Bull. Aust. Math. Soc. **114** (2026), 536–549
doi:10.1017/S0004972726101233

A FREE BOUNDARY MODEL IN RIVER NETWORKS

YING JI, GUIHONG FAN and NINGKUI SUN✉

(Received 25 January 2026 UTC; accepted 20 February 2026 UTC; first published online 14 May 2026 UTC)

Abstract

We study the long-time behaviour of solutions to a free boundary model in river networks with small water flow speeds. First, we establish the well-posedness of our model. Under suitable assumptions, we then establish a spreading–vanishing dichotomy result for solutions to our model. By introducing a parameter σ in the initial data, we derive sharp threshold behaviours.

2020 *Mathematics subject classification*: primary 35K55; secondary 35B40, 35P15, 35R35.

Keywords and phrases: reaction-diffusion equation, river network, long-time behaviour, free boundary.

1. Introduction

In the pioneering work [1], the authors considered the problem

$$u_t = u_{xx} + u(1-u),\ t>0,\ x\in\mathbb{R}, \qquad u(0,x)\geq,\not\equiv 0,\ x\in\mathbb{R},$$

and obtained the so-called hair-trigger effect, which states that *spreading* always happens (that is, $\lim_{t\to\infty} u(t,x)=1$) for the solution starting from any nonnegative initial data, no matter how small it is. The dynamics of reaction-diffusion equations in both bounded and unbounded domains have been studied extensively (see [10]).

Du and Lin [6] introduced the free boundary problem

$$\begin{cases} u_t = u_{xx} + u(1-u), & t>0,\ x\in(g(t),h(t)),\\ u(t,g(t))=0,\ g'(t)=-\mu u_x(t,g(t)), & t>0,\\ u(t,h(t))=0,\ h'(t)=-\mu u_x(t,h(t)), & t>0, \end{cases}$$

where $h(0)=h_0=-g(0)$ with $h_0>0$ and $u(0,x)>0$ in $[-h_0,h_0]$. The free boundaries $x=h(t)$ and $x=g(t)$ represent the spreading fronts of the population whose density is denoted by $u(t,x)$. They proved that either spreading happens, that is, $-g(t),\ h(t)\to\infty$, $u\to 1$ locally uniformly in $\mathbb{R}$ as $t\to\infty$, or vanishing happens, that is, $g(t)\to g_\infty$, $h(t)\to h_\infty$ with $h_\infty-g_\infty\leq\pi$, $u\to 0$ uniformly in $[g_\infty,h_\infty]$ as $t\to\infty$.

This research was supported by Shandong Provincial Natural Science Foundation of China (no. ZR2023YQ002).

© The Author(s), 2026. Published by Cambridge University Press on behalf of Australian Mathematical Publishing Association Inc.

https://doi.org/10.1017/S0004972726101233 Published online by Cambridge University Press

Subsequently, Gu *et al.* [9] extended this model to advective environments by adding an advection term, leading to the free boundary problem

$$\begin{cases} u_t = u_{xx} - \beta u_x + u(1-u), & t > 0,\ x \in (g(t), h(t)), \\ u(t, g(t)) = 0,\ g'(t) = -\mu u_x(t, g(t)), & t > 0, \\ u(t, h(t)) = 0,\ h'(t) = -\mu u_x(t, h(t)), & t > 0. \end{cases}$$

They showed that if $\beta \in (0, 2)$, there is a spreading–vanishing dichotomy result, while if $\beta \geq 2$, spreading does not happen. A rather complete description of the asymptotic behaviour of solutions is obtained. Relevant background on the dynamics of free boundary problems can be found in [4].

In 2020, Du *et al.* [8] studied the problem

$$\begin{cases} \partial_t v_U - \partial_{xx} v_U + \beta_U \partial_x v_U = v_U - v_U^2, & x \in \mathbb{R}_U,\ t > 0, \\ \partial_t v_L - \partial_{xx} v_L + \beta_L \partial_x v_L = v_L - v_L^2, & x \in \mathbb{R}_L,\ t > 0, \\ v_U(t, 0) = v_L(t, 0),\ a_U \partial_x v_U(t, 0) = a_L \partial_x v_L(t, 0), & t > 0, \end{cases}$$

where β_U, β_L, a_U and a_L are positive constants. Let U and L denote the upstream and downstream branches, respectively. The subscripts U and L are used in parameters such as v_U, v_L, β_U, β_L, a_U and a_L. Thus, β_U and β_L denote the advection coefficients in each respective branch. They found that when $\beta_U \in (0, 2)$, $(v_U, v_L) \to (1, 1)$ locally uniformly as $t \to \infty$; when $\beta_U,\ \beta_L \geq 2$, $(v_U, v_L) \to (0, 0)$ locally uniformly as $t \to \infty$; when $0 < \beta_L < 2 \leq \beta_U$, $(v_U, v_L) \to (\Psi_U, \Psi_L)$ locally uniformly as $t \to \infty$, where (Ψ_U, Ψ_L) is a stationary solution of the Cauchy problem (see [8] for details).

To describe the biological invasion of species in a two-branch river system, we consider a free boundary model with small advection speeds:

$$\begin{cases} \partial_t w_U - \partial_{xx} w_U + \beta_U \partial_x w_U = w_U - w_U^2, & g(t) < x < 0,\ t > 0, \\ \partial_t w_L - \partial_{xx} w_L + \beta_L \partial_x w_L = w_L - w_L^2, & 0 < x < h(t),\ t > 0, \\ g(0) = -h_0,\ w_U(t, x) = 0,\ g'(t) = -\mu \partial_x w_U(t, x), & t > 0,\ x = g(t), \\ h(0) = h_0,\ w_L(t, x) = 0,\ h'(t) = -\mu \partial_x w_L(t, x), & t > 0,\ x = h(t), \\ w_U(t, 0) = w_L(t, 0), & t > 0, \\ a_U \partial_x w_U(t, 0) = a_L \partial_x w_L(t, 0), & t > 0, \\ w_U(0, x) = w_0(x), w_L(0, y) = w_0(y), & x \in [-h_0, 0],\ y \in [0, h_0]. \end{cases} \tag{P}$$

In our model, U and L denote the upstream and downstream branches. The functions $w_U(t, x)$ and $w_L(t, x)$ represent the population density in $[g(t), 0]$ and $[0, h(t)]$, respectively, the constant $\mu > 0$ measures the ability for spreading into the new habitat of this species (see [2]), and $\beta_L,\ \beta_U \in (0, 2)$. By the conservation of flow at $x = 0$, we have $a_L \beta_L = a_U \beta_U$, where $a_U,\ a_L$ are positive constants. The fifth line represents the natural continuity connection condition and the sixth line is the Kirchhoff law, which is given by the continuity connection condition and the conservation of flow at the junction

https://doi.org/10.1017/S0004972726101233 Published online by Cambridge University Press

point 0 (see [8]). The initial function w_0 belongs to $\mathscr{X}(h_0)$ for some $h_0 > 0$, where

$$\mathscr{X}(h_0) := \{\phi \in C^2([-h_0, h_0]) : \phi(\pm h_0) = 0,\ \phi(x) > 0 \text{ in } (-h_0, h_0)\}.$$

The main purpose of this paper is to study the long-time behaviour of solutions to (*P*). In this paper, we only consider $\beta_U,\ \beta_L \in (0, 2)$; a forthcoming paper [11] considers the case $\beta_U > 2$ or $\beta_L > 2$. The first result is the well-posedness of the problem.

THEOREM 1.1 (Well-posedness). *Assume $\beta_L,\ \beta_U > 0$. For any $w_0 \in \mathscr{X}(h_0)$ with $h_0 > 0$ and any $\alpha \in (0, 1)$, problem (P) admits a unique solution pair (w_U, w_L, g, h) with*

$$w_U(t, x) \in C^{1+\alpha/2, 2+\alpha}((0, \infty) \times [g(t), 0]), \quad g(t) \in C^{1+\alpha/2}([0, \infty)),$$

$$w_L(t, x) \in C^{1+\alpha/2, 2+\alpha}((0, \infty) \times [0, h(t)]), \quad h(t) \in C^{1+\alpha/2}([0, \infty)).$$

Throughout this paper, for constants $a < b$ and functions $l(t) < r(t)$, we define

$$[a, b] \times [l, r] := \{(t, x) : t \in [a, b],\ x \in [l(t), r(t)]\}.$$

The sets $(a, b) \times [l, r]$, $[a, b] \times (l, r)$ are defined similarly. Denote

$$g_\infty := \lim_{t\to\infty} g(t), \quad h_\infty := \lim_{t\to\infty} h(t) \quad \text{and} \quad I_\infty := (g_\infty, h_\infty).$$

The long-time behaviour of problem (*P*) is determined in the following theorem.

THEOREM 1.2. *Assume $\beta_L,\ \beta_U \in (0, 2)$ and (w_U, w_L, g, h) is a time-global solution of (P) with $w_0 = \sigma\phi$ for some $\sigma \geq 0$. There is $\sigma^* = \sigma^*(h_0, \phi, \beta_U, \beta_L) \in [0, \infty]$ such that:*

(i) *when $\sigma > \sigma^*$, spreading happens in the sense that $I_\infty = \mathbb{R}$ and $(w_U, w_L) \to (1, 1)$ locally uniformly as $t \to \infty$;*
(ii) *when $0 \leq \sigma \leq \sigma^*$, vanishing happens in the sense that I_∞ is a bounded interval, $\|w_U(t, \cdot)\|_{L^\infty([g(t),0])} \to 0$ and $\|w_L(t, \cdot)\|_{L^\infty([0,h(t)])} \to 0$ as $t \to \infty$.*

Moreover, if $h_0 \geq \hbar^*$, then $\sigma^* = 0$, where $\hbar^*$ is given in Lemma 3.1.

Here and in what follows, we say (w_U, w_L) converges locally uniformly if w_U converges locally uniformly in $(-\infty, 0]$, and w_L converges locally uniformly in $[0, +\infty)$.

The rest of the paper is organised as follows. In Section 2, we give some basic results including the comparison principle and well-posedness for (*P*). Section 3 covers the long-time behaviour of the solution of (*P*) and the proof of Theorem 1.2.

2. Comparison principle and well-posedness

In this section, we give some basic results which are used later in the paper.

2.1. Comparison principle. In this subsection, we establish the comparison principle, which is used in the rest of this paper.

https://doi.org/10.1017/S0004972726101233 Published online by Cambridge University Press

LEMMA 2.1. *Assume that* $T \in (0,\infty)$, $\hat{g}, \hat{h} \in C^1([0,T])$, $\hat{w}_U \in C(\overline{D_U^T}) \cap C^{1,2}(D_U^T)$ *and* $\hat{w}_L \in C(\overline{D_L^T}) \cap C^{1,2}(D_L^T)$ *with* $D_U^T := (0,T] \times (\hat{g}, 0)$, $D_L^T := (0,T] \times (0, \hat{h})$. *Assume*

$$\begin{cases} \partial_t \hat{w}_U - \partial_{xx}\hat{w}_U + \beta_U \partial_x \hat{w}_U \geq \hat{w}_U - \hat{w}_U^2, & \hat{g}(t) < x < 0,\ t \in (0,T], \\ \partial_t \hat{w}_L - \partial_{xx}\hat{w}_L + \beta_L \partial_x \hat{w}_L \geq \hat{w}_L - \hat{w}_L^2, & 0 < x < \hat{h}(t),\ t \in (0,T], \\ \hat{w}_U(t, \hat{g}(t)) = 0,\ \hat{g}'(t) \leq -\mu \partial_x \hat{w}_U(t, \hat{g}(t)), & t \in (0,T], \\ \hat{w}_L(t, \hat{h}(t)) = 0,\ \hat{h}'(t) \geq -\mu \partial_x \hat{w}_L(t, \hat{h}(t)), & t \in (0,T], \\ \hat{w}_U(t,0) = \hat{w}_L(t,0),\ a_U \partial_x \hat{w}_U(t,0) \geq a_L \partial_x \hat{w}_L(t,0), & t \in (0,T]. \end{cases}$$

If $[-h_0, h_0] \subseteq [\hat{g}(0), \hat{h}(0)]$ *and* $(w_U(0,\cdot), w_L(0,\cdot)) \leq (\hat{w}_U(0,\cdot), \hat{w}_L(0,\cdot))$, *where the solution pair* (w_U, w_L, g, h) *is a solution to problem* (*P*), *then*

$$g(t) \geq \hat{g}(t), \quad h(t) \leq \hat{h}(t), \quad (w_U(t,\cdot), w_L(t,\cdot)) \leq (\hat{w}_U(t,\cdot), \hat{w}_L(t,\cdot)) \quad \text{for } t \in (0,T].$$

The above lemma can be proved by similar arguments to those of [6, 8, 12], so the details are omitted.

REMARK 2.2. The pair $(\hat{w}_U, \hat{w}_L)$, or $(\hat{w}_U, \hat{w}_L, \hat{g}, \hat{h})$, is often called a supersolution of (*P*). A subsolution can be defined analogously by reversing all the inequalities.

2.2. Well-posedness. We first establish the local existence of a solution to problem (*P*) for any β_U, $\beta_L > 0$, and prove its uniqueness. Then, we show that the local solution can be extended to all time by an estimate on the free boundaries.

THEOREM 2.3. *For any given* β_U, $\beta_L > 0$ *and any fixed* $\alpha \in (0,1)$, *there exists a* $T > 0$ *such that problem* (*P*) *with* $w_0 \in \mathscr{X}(h_0)$ *for some* $h_0 > 0$ *admits a unique solution* (w_U, w_L, g, h), *which satisfies* $g \in C^{1+\alpha/2}([0,T])$, $h \in C^{1+\alpha/2}([0,T])$ *and*

$$w_U \in C^{1+\alpha/2,2+\alpha}((0,T] \times [g(t), 0]), \quad w_L \in C^{1+\alpha/2,2+\alpha}((0,T] \times [0, h(t)]).$$

PROOF. First, we transform (*P*) into a fixed boundary problem as in [3, 6]. Let $\xi_1(y)$ and $\xi_2(y)$ in $C^3(\mathbb{R})$ be two nonnegative functions with $\xi_2(y) = \xi_1(-y)$ and

$$\xi_1(y) = 1 \ \text{ if } |y - h_0| < h_0/4, \quad \xi_1(y) = 0 \ \text{ if } |y - h_0| > h_0/2, \quad |\xi_1'(y)| < 6/h_0 \ \text{ for } y \in \mathbb{R}.$$

If we set

$$\begin{cases} v_U(t,y) := w_U(t, y + \xi_2(y)(g(t) + h_0)) = w_U(t,x), & y < 0, \\ v_L(t,y) := w_L(t, y + \xi_1(y)(h(t) - h_0)) = w_L(t,x), & y > 0, \end{cases}$$

then the free boundary problem becomes

$$\begin{cases} \partial_t v_U - A_U \partial_{yy} v_U - (B_U + C_U - \beta_U \sqrt{A_U}) \partial_y v_U = f(v_U), & t > 0,\ y \in (-h_0, 0), \\ \partial_t v_L - A_L \partial_{yy} v_L - (B_L + C_L - \beta_L \sqrt{A_L}) \partial_y v_L = f(v_L), & t > 0,\ y \in (0, h_0), \\ v_U(t,0) = v_L(t,0), a_U \partial_y v_U(t,0) = a_L \partial_y v_L(t,0), & t > 0, \\ v_U(0,z) = w_0(z), v_L(0,y) = w_0(y), & z \in [-h_0, 0],\ y \in [0, h_0], \end{cases} \tag{2.1}$$

https://doi.org/10.1017/S0004972726101233 Published online by Cambridge University Press

where, for $y < 0$, $t > 0$,

$$\sqrt{A_U(t,y)} := \frac{1}{1+\xi_2'(y)(g(t)+h_0)},$$
$$B_U(t,y) := -\frac{\xi_2''(y)(g(t)+h_0)}{[1+\xi_2'(y)(g(t)+h_0)]^3},$$
$$C_U(t,y) := \frac{\xi_2(y)g'(t)}{1+\xi_2'(y)(g(t)+h_0)}$$

and, for $y > 0$, $t > 0$,

$$\sqrt{A_L(t,y)} := \frac{1}{1+\xi_1'(y)(h(t)-h_0)},$$
$$B_L(t,y) := -\frac{\xi_1''(y)(h(t)-h_0)}{[1+\xi_1'(y)(h(t)-h_0)]^3},$$
$$C_L(t,y) := \frac{\xi_1(y)h'(t)}{1+\xi_1'(y)(h(t)-h_0)}.$$

Denote $h_1 = -\mu w_0'(h_0)$, $g_1 = \mu w_0'(-h_0)$. For $0 < T \le h_0/16(1+h_1+g_1)+1 := T_1$, define

$$D_{g,T} := \{g \in C^1([0,T]) : g(0) = -h_0, g'(0) = -g_1, \|g' + g_1\|_{C([0,T])} \le 1\},$$
$$D_{h,T} := \{h \in C^1([0,T]) : h(0) = h_0, h'(0) = h_1, \|h' - h_1\|_{C([0,T])} \le 1\}.$$

Then, $D_T := D_T^g \times D_T^h$ is a bounded and closed convex set of $C^1([0,T]) \times C^1([0,T])$. For $T \in (0, T_1)$, we define a subspace of D_{T_1} by $D_{T_1}^T = D_{g,T_1}^T \times D_{h,T_1}^T$ with

$$D_{g,T_1}^T := \{g \in D_{g,T_1} : g(t) = g(T) \text{ in } [T,T_1]\},$$
$$D_{h,T_1}^T := \{h \in D_{h,T_1} : h(t) = h(T) \text{ in } [T,T_1]\}.$$

Clearly, we can extend each $(g,h) \in D_T$ to be a member of $D_{T_1}^T$ as indicated above. In what follows, we will always identify $D_{T_1}^T$ with D_T. For each $(g,h) \in D_T = D_{T_1}^T \subseteq D_{T_1}$, we apply the extension trick of [15, 16], the L^p estimates and the Sobolev embedding theorem to show that problem (2.1) has a unique solution (v_U, v_L) with

$$\|v_U\|_{C^{(1+\alpha)/2,1+\alpha}(\Omega_{T_1}^U)} + \|v_L\|_{C^{(1+\alpha)/2,1+\alpha}(\Omega_{T_1}^L)} \le C_{T_1}(\|v_U\|_{W_p^{1,2}(\Omega_{T_1}^U)} + \|v_L\|_{W_p^{1,2}(\Omega_{T_1}^L)}) \le C_1,$$

where $\Omega_{T_1}^U = [0,T_1] \times [-h_0, 0]$, $\Omega_{T_1}^L = [0,T_1] \times [0,h_0]$, C_1 is a constant dependent on μ, h_0, and $\|w_0\|_{C^2([-h_0,h_0])}$, C_{T_1} and $\bar{C}_{T_1}$ depend on h_0, T_1 and α.

For $t \in [0,T_1]$, define

$$\hat{h}(t) = h_0 - \int_0^t \mu \partial_y v_L(\tau, h_0)\, d\tau, \quad \hat{g}(t) = -h_0 - \int_0^t \mu \partial_y v_U(\tau, -h_0)\, d\tau.$$

Then, $\hat{h}'(t) = -\mu \partial_y v_L(t, h_0)$ and $\hat{h}'(0) = -\mu w_0'(h_0) = h_1$, so that $\hat{h}' \in C^{\alpha/2}([0,T_1])$ with $\|\hat{h}'\|_{C^{\alpha/2}([0,T_1])} \le \mu C_1 =: C_2$. Similarly, $\hat{g}' \in C^{\alpha/2}([0,T_1])$ with $\|\hat{g}'\|_{C^{\alpha/2}([0,T_1])} \le C_2$.

https://doi.org/10.1017/S0004972726101233 Published online by Cambridge University Press

Next, define the mapping $\mathcal{F}: D_T = D_{T_1}^T \to C^1([0,T_1]) \times C^1([0,T_1])$ by $\mathcal{F}(g,h) = (\hat{g},\hat{h})$. We show that if $T>0$ is small, then $\mathcal{F}$ has a fixed point by the Schauder fixed point theorem. For any $T \in (0,T_1)$, by the above results,

$$\|\hat{h}' - h_1\|_{C([0,T])} \le C_2 T^{\alpha/2} \quad \text{and} \quad \|\hat{g}' + g_1\|_{C([0,T])} \le C_2 T^{\alpha/2}.$$

Taking $T < T_2 := \min\{T_1, C_2^{-2/\alpha}\}$, we conclude that $\mathcal{F}$ maps D_T into itself. Consequently, by the Schauder fixed point theorem, $\mathcal{F}$ has at least one fixed point which also means that problem (*P*) admits at least one solution (w_U, w_L, g, h).

Next, we show the uniqueness of the solution to problem (*P*). For this purpose, let (w_U^i, w_L^i, g_i, h_i) (for $i=1,2$) be two solutions of problem (*P*) for $t \in [0,T]$, and set

$$\begin{cases} v_U^i(t,y) := w_U^i(t, y + \xi_2(y)(g_i(t)+h_0)), & y<0, \\ v_L^i(t,y) := w_L^i(t, y + \xi_1(y)(h_i(t)-h_0)), & y>0. \end{cases}$$

From the above estimates,

$$\|v_U^i\|_{C^{(1+\alpha)/2,1+\alpha}(\Omega_T^U)} + \|v_L^i\|_{C^{(1+\alpha)/2,1+\alpha}(\Omega_T^L)} \le C_1, \quad \|{g_i}'\|_{C^{\alpha/2}([0,T])} \le C_2, \quad \|{h_i}'\|_{C^{\alpha/2}([0,T])} \le C_2.$$

Define $\tilde{g}(t) := g_1(t) - g_2(t)$, $\tilde{h}(t) := h_1(t) - h_2(t)$, $V_U(t,y) := v_U^1(t,y) - v_U^2(t,y)$ and $V_L(t,y) := v_L^1(t,y) - v_L^2(t,y)$. Since $T \le T_1$, we can check that for any $t \in [0,T]$,

$$\begin{cases} \partial_t V_U - A_U^2 \partial_{yy} V_U - (B_U^2 + C_U^2 - \beta_U \sqrt{A_U^2})\partial_y V_U = \tilde{f}_U(t,y), & t \in (0,T],\ y \in (-h_0, 0), \\ \partial_t V_L - A_L^2 \partial_{yy} V_L - (B_L^2 + C_L^2 - \beta_L \sqrt{A_L^2})\partial_y V_L = \tilde{f}_L(t,y), & t \in (0,T],\ y \in (0, h_0), \\ V_U(t,-h_0) = 0 = V_L(t,h_0), & t \in (0,T], \\ V_U(t,0) = V_L(t,0), a_U \partial_y V_U(t,0) = a_L \partial_y V_L(t,0), & t \in (0,T], \end{cases}$$

with $V_U(0,z) = V_L(0,y) = 0$ for $z \in [-h_0,0]$, $y \in [0,h_0]$ and

$$\begin{aligned} \tilde{f}_U(t,y) &= f(v_U^1) - f(v_U^2) + (A_U^1 - A_U^2)\partial_{yy} v_U^1 \\ &\quad + \left[B_U^1 - B_U^2 + C_U^1 - C_U^2 - \beta_U\left(\sqrt{A_U^1} - \sqrt{A_U^2}\right)\right]\partial_y v_U^1, \\ \tilde{f}_L(t,y) &= f(v_L^1) - f(v_L^2) + (A_L^1 - A_L^2)\partial_{yy} v_L^1 \\ &\quad + \left[B_L^1 - B_L^2 + C_L^1 - C_L^2 - \beta_L\left(\sqrt{A_L^1} - \sqrt{A_L^2}\right)\right]\partial_y v_L^1, \end{aligned}$$

where A_j^i, B_j^i and C_j^i are the coefficients of problem (2.1) with (g_i, h_i) in place of (g,h) for $j \in \{U, L\}$. We apply the extension trick of [15, 16], the L^p estimates for parabolic equations and the Sobolev embedding theorem to see that

$$\|V_U\|_{C^{(1+\alpha)/2,1+\alpha}(\Omega_T^U)} + \|V_L\|_{C^{(1+\alpha)/2,1+\alpha}(\Omega_T^L)} \le C_3(\|\tilde{g}\|_{C^1([0,T])} + \|\tilde{h}\|_{C^1([0,T])}), \tag{2.2}$$

where C_3 depends only on h_0, $\|w_0\|_{C^2}$, C_1, C_2 and α, but is independent of $T \in (0,T_1]$.

Recalling that $\tilde{g}'(0) = g_1'(0) - g_2'(0)$, from (2.2),

$$\|\tilde{g}\|_{C^1([0,T])} \le 2T^{\alpha/2}\|\tilde{g}'\|_{C^{\alpha/2}([0,T])} \le C_4 T^{\alpha/2}(\|\tilde{g}\|_{C^1([0,T])} + \|\tilde{h}\|_{C^1([0,T])}),$$

https://doi.org/10.1017/S0004972726101233 Published online by Cambridge University Press

where $C_4 := 2\mu C_3$. Similarly, $\|\tilde{h}\|_{C^1([0,T])} \le C_4 T^{\alpha/2}(\|\tilde{g}\|_{C^1([0,T])} + \|\tilde{h}\|_{C^1([0,T])})$. Consequently,

$$\|\tilde{g}\|_{C^1([0,T])} + \|\tilde{h}\|_{C^1([0,T])} \le 2C_4 T^{\alpha/2}(\|\tilde{g}\|_{C^1([0,T])} + \|\tilde{h}\|_{C^1([0,T])}). \tag{2.3}$$

If we take $T < \min\{T_1,\ C_2^{-2/\alpha},\ (4C_4)^{-2/\alpha}\}$, it then follows from (2.3) that

$$\|\tilde{g}\|_{C^1([0,T])} + \|\tilde{h}\|_{C^1([0,T])} \le \tfrac{1}{2}(\|\tilde{g}\|_{C^1([0,T])} + \|\tilde{h}\|_{C^1([0,T])}).$$

This implies that $\tilde{g} \equiv 0 \equiv \tilde{h}$, which implies that $V_U \equiv 0 \equiv V_L$, establishing uniqueness of local solutions of problem (*P*). By the Schauder estimates, $w_U(t,x) \in C^{1+\alpha/2,2+\alpha}((0,T]\times[g(t),0])$ and $w_L(t,x) \in C^{1+\alpha/2,2+\alpha}((0,T]\times[0,h(t)])$. □

LEMMA 2.4. *For any given $\beta_U,\ \beta_L > 0$, problem (P) admits a unique positive solution (w_U, w_L, g, h) which exists for all $t \in (0,\infty)$.*

PROOF. Let $[0, T_{\max})$ be the maximal time interval of existence for the solution. We assume $T_{\max} < \infty$. Let $K_0 := \|w_0\|_{L^\infty} + 1$. From Lemma 2.1,

$$0 \le w_U(t,x), w_L(t,y) \le K_0 \quad \text{for } t \in (0, T_{\max}),\ x \in [g(t),0],\ y \in [0,h(t)].$$

Furthermore, $f(s) = s - s^2 \le K_0$ for $s \in [0,K_0]$. For $(t,x) \in (0,T_{\max})\times(h(t) - M^{-1}, h(t))$, define $u_L(t,x) := K_0[2M(h(t)-x) - M^2(h(t)-x)^2]$, where

$$M := \max\left\{h_0^{-1}, \sqrt{\frac{1}{2}\beta_L} + \sqrt{\frac{1}{4}\beta_L^2 + 1}, \sqrt{\frac{1}{2}\beta_U} + \sqrt{\frac{1}{4}\beta_U^2 + 1}, \frac{4\|w_0\|_{C^1([0,h_0])}}{3K_0}\right\}.$$

From direct calculations and the comparison principle,

$$-g'(t), h'(t) \le 2\mu K_0 M := K_1 \quad \text{for } t \in (0, T_{\max}).$$

Similar to the proof of Theorem 2.3, by the L^p theory,

$$(w_U, w_L) \in W_p^{1,2}([0,T]\times[g(t),0]) \times W_p^{1,2}([0,T]\times[0,h(t)])$$

for any $p > 1$ and $T \in (T_{\max}/2, T_{\max})$. From the Sobolev embedding theorem, for any $\alpha \in (0,1)$ and some large enough $p > 1$ depending on α,

$$\|w_L\|_{C^{(1+\alpha)/2,1+\alpha}([0,T]\times[0,h(t)])} + \|w_U\|_{C^{(1+\alpha)/2,1+\alpha}([0,T]\times[g(t),0])} \le C_{p,\alpha}, \tag{2.4}$$

where $C_{p,\alpha}$ is a positive constant independent of $T \in (T_{\max}/2, T_{\max})$.

Choose $t_n \in (0, T_{\max})$ with $t_n \nearrow T_{\max}$, and regard $(w_U(t_n,\cdot), w_L(t_n,\cdot), g(t_n), h(t_n))$ as the initial data. By (2.4) and the properties of g and h, we can repeat the proof of Theorem 2.3 to conclude that there is $s_0 > 0$ depending on $C_{p,\alpha}$ but independent of n such that problem (*P*) has a unique solution (w_U, w_L, g, h) for $t \in [t_n, t_n + s_0]$. It follows that problem (*P*) admits a solution (w_U, w_L, g, h), which is defined for $t \in [0, t_n + s_0]$. As $t_n + s_0 > T_{\max}$ for $n \gg 1$, this contradicts the definition of $T_{\max}$, and so we see that $T_{\max} = \infty$, as desired. The proof is complete. □

https://doi.org/10.1017/S0004972726101233 Published online by Cambridge University Press

3. Classification of dynamical behaviour

This section deals with the long-time dynamical behaviour of our problem and the proof of Theorem 1.2.

3.1. Linear eigenvalue problem. For any $\beta_L,\ \beta_U,\ l_1$ and $l_2 > 0$, it is known that

$$\begin{cases} -\varphi_U'' + \beta_U \varphi_U' - \varphi_U = \lambda \varphi_U, & -l_1 < x < 0, \\ -\varphi_L'' + \beta_L \varphi_L' - \varphi_L = \lambda \varphi_L, & 0 < x < l_2, \\ \varphi_U(0) = \varphi_L(0), \quad a_U \varphi_U'(0) = a_L \varphi_L'(0), \\ \varphi_U(-l_1) = 0 = \varphi_L(l_2), \end{cases} \tag{3.1}$$

admits a principal eigenvalue $\lambda_1(l_1, l_2)$. We obtain the following result.

LEMMA 3.1. *For any given $l_1,\ l_2 > 0$, $0 < \beta_L,\ \beta_U < 2$, the following statements hold.*

(i) *If $l_2 \geq 2\pi/\sqrt{4-\beta_L^2}$, then $\lambda_1(l_1, l_2) \leq 0$ for all $l_1 > 0$.*

(ii) *If $l_2 < 2\pi/\sqrt{4-\beta_L^2}$, then there exists*

$$\ell^*(l_2) = -\frac{2}{\sqrt{4-\beta_U^2}} \arctan\left(\frac{\beta_L \sqrt{4-\beta_U^2}}{\beta_U \sqrt{4-\beta_L^2}} \tan l_2 \sqrt{1 - \frac{1}{4}\beta_L^2} \right) \tag{3.2}$$

such that $\lambda_1(l_1, l_2)$ is negative, 0 or positive according as $l_1 > \ell^*(l_2)$, $l_1 = \ell^*(l_2)$ or $l_1 < \ell^*(l_2)$. There is a unique $\hbar^* \in (0, 2\pi/\sqrt{4-\beta_L^2})$ satisfying $\ell^*(\hbar^*) = \hbar^*$.

PROOF. (i) We can check directly that if $l_2 \geq 2\pi/\sqrt{4-\beta_L^2}$, then the principal eigenvalue $\tilde{\lambda}_1$ of

$$-\varphi'' + \beta_L \varphi' - \varphi = \lambda \varphi \ \text{ in } \ (0, l_2), \quad \varphi(0) = 0 = \varphi(l_2),$$

satisfies $\tilde{\lambda}_1 \leq 0$. By the comparison principle, we obtain the desired result.

(ii) Given $0 < l_1^1 < l_1^2 < 2\pi/\sqrt{4-\beta_L^2}$, let $(\varphi_{L,i}, \varphi_{U,i})$ be the positive principal eigenfunction pair of problem (3.1) corresponding to λ_1^i with l_1 replaced by l_1^i for $i = 1,\ 2$. Then, $\varphi_{U,i}(-l_1^i) = 0 = \varphi_{L,i}(l_2)$ and

$$\begin{cases} -\varphi_{U,i}'' + \beta_U \varphi_{U,i}' - \varphi_{U,i} = \lambda_1^i \varphi_{U,i}, & -l_1^i < x < 0, \\ -\varphi_{L,i}'' + \beta_L \varphi_{L,i}' - \varphi_{L,i} = \lambda_1^i \varphi_{L,i}, & 0 < x < l_2, \\ \varphi_{U,i}(0) = \varphi_{L,i}(0), \quad a_U \varphi_{U,i}'(0) = a_L \varphi_{L,i}'(0). \end{cases}$$

Motivated by [13, 17], multiply the equation for $\varphi_{L,1}$ by $\varphi_{L,2}$ and the equation for $\varphi_{L,2}$ by $\varphi_{L,1}$, subtract the resulting equations, and integrate over $(0, l_2)$ to obtain

$$(\lambda_1^1 - \lambda_1^2) \int_0^{l_2} \varphi_{L,1} \varphi_{L,2}\, dx = \varphi_{L,2}(0) \varphi_{L,1}'(0) - \varphi_{L,1}(0) \varphi_{L,2}'(0). \tag{3.3}$$

https://doi.org/10.1017/S0004972726101233 Published online by Cambridge University Press

Similarly,

$$(\lambda_1^1 - \lambda_1^2)\int_{-l_1^1}^{0} \varphi_{U,1}\varphi_{U,2}\,dx = \varphi_{U,1}(0)\varphi'_{U,2}(0) - \varphi_{U,2}(0)\varphi'_{U,1}(0) + \varphi_{U,2}(-l_1^1)\varphi'_{U,1}(-l_1^1). \tag{3.4}$$

We claim that $\Phi = \varphi_{L,2}(0)\varphi'_{L,1}(0) - \varphi_{L,1}(0)\varphi'_{L,2}(0) > 0$. Otherwise, $\Phi \le 0$. It follows from (3.3) that $\lambda_1^1 \le \lambda_1^2$. Moreover,

$$\varphi_{U,2}(0)\varphi'_{U,1}(0) - \varphi_{U,1}(0)\varphi'_{U,2}(0) = a_L a_U^{-1}[\varphi_{L,2}(0)\varphi'_{L,1}(0) - \varphi_{L,1}(0)\varphi'_{L,2}(0)] \le 0.$$

This, together with (3.4), implies that $\lambda_1^1 > \lambda_1^2$, which is a contradiction. This proves our claim and that $\lambda_1^1 > \lambda_1^2$. The monotonicity of $\lambda_1(l_1, l_2)$ in $l_1 > 0$ is established.

Next, we prove that $\lambda_0 := \lim_{l_1 \to 0} \lambda_1(l_1, l_2) > 0$. The monotonicity of $\lambda_1(l_1, l_2)$ in $l_1 > 0$ implies the existence of the limit λ_0. Letting $l_1 \to 0$ in problem (3.1) gives

$$-\varphi_L'' + \beta_L\varphi_L' - \varphi_L = \lambda_0\varphi_L \text{ for } x \in (0, l_2), \quad \varphi_L(0) = 0 = \varphi_L(l_2).$$

Since $l_2 < 2\pi/\sqrt{4-\beta_L^2}$, we see that $\lambda_0 > 0$. Moreover, since $\beta_U \in (0, 2)$, it follows from a similar argument to that in item (i) that $\lambda(l_1, l_2) < 0$ when $l_1 \ge 2\pi/\sqrt{4-\beta_U^2}$. This, together with the above arguments, implies the existence and uniqueness of $\ell^*(l_2)$.

Finally, setting $\varphi_j(x) := e^{\beta_j x/2}\psi_j(x)$ for $j = U,\ L$ and $l_1 = \ell^*(l_2)$ in (3.1), we see that there are four constants $C_1 > 0$, $C_2 > 0$, θ_1 and θ_2 such that

$$\psi_L(x) = C_1 \cos\sqrt{1 - \tfrac{1}{4}\beta_L^2}(x - \theta_1), \quad \psi_U(x) = C_2 \cos\sqrt{1 - \tfrac{1}{4}\beta_U^2}(x - \theta_2),$$

on the respective intervals $(0, l_2)$ and $(-\ell^*(l_2), 0)$. Since $\psi_U(-\ell^*(l_2)) = \psi_L(l_2) = 0$, then

$$(l_2 - \theta_1)\sqrt{1 - \tfrac{1}{4}\beta_L^2} = (\pi/2) = (\ell^*(l_2) + \theta_2)\sqrt{1 - \tfrac{1}{4}\beta_U^2}.$$

Recalling that $a_U\psi'_U(0) = a_L\psi'_L(0)$ and $\psi_L(0) = \psi_U(0)$, a direct calculation yields

$$a_U\sqrt{4-\beta_U^2}\tan\theta_2\sqrt{1 - \tfrac{1}{4}\beta_U^2} = a_L\sqrt{4-\beta_L^2}\tan\theta_1\sqrt{1 - \tfrac{1}{4}\beta_L^2}.$$

This, together with the above results, proves (3.2). Further, $\ell^*(l_2)$ is decreasing in l_2, $\ell^*(l_2) \to 0$ as $l_2 \to 2\pi/\sqrt{4-\beta_L^2}$ and $\ell^*(l_2) \to 2\pi/\sqrt{4-\beta_U^2}$ as $l_2 \to 0$. The monotonicity of $\ell^*(l_2)$ yields the existence and uniqueness of $\hbar^*$. □

3.2. Vanishing phenomenon. We start with the following result on vanishing.

LEMMA 3.2. *Assume that $\beta_U,\ \beta_L \in (0, 2)$, and (w_U, w_L, g, h) is a solution of problem (P) with $w_0 \in \mathscr{X}(h_0)$ and $h_0 > 0$. Then, the following three assertions are equivalent:*

(i) *h_∞ or g_∞ is finite*; (ii) $\lambda_1(-g_\infty, h_\infty) \ge 0$; (iii) $(w_U, w_L) \to (0, 0)$ *uniformly as* $t \to \infty$,

where $\lambda_1(-g_\infty, h_\infty)$ is the principal eigenvalue of (3.1) with $l_1 = -g_\infty$, $l_2 = h_\infty$.

https://doi.org/10.1017/S0004972726101233 Published online by Cambridge University Press

PROOF. (i)⇒(ii) Without loss of generality, assume that $h_\infty < \infty$. Aiming for a contradiction, assume that $\lambda_1(-g_\infty, h_\infty) < 0$. Then, there is $t_1 > 0$ large enough such that $\lambda_1(-g(t_1), h(t_1)) < 0$. Now, we consider the auxiliary problem:

$$\begin{cases} \partial_t \underline{w}_U - \partial_{xx}\underline{w}_U + \beta_U \partial_x \underline{w}_U = \underline{w}_U - \underline{w}_U^2, & g(t_1) < x < 0,\ t > t_1, \\ \partial_t \underline{w}_L - \partial_{xx}\underline{w}_L + \beta_L \partial_x \underline{w}_L = \underline{w}_L - \underline{w}_L^2, & 0 < x < \xi(t),\ t > t_1, \\ \underline{w}_L(t, \xi(t)) = 0 = \underline{w}_U(t, g(t_1)), & t > t_1, \\ \xi(t_1) = h(t_1),\ \xi'(t) = -\mu \partial_x \underline{w}_L(t, \xi(t)), & t > t_1, \\ \underline{w}_U(t, 0) = \underline{w}_L(t, 0),\ a_U \partial_x \underline{w}_U(t, 0) = a_L \partial_x \underline{w}_L(t, 0), & t > t_1, \\ \underline{w}_L(t_1, x) = w_L(t_1, x),\ \underline{w}_U(t_1, y) = w_U(t_1, y), & x \in [0, h(t_1)],\ y \in [g(t_1), 0]. \end{cases}$$

Clearly, $(\underline{w}_U, \underline{w}_L)$ is a subsolution of problem (*P*). Thus, $\xi(t) \le h(t)$ and $\xi(\infty) < \infty$. It follows from a similar argument to that in [5, Lemma 2.2] that

$$\|\underline{w}_L(t, \cdot) - \Phi_L(\cdot)\|_{C^2([h_0, \xi(t)])} \to 0, \quad \text{as } t \to \infty,$$

where (Φ_U, Φ_L) is the solution pair of

$$\begin{cases} -\Phi_U'' + \beta_U \Phi_U' = \Phi_U - \Phi_U^2, & g(t_1) < x < 0, \\ -\Phi_L'' + \beta_L \Phi_L' = \Phi_L - \Phi_L^2, & 0 < x < \xi(\infty), \\ \Phi_L(\xi(\infty)) = 0 = \Phi_U(g(t_1)), & \\ \Phi_U(0) = \Phi_L(0),\ a_U \Phi_U'(0) = a_L \Phi_L'(0). & \end{cases}$$

It follows that $\xi'(t) \to -\mu \Phi_L'(\xi(\infty)) > 0$, which contradicts $\xi(\infty) < \infty$.

(ii)⇒(i) Since $\beta_L,\ \beta_U \in (0, 2)$, if assertion (ii) holds, we can check that $h_\infty < 2\pi/\sqrt{4 - \beta_L^2}$ and $g_\infty > -2\pi/\sqrt{4 - \beta_U^2}$, which implies assertion (i).

(iii)⇒(ii) We use a proof by contradiction and assume that, for some small $\varepsilon > 0$, there exists $t_2 > 0$ large such that $\lambda_\varepsilon := \lambda_1(-g(t_2) - \varepsilon, h(t_2) - \varepsilon) < 0$. Let (φ_U, φ_L) be the corresponding positive eigenfunction pair with $\max\{\|\varphi_U\|_{L^\infty},\ \|\varphi_L\|_{L^\infty}\} = 1$. Set

$$\begin{cases} u_U(t, x) = \delta \varphi_U(x) & \text{for } t \ge 0,\ x \in [g(t_2) + \varepsilon, 0], \\ u_L(t, x) = \delta \varphi_L(x) & \text{for } t \ge 0,\ x \in [0, h(t_2) - \varepsilon], \end{cases}$$

where $0 < \delta < \min\{-\lambda_\varepsilon, \min_{x \in [0, h(t_2) - \varepsilon]} w_L(t_2, x), \min_{x \in [g(t_2) + \varepsilon, 0]} w_U(t_2, x)\}$. It follows from Lemma 2.1 that $w_U(t + t_2, x) \ge u_U(t, x)$ for $(t, x) \in [0, \infty) \times [0, h(t_2) - \varepsilon]$, which contradicts assertion (iii).

(ii)⇒(iii) As $\beta_L,\ \beta_U \in (0, 2)$, it follows from the proof of Lemma 3.1 and $\lambda_1(-g_\infty, h_\infty) \ge 0$ that h_∞ and g_∞ are both bounded. This, together with [7, Theorem 1.1], implies assertion (iii) and completes the proof. □

We end this part with a sufficient condition for vanishing.

LEMMA 3.3. *Assume that $\beta_U,\ \beta_L \in (0, 2)$ and (w_U, w_L, g, h) is a solution of problem (P) with $w_0 \in \mathscr{X}(h_0)$ for some $h_0 > 0$. Let $\hbar^*$ be given in Lemma 3.1. If $h_0 < \hbar^*$ and $\|w_0\|_{L^\infty([-h_0, h_0])}$ is sufficiently small, then vanishing happens.*

https://doi.org/10.1017/S0004972726101233 Published online by Cambridge University Press

PROOF. Choosing $h_1 \in (h_0, \hbar^*)$, it follows from Lemma 3.1 that (3.1) with $l_1 = l_2 = h_1$ admits a positive principal eigenvalue, denoted by λ^*, whose corresponding positive eigenfunction pair, denoted by $(\varphi_U^*, \varphi_L^*)$, can be normalised by $\|\varphi_U^*\|_{L^\infty} + \|\varphi_L^*\|_{L^\infty} = 1$. There exists $x^* \in (h_1/2, h_1)$ such that

$$(\varphi_U^*)'(x) > 0 > (\varphi_L^*)'(y) \quad \text{for } x \in [-h_1, -x^*],\ y \in [x^*, h_1].$$

Set $\delta := \min\{h_1/h_0 - 1, \lambda_*, 1\}$, $\eta := \max\{h_1 - (\delta/2)h_0, h_0, x^*\}$, $\varepsilon_0 := \min\{\varphi_U^*(-\eta), \varphi_L^*(\eta)\}$. Then, $\varepsilon_0 \le 1$ and there is $\varepsilon_1 = \varepsilon_1(\delta) > 0$ such that $2\mu\varepsilon_1[(\varphi_U^*)'(-h_1) - (\varphi_L^*)'(h_1)] < \delta^2 h_0$. Define

$$\begin{cases} v_U(t,x) = \varepsilon_0\varepsilon_1 e^{-\delta t}\varphi_U^*(x) & \text{for } t \ge 0,\ x \in [-h_1, 0], \\ v_L(t,x) = \varepsilon_0\varepsilon_1 e^{-\delta t}\varphi_L^*(x) & \text{for } t \ge 0,\ x \in [0, h_1]. \end{cases}$$

If w_0 is chosen to be sufficiently small such that $w_0 \le \varepsilon_0\varepsilon_1\varphi_U^*$ in $[-h_0, 0]$, and $w_0 \le \varepsilon_0\varepsilon_1\varphi_L^*$ in $[0, h_0]$, it follows from the comparison principle that

$$\begin{cases} w_U(t,x) \le v_U(t,x) & \text{for } t \in [0,\tau),\ x \in [g(t), 0], \\ w_L(t,x) \le v_L(t,x) & \text{for } t \in [0,\tau),\ x \in [0, h(t)], \end{cases}$$

where $\tau := \sup\{t > 0 : h(t) < h_1,\ \ g(t) > -h_1\}$. We prove that $\tau = \infty$. Once this is done, we see that (g_∞, h_∞) is bounded, and that $\|w_U(t,\cdot)\|_{L^\infty([g(t),0])} \to 0$ and $\|w_L(t,\cdot)\|_{L^\infty([0,h(t)])} \to 0$ as $t \to \infty$, so the vanishing conclusion is established.

In what follows, we prove $\tau = \infty$ by contradiction. Assume that $\tau < \infty$. Without loss of generality, we may assume that $h(\tau) = h_1$. Motivated by [14], define

$$k(t) := h_0(1 + \delta - \tfrac{1}{2}\delta e^{-\delta t}), \quad \overline{w_L}(t,x) := \varepsilon_1 e^{-\delta t}\varphi_L^*(x - k(t) + h_1)$$

for $t \ge 0$, $x \in I_1(t) := [\eta + k(t) - h_1, k(t)]$. We check that $\eta + k(t) - h_1 \ge h_0$ and

$$\begin{cases} \partial_t\overline{w_L} - \partial_{xx}\overline{w_L} + \beta_L\partial_x\overline{w_L} - \overline{w_L} + \overline{w_L}^2 \ge 0, & t \ge 0,\ x \in I_1(t), \\ k'(t) = \dfrac{\delta^2 h_0}{2}e^{-\delta t} \ge -\mu\varepsilon_1 e^{-\delta t}(\varphi_L^*)'(h_1) = -\mu\partial_x\overline{w_L}(t, k(t)), & t > 0. \end{cases}$$

We claim that $k(t) \ge h(t)$ for $t \in [0,\tau]$. As $\eta \le h_1$, the desired conclusion is true when $\eta + k(t) - h_1 \ge h(t)$. We consider the case where $\{0 \le t \le \tau : h(t) > \eta + k(t) - h_1\} \ne \emptyset$ consists of some intervals and $[s_1, s_2]$ is one of them. By the choice of η, we see that $s_1 > 0$, $h(s_1) = \eta + k(s_1) - h_1$. As $\eta + k(t) - h_1 > h_0$ for $t \ge 0$,

$$w_L(t, \eta + k(t) - h_1) \le \varepsilon_0\varepsilon_1 e^{-\delta t} \le \varepsilon_1 e^{-\delta t}\varphi_L^*(t,\eta) \equiv \overline{w_L}(t, \eta + k(t) - h_1), \quad t \in [s_1, s_2].$$

Hence, by comparison, $h(t) < k(t)$ in $[s_1, s_2]$. This proves our claim and shows that

$$h(\tau) \le k(\tau) < k(\infty) \le h_1,$$

which contradicts $h(\tau) = h_1$. Thus, $\tau = \infty$, which completes the proof. □

3.3. Spreading phenomenon. We give some sufficient conditions for spreading.

LEMMA 3.4. *Assume β_U, $\beta_L \in (0,2)$ and (w_U, w_L, g, h) is a solution of problem (P) for $w_0 \in \mathscr{X}(h_0)$ with $h_0 > 0$. Let $\hbar^*$ be given in Lemma 3.1. If $h_0 \ge \hbar^*$, spreading happens.*

https://doi.org/10.1017/S0004972726101233 Published online by Cambridge University Press

PROOF. We first check that $[-\hbar^*, \hbar^*] \subset [g(t), h(t)]$ for any $t > 0$. By Lemma 3.2, $-g_\infty = h_\infty = \infty$ and $(w_U, w_L) \twoheadrightarrow (0, 0)$ as $t \to \infty$ locally uniformly in $(-\infty, 0] \times [0, \infty)$.

In what follows, we prove that (w_U, w_L) converges to $(1, 1)$ locally uniformly as $t \to \infty$. Setting $\bar{w}(t) = 1 + \|w_0\|_{L^\infty([-h_0, h_0])} e^{-t}$, we can check that

$$\begin{cases} \limsup_{t\to\infty} \|w_U(t,\cdot)\|_{L^\infty([g(t),0])} \le \lim_{t\to\infty} \bar{w}(t) = 1, \\ \limsup_{t\to\infty} \|w_L(t,\cdot)\|_{L^\infty([0,h(t)])} \le \lim_{t\to\infty} \bar{w}(t) = 1. \end{cases} \tag{3.5}$$

As $h_0 \ge \hbar^*$, there is $\epsilon > 0$ small such that $[-\hbar^* - \epsilon, \hbar^* + \epsilon] \subseteq (g(1) + \epsilon, h(1) - \epsilon)$. It follows from Lemma 3.1 that (3.1) with $l_1 = l_2 = \hbar^* + \epsilon$ admits a negative principal eigenvalue, whose corresponding positive eigenfunction pair $(\varphi_U^\epsilon, \varphi_L^\epsilon)$, can be normalised by $\|\varphi_U^\epsilon\|_{L^\infty} + \|\varphi_L^\epsilon\|_{L^\infty} = 1$. We can find $\rho > 0$ small such that

$$w_U(1, x) \ge \rho\varphi_U^\epsilon(x), \quad w_L(1, y) \ge \rho\varphi_L^\epsilon(y) \quad \text{for } x \in [-\hbar^* - \epsilon, 0],\ y \in [0, \hbar^* + \epsilon]. \tag{3.6}$$

Let $(\underline{w}_U, \underline{w}_L, \underline{g}, \underline{h})$ be the solution of (P) with initial function $(\rho\varphi_U^\epsilon, \rho\varphi_L^\epsilon)$. We can check directly that $-\underline{g}_\infty = \underline{h}_\infty = \infty$. By the standard parabolic comparison principle,

$$\underline{w}_U(t, x) \ge \rho\varphi_U^\epsilon(x), \quad \underline{w}_L(t, y) \ge \rho\varphi_L^\epsilon(y) \quad \text{for } t > 0,\ x \in [-\hbar^* - \epsilon, 0],\ y \in [0, \hbar^* + \epsilon].$$

It follows from Lemma 2.1 that $(\underline{w}_U, \underline{w}_L)$ is nondecreasing with respect to t. Define $(\underline{w}_{U,\infty}(x), \underline{w}_{L,\infty}(x)) := (\lim_{t\to\infty} \underline{w}_U(t, x), \lim_{t\to\infty} \underline{w}_L(t, x))$. Note that $-\underline{g}_\infty = \underline{h}_\infty = \infty$. By a similar argument to that in [8], $(\underline{w}_{U,\infty}(x), \underline{w}_{L,\infty}(x))$ is a positive solution of

$$\begin{cases} -\partial_{xx} W_U + \beta_U \partial_x W_U - W_U + W_U^2 = 0, & -\infty < x < 0, \\ -\partial_{xx} W_L + \beta_L \partial_x W_L - W_L + W_L^2 = 0, & 0 < x < \infty, \\ W_U(0) = W_L(0),\ a_U \partial_x W_U(0) = a_L \partial_x W_L(0), \end{cases}$$

and satisfies $0 < \underline{w}_{U,\infty} \le 1$, $0 < \underline{w}_{L,\infty} \le 1$. Thanks to [8, Theorem 1.1], the above problem admits a unique positive solution pair $(1, 1)$. Thus,

$$(\underline{w}_U(t,\cdot), \underline{w}_L(t,\cdot)) \to (1, 1) \quad \text{locally uniformly as } t \to \infty. \tag{3.7}$$

Moreover, from (3.6) and Lemma 2.1,

$$\underline{w}_U(t, x) \le w_U(t + 1, x), \quad \underline{w}_L(t, y) \le w_L(t + 1, y) \quad \text{for } t > 0,\ x \in [\underline{g}(t), 0],\ y \in [0, \underline{h}(t)].$$

From these estimates and (3.7),

$$\begin{cases} \liminf_{t\to\infty} w_U(t, x) \ge 1 & \text{locally uniformly in } (-\infty, 0], \\ \liminf_{t\to\infty} w_L(t, x) \ge 1 & \text{locally uniformly in } [0, +\infty). \end{cases}$$

From this, together with (3.5), $(w_U, w_L) \to (1, 1)$ locally uniformly as $t \to \infty$, which completes the proof of this lemma. □

3.4. Proof of Theorem 1.2. By Lemmas 3.2–3.4, either g_∞, h_∞ are both bounded and $(w_U, w_L) \to (0, 0)$ uniformly as $t \to \infty$, or $h_\infty = -g_\infty = \infty$ and $(w_U, w_L) \to (1, 1)$ locally uniformly as $t \to \infty$, which implies a spreading–vanishing dichotomy result. In the following, we show the spreading threshold is sharp.

https://doi.org/10.1017/S0004972726101233 Published online by Cambridge University Press

Set

$$\sigma^* := \sup\{\sigma_0 \geq 0 : \text{vanishing happens for } \sigma \in (0, \sigma_0]\}.$$

If $h_0 \geq \hbar^*$, it follows from Lemma 3.4 that $\sigma^* = 0$. We now assume $h_0 < \hbar^*$. From Lemma 3.3, vanishing happens for all small $\sigma > 0$, so $\sigma^* \in (0, +\infty]$. If $\sigma^* = \infty$, the proof is done. Let us assume $\sigma^* \in (0, \infty)$. The comparison principle shows that vanishing happens for $\sigma \in [0, \sigma^*)$. If vanishing does not happen for $\sigma = \sigma^*$, from the spreading–vanishing dichotomy result, spreading must happen for $\sigma = \sigma^*$. Thus, there is $t_0 \gg 1$ such that $h(t_0) - g(t_0) > 2\hbar^* + 1$. Due to the continuous dependence of the solution of (*P*) on its initial values, there exists $\epsilon > 0$ small such that the solution $(w_U^\epsilon, w_L^\epsilon, g^\epsilon, h^\epsilon)$ of problem (*P*) with $w_0 = (\sigma^* - \epsilon)\phi$ satisfies $h^\epsilon(t_0) - g^\epsilon(t_0) > 2\hbar^*$. It follows from Lemma 3.4 that spreading happens for $\sigma^* - \epsilon$, which contradicts the definition of σ^*. Hence, vanishing happens for $\sigma = \sigma^*$. From the spreading–vanishing dichotomy result and the comparison principle, spreading happens for $\sigma > \sigma^*$. This completes the proof.

References

[1] D. Aronson and H. Weinberger, 'Multidimensional nonlinear diffusion arising in population genetics', *Adv. Math.* **30** (1978), 33–76.

[2] G. Bunting, Y. Du and K. Krakowski, 'Spreading speed revisited: analysis of a free boundary model', *Netw. Heterog. Media* **7** (2012), 583–603.

[3] X. F. Chen and A. Friedman, 'A free boundary problem arising in a model of wound healing', *SIAM J. Math. Anal.* **32** (2000), 778–800.

[4] Y. Du, 'Propagation and reaction-diffusion models with free boundaries', *Bull. Math. Sci.* **12** (2022), 2230001.

[5] Y. Du and Z. Guo, 'Spreading-vanishing dichotomy in the diffusive logistic model with a free boundary II', *J. Differential Equations* **250** (2011), 4336–4366.

[6] Y. Du and Z. Lin, 'Spreading-vanishing dichotomy in the diffusive logistic model with a free boundary', *SIAM J. Math. Anal.* **42** (2010), 377–405.

[7] Y. Du and B. Lou, 'Spreading and vanishing in nonlinear diffusion problems with free boundaries', *J. Eur. Math. Soc. (JEMS)* **17** (2015), 2673–2724.

[8] Y. Du, B. Lou, R. Peng and M. Zhou, 'The Fisher-KPP equation over simple graphs: varied persistence states in river networks', *J. Math. Biol.* **80** (2020), 1559–1616.

[9] H. Gu, B. Lou and M. Zhou, 'Long time behavior of solutions of Fisher-KPP equation with advection and free boundaries', *J. Funct. Anal.* **269** (2015), 1714–1768.

[10] F. Hamel, F. Lutscher and M. Zhang, 'Propagation and blocking in a two-patch reaction-diffusion model', *J. Math. Pures Appl. (9)* **168** (2022), 213–267.

[11] Y. Ji and N. Sun, 'Dynamics of solutions of a free boundary model in river networks', Preprint.

[12] Y. Jin, R. Peng and J. Shi, 'Population dynamics in river networks', *J. Nonlinear Sci.* **29** (2019), 2501–2545.

[13] Y. Meng, J. Ge and Z. Lin, 'Dynamics of a free boundary problem modelling species invasion with impulsive harvesting', *Discrete Contin. Dyn. Syst. Ser. B* **27** (2022), 7689–7720.

[14] N. Sun, B. Lou and M. Zhou, 'Fisher-KPP equation with free boundaries and time-periodic advections', *Calc. Var. Partial Differential Equations* **56** (2017), 36–61.

[15] M. Wang, 'Existence and uniqueness of solutions of free boundary problems in heterogeneous environments', *Discrete Contin. Dyn. Syst.* **24** (2019), 415–421.

[16] R. Wang and Y. Du, 'Long-time dynamics of a diffusive epidemic model with free boundaries', *Discrete Contin. Dyn. Syst.* **26** (2021), 2201–2238.

[17] H. Xu, Z. Lin and C. Santos, 'Interaction between harvesting intervention and birth perturbation in an age-structured model', *J. Math. Biol.* **91**(2) (2025), 12.

https://doi.org/10.1017/S0004972726101233 Published online by Cambridge University Press

YING JI, School of Mathematics and Statistics,
Shandong Normal University, Jinan 250358, PR China
e-mail: 15753982488@163.com

GUIHONG FAN, School of Department of Mathematics,
Columbus State University, Columbus, GA 31907, USA
e-mail: fan_guihong1@columbusstate.edu

NINGKUI SUN, School of Mathematics and Statistics,
Shandong Normal University, Jinan 250358, PR China
e-mail: sunnk@sdnu.edu.cn

https://doi.org/10.1017/S0004972726101233 Published online by Cambridge University Press