \leq p_{T,i} < p_{T,i}^{\max}$, $\eta_i^{\min} \leq \eta_i < \eta_i^{\max}$, and $\phi_i^{\min} \leq \phi_i < \phi_i^{\max}$ centered at $\vec{p}_i$. Given the stochastic nature of particle production, the yields N_i are expected to fluctuate event-by-event. For a given type of particle (and a specific projectile, target, collision energy, and possibly several other collision system parameters), the yields will have average values $\langle N_i \rangle$ that depend on the cross section of the process considered:

$$\langle N_i \rangle = \int_{\Omega_i} \frac{d^3 N_i}{dp_T d\phi d\eta} \, dp_T \, d\phi \, d\eta. \tag{10.1}$$

Fluctuations about each of these means $\langle N_i \rangle$ are characterized by the variances

$$\mathrm{Var}[N_i] = \langle N_i^2 \rangle - \langle N_i \rangle^2. \tag{10.2}$$

However, it is usually more informative to study the covariance of these two yields:

$$\mathrm{Cov}[N_1, N_2] = \langle N_1 N_2 \rangle - \langle N_1 \rangle \langle N_2 \rangle \tag{10.3}$$

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The covariance $\mathrm{Cov}[N_1, N_2]$ is quite obviously a function of the size of the bins Δ_1 and Δ_2 used to measure the yields N_1 and N_2, respectively, as well as of the coordinates $\vec{p}_1$ and $\vec{p}_2$ at which the particle emission is considered. It is thus natural to introduce the notion of **correlation function**

$$C(\vec{p}_1, \vec{p}_2) = \frac{1}{\Omega_1 \Omega_2} \{\langle N(\vec{p}_1) N(\vec{p}_2)\rangle - \langle N(\vec{p}_1)\rangle \langle N\vec{p}_2)\rangle\}, \tag{10.4}$$

defined in the limit in which the bin sizes Ω_1 and Ω_2 vanish. For finite bin sizes, the ratios $\langle N(\vec{p}_i)\rangle/\Delta_i$ provide estimates $\hat{\rho}_1(\vec{p}_i)$ of the single particle density $\rho_1(\vec{p}_i)$, defined as

$$\rho_1(\vec{p}_i) = \frac{d^3 N_i}{p_T dp_T d\phi d\eta}(\vec{p}_i). \tag{10.5}$$

The term $\langle N(\vec{p}_1) N\vec{p}_2)\rangle$ represents the average number of particle pairs detected jointly, that is, in the same event, at momenta $\vec{p}_1$ and $\vec{p}_2$, in bins of size Δ_1 and Δ_2, respectively. The ratio $\langle N(\vec{p}_1) N\vec{p}_2)\rangle/(\Delta_1 \Delta_2)$ thus provides an estimate $\hat{\rho}_2(\vec{p}_1, \vec{p}_2)$ of the two-particle density, $\rho_2(\vec{p}_1, \vec{p}_2)$, defined as

$$\rho_2(\vec{p}_1, \vec{p}_2) = \frac{d^6 N_{\mathrm{pairs}}}{dp_{T,1} d\phi_1 d\eta_1 dp_{T,2} d\phi_2 d\eta_2}(\vec{p}_1, \vec{p}_2). \tag{10.6}$$

In the limit $\Delta_1, \Delta_2 \to 0$, one thus gets

$$C(\vec{p}_1, \vec{p}_2) = \rho_2(\vec{p}_1, \vec{p}_2) - \rho_1(\vec{p}_1)\rho_1(\vec{p}_2). \tag{10.7}$$

In its most general form, the two-particle correlation function $C(\vec{p}_1, \vec{p}_2)$ is defined as a function of six coordinates, that is, three momentum coordinates for each particle. However, a measurement of correlation function can obviously be reduced to a smaller number of coordinates of interest by integrating (marginalization) over variables that are not of interest. For instance, it is common to study the correlation functions of produced particles as a function of the relative angle $\Delta\phi = \phi_1 - \phi_2$, or the difference in pseudorapidity $\Delta\eta = \eta_1 - \eta_2$, or both, for specific types of particles (e.g., all charge hadrons, positive particles only, or only pions, etc.), and within a specific range of transverse momentum, and for events (i.e., collisions) satisfying specific conditions. We will discuss examples of such correlation functions in §§11.1 and 11.2. However, it is appropriate to first define correlation functions formally without specific recourse to the notion of covariance, and we do so in §10.2.

It is important to realize that while particle yields are by definition nonnegative (i.e., positive or null), the function $C(\vec{p}_1, \vec{p}_2)$ may be positive, null, or even negative. As for covariances, a positive value indicates that a rise of the particle yield at $\vec{p}_1$ is, on average, accompanied by a rise of the yield at $\vec{p}_2$, whereas a negative value corresponds to anticorrelation, so that the rise of the yield at one momentum is accompanied by a decline at the other momentum point. A null value, of course, implies that the two yields, at the given momenta $\vec{p}_1$ and $\vec{p}_2$, are seemingly independent.

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10.2 Correlation Function Cumulants

10.2.1 Multiparticle Densities and Factorial Moments

Let us consider the distribution of particles produced by a beam a and a target b at a specific collisional energy $\sqrt{s}$ in a specific subvolume Ω of the total phase space Ω_{tot}. For convenience, and without loss of generality, we will here denote all particle kinematic variables (e.g., p_x, p_T, η, ϕ, etc.) of a given particle with a single variable y, which we shall use to identify its position in Ω. For simplicity's sake, we will first treat all particles as if they were of the same type or species, and the kinematic variables of different particles will be denoted y_1, y_2, and so on. We will further assume that the probability of finding m particles at specific values $y_1, y_2, \ldots, y_m$, can be described with an **exclusive** continuous probability density function $P_m(y_1, y_2, \ldots, y_m)$. Since the particles are assumed to be of the same species, the PDF $P_m(y_1, y_2, \ldots, y_m)$ will be taken to be fully symmetric in the coordinates $y_1, y_2, \ldots, y_m$. This means that the value of the probability should remain invariant under exchange of any two variables, for instance:

$$P_m(y_1, y_3, y_2, \ldots, y_m) = P_m(y_1, y_2, y_3, \ldots, y_m). \tag{10.8}$$

The next step is to connect the joint-probabilities $P_m(y_1, y_3, y_2, \ldots, y_m)$ to particle densities or invariant cross sections. By construction, inclusive number densities ρ_n yield a sequence of inclusive differential functions:

$$\begin{aligned} \frac{1}{\sigma_{\text{inel}}} d\sigma &= \rho_1(y)\, dy, \\ \frac{1}{\sigma_{\text{inel}}} d^2\sigma &= \rho_2(y_1, y_2)\, dy_1\, dy_2, \end{aligned} \tag{10.9}$$

and so on.

Let us first consider the lowest order $n = 1$. Clearly, the single particle density can be factorized in terms of the average number of particles emitted, $\langle N \rangle$, and the probability of finding a given particle at a specific point y (recall that y here stands for (y, ϕ, p_T). One can then write

$$\rho_1(y) = \langle N \rangle P_1(y). \tag{10.10}$$

The pair density can be likewise factorized into the average number of pairs and the probability of finding particles at y_1 and y_2 (see §10.3 for alternative derivation of the preceding and following relations):

$$\rho_2(y_1, y_2) = \langle N(N-1) \rangle P_2(y_1, y_2). \tag{10.11}$$

Similarly, for higher-order densities, one can write

$$\rho_n(y_1, \ldots, y_n) = \langle N(N-1)\cdots(N-n+1) \rangle P_n(y_1, \ldots, y_n). \tag{10.12}$$

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Integration of densities over the momentum volume Ω thus yields the following important relations:

$$\int_{\Omega} \rho_1(y)\, dy = \langle N \rangle \tag{10.13}$$

$$\int_{\Omega}\int_{\Omega} \rho_2(y_1, y_2)\, dy_1\, dy_2 = \langle N(N-1) \rangle \tag{10.14}$$

$$\cdots \tag{10.15}$$

$$\int \cdots \int_{\Omega} \rho_n(y_1, \ldots, y_n)\, dy_1 \cdots dy_n = \langle N(N-1)\cdots(N-n+1) \rangle \tag{10.16}$$

which provide the average number of particles, the average number of pairs of particles, and the average number of n-tuplets of particles produced in the volume Ω, respectively. The averages $\langle N(N-1)\cdots(N-n+1)\rangle$ are commonly known as **factorial moments** of order n.

10.2.2 Definition of Cumulants

In general, inclusive n-particle densities $\rho_n(y_1, \ldots, y_n)$ are the result of a superposition of several subprocesses. Indeed, although the n particles might be produced by a single and specific subprocess, it is also quite possible that they originate from two or more distinct subprocesses. It is in fact possible that the n particles originate from n distinct and uncorrelated subprocesses. The measured n-tuplets of particles may then feature a broad variety of correlation sources associated with a plurality of dynamic processes. It is thus a common goal of multiparticle production measurements to identify and study these correlated emission as distinct (sub)processes. This is best accomplished by invoking correlation functions known as (factorial) cumulant functions, expressed either in terms of integral correlators or as differential functions of one or more particle coordinates. Cumulants of order m, hereafter noted C_m, are defined as m-particle densities representing the emission (production) of m correlated particles originating from a common production process. The emission of n particles, with $n > m$, can thus be regarded as a superposition (sum) of several processes that together concur to produce a total of n particles. There are obviously several ways to cluster n particles. An n-particle density can then be expressed as a sum of several terms yielding n particles, but each with its own **cluster decomposition** into products of m-cumulants, as illustrated schematically in Figure 10.2.

Particle densities of the two lowest orders can be expressed in terms of correlation functions cumulants as follows:

$$\rho_1(1) = C_1(1) \tag{10.17}$$

$$\rho_2(1,2) = C_1(1)C_1(2) + C_2(1,2). \tag{10.18}$$

For orders 3 and 4, one gets the much lengthier expressions:

$$\begin{aligned}\rho_3(1,2,3) = {} & C_1(1)C_1(2)C_1(3) + C_1(1)C_2(2,3) \\ & + C_1(2)C_2(1,3) + C_1(3)C_2(1,2) + C_3(1,2,3)\end{aligned} \tag{10.19}$$

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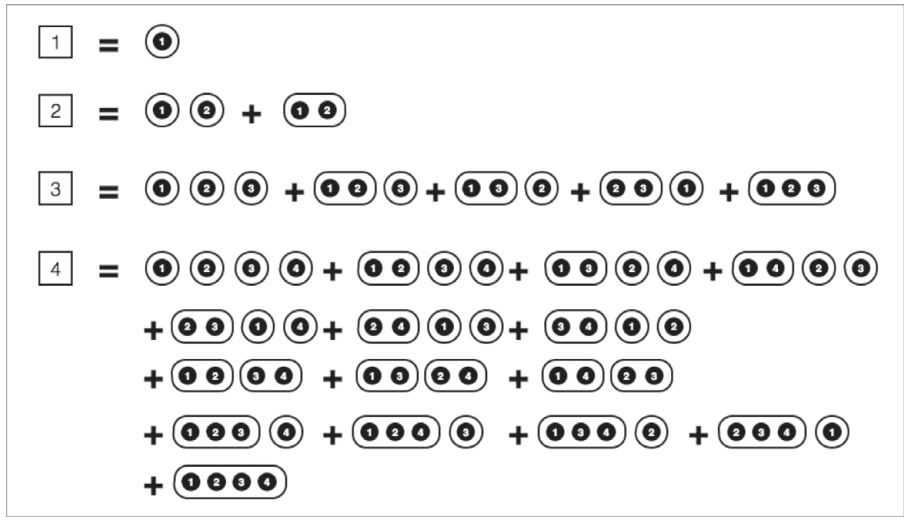


Fig. 10.2 Diagrammatic expression of n-particle densities in terms of cumulants. n-particle densities are represented as squares, while cumulants are denoted by circles and ovals.

$$\begin{aligned}\rho_4(1,2,3,4) = {} & C_1(1)C_1(2)C_1(3)C_1(4) \\ & + C_2(1,2)C_1(3)C_1(4) + C_2(1,3)C_1(2)C_1(4) \\ & + C_2(1,4)C_1(2)C_1(3) + C_2(2,3)C_1(1)C_1(4) \\ & + C_2(2,4)C_1(1)C_1(3) + C_2(3,4)C_1(1)C_1(2) \\ & + C_2(1,2)C_2(3,4) + C_2(1,3)C_2(2,4) \\ & + C_2(1,4)C_2(2,3) + C_3(1,2,3)C_1(4) \\ & + C_3(1,2,4)C_1(3) + C_3(1,3,4)C_1(2) \\ & + C_3(2,3,4)C_1(1) + C_4(1,2,3,4),\end{aligned} \tag{10.20}$$

where we used a shorthand notation indicating the index of the particles rather than kinematical variables y_i. Higher-order densities may be obtained based on the following expression:

$$\begin{aligned}\rho_m(1,\ldots,m) = {} & C_m(1,\ldots,m) + \sum_{\text{perm}} C_1(1)C_{m-1}(2,\ldots,m) \\ & + \sum_{\text{perm}} C_1(1)C_1(2)C_{m-2}(3,\ldots,m) \\ & + \sum_{\text{perm}} C_2(1,2)C_{m-2}(3,\ldots,m) + \cdots + \prod_{i=1}^{m} C_1(i)\end{aligned} \tag{10.21}$$

where "perm" indicates permutations of all particle indexes yielding distinct terms. Theoretically, cumulants naturally arise as a byproduct of calculations of the cross section of specific processes yielding specific particle multiplicities. They can be calculated, in particular, on the basis of cumulant generating functions, such as those introduced in §2.13 and

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discussed in more details in §10.4 (see also [56] and references therein). Experimentally, the extraction of cumulants is not as direct. They are not readily available experimentally as **raw** measurements but must be derived from (efficiency corrected) measurements of n-particle densities. One finds cumulants of order n must be recursively calculated from lower order cumulants and densities. For instance, from Eqs. (10.17) and (10.18), one gets

$$C_2(1,2) = \rho_2(1,2) - \rho_1(1)\rho_1(2), \tag{10.22}$$

where, once again, we used a shorthand notation indicating the index of the particles rather than kinematical variables y_i. Inserting the preceding expression for C_2 into Eq. (10.19), and accounting for particle indices, one gets

$$C_3(1,2,3) = \rho_3(1,2,3) - \sum_{(3)} \rho_1(1)\rho_2(2,3) + 2\rho_1(1)\rho_1(2)\rho_1(3). \tag{10.23}$$

The symbol (3) is used to indicate that the sum is carried over three permutations of the indices.[1] The fourth-order cumulant is obtained by substitution of second- and third-order cumulants into Eq. (10.20). One gets

$$\begin{aligned} C_4(1,2,3,4) = \rho_4(1,2,3,4) &- \sum_{(4)} \rho_1(1)\rho_3(2,3,4) \\ &- \sum_{(3)} \rho_2(1,2)\rho_2(3,4) + 2\sum_{(6)} \rho_1(1)\rho_1(2)\rho_2(3,4) \\ &- 6\rho_1(1)\rho_1(2)\rho_1(3)\rho_1(4), \end{aligned} \tag{10.24}$$

where the symbols (q) indicate the number of indices permutations over which the sums are taken. The preceding expressions for cumulants C_2, C_3, and C_4 are illustrated diagrammatically in Figure 10.3. Expressions for higher-order cumulants can be obtained recursively or from generic relations such as those provided in ref. [123].

In closing this section, it is important to reiterate that cumulants may be negative as well as positive, or null. Indeed, while the n-densities $\rho_n(1,\ldots,n)$ are by definition either positive or null, their combinations into cumulants may yield negative values as well as positive or null values. As for covariances, positive values indicate the yields at coordinates $y_1,\ldots,y_m$ are correlated, that is, collectively grow or decrease together, while negative values imply that upward fluctuations of the yields at some coordinate y_i are on average accompanied by downward fluctuations at other points $y_{j\neq i}$.

10.2.3 Cumulants Scaling with Source Multiplicity

The cumulants $C_n(y_1,\ldots,y_n)$ feature a simple scaling property for collision systems consisting of a superposition of m_s independent (but otherwise identical) subsystems. To demonstrate this property, consider, for instance, a collision of two large nuclei (A–A collisions) at a specific energy, and let us assume that it can be reduced, to first-order approximation, to a superposition of m_s proton-proton (p–p) interactions, which each produce clusters consisting of n correlated particles. Let us further assume that the production of

[1] Permutations are $\rho_1(1)\rho_2(2,3)$, $\rho_1(2)\rho_2(1,3)$, and $\rho_1(3)\rho_2(1,2)$.

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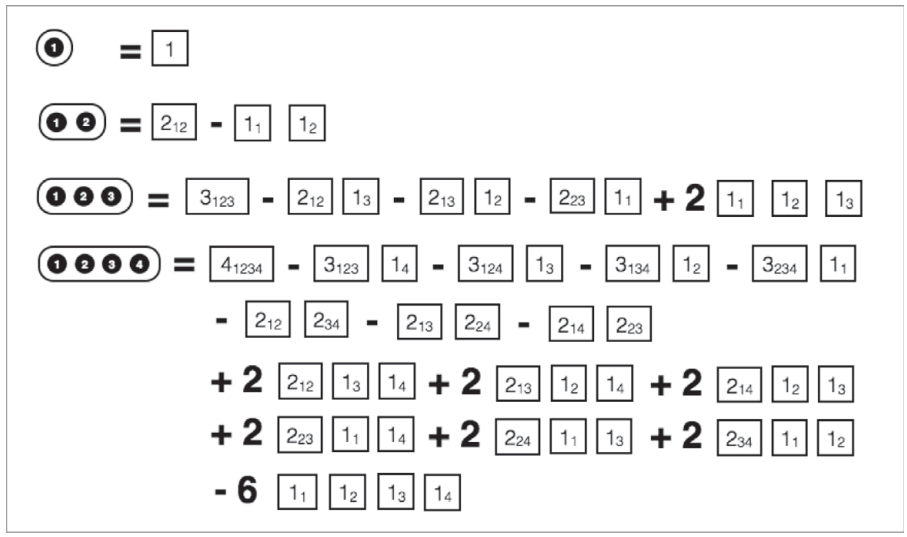


Fig. 10.3 Diagrammatic expression of cumulants in terms of particle densities. Cumulants are denoted by circles and ovals, and densities by squares. Sub-labels indicate particle indices.

such clusters in p–p may be described by cumulants C_n^{pp}. At a given impact parameter, A–A collisions should involve an average of $\langle m_s \rangle$ p–p interactions. This number m_s is obviously anticipated to fluctuate collision by collision, but for a given value of m_s, one expects that the number of clusters of correlated particles of size n should be, on average, m_s times larger than in p–p collisions. The n-cumulant for A–A collisions, at fixed m_s, may thus be written

$$C_n^{AA}(y_1, \ldots, y_n | m_s) = m_s C_n^{\mathrm{pp}}(y_1, \ldots, y_n) \tag{10.25}$$

Given that m_s fluctuates event by event, averaging over all A–A collisions consequently yields

$$C_n^{AA}(y_1, \ldots, y_n) = \langle m_s \rangle C_n^{\mathrm{pp}}(y_1, \ldots, y_n) \tag{10.26}$$

for A–A collisions consisting of a superposition of independent and unmodified p–p collisions, and such that produced particles do not interact with one another.

The total multiplicity of particles produced in A–A collisions consisting of m_s independent and unmodified p–p collisions also features the same simple scaling with m_s. Indeed, if the m_s p–p collisions are independent and unmodified, the average multiplicity obtained in A–A for a given (fixed) value of m_s should simply be the product of m_s by the average particle multiplicity produced in p–p, which we can thus write

$$\rho_1^{AA}(y) = m_s \rho_1^{\mathrm{pp}}(y) \tag{10.27}$$

and

$$\langle n \rangle_{AA} = m_s \langle n \rangle_{\mathrm{pp}}. \tag{10.28}$$

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This simple property arises because $\rho_1(y) = C_1(y)$ and the scaling of the first cumulant $C_1^{AA}(y_1) = m_s C_1^{\rm pp}(y_1)$ is trivial. The treatment of pairs, triplets, and higher n-tuplets is, however, more complicated because higher-order densities, ρ_n, involve sums of several terms with varied products of cumulants.

Let us first consider pairs of particles. In an A–A collision consisting of m_s independent p–p interactions, one can form m_s times the pairs from individual p–p collisions. But one can also mix particles from different p–p interactions. Since there are $m_s(m_s - 1)$ ways of doing that, one can write

$$\rho_2^{AA}(y_1, y_2) = m_s \rho_2^{\rm pp}(y_1, y_2) + m_s(m_s - 1)\rho_1^{\rm pp}(y_1)\rho_1^{\rm pp}(y_2). \tag{10.29}$$

Note that one can obtain the same result using the cumulant decomposition shown in Eq. (10.22), as follows:

$$\begin{aligned}\rho_2^{AA}(y_1, y_2) &= C_1^{AA}(y_1)C_1^{AA}(y_2) + C_2^{AA}(y_1, y_2) \\ &= m_s^2 C_1^{\rm pp}(y_1)C_1^{\rm pp}(y_2) + m_s C_2^{\rm pp}(y_1, y_2) \\ &= m_s^2 \rho_1^{\rm pp}(y_1)\rho_1^{\rm pp}(y_2) + m_s\left[\rho_2^{\rm pp}(y_1, y_2) - \rho_1^{\rm pp}(y_1)\rho_1^{\rm pp}(y_2)\right] \\ &= m_s(m_s - 1)\rho_1^{\rm pp}(y_1)\rho_1^{\rm pp}(y_2) + m_s\rho_2^{\rm pp}(y_1, y_2).\end{aligned} \tag{10.30}$$

At fixed value of m_s, integration over y_1 and y_2 yields

$$\langle n(n-1)\rangle_{AA} = m_s\langle n(n-1)\rangle_{\rm pp} + m_s(m_s - 1)\langle n\rangle_{\rm pp}^2. \tag{10.31}$$

For large m_s, the scaling of the number of pairs produced in A–A is thus dominated by the term in $m_s(m_s - 1)$, which involves uncorrelated, combinatorial pairs from particles produced by different p–p collisions.

Based on the assumption that A–A collisions consist of m_s independent p–p interaction, it might be tempting to seek a measure of particle correlation in A–A by writing

$$Z_2^{AA}(y_1, y_2) = \rho_2^{AA}(y_1, y_2) - k\rho_1^{\rm pp}(y_1)\rho_1^{\rm pp}(y_2), \tag{10.32}$$

where the constant k is adjusted so the maximum of the product $\rho_1^{\rm pp}(y_1)\rho_1^{\rm pp}(y_2)$ matches the minimum of $\rho_2^{AA}(y_1, y_2)$. A particular application of this approach is known, in the recent literature, as zero yield at minimum (ZYAM) approximation [16]. However, it should be clear from the preceding discussion that although $Z_2(y_1, y_2)$, and in particular the ZYAM approximation, do provide a technique to assess whether $\rho_2^{AA}(y_1, y_2)$ differs from what is expected from a trivial scaling, it does NOT constitute a logically consistent measure of two particle correlations in A–A collisions. Only $C_2^{AA}(y_1, y_2)$ defined by Eq. (10.22) does, although there are several different ways of expressing and measuring C_2, which we discuss in the next section.

Let us next consider the scaling of triplets. We use the decomposition (10.19) and our shorthand notation to write

$$\begin{aligned}\rho_3^{AA}(1, 2, 3) = {} & C_1^{AA}(1)C_1^{AA}(2)C_1^{AA}(3) \\ & + C_1^{AA}(1)C_2^{AA}(1, 2) + C_1^{AA}(2)C_2^{AA}(1, 3) \\ & + C_1^{AA}(3)C_2^{AA}(2, 3) + C_3^{AA}(1, 2, 3).\end{aligned} \tag{10.33}$$

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We next apply the scaling property (10.25) and the expressions (10.22) and (10.23) to get

$$\begin{aligned}\rho_3^{AA}(1,2,3) &= m_s^3 C_1^{\mathrm{pp}}(1) C_1^{\mathrm{pp}}(2) C_1^{\mathrm{pp}}(3) \\ &\quad + m_s^2 \sum_{\text{perms}} C_1^{\mathrm{pp}}(1) C_2^{\mathrm{pp}}(2,3) + m_s C_3^{\mathrm{pp}}(1,2,3) \\ &= \left(m_s^3 - m_s^2 + 2m_s\right) \rho_1^{\mathrm{pp}}(1) \rho_1^{\mathrm{pp}}(2) \rho_1^{\mathrm{pp}}(3) \\ &\quad + \left(m_s^2 - m_s\right) \sum_{\text{perms}} \rho_1^{\mathrm{pp}}(1) \rho_2^{\mathrm{pp}}(2,3) + m_s \rho_3^{\mathrm{pp}},\end{aligned} \tag{10.34}$$

where the sums are carried over distinct permutations of the three particle coordinates. Integration over the coordinates y_1, y_2, and y_3 yields

$$\begin{aligned}\langle n(n-1)(n-2)\rangle_{AA} &= \left(m_s^3 - m_s^2 + 2m_s\right) \langle n\rangle_{\mathrm{pp}} \\ &\quad + 3\left(m_s^2 - m_s\right) \langle n(n-1)\rangle_{\mathrm{pp}} \langle n\rangle_{\mathrm{pp}} \\ &\quad + m_s \langle n(n-1)(n-2)\rangle_{\mathrm{pp}}\end{aligned} \tag{10.35}$$

The average number of triplets in A–A collisions is thus determined mostly by combinatorics and essentially scales as $m_s^3 \langle n\rangle_{\mathrm{pp}}^3$, while the role of correlated triplets (from individual p–p collisions) is suppressed by m_s^2. By extension, we conclude that the average number of n-tuplets in A–A collisions scales as the nth power of m_s times the average multiplicity measured in p–p interactions, and the influence of truly correlated n-tuplets is suppressed by a factor m_s^{n-1}.

10.2.4 Normalized Cumulants and Normalized Factorial Moments

It is convenient to divide the densities ρ_n and cumulants C_n by products of one-particle densities. This leads to the definition of **normalized inclusive densities** and **normalized cumulants** as follows:

$$r_n(y_1, \ldots, y_n) = \frac{\rho_n(y_1, \ldots, y_n)}{\rho_1(y_1) \cdots \rho_1(y_n)} \tag{10.36}$$

$$R_n(y_1, \ldots, y_n) = \frac{C_n(y_1, \ldots, y_n)}{\rho_1(y_1) \cdots \rho_1(y_n)} \tag{10.37}$$

The use of ratios of factorial moments $\langle N(N-1)\cdots(N-n+1)\rangle$ to the nth power of the average particle multiplicity $\langle N\rangle$ is also common:

$$f_n = \frac{\langle N(N-1)\cdots(N-n+1)\rangle}{\langle N\rangle^n} \tag{10.38}$$

Unfortunately, there is no universally agreed upon notation for either of these quantities. Indeed, while the quantities are unique and well defined, the labels or notations used to describe them typically vary from one author to the next. The functions r_n and R_n are often referred to as **reduced densities** and **reduced cumulants**, respectively. Likewise, the **normalized factorial moments** are often called **reduced factorial moments**. In this text, we will use **normalized** as the qualifier.

It is interesting to consider the scaling behavior of the normalized cumulants $R_n(y_1, \ldots, y_n)$ for systems consisting of a superposition of m identical subprocesses. Based

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on the scaling property (10.25), one gets

$$R_n^{(m)}(y_1,\ldots,y_n) = \frac{C_n^{(m)}(y_1,\ldots,y_n)}{\rho_1^{(m)}(y_1)\cdots\rho_1^{(m)}(y_n)} = \frac{1}{m^{n-1}} R_n^{(1)}(y_1,\ldots,y_n). \tag{10.39}$$

The normalized n-cumulant of a composite system consisting of m (identical) subsystems scales inversely as the power $m^{(n-1)}$ times the n-cumulant of the subsystems. The normalized cumulants $R_n^{(m)}$ are consequently said to be **diluted** by a power m^{n-1} relative to the subsystems' normalized cumulants $R_n^{(1)}$. This dilution stems from the fact that the cumulant of order n scales as the number of distinct sources or subsystems while the denominator is proportional to m^n. For instance, the strength of a two-particle cumulant measured in A–A collisions consisting of m sources is m times the strength of the two-particle cumulant of each of these sources while the total number of pairs scales as m^2. The ratio of C_2 by $\rho_1 \otimes \rho_1$ thus scales as $1/m$. This implies that, in general, measurements of correlation functions in large colliding systems require a level of statistical precision commensurate with this dilution effect. Obviously, the impact of combinatorial terms increases as powers of m. Measurements of higher-order cumulants thus require very large datasets.

It is also interesting to consider the scaling behavior of the normalized factorial moments f_n defined by Eq. (10.38). The scaling of the number of n-tuplets involves a combination of several terms. We have seen, however, from Eqs. (10.31) and (10.35) that the scaling is dominated by combinatorial effects that scale as m^n. The normalized factorial moments f_n are thus expected to converge toward unity for increasing values of the number of subsystems m contributing to an A–A collision.

We will show in §12.4 that the normalized densities r_n, the normalized cumulants R_n, and the normalized factorial moments all share the property of being robust under particle losses associated with particle detection efficiencies. We thus recommend their use, experimentally, against other observables.

It is useful to also note that a simple relationship exists between the normalized densities r_n and the normalized R_n. Indeed, based on the definitions of the normalized densities and normalized cumulants, one finds

$$r_2(1,2) = 1 + R_2(1,2) \tag{10.40}$$

$$r_3(1,2,3) = 1 + \sum_{(3)} R_2(1,2) + R_3(1,2,3) \tag{10.41}$$

$$r_4(1,2,3,4) = 1 + \sum_{(6)} R_2(1,2) + \sum_{(3)} R_2(1,2)R_2(3,4) + \sum_{(3)} R_3(1,2,3) + R_4(1,2,3,4) \tag{10.42}$$

$$r_5(1,2,3,4,5) = 1 + \sum_{(10)} R_2(1,2) + \sum_{(15)} R_2(1,2)R_2(3,4) + \sum_{(10)} R_3(1,2,3)R_2(4,5) + \sum_{(10)} R_3(1,2,3) + \sum_{(5)} R_4(1,2,3,4) + R_5(1,2,3,4,5), \tag{10.43}$$

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with similar expressions for higher-order densities. Sums are to be carried on the indicated number, (m), of permutations of the particle indices.

10.2.5 Particle Probability Densities

The integration (10.13) of particle densities $\rho_n(y_1, \ldots, y_n)$ over the momentum volume Ω provides a natural and convenient normalization to define particle probability densities:

$$P_n(y_1, \ldots, y_n) = \frac{\rho_n(y_1, \ldots, y_n)}{\langle N(N-1)\cdots(N-n+1)\rangle}, \tag{10.44}$$

which expresses the probability (per unit volume) of finding n particles jointly, or simultaneously, at coordinates $y_1, \ldots, y_n$. It is also interesting to consider the normalization of these probability densities by products of single particle probability densities $P_1(y)$, as follows:

$$q_n(y_1, \ldots, y_n) = \frac{P_n(y_1, \ldots, y_n)}{P_1(y_1)\cdots P_1(y_n)}. \tag{10.45}$$

The functions $q_n(y_1, \ldots, y_n)$ equal unity if the production of particles 1 to n is statistically independent, that is, if the probability densities $P_n(y_1, \ldots, y_n)$ factorize into products $P_1(y_1)\cdots P_1(y_n)$. Substituting the expression (10.44) for n-particles and single particle densities into the expression of normalized densities, one gets

$$r_n(y_1, \ldots, y_n) = \frac{\langle N(N-1)\cdots(N-n+1)\rangle}{\langle N\rangle^n} q_n(y_1, \ldots, y_n), \tag{10.46}$$

which tells us that correlated particle production occurs in part because of genuine correlations, that is, for $q_n(y_1, \ldots, y_n) \neq 1$ and in part because of simple multiplicity fluctuations, that is, for $\langle N(N-1)\cdots(N-n+1)\rangle / \langle N\rangle^n \neq 1$. Evidently, $q_n \neq 1$ is required, no matter what, to yield nonvanishing normalized cumulants R_n. As an example, consider that the normalized two-particle cumulant may be written

$$R_2(y_1, y_2) = \frac{\langle N(N-1)\rangle}{\langle N\rangle^2} q_2(y_1, y_2) - 1. \tag{10.47}$$

The strength of two-particle correlations is thus determined both by the function $q_2(y_1, y_2)$ and the amplitude of multiplicity fluctuations, which yield $\langle N(N-1)\rangle / \langle N\rangle^2 \neq 1$.

10.3 Semi-inclusive Correlation Functions

Elementary hadron collision processes are typically rather complex and lead to stochastic fluctuations of the produced particle multiplicity, m. The probability P_m of encountering an event with a specific value of multiplicity m depends on the cross section of the process that produces the m particles relative to processes that lead to other values of multiplicity:

$$P_m = \sigma_m / \sum_{m'} \sigma_{m'}. \tag{10.48}$$

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Given the laws of momentum and energy conservation, charge conservation, and other quantum numbers, it is reasonable to expect that the n-particle densities associated with collisions that produce different multiplicities, m, might be quite different. It is thus of interest to consider n-particle densities for collisions that produce multiplicities m explicitly for "all" values of m. This leads to the notion of **semi-inclusive** n-densities and n-cumulants measured for specific values of m. For simplicity's sake, we restrict our discussion to single- and two-particle densities only, but the notions introduced here can be extended straightforwardly to all orders n.

As in prior sections, we use the symbol y_i to represent all momentum coordinates of a particle (i.e., transverse momentum, rapidity, and azimuthal angle) in aggregate. But, again for simplicity's sake and without loss of generality, one may also regard the variables y_i as representing rapidity variables exclusively.

Single- and two-particle density distribution, expressed as a function of y_i, may be written

$$\rho_1^{(m)}(y) = \frac{1}{\sigma_m}\frac{d\sigma_m}{dy} \tag{10.49}$$
$$\rho_2^{(m)}(y_1, y_2) = \frac{1}{\sigma_m}\frac{d^2\sigma_m}{dy_1 dy_2}.$$

The inclusive single- and two-particle densities are averages of the semi-inclusive distributions determined by the relative probabilities of these processes:

$$\rho_1(y) = \sum_m P_m \rho_1^{(m)}(y) \tag{10.50}$$
$$\rho_2(y_1, y_2) = \sum_m P_m \rho_2^{(m)}(y_1, y_2).$$

The two-cumulants of the semi-inclusive processes of multiplicity m are by definition

$$C_2^{(m)}(y_1, y_2) = \rho_2^{(m)}(y_1, y_2) - \rho_1^{(m)}(y_1)\rho_1^{(m)}(y_2). \tag{10.51}$$

It is convenient to define density deviates $\Delta\rho^{(m)}(y_i) = \rho_1^{(m)}(y_i) - \rho_1(y_i)$, and introduce two new functions, noted $C_S(y_1, y_2)$ and $C_L(y_1, y_2)$, defined as

$$C_S(y_1, y_2) = \sum_m P_m C_2^{(m)}(y_1, y_2) \tag{10.52}$$

$$C_L(y_1, y_2) = \sum_m P_m \Delta\rho^{(m)}(y_1)\Delta\rho^{(m)}(y_2). \tag{10.53}$$

It is then straightforward to verify (see Problem 10.2) that the inclusive two-cumulant $C_2(y_1, y_2)$ may be written

$$C_2(y_1, y_2) = C_S(y_1, y_2) + C_L(y_1, y_2). \tag{10.54}$$

The function $C_S(y_1, y_2)$ corresponds to the average of the semi-inclusive two-cumulants and as such is quite sensitive to the actual collision dynamics, while the term $C_L(y_1, y_2)$ arises from the mixing of different event topologies (multiplicity) in the two-cumulant calculation and has consequently little to do with actual particle correlations. The label S

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and L have a historical origin and are often misleadingly referred to as "short-" and "long-" range correlation terms.

Given the expression (10.52), a normalized form of C_S may be defined as

$$R_S(y_1, y_2) = \frac{C_S(y_1, y_2)}{\sum_m P_m \rho_1^{(m)}(y_1)\rho_1^{(m)}(y_2)} = \frac{\sum_m P_m \rho_2^{(m)}(y_1, y_2)}{\sum_m P_m \rho_1^{(m)}(y_1)\rho_1^{(m)}(y_2)} - 1. \tag{10.55}$$

Analogous expressions may be derived for three- or more particle correlations [128].

10.4 Factorial and Cumulant Moment-Generating Functions

We saw in §2.10 that it is useful to introduce the notions of moment-generating function and characteristic function to facilitate the calculation of the moments of probability density functions. Indeed, recall that the moment-generating function

$$M_x(t) = E\left[e^{tx}\right] = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx, \qquad t \in \mathrm{R} \tag{10.56}$$

yields moments of the function $f(x)$ at all orders:

$$\mu_k' = \frac{d^k}{dt^k} E\left[e^{tx}\right]\Big|_{t=0} = \frac{d^k}{dt^k} M_x(t)\Big|_{t=0}. \tag{10.57}$$

It is also convenient to introduce generating functions for factorial moments and for cumulant moments. We derive expressions for these moments in the following paragraphs. Examples of application are discussed later in this section.

The probability P_m of a semi-inclusive process yielding m particles is determined by its cross-section σ_m relative to the total inelastic cross section

$$P_m \equiv \frac{\sigma_m}{\sum_m \sigma_m} = \frac{\sigma_m}{\sigma_{\text{inel}}}. \tag{10.58}$$

Inclusive densities of order n may consequently be written

$$\rho_n(y_1, \ldots, y_n) = \sum_m P_m \rho_n^{(m)}(y_1, \ldots, y_n) \tag{10.59}$$

where the functions $\rho_n^{(m)}(y_1, \ldots, y_n)$ are n-particle densities for processes that produce exactly m particles ($m \geq n$). Integration of the single density $\rho_1^{(m)}(y)$ over the fiducial range y is by construction equal to the multiplicity m:

$$\int_\Omega \rho_1^{(m)}(y)\, dy = m. \tag{10.60}$$

Integration of the inclusive density over the range Ω consequently yields

$$\int_\Omega \rho_1(y)\, dy = \sum_m P_m \int_\Omega \rho_1^{(m)}(y)\, dy = \sum_m P_m m = \langle m \rangle. \tag{10.61}$$

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Likewise, integration of two-particle densities $\rho_2^{(m)}(y_1, y_2)$ yields $m(m-1)$, which is the number of particle pairs one can form with m particles. And for higher-order densities, one gets

$$\int_\Omega \rho_n^{(m)}(y_1, \ldots, y_n)\, dy_1 \cdots dy_n = m(m-1)\cdots(m-n+1). \tag{10.62}$$

Integration of inclusive functions consequently yield the averages

$$\tilde{F}_n \equiv \int_\Omega \rho_n(y_1, \ldots, y_n)\, dy_1 \cdots dy_n \tag{10.63}$$

$$= \sum_m P_m \int_\Omega \rho_n^{(m)}(y_1, \ldots, y_n)\, dy_1 \cdots dy_n \tag{10.64}$$

$$= \sum_m P_m m(m-1)\cdots(m-n+1) \tag{10.65}$$

$$= \langle m(m-1)\cdots(m-n+1)\rangle \tag{10.66}$$

$$\equiv \langle m^{[n]}\rangle, \tag{10.67}$$

where we have introduced the alternative notation $\langle m^{[n]}\rangle$ for (nonnormalized) factorial moments. The aforementioned development suggests the introduction of a moment-generating function as follows:

$$G(z) = 1 + \sum_{n=1}^{\infty} \frac{z^n}{n!} \tilde{F}_n \tag{10.68}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{z^n}{n!} \int_\Omega \rho_n(y_1, \ldots, y_n)\, dy_1 \cdots dy_n \tag{10.69}$$

Given a generating function $G(z)$, the moments $\tilde{F}_n$ may then be obtained from (see Problem 10.4)

$$\tilde{F}_n = \left.\frac{d^n G(z)}{dz^n}\right|_{z=0} \tag{10.70}$$

We next use Eq. (10.65) to derive expressions for the probabilities P_n. Assuming there is a value $n = N$ beyond which all probabilities P_n vanishes, one can write

$$P_N = \frac{\tilde{F}_N}{N!} \tag{10.71}$$

since terms in $P_{n<N}$ cannot contribute to $\tilde{F}_N$ and by hypothesis, there are no terms with $n > N$. The $(N-1)$th factorial moment may be written

$$\tilde{F}_{N-1} = P_{N-1}(N-1)! + P_N \frac{N!}{1!} \tag{10.72}$$

from which we find

$$P_{N-1} = \frac{1}{(N-1)!}\left\{\tilde{F}_{N-1} - \frac{\tilde{F}_N}{1!}\right\} \tag{10.73}$$

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Proceeding recursively, one finds

$$P_n = \frac{1}{n!}\sum_{k=0}^{N-n}(-1)^k\frac{\tilde{F}_{k+n}}{k!} \quad \text{for } n = 0, 1, \ldots, N. \tag{10.74}$$

Introducing factorial cumulants

$$f_n = \int_\Omega dy_1 \cdots \int_\Omega dy_n C_n(y_1, \ldots, y_n) \tag{10.75}$$

where $C_n(y_1, \ldots, y_n)$ are cumulants of order n, one can show that that factorial moments $\tilde{F}_n$ of order $n \leq 5$ may be written

$$\begin{aligned}
\tilde{F}_1 &= f_1 \\
\tilde{F}_2 &= f_2 + f_1^2 \\
\tilde{F}_3 &= f_3 + 3f_2f_1 + f_1^3 \\
\tilde{F}_4 &= f_4 + 4f_3f_1 + 3f_2^2 + 6f_2f_1^2 + f_1^4 \\
\tilde{F}_5 &= f_5 + 5f_4f_1 + 10f_3f_2 + 10f_3f_1^2 + 15f_2^2f_1 + 10f_2f_1^3 + f_1^5
\end{aligned} \tag{10.76}$$

and for higher moments,

$$\tilde{F}_n = n!\sum_{\{l_i\}_n}\prod_{j=1}^{n}\left(\frac{f_j}{j!}\right)^{l_j}\frac{1}{l_j!} \tag{10.77}$$

where the summation is done for permutations satisfying $\sum_{l=1}^{q} il_i = q$.

Factorial cumulant generating functions can be introduced similarly to the generating functions of basic cumulants discussed in §2.13.1. One writes

$$\ln G(z) = \langle n\rangle z + \sum_{k=2}^{\infty}\frac{z^k}{k!}f_k \tag{10.78}$$

and consequently,

$$f_n = \left.\frac{d^n \ln G(z)}{dz^n}\right|_{z=0} \tag{10.79}$$

The preceding equations are useful if the generating function $G(z)$ can be calculated directly. For instance, for a Poisson distribution [128],

$$P_n = e^{-\langle n\rangle}\frac{\langle n\rangle^n}{n!}, \tag{10.80}$$

the generating function is

$$G(z) = \sum_{n=0}^{\infty}P_n(1+z)^n = e^{-\langle n\rangle}\sum_{n=0}^{\infty}\frac{\langle n\rangle^n}{n!}(1+z)^n \tag{10.81}$$

$$= \exp\left(\langle n\rangle z\right). \tag{10.82}$$

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Equation (10.70) then yields

$$\tilde{F}_m = \langle n \rangle^m. \tag{10.83}$$

One can also verify from (10.79) that $f_1 = \langle n \rangle$ and that for $m > 1$, one gets $f_m \equiv 0$, as expected, since Poisson statistics implies the production of uncorrelated particles and cumulants of order $m \geq 2$ must vanish.

10.5 Multivariate Factorial Moments

The factorial moments $\tilde{F}_n$ and cumulant moments f_n characterize multiplicity fluctuations in a single phase-space region or cell Ω. Short of considering fully differential correlation functions such as those presented in §10.2, one can also consider multivariate factorial moments based on the multiplicities measured in two or more phase space regions. For nonoverlapping regions Ω_m and $\Omega_{m'}$, one may thus define twofold factorial moments, or correlators, as

$$\tilde{F}_{pq} = \left\langle n_m^{[p]} n_{m'}^{[q]} \right\rangle, \tag{10.84}$$

where n_m and $n_{m'}$ are the number of particles in region m and m', respectively. Normalized correlators are defined according to

$$\tilde{f}_{pq} = \frac{\left\langle n_m^{[p]} n_{m'}^{[q]} \right\rangle}{\tilde{F}_p \tilde{F}_q}. \tag{10.85}$$

Extensions to higher-order correlators are also possible.

Multifold correlators of this type find applications in high-energy physics for the study of self-similar particle production behavior (intermittency), in radio- and radar-physics as well as in quantum optics. They are discussed in more detail in ref. [128].

10.6 Correlation Functions of Nonidentical Particles

We have so far considered correlation cumulants, factorial moments, and normalized cumulants for the study of correlation of identical or indistinguishable particles. However, these functions may also be used for the study of correlations of nonidentical particles. In the context of data analyses, nonidentical particles could be particles of different species, different charges, or those belonging to distinct kinematical ranges. For instance, one may be interested in studying correlations between pions and kaons, or correlations between positively and negatively charged particles, or even correlations between low p_T and high p_T particles.

Multiparticle densities are once again denoted $\rho_n(y_1, \ldots, y_n)$, but it is now understood that the coordinates $y_1, \ldots, y_n$ apply to distinguishable particles, either because of their

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species type or because of their kinematical variable attribute or range.[2] As such, the densities $\rho_n(y_1, \ldots, y_n)$ are no longer symmetric under interchange or permutation of the coordinates of the $y_1, \ldots, y_n$. One must in particular recognize the functions $\rho_1(y_1)$ and $\rho_1(y_2)$ as distinct single particle densities. Their integration over fiducial volumes Ω_1 and Ω_2 yields the average number of particles of type 1 and type 2, respectively,

$$\int_{\Omega_1} \rho_1(y_1)\, dy_1 = \langle N_1 \rangle \qquad (10.86)$$
$$\int_{\Omega_2} \rho_1(y_2)\, dy_2 = \langle N_2 \rangle.$$

Similarly, integration of the two-particle density $\rho_2(y_1, y_2)$ yields the average number of pairs

$$\int_{\Omega_1} dy_1 \int_{\Omega_2} dy_2 \rho_2(y_1, y_2) = \langle N_1 N_2 \rangle \qquad (10.87)$$

produced within the acceptance $\Omega_1 \otimes \Omega_2$. Likewise, integration of higher-order densities $\rho_n(y_1, \ldots, y_n)$ yield the average number of n-tuplets. For triplets, one has

$$\int_{\Omega_1} dy_1 \int_{\Omega_2} dy_2 \int_{\Omega_3} dy_3 \rho_3(y_1, y_2, y_3) = \langle N_1 N_2 N_3 \rangle, \qquad (10.88)$$

where N_1, N_2, and N_3 represent the number of particles of type 1, 2, and 3 measured event by event. Note, however, that it may often be of interest to mix distinguishable and indistinguishable particles. For instance, one might wish to study the correlation function of particle triplets involving one high p_T particle, emitted above a specific minimum threshold, and two low p_T particles. The integration of $\rho_3(y_1, y_2, y_3)$ would then yield

$$\int_{\Omega_1} dy_1 \int_{\Omega_2} dy_2 \int_{\Omega_2} dy_3 \rho_3(y_1, y_2, y_3) = \langle N_1 N_2 (N_2 - 1) \rangle, \qquad (10.89)$$

where N_1 and $N_2(N_2 - 1)$ represent the number of high p_T particles and the number of low p_T particle pairs measured event by event, respectively. Similar considerations apply to higher-order densities.

Correlation cumulants for distinguishable particles of all orders may be defined similarly to those of indistinguishable particles. Second-order cumulants are written

$$C_2(y_1, y_2) = \rho_2(y_1, y_2) - \rho_1(y_1)\rho_1(y_2), \qquad (10.90)$$

where it is now understood that the variables y_1 and y_2 apply to particles of different species, different types (e.g., positive and negative charges), or different kinematic ranges (e.g., forward and backward production, or high and low transverse momentum).

[2] Once again in this section, and unless otherwise noted, y_i is meant to represent all kinematical variables of particles, such as p_T, η, and ϕ.

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Normalized densities and cumulants are defined as for distinguishable particles as follows:

$$r_n(y_1,\ldots,y_n) = \frac{\rho_n(y_1,\ldots,y_n)}{\rho_1(y_1)\cdots\rho_1(y_n)} \qquad (10.91)$$
$$R_n(y_1,\ldots,y_n) = \frac{C_n(y_1,\ldots,y_n)}{\rho_1(y_1)\cdots\rho_1(y_n)}.$$

Given their identical structure, the functions $C_n(y_1,\ldots,y_n)$ and $R_n(y_1,\ldots,y_n)$ obey the same scaling relations with the number of sources as for distinguishable particles:

$$C_n^{(m)}(y_1,\ldots,y_n) = mC_n^{(1)}(y_1,\ldots,y_n) \qquad (10.92)$$
$$R_n^{(m)}(y_1,\ldots,y_n) = \frac{1}{m^{n-1}}R_n^{(1)}(y_1,\ldots,y_n),$$

where m stands for the number of identical sources producing correlated n-tuplets.

All in all, correlation functions of indistinguishable particles have very similar properties and behaviors as those constructed for distinguishable particles. However, there is also a common practice of defining so-called "per trigger" correlation functions. These are usually obtained by counting the number of particles in a given low p_T range associated with a "trigger" particle found in a higher p_T range as a function of some relative kinematic variable such as $\Delta\phi$, which is the difference between the azimuthal angle of the particles, or $\Delta\eta$, corresponding to the difference between their pseudorapidities. A measurement of per trigger correlation thus amounts to a number of trigger–associate pairs, N_{pairs}, normalized by the number of trigger particles, N_T. This can be done in two ways. The event-by-event (ebye) method consists in calculating the number of pairs per trigger on an event-by-event basis:

$$C_{T,\text{ebye}} = \left\langle \frac{N_{\text{pairs}}}{N_T} \right\rangle. \qquad (10.93)$$

By contrast, the *inclusive* method involves a ratio of averages.

$$C_{T,\text{inc}} = \frac{\langle N_{\text{pairs}} \rangle}{\langle N_T \rangle}. \qquad (10.94)$$

While the event-by-event method seems more intuitive, it hides and complicates the fact that not all particle pairs are correlated. The inclusive method, on the other hand, readily translates into simple integrals of the single- and two-particle cross sections. For instance,

$$C_{T,\text{inc}}(y_T, y_A) = \frac{\int_{\Omega_T} dx_T \int_{\Omega_A} dx_A \rho_2(x_T, y_T, x_A, y_A)}{\int_{\Omega_T} dx_T dy_T \rho_1(x_T, y_T)} \qquad (10.95)$$

is a per trigger measure of the average number of pairs of trigger (T) and associate (A) particles per trigger, expressed as a function of the coordinates y_T and y_A. The number of pairs is integrated over kinematic variables x_T and x_A, and the number of triggers is obtained by integration over both y_T and x_T. In a practical case, y could stand for the pseudorapidity and x for both p_T and azimuthal angle ϕ of the particles.

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The inclusive method has the additional advantage of being trivially related to simple integrals of single- and two-particle density functions defined earlier in this section. For instance, the per trigger function (10.95) may be written

$$C_{T,\mathrm{inc}}(y_T, y_A) = \frac{\rho_2(y_T, y_A)}{\langle N_T \rangle} \tag{10.96}$$
$$C_{T,\mathrm{inc}}(y_T, y_A) = \frac{\rho_1(y_T)\rho_1(y_A) r_n(y_T, y_A)}{\langle N_T \rangle}.$$

It is to be noted, however, that neither $C_{T,\mathrm{inc}}$ nor $C_{T,\mathrm{ebye}}$ are actual correlation functions since they are not based on cumulants, and as such do not account for nor remove combinatorial contributions to the pair production cross section. In other words, both functions could be nonvanishing even in the absence of actual correlations. Additionally, note that the common practice of subtracting the minimal value of $C_{T,\mathrm{inc}}(y_T, y_A)$ from the function to seek an estimate of the "true" correlation is *not* logically consistent, since it may actually over- or undersubtract the combinatorial contributions. A better and logically consistent estimate of the number of correlated associates is obtained by using cumulants. One can, for instance, define the per trigger cumulant

$$K_{T,\mathrm{inc}}(y_T, y_A) = \frac{C_2(y_T, y_A)}{\langle N_T \rangle} \tag{10.97}$$

where the cumulant $C_2(y_T, y_A)$ is obtained according to Eq. (10.90). Note, however, that since $C_2(y_T, y_A)$ is not necessarily positive definite, $K_{T,inc}(y_T, y_A)$ may also end up being negative for selected ranges of the variables y_T and y_A. That simply reflects the fact that the notion of per trigger correlated associates is not a very well defined quantity in general, although it may apply in the context of two-component particle production models.

The notion of per trigger correlation function may be generalized to higher-order densities or cumulants:

$$K_n(y_1, y_2, \ldots, y_n) = \frac{C_n(y_1, y_2, \ldots, y_n)}{\langle N_1 \rangle} \tag{10.98}$$

where, for instance, particle 1 is assumed to be the **trigger** and particles 2 to n are the **associates**.

10.7 Charge-Dependent and Charge-Independent Correlation Functions

Conservation laws play a central role in shaping and constraining particle production. Charge conservation, in particular, dictates that the production (or destruction) of a negative particle must necessarily be accompanied by the production (or destruction) of a positive particle. It applies both locally, where particles are created in space, as well as globally (i.e., for an entire collision). While the dynamics of collisions (momentum transfer, pressure, and so on) largely drive the transverse momenta, rapidity, and angle at which

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particles are produced, they cannot erase the common origin of positive and negative particle pairs. It is thus interesting to define **charge-independent** (CI) and **charge-dependent** (CD) correlation functions cumulants. In the study of two-particle correlations, one may separate the pairs according to their (electric) charge, and measure two-particle densities and cumulants for $(+,+)$, $(-,-)$, and $(+,-)$ separately. The CI and CD cumulants are thus defined as follows:

$$\begin{aligned} C_{\mathrm{CI}}(y_1,y_2) &= \frac{1}{4}\{C_{+-}(y_1,y_2)+C_{-+}(y_1,y_2) \\ &\quad + C_{++}(y_1,y_2)+C_{--}(y_1,y_2)\} \\ C_{\mathrm{CD}}(y_1,y_2) &= \frac{1}{4}\{C_{+-}(y_1,y_2)+C_{-+}(y_1,y_2) \\ &\quad - C_{++}(y_1,y_2)-C_{--}(y_1,y_2)\}\,. \end{aligned} \tag{10.99}$$

One may likewise define CI and CD normalized cumulants as follows:

$$\begin{aligned} R_{\mathrm{CI}}(y_1,y_2) &= \frac{1}{4}\{R_{+-}(y_1,y_2)+R_{-+}(y_1,y_2) \\ &\quad + R_{++}(y_1,y_2)+R_{--}(y_1,y_2)\} \\ R_{\mathrm{CD}}(y_1,y_2) &= \frac{1}{4}\{R_{+-}(y_1,y_2)+R_{-+}(y_1,y_2) \\ &\quad - R_{++}(y_1,y_2)-R_{--}(y_1,y_2)\} \end{aligned} \tag{10.100}$$

In very high-energy collisions, particle production at mid-rapidities ($y \approx 0$) yields approximately equal numbers of positive and negative particles, $\rho_+(y) \approx \rho_-(y)$. Defining $\rho_1(y) = \rho_+(y) + \rho_-(y)$, one expects

$$\begin{aligned} R_{\mathrm{CI}}(y_1,y_2) &\approx \frac{1}{\rho_1(y_1)\rho_1(y_2)}\{C_{+-}(y_1,y_2)+C_{-+}(y_1,y_2) \\ &\quad + C_{++}(y_1,y_2)+C_{--}(y_1,y_2)\} \\ &= \frac{C_2(y_1,y_2)}{\rho_1(y_1)\rho_1(y_2)} = R_2(y_1,y_2). \end{aligned} \tag{10.101}$$

where C_2 is obtained by summing over all pairs, i.e., $(+,+)$, $(+,-)$, $(-,+)$, and $(-,-)$. This relationship is, however, not strictly valid at "low" energy or in regions of momentum space where the yields of positive and negative particles differ appreciably.

The functions C_{CI} and C_{CD} have properties and scaling behaviors similar to those of the more basic functions C_2. In particular, their amplitude is expected to be proportional to the number of sources m in a system consisting of independent sources. Consequently, the functions R_{CI} and R_{CD} are expected to scale as $1/m$, similarly to the generic normalized cumulant R_2 functions.

It is also of common practice to study **balance functions** to determine how the production of positively charged particles is balanced by the production of negatively charged particles. Balance functions are typically defined and integrated in a fixed kinematic range (e.g., the pseudorapidity η). Steffen et al. [30], for instance defined a balance function as

$$B(\Delta\eta) = \frac{1}{2}\left\{\frac{\langle N_{+-}(\Delta\eta)\rangle - \langle N_{++}(\Delta\eta)\rangle}{\langle N_+\rangle} + \frac{\langle N_{-+}(\Delta\eta)\rangle - \langle N_{--}(\Delta\eta)\rangle}{\langle N_-\rangle}\right\}, \tag{10.102}$$

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where $\langle N_{\alpha\beta}(\Delta\eta)\rangle$ is the number of particle pairs of type (α, β) measured at a given pseudorapidity interval $\Delta\eta$, while $\langle N_+\rangle$ and $\langle N_-\rangle$ are the average number of positive and negative particles produced in that same interval, respectively. The balance function $B(\Delta\eta)$ can be expressed in terms of the normalized cumulant $R_{\mathrm{CD}}(y_1, y_2)$ (see Problem 10.1). A more detailed discussion of the balance function is presented in §11.1.3.

While the correlation functions C_{CD} and R_{CD} and the balance function B were first introduced for the study of charged particles, their use can be narrowed to specific particle species, that is, π^+ relative to π^-, or K^+ vs. K^-. They can also be extended to any conserved quantities, for instance, the net strangeness or the net baryon number. For baryons, one could, for instance, write

$$R_{\mathrm{Net-B}}(y_1, y_2) = \frac{1}{4}\left\{R_{B\overline{B}}(y_1, y_2) + R_{\overline{B}B}(y_1, y_2) - R_{BB}(y_1, y_2) - R_{\overline{B}\overline{B}}(y_1, y_2)\right\} \quad (10.103)$$

$R_{\mathrm{Net-B}}(y_1, y_2)$ quantifies the correlation between produced baryons (e.g., protons) and antibaryons (e.g., anti-protons). As such, it provides an invaluable tool to study both the production and transport of baryons in nuclear collisions.

10.8 Generalized (Weighted) Correlation Functions

Just as it is possible and meaningful to study the expectation value of a function $g(x)$ of a continuous random variable x determined by a PDF $f(x)$

$$\mathrm{E}[g(x)] = \int g(x) f(x)\, dx, \quad (10.104)$$

one can also study generalized correlation functions that involve functions $g(x_1, \ldots, x_n)$ of the particle coordinates $x_1, x_2, \ldots, x_n$. Effectively, such generalized correlation functions are a measure of the average of the function $g(x_1, \ldots, x_n)$. Since it is not necessary to integrate all particle coordinates, we label coordinates to be integrated over as $x_1, \ldots, x_n$, and those not to be integrated over as $y_1, \ldots, y_n$. By strict definition of an average, one can consequently define weighted densities functions as follows:

$$\langle g(y_1, \ldots, y_n)\rangle = \frac{\int_\Omega g(x_1, \ldots, x_n)\rho_n(x_1, y_1, \ldots, x_n, y_n)\prod_i^n dx_i}{\int_\Omega \rho_n(x_1, y_1, \ldots, x_n, y_n)\prod_i^n dx_i}. \quad (10.105)$$

If the integration is taken over all particle coordinates, that is, for both the x_i and y_i coordinates, the preceding function become an integral correlator defined over the fiducial volume Ω:

$$\langle g\rangle_n = \frac{\int_\Omega g(x_1, \ldots, x_n)\rho_n(x_1, y_1, \ldots, x_n, y_n)\prod_i^n dx_i dy_i}{\langle N(N-1)\cdots(N-n+1)\rangle}. \quad (10.106)$$

As a practical example of a generalized correlation function, consider a two-particle average of the product $\Delta x_1 \Delta x_2$, in other words, with

$$g(x_1, x_2) = \Delta x_1 \Delta x_2, \quad (10.107)$$

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where $\Delta x_i = x_i - \langle x \rangle(y_1, y_2)$, and $\langle x_i \rangle(y_1, y_2)$ stands for the average value of x evaluated as

$$\langle x_i \rangle(y_1, y_2) = \frac{\int_\Omega x_i \rho_2(x_1, y_1, x_2, y_2)\, dx_1\, dx_2}{\int_\Omega \rho_2(x_1, y_1, x_2, y_2)\, dx_1\, dx_2}. \tag{10.108}$$

Additionally, defining $\langle x_1 x_2 \rangle(y_1, y_2)$ as

$$\langle x_1 x_2 \rangle(y_1, y_2) = \frac{\int_\Omega x_1 x_2 \rho_2(x_1, y_1, x_2, y_2)\, dx_1\, dx_2}{\int_\Omega \rho_2(x_1, y_1, x_2, y_2)\, dx_1\, dx_2}, \tag{10.109}$$

one finds that Eq. (10.105) may be written, for $n = 2$, and $g(x_1, x_2) = \Delta x_1 \Delta x_2$,

$$\langle \Delta x_1 \Delta x_2 \rangle(y_1, y_2) = \langle x_1 x_2 \rangle(y_1, y_2) - \langle x_1 \rangle(y_1, y_2) \langle x_2 \rangle(y_1, y_2), \tag{10.110}$$

which defines the covariance of variables x_1 and x_2 evaluated as a function of the particle coordinates y_1 and y_2. Integration over y_1 and y_2 yields the system wide covariance $\langle \Delta x_1 \Delta x_2 \rangle$. Practical implementations of differential and integral correlation function, with $x \equiv p_T$, are discussed in §§11.1.5 and 11.3.5.

10.9 Autocorrelations and Time-Based Correlation Functions

Autocorrelation and time-based correlation functions are commonly used for the study of signals that vary with time and to identify repetitive behaviors, time ordering, and/or dependencies.

An autocorrelation is usually defined as the cross-correlation between a signal and itself at prior time. Consider a random process X that may be measured repeatedly in several "runs" indexed by an integer i. Further assume the process is measured as a function of time and yields a signal $x_i(t)$, where the time t may be expressed either in discrete steps or as a continuous variable. Let $\mu(t)$ and $\sigma(t)$ represent the expectation value and the standard deviation of the signal X at a given time t (or in a given time step), respectively. The autocorrelation of the signal between two times t_1 and t_2 is defined as

$$R_x(t_1, t_2) = \frac{E\left[(x(t_2) - \mu(t_2))\,(x(t_1) - \mu(t_1))\right]}{\sigma(t_1)\sigma(t_2)}. \tag{10.111}$$

Obviously, this definition is meaningful provided the variances $\sigma(t_1)$ and $\sigma(t_2)$ are finite. Variants of this definition includes applications where the means are not subtracted, and no division by the standard deviations is carried out. If the signal is stable, the strength of the correlation becomes exclusively a function of the time difference .

A time-based correlation function between two signals $X(t)$ and $Y(t)$ is similarly defined as

$$R_{xy}(t_1, t_2) = \frac{E\left[(x(t_2) - \mu_x(t_2))\,(y(t_1) - \mu_y(t_1))\right]}{\sigma_x \sigma_y}, \tag{10.112}$$

where $\mu_x(t)$ and $\mu_y(t)$ are the expectation values of the signals x and y, respectively.

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Exercises

10.1 Express the balance function $B(\Delta\eta)$ given by Eq. (10.102) in terms of the normalized cumulant $R_{\mathrm{CD}}(y_1, y_2)$ given by Eq. 10.101), and single particle densities $\rho_1(y_i)$, or their integrals.

10.2 Derive the expression of the inclusive two-cumulant, Eq. (10.54), in terms of the functions C_S and C_L defined by Eqs. (10.52) and (10.53).

10.3 Derive an expression for the inclusive three-cumulant.

10.4 Verify that the expression (10.68) yields the factorial moments $\tilde{F}_n$ by explicitly calculating the lowest four-order moments.

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