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TWO-CARDINAL DERIVED TOPOLOGIES, INDESCRIBABILITY AND RAMSEYNESS

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Abstract. We introduce a natural two-cardinal version of Bagaria's sequence of derived topologies on ordinals. We prove that for our sequence of two-cardinal derived topologies, limit points of sets can be characterized in terms of a new iterated form of pairwise simultaneous reflection of certain kinds of stationary sets, the first few instances of which are often equivalent to notions related to strong stationarity, which has been studied previously in the context of strongly normal ideals. The non-discreteness of these two-cardinal derived topologies can be obtained from certain two-cardinal indescribability hypotheses, which follow from local instances of supercompactness. Additionally, we answer several questions posed by the first author, Holy and White on the relationship between Ramseyness and indescribability in both the cardinal context and in the two-cardinal context.

§1. Introduction. The *derived set* of a subset A of a topological space (X,τ) is the collection $d(A)$ of all limit points of A in the space. We refer to the function d as the Cantor derivative of the space (X,τ). Recently, Bagaria showed [2] that the *derived topologies on ordinals*, whose definition we review now, are closely related to certain widely studied stationary reflection properties and large cardinal notions. Suppose δ is an ordinal and τ_0 is the order topology on δ. That is, τ_0 is the topology on δ generated by $B_0=\{\{0\}\}\cup\{(\alpha,\beta)\mid \alpha<\beta\leq\delta\}$. For a set $A\subseteq\delta$, it easily follows that the collection $d_0(A)$ of all limit points of A in the space (δ,τ_0), is equal to $\{\alpha<\delta\mid\ A \text{ is unbounded in } \alpha\}$. Beginning with the interval topology on δ and declaring more and more derived sets to be open, Bagaria [2] introduced the sequence of derived topologies $\langle\tau_\xi\mid\xi<\delta\rangle$ on δ. For example, τ_1 is the topology on δ generated by $B_1=B_0\cup\{d_0(A)\mid A\subseteq\delta\}$, and τ_2 is the topology on δ generated by $B_2=B_1\cup\{d_1(A)\mid A\subseteq\delta\}$ where d_1 is the Cantor derivative of the space (δ,τ_1). Bagaria showed that limit points of sets in the spaces (δ,τ_ξ), for $\xi\in\{1,2\}$, can be characterized as follows. For $A\subseteq\delta$ and $\alpha<\delta$: α is a limit point of A in (δ,τ_1) if and only if A is stationary in α, and α is a limit point of A in (δ,τ_2) if and only if whenever S and T are stationary subsets of α there is a $\beta\in A$ such that $S\cap\beta$ and $T\cap\beta$ are stationary subsets of β. Furthermore, Bagaria proved that limit points of sets in the spaces (δ,τ_ξ) for $\xi>2$ can be characterized in terms of an iterated form of pairwise simultaneous stationary reflection called ξ-s-stationarity.

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In this article we address the following natural question: is there some analogue of the sequence of derived topologies on an ordinal in the two-cardinal setting? Specifically, suppose κ is an uncountable cardinal and X is a set of ordinals with $\kappa \subseteq X$. Is there a topology τ on $P_\kappa X$ such that, for all $A \subseteq P_\kappa X$, the limit points of A in the space $(P_\kappa X, \tau)$ are precisely the points $x \in P_\kappa X$ such that the set A satisfies:

- Some *unboundedness* condition at x?
- Some *stationarity* condition at x?
- Some *pairwise simultaneous stationary reflection-like* condition at x?

Recall that, for an infinite cardinal κ and a set X, $P_\kappa X = \{x \subseteq X \mid |x| < \kappa\}$. Given $x \in P_\kappa X$, we denote $|x \cap \kappa|$ by κ_x. For $x, y \in P_\kappa X$ we say that x is a *strong subset* of y and write $x \prec y$ if $x \subseteq y$ and $|x| < \kappa_y$. Let us note that the ordering $\prec$, and its variants, are used in the context of supercompact Prikry forcings [19]. In Section 3.1, we show that the ordering $\prec$ induces a natural topology τ_0 on $P_\kappa X$ analogous to the order topology on an ordinal δ. Furthermore, beginning with τ_0 and following the constructions of [2], in Section 3.2 we define a sequence of *derived topologies* $\langle \tau_\xi \mid \xi < \kappa \rangle$ on $P_\kappa X$. Let us note that after submitting the current article, the authors learned that Torres, working under the supervision of Bagaria, simultaneously and independently defined a sequence of two-cardinal derived topologies and obtained results similar to those in Sections 3.2–3.6 involving the relationship between various two-cardinal notions of ξ-s-stationarity and two-cardinal derived topologies.

We show (see Propositions 3.10 and 3.14) that in the space $(P_\kappa X, \tau_1)$, for $x \in P_\kappa X$ with $\kappa_x = x \cap \kappa$ an inaccessible cardinal, x is a limit point of a set $A \subseteq P_\kappa X$ if and only if A is *strongly stationary in* $P_{x \cap \kappa} x$ (see Section 2 for the definition of strongly stationary set). Let us note that although the notion of strong stationarity is distinct from the widely popular notion of two-cardinal stationarity introduced by Jech [23] (see [11, Lemma 2.2]), it has previously been studied by several authors [10, 11, 25, 26, 31]. The analogy with the case of derived topologies on ordinals continues: in the space $(P_\kappa X, \tau_2)$, when $x \in P_\kappa X$ is such that $\kappa_x = x \cap \kappa < \kappa$ and $P_{x \cap \kappa} x$ satisfies a two-cardinal version of Π^1_1-indescribability, x is a limit point of a set $A \subseteq P_\kappa X$ if and only if for every pair S, T of strongly stationary subsets of $P_{\kappa \cap x} x$ there is a $y \prec x$ in A with $y \cap \kappa < x \cap \kappa$ such that S and T are both strongly stationary in $P_{y \cap \kappa} y$ (see Proposition 3.29). Additionally, using a different method, we show (see Corollary 3.35) that if κ is weakly inaccessible and X is a set of ordinals with $\kappa \subseteq X$, then there is a topology on $P_\kappa X$ such that for $A \subseteq P_\kappa X$, $x \in P_\kappa X$ is a limit point of A if and only if κ_x is weakly inaccessible and A is stationary in $P_{\kappa_x} x$ in the sense of Jech [23].

In order to prove the characterizations of limit points of sets in the spaces $(P_\kappa X, \tau_\xi)$ (Theorem 3.16(1)), we introduce new iterated forms of two-cardinal stationarity and two-cardinal pairwise simultaneous stationary reflection, which we refer to as ξ-strong stationarity and ξ-s-strong stationarity (see Definition 3.7). Let us note that the notions of ξ-strong stationarity and ξ-s-strong stationarity introduced here are natural generalizations of notions previously studied in the cardinal context by Bagaria, Magidor, and Sakai [4], Bagaria [2], and by Brickhill and Welch [8], as well as those previously studied in the two-cardinal context by Sakai [28], by Torres [30], as well as by Benhamou and the third author [7].

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We establish some basic properties of the ideals associated with ξ-strong stationarity and ξ-s-strong stationarity and introduce notions of ξ-weak club and ξ-s-weak club which provide natural filter bases for the corresponding filters (see Corollary 3.19). The consistency of the non-discreteness of the derived topologies τ_ξ on $P_\kappa X$ is obtained using various two-cardinal indescribability hypotheses, all of which follow from appropriate local instances of supercompactness (see Section 3.5). We also show that by restricting our attention to a certain natural club subset of $P_\kappa X$, some questions about the resulting spaces, such as questions regarding when particular subbases are in fact bases, become more tractable (see Section 3.6).

Additionally, in Section 4, we answer several questions asked by the first author and Holy [15] and the first author and White [16] concerning the relationship between Ramseyness and indescribability. Suppose κ is an uncountable cardinal. For example, answering [15, Question 10.9] in the affirmative, we show that the existence of an uncountable cardinal κ such that for every regressive function $f : [\kappa]^{<\omega} \to \kappa$ there is a set $H \subseteq \kappa$ which is positive for the Ramsey ideal and homogeneous for f, is strictly stronger in consistency strength than the existence of a cardinal κ such that for every regressive function $f : [\kappa]^{<\omega} \to \kappa$ there is a set $H \subseteq \kappa$ that is positive for the Π^1_1-indescribability ideal and homogeneous for f.

§2. Strong stationarity and weak clubs. An ideal I on $P_\kappa X$ is *strongly normal* if whenever $S \in I^+$ and $f : S \to P_\kappa X$ is such that $f(x) \prec x$ for all $x \in S$, then there is some $T \in P(S) \cap I^+$ such that $f \restriction T$ is constant. It is easy to see that an ideal I is strongly normal if and only if the dual filter I^* is closed under $\prec$-diagonal intersections in the following sense: whenever $A_x \in I^*$ for all $x \in P_\kappa X$, the $\prec$-diagonal intersection

$$\triangle_\prec \{A_x \mid x \in P_\kappa X\} = \{y \in P_\kappa X \mid y \in \bigcap_{x \prec y} A_x\}$$

is in I^*. Carr, Levinski, and Pelletier [10] showed that there is a strongly normal ideal on $P_\kappa X$ if and only if κ is a Mahlo cardinal or $\kappa = \mu^+$ for some cardinal μ with $\mu^{<\mu} = \mu$. Furthermore, they proved that when a strongly normal ideal exists on $P_\kappa X$, the minimal such ideal is that consisting of the non-strongly stationary subsets of $P_\kappa X$, which are defined as follows. Given a function $f : P_\kappa X \to P_\kappa X$ we let

$$B_f = \{x \in P_\kappa X \mid x \cap \kappa \neq \emptyset \wedge f[P_{\kappa_x} x] \subseteq P(x)\}.$$

A set $S \subseteq P_\kappa X$ is *strongly stationary in* $P_\kappa X$ if for all $f : P_\kappa X \to P_\kappa X$ we have $S \cap B_f \neq \emptyset$. The non-strongly stationary ideal on $P_\kappa X$ is the collection

$$\mathrm{NSS}_{\kappa,X} = \{X \subseteq P_\kappa X \mid \ X \text{ is not strongly stationary}\}.$$

Thus, when κ is Mahlo or $\kappa = \mu^+$ where $\mu^{<\mu} = \mu$, the ideal $\mathrm{NSS}_{\kappa,X}$ is the minimal strongly normal ideal on $P_\kappa X$.

When κ is Mahlo, we can identify a filter base for the filter dual to $\mathrm{NSS}_{\kappa,X}$ consisting of sets which are, in a sense, cofinal in $P_\kappa X$ and satisfy a certain natural closure property. We say that a set $C \subseteq P_\kappa X$ is *$\prec$-cofinal in* $P_\kappa X$ if for all $x \in P_\kappa X$ there is a $y \in C$ such that $x \prec y$. A set $C \subseteq P_\kappa X$ is said to be a *weak club in* $P_\kappa X$

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if C is $\prec$-cofinal in $P_\kappa X$ and $\prec$-*closed in* $P_\kappa X$, meaning that for all $x \in P_\kappa X$, if C is $\prec$-cofinal in $P_{\kappa_x} x$ then $x \in C$. Given a function $f : P_\kappa X \to P_\kappa X$ let

$$C_f = \{x \in P_\kappa X \mid x \cap \kappa \neq \emptyset \wedge f[P_{\kappa_x} x] \subseteq P_{\kappa_x} x\}.$$

FACT 2.1. *If κ is a Mahlo cardinal, the sets*

$$\mathcal{C}_0 = \{B_f \mid f : P_\kappa X \to P_\kappa X\},$$
$$\mathcal{C}_1 = \{C_f \mid f : P_\kappa X \to P_\kappa X\}$$

and

$$\mathcal{C}_2 = \{C \subseteq P_\kappa X \mid \ C \text{ is a weak club in } P_\kappa X\}$$

generate the same filter on $P_\kappa X$, namely, the filter $\mathrm{NSS}^*_{\kappa,X}$ dual to the ideal $\mathrm{NSS}_{\kappa,X}$. Hence, when κ is a Mahlo cardinal, a set $S \subseteq P_\kappa X$ is strongly stationary in $P_\kappa X$ if and only if $S \cap C \neq \emptyset$ for every C which is a weak club in $P_\kappa X$.

PROOF. By definition, the filter on $P_\kappa X$ generated by $\mathcal{C}_0$ is $\mathrm{NSS}^*_{\kappa,X}$.

Let us show that the filter generated by $\mathcal{C}_1$ equals that generated by $\mathcal{C}_2$. Suppose $C \in \mathcal{C}_2$ is a weak club in $P_\kappa X$. Define $f : P_\kappa X \to P_\kappa X$ by letting $f(x)$ be some member of C with $x \prec f(x)$. Then $C_f \subseteq C$ because if $x \in C_f$ then C is $\prec$-cofinal in $P_{\kappa_x} x$ and hence $x \in C$.

For the other direction, we fix a function $g : P_\kappa X \to P_\kappa X$ and show that C_g is a weak club in $P_\kappa X$. First let us check that C_g is $\prec$-cofinal in $P_\kappa X$. Fix $x \in P_\kappa X$. We define an increasing chain $\langle x_\eta \mid \eta < \kappa\rangle$ in $P_\kappa X$ as follows. Let $x_0 = x$. Given x_η we choose $x_{\eta+1} \in P_\kappa X$ with $\kappa_{x_{\eta+1}} = x_{\eta+1} \cap \kappa$ and $\bigcup g[P_{\kappa_{x_\eta}} x_\eta] \prec x_{\eta+1}$. When $\eta < \kappa$ is a limit ordinal we let $x_\eta = \bigcup_{\alpha<\eta} x_\alpha$. Then $\langle \kappa_{x_\eta} \mid \eta < \kappa\rangle$ is a strictly increasing sequence in κ and the set

$$C = \{\eta < \kappa \mid (\forall \zeta < \eta)\kappa_{x_\zeta} < \eta\} = \{\eta < \kappa \mid \kappa_{x_\eta} = \eta\}$$

is a club in κ. Since κ is Mahlo, we can fix some regular $\kappa_{x_\eta} = \eta \in C$. Clearly $x \prec x_\eta$. Let us show that $x_\eta \in C_g$. Suppose $a \in P_{\kappa_{x_\eta}} x_\eta$. Since $\kappa_{x_\eta} = \eta$ is regular, $|a| < \kappa_{x_\eta}$ implies that $a \in P_{\kappa_{x_\zeta}} x_\zeta$ for some $\zeta < \eta$, and therefore

$$g(a) \subseteq \bigcup g[P_{\kappa_{x_\zeta}} x_\zeta] \prec x_{\zeta+1},$$

which implies $g(a) \in P_{\kappa_{x_\eta}} x_\eta$ and hence $x_\eta \in C_g$. Since $x \prec x_\eta$, it follows that C_g is $\prec$-cofinal.

Now we verify that C_g is $\prec$-closed in $P_\kappa X$. Suppose $C_g \cap P_{\kappa_x} x$ is $\prec$-cofinal in $P_{\kappa_x} x$. We must show that $x \in C_g$. Suppose $y \in P_{\kappa_x} x$. Then there is some $z \in C_g$ with $y \prec z \prec x$. Thus $g(y) \prec z \prec x$ and hence $x \in C_g$.

Now let us verify that the filter generated by $\mathcal{C}_0$ equals that generated by $\mathcal{C}_1$. For any function $f : P_\kappa X \to P_\kappa X$ we have $C_f \subseteq B_f$, so the filter generated by $\mathcal{C}_0$ is contained in the filter generated by $\mathcal{C}_1$. Let us fix a function $g : P_\kappa X \to P_\kappa X$. We must show that there is a function $h : P_\kappa X \to P_\kappa X$ such that $B_h \subseteq C_g$. Define $h : P_\kappa X \to P_\kappa X$ by letting $h(x)$ be some member of C_g with $g(x) \prec h(x)$, for all $x \in P_\kappa X$. Suppose $x \in B_h$. To show $x \in C_g$, suppose $y \prec x$. Then it follows that $g(y) \prec h(y) \subseteq x$, which implies $g(y) \prec x$ and thus $x \in C_g$. Therefore $B_h \subseteq C_g$ and hence the filter generated by $\mathcal{C}_0$ equals the filter generated by $\mathcal{C}_1$. ⊣

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We end this section by discussing the more common variants of "club" and "stationary" subsets of $P_\kappa X$, introduced by Jech in [23]. Recall that, for a regular cardinal κ and a set $X \supseteq \kappa$, a set $C \subseteq P_\kappa X$ is said to be a *club* in $P_\kappa X$ if it is $\subseteq$-cofinal in $P_\kappa X$ and, whenever $D \subseteq C$ is a $\subseteq$-linearly ordered set of cardinality less than κ, we have $\bigcup D \in C$. This latter requirement is equivalent to the following formal strengthening: whenever $D \subseteq C$ is $\subseteq$-directed and $|D| < \kappa$, we have $\bigcup D \in C$. We then say that a set $S \subseteq P_\kappa X$ is *stationary* if, for every club C in $P_\kappa X$, we have $S \cap C \neq \emptyset$. The following basic observation justifies the use of the name "weak club" for the notion thusly designated above.

PROPOSITION 2.2. *If κ is weakly inaccessible, $X \supseteq \kappa$ is a set of ordinals, and C is a club in $P_\kappa X$, then C is a weak club in $P_\kappa X$.*

PROOF. Suppose that C is a club in $P_\kappa X$. Since κ is a limit cardinal, the fact that C is $\subseteq$-cofinal implies that it is also $\prec$-cofinal. To verify closure, fix $x \in P_\kappa X$ such that C is $\prec$-cofinal in $P_{\kappa_x} x$. Then it is straightforward to construct a set $D \subseteq C \cap P_{\kappa_x} x$ such that:

- D is $\subseteq$-directed;
- $|D| \leq |x| < \kappa$;
- $\bigcup D = x$.

Since C is a club, it follows that $\bigcup D = x \in C$. Thus, C is a weak club. ⊣

§3. Two-cardinal derived topologies and ξ-strong stationarity. Fix for this section an arbitrary regular uncountable cardinal κ and a set of ordinals $X \supseteq \kappa$. We will investigate a sequence of derived topologies $\langle \tau_\xi \mid \xi < \kappa \rangle$ on $P_\kappa X$, simultaneously isolating a hierarchy of stationary reflection principles that characterize the existence of limit points with respect to these topologies. We emphasize that all definitions and arguments in this section are in the context of the ambient space of $P_\kappa X$. We begin by describing τ_0, a generalization of the order topology.

3.1. A generalization of the order topology to $P_\kappa X$. Given $x, y \in P_\kappa X$ with $x \prec y$, let

$$(x, y] = \{z \in P_\kappa X \mid x \prec z \prec y \vee z = y\}$$

and

$$(x, y) = \{z \in P_\kappa X \mid x \prec z \prec y\}.$$

Let τ_0 be the topology on $P_\kappa X$ generated by

$$\mathcal{B}_0 = \{(x, y] \mid x, y \in P_\kappa X \wedge x \prec y\} \cup \{\{y\} \mid y \in P_\kappa X \wedge \kappa_y = 0\}.$$

It is easy to see that $\mathcal{B}_0$ is a base for τ_0: If $y \in P_\kappa X$ and $\kappa_y = 0$, then $\{y\} \in \mathcal{B}_0$. If $0 < \kappa_y < \omega$, then, letting x be any subset of y of cardinality $\kappa_y - 1$, we have $\{y\} = (x, y] \in \mathcal{B}_0$. Finally, if κ_y is infinite and $y \in (a_0, b_0] \cap \cdots \cap (a_{n-1}, b_{n-1}]$, then $y \in (a, y] \subseteq (a_0, b_0] \cap \cdots \cap (a_{n-1}, b_{n-1}]$ where $a = \bigcup_{i<n} a_i$.

For $A \subseteq P_\kappa X$, let

$$d_0(A) = \{x \in P_\kappa X \mid \ x \text{ is a limit point of } A \text{ in } (P_\kappa X, \tau_0)\}.$$

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PROPOSITION 3.1. *For every* $A \subseteq P_\kappa X$,

$$d_0(A) = \{x \in P_\kappa X \mid A \text{ is } \prec\text{-cofinal in } P_{\kappa_x} x\}.$$

PROOF. Fix $A \subseteq P_\kappa X$ and $x \in d_0(A)$. By the above argument that $\mathcal{B}_0$ is a base for τ_0, it follows that $\kappa_x \geq \omega$. Suppose that $y \in P_{\kappa_x} x$. Since $(y, x]$ is an open neighborhood of x, we can choose a $z \in (y, x] \cap A$ with $z \neq x$. This implies $z \in (y, x) \cap A$, and hence A is $\prec$-cofinal in $P_{\kappa_x} x$. Conversely, suppose A is $\prec$-cofinal in $P_{\kappa_x} x$ and let $(a, b]$ be a basic open neighborhood of x. Then $a \in P_{\kappa_x} x$ and we may choose some $y \in A$ with $a \prec y \in P_{\kappa_x} x$. Hence $y \in (a, b] \cap A \setminus \{x\}$. ⊣

COROLLARY 3.2. *A point* $x \in P_\kappa X$ *is not isolated in* τ_0 *if and only if* $\kappa_x = |x \cap \kappa|$ *is a limit cardinal.*

The following proposition connects the order topology τ_0 on $P_\kappa X$ to the notion of weak club discussed in Section 2, in the case where κ is a weakly Mahlo cardinal.

PROPOSITION 3.3. *If* κ *is weakly Mahlo and* A *is* $\prec$*-cofinal in* $P_\kappa X$ *then* $d_0(A)$ *is a weak club in* $P_\kappa X$.

PROOF. First let us show that $d_0(A)$ is $\prec$-closed. Suppose $d_0(A)$ is $\prec$-cofinal in $P_{\kappa_x} x$. Fix $a \in P_{\kappa_x} x$ and let $b \in d_0(A) \cap (a, x)$. Then A is $\prec$-cofinal in $P_{\kappa_b} b$, so we may choose $c \in A \cap (a, b) \subseteq A \cap (a, x)$ and hence $x \in d_0(A)$.

Let us show that $d_0(A)$ is $\prec$-cofinal in $P_\kappa X$. Fix $x \in P_\kappa X$. We define an increasing chain $\langle x_\eta \mid \eta < \kappa \rangle$ in $P_\kappa X$ as follows. Let $x_0 = x$. Given x_η choose $x_{\eta+1} \in A$ with $x_\eta \prec x_{\eta+1}$. If $\eta < \kappa$ is a limit let $x_\eta = \bigcup_{\zeta<\eta} x_\zeta$. Then $\langle \kappa_{x_\eta} \mid \eta < \kappa \rangle$ is a strictly increasing sequence in κ and the set

$$C = \{\eta < \kappa \mid (\forall \zeta < \eta)\, \kappa_{x_\zeta} < \eta\} = \{\eta < \kappa \mid \kappa_{x_\eta} = \eta\}$$

is a club in κ. Thus, since κ is weakly Mahlo, we can fix some regular $\kappa_{x_\eta} = \eta \in C$. Let us show that $x_\eta \in d_0(A)$. Suppose $a \in P_{\kappa_{x_\eta}} x_\eta$. Since $\kappa_{x_\eta} = \eta$ is regular, $|a| < \kappa_{x_\eta}$ implies $a \in P_{\kappa_{x_\zeta}} x_\zeta$ for some $\zeta < \eta$, and therefore $a \in P_{\kappa_{x_{\zeta+1}}} x_{\zeta+1}$ which entails that $a \prec x_{\zeta+1} \in A$. ⊣

Recall that an ordinal δ has uncountable cofinality if and only if for every $A \subseteq \delta$ which is unbounded in δ, there is an $\alpha < \delta$ such that A is unbounded in α. The following proposition is the analogous result for the notion of $\prec$-cofinality in $P_\kappa X$ when κ is weakly inaccessible.

PROPOSITION 3.4. *If* κ *is weakly inaccessible, then the following are equivalent.*

(1) κ *is a weakly Mahlo cardinal.*
(2) *For all* $A \subseteq P_\kappa X$ *if* A *is* $\prec$*-cofinal in* $P_\kappa X$ *then there is an* $x \in P_\kappa X$ *such that* A *is* $\prec$*-cofinal in* $P_{\kappa_x} x$.

PROOF. The fact that (1) implies (2) follows from Proposition 3.3. Let us show that (2) implies (1). We assume (2) holds, and that κ is weakly inaccessible but not weakly Mahlo. Let $C \subseteq \kappa$ be a club consisting of singular cardinals, and let $D = \{x \in P_\kappa X \mid x \cap \kappa \in C\}$. Then D is $\prec$-cofinal in $P_\kappa X$. By (2), there is a $y \in P_\kappa X$ such that D is $\prec$-cofinal in $P_{\kappa_y} y$. Then κ_y is a limit cardinal and C is cofinal in κ_y, so $\kappa_y \in C$ is a singular cardinal.

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Let us argue that $y \cap \kappa$ is an ordinal less than κ. Suppose $\alpha \in y \cap \kappa$. Since D is $\prec$-cofinal in $P_{\kappa_y} y$ there is some $z \in D \cap P_{\kappa_y} y$ such that $\{\alpha\} \prec z$. Thus $\alpha \in z$ and $z \cap \kappa$ is an ordinal, which implies $\alpha \subseteq z \cap \kappa \subseteq y \cap \kappa$. Hence $y \cap \kappa$ is transitive.

Let $a \subseteq \kappa_y$ be cofinal in κ_y with $|a| = \mathrm{cf}(\kappa_y) < \kappa_y$. Since $y \cap \kappa$ is an ordinal we have $a \subseteq \kappa_y = |y \cap \kappa| \subseteq y \cap \kappa$ and thus $a \in P_{\kappa_y} y$. However, there is no $x \in D \cap P_{\kappa_y} y$ with $a \prec x$ because for such an x, $\kappa \cap x \in C$ would be an ordinal containing the set a which is cofinal in κ_y, and hence $\kappa_x \geq \kappa_y$. ⊣

We note that the assumption that κ is weakly inaccessible is necessary in Proposition 3.4, but only for the somewhat trivial reason that, if κ is a successor cardinal, then there are no $\prec$-cofinal subsets of $P_\kappa X$.

3.2. Definitions of derived topologies and iterated stationarity in $P_\kappa X$. With the topology τ_0 on $P_\kappa X$, the base $\mathcal{B}_0$ for τ_0 and the Cantor derivative d_0 in hand, we can now define the derived topologies on $P_\kappa X$ as follows. Given τ_ξ, $\mathcal{B}_\xi$ and d_ξ, we let

$$\mathcal{B}_{\xi+1} = \mathcal{B}_\xi \cup \{d_\xi(A) \mid A \subseteq P_\kappa X\},$$

we let $\tau_{\xi+1}$ be the topology generated by $\mathcal{B}_{\xi+1}$ and we let $d_{\xi+1}$ be defined by

$$d_{\xi+1}(A) = \{x \in P_\kappa X \mid \ x \text{ is a limit point of } A \text{ in } (P_\kappa X, \tau_{\xi+1})\}$$

for $A \subseteq P_\kappa X$. When ξ is a limit ordinal we let τ_ξ be the topology generated by $\mathcal{B}_\xi := \bigcup_{\zeta<\xi} \mathcal{B}_\zeta$ and we let d_ξ be the Cantor derivative of the space $(P_\kappa X, \tau_\xi)$.

Since $\mathcal{B}_0$ is a base for τ_0, it easily follows that the sets of the form

$$I \cap d_{\xi_0}(A_0) \cap \cdots \cap d_{\xi_{n-1}}(A_{n-1}),$$

where $I \in \mathcal{B}_0$, $n < \omega$, $\xi_i < \xi$ and $A_i \subseteq P_\kappa X$ for $i < n$, form a base for τ_ξ whenever $\xi < \kappa$. We return to the question of whether or not $\mathcal{B}_\xi$ forms a base for τ_ξ in Theorem 3.21, as well as in Section 3.6.

Let us note here that the next two lemmas can easily be established using arguments similar to those for [2, Proposition 2.1 and Corollary 2.2].

LEMMA 3.5. *For all $\zeta < \xi$ and all $A_0, \ldots, A_n \subseteq P_\kappa X$,*

$$d_\zeta(A_0) \cap \cdots \cap d_\zeta(A_{n-1}) \cap d_\xi(A_n) = d_\xi(d_\zeta(A_0) \cap \cdots \cap d_\zeta(A_{n-1}) \cap A_n).$$

LEMMA 3.6. *For every ordinal ξ, the sets of the form*

$$I \cap d_\zeta(A_0) \cap \cdots \cap d_\zeta(A_{n-1}),$$

where $I \in \mathcal{B}_0$, $n < \omega$, $\zeta < \xi$ and $A_i \subseteq P_\kappa X$ for $i < n$, form a base for τ_ξ.

In the next few sections, we will characterize the non-isolated points of the spaces $(P_\kappa X, \tau_\xi)$ in terms of the following two-cardinal notions of ξ-s-strong stationarity.

DEFINITION 3.7. (1) For $A \subseteq P_\kappa X$ and $x \in P_\kappa X$, we say that A is 0-*strongly stationary in* $P_{\kappa_x} x$ if and only if A is $\prec$-cofinal in $P_{\kappa_x} x$. For an ordinal $\xi > 0$, we say that A is ξ-*strongly stationary in* $P_{\kappa_x} x$ if and only if κ_x is a limit cardinal[1]

[1]The requirement that κ_x be a limit cardinal in order for A to be ξ-strongly stationary in $P_{\kappa_x} x$ is necessary because otherwise, when κ_x is a successor cardinal there are *no* 0-strongly stationary subsets of $P_{\kappa_x} x$ and hence *every* subset of $P_{\kappa_x} x$ would be 1-strongly stationary.

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and, whenever $\zeta < \xi$ and $S \subseteq P_{\kappa_x} x$ is ζ-strongly stationary in $P_{\kappa_x} x$, there is some $y \in A \cap P_{\kappa_x} x$ such that S is ζ-strongly stationary in $P_{\kappa_y} y$.

(2) A set $C \subseteq P_\kappa X$ is a 0-*weak club in* $P_{\kappa_x} x$ if and only if it is $\prec$-cofinal and $\prec$-closed in $P_{\kappa_x} x$. For an ordinal $\xi > 0$, we say that C is a ξ-*weak club in* $P_{\kappa_x} x$ if and only if it is ξ-strongly stationary in $P_{\kappa_x} x$ and it is ξ-*strongly stationary closed in* $P_{\kappa_x} x$, meaning that whenever $y \prec x$ and C is ξ-strongly stationary in $P_{\kappa_y} y$ we have $y \in C$.

(3) We say that A is 0-*simultaneously strongly stationary in* $P_{\kappa_x} x$ or (0-*s-strongly stationary in* $P_{\kappa_x} x$ for short) if and only if A is $\prec$-cofinal in $P_{\kappa_x} x$. For an ordinal $\xi > 0$, we say that A is ξ-*simultaneously strongly stationary in* $P_{\kappa_x} x$ (or ξ-*s-strongly stationary in* $P_{\kappa_x} x$ for short) if and only if κ_x is a limit cardinal and, whenever $\zeta < \xi$ and $S, T \subseteq P_\kappa X$ are ζ-s-strongly stationary in $P_{\kappa_x} x$ there is some $y \in A \cap P_{\kappa_x} x$ such that S and T are both ζ-s-strongly stationary in $P_{\kappa_y} y$.

(4) A set $C \subseteq P_\kappa X$ is a 0-*s-weak club in* $P_{\kappa_x} x$ if and only if it is $\prec$-cofinal and $\prec$-closed in $P_{\kappa_x} x$. For an ordinal $\xi > 0$, we say that C is a ξ-*s-weak club in* $P_{\kappa_x} x$ if and only if it is ξ-s-strongly stationary in $P_{\kappa_x} x$ and it is ξ-*s-closed in* $P_{\kappa_x} x$, meaning that whenever $y < x$ and C is ξ-s-strongly stationary in $P_{\kappa_y} y$ we have $y \in C$.

In what follows, given $x \in P_\kappa X$ and $\xi < \kappa$, we will simply say that, e.g., $P_{\kappa_x} x$ is ξ-s-strongly stationary to mean that it is ξ-s-strongly stationary *in* $P_{\kappa_x} x$. Let us first note the following simple proposition, which justifies the restriction of our attention to values of ξ less than κ. By the results of Section 3.5, the proposition is sharp, at least assuming the consistency of certain large cardinals.

PROPOSITION 3.8. *For all* $x \in P_\kappa X$, $P_{\kappa_x} x$ *is not* $(\kappa_x + 1)$-*strongly stationary.*

PROOF. Suppose otherwise, and let $x \in P_\kappa X$ be a counterexample such that κ_x is minimal among all counterexamples. Since $P_{\kappa_x} x$ is $(\kappa_x + 1)$-strongly stationary, it is *a fortiori* κ_x-strongly stationary. Therefore, by the definition of $(\kappa_x + 1)$-strong stationarity, we can find $y \in P_{\kappa_x} x$ such that $P_{\kappa_x} x$ is κ_x-strongly stationary in $P_{\kappa_y} y$. Since $\kappa_x > \kappa_y$, this implies that $P_{\kappa_y} y$ is $(\kappa_y + 1)$-strongly stationary, contradicting the minimality of κ_x. ⊣

Considering the previous proposition, it is natural to wonder whether the definitions of ξ-strong stationarity and ξ-s-strong stationarity can be modified using canonical functions to allow for settings in which some $x \in P_\kappa X$ can be ξ-strongly stationary for $\kappa_x < \xi < |x|^+$; this was done in the cardinal setting by the first author in [13]. See the discussion before Questions 5.7 and 5.8 for more information.

Definition 3.7 leads naturally to the definition of the following ideals, which can be strongly normal under a certain large cardinal hypothesis by Proposition 3.30 below.

DEFINITION 3.9. Suppose that $x \in P_\kappa X$. We define

$$\overline{\mathrm{NS}}^{\xi}_{\kappa_x, x} = \{A \subseteq P_\kappa X \mid A \text{ is not } \xi\text{-strongly stationary in } P_{\kappa_x} x\}$$

and

$$\mathrm{NS}^{\xi}_{\kappa_x, x} = \{A \subseteq P_\kappa X \mid A \text{ is not } \xi\text{-s-strongly stationary in } P_{\kappa_x} x\}.$$

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Let us show that for the x's in $P_\kappa X$ that we will care most about, namely those for which κ_x is regular, 1-strong stationarity and 1-s-strong stationarity are equivalent in $P_{\kappa_x} x$; moreover, if κ_x is inaccessible, then these notions are equivalent to strong stationarity in $P_{\kappa_x} x$ plus the Mahloness of κ_x.

PROPOSITION 3.10. *Suppose $A \subseteq P_\kappa X$ and $x \in P_\kappa X$ with κ_x regular. Then the following are equivalent, and both imply that κ_x is weakly Mahlo.*

(1) *A is 1-strongly stationary in $P_{\kappa_x} x$.*
(2) *A is 1-s-strongly stationary in $P_{\kappa_x} x$.*

If, moreover, κ_x is strongly inaccessible, then these two statements are also equivalent to the following:

(3) *κ_x is Mahlo and A is strongly stationary in $P_{\kappa_x} x$.*

PROOF. Note that, if A is 1-strongly stationary in $P_{\kappa_x} x$, then κ_x is a limit cardinal and hence weakly inaccessible. We can thus assume that this is the case. (2) $\implies$ (1) is trivial. Let us now assume that A is 1-strongly stationary in $P_{\kappa_x} x$. By Proposition 3.4, it follows that κ_x is weakly Mahlo. To see that A is 1-s-strongly stationary in $P_{\kappa_x} x$, fix sets $S_0, S_1 \subseteq P_\kappa X$ that are both $\prec$-cofinal in $P_{\kappa_x} x$. Let T be the set of $y \in P_{\kappa_x} x$ such that S_0 and S_1 are both $\prec$-cofinal in $P_{\kappa_y} y$. We claim that T is $\prec$-cofinal in $P_{\kappa_x} x$. To see this, fix an arbitrary $y_0 \in P_{\kappa_x} x$. Define a continuous, $\prec$-increasing sequence $\langle y_\eta \mid \eta < \kappa_x \rangle$ in $P_{\kappa_x} x$ as follows. The set y_0 is already fixed. Given y_η, find $z^0_\eta \in S_0$ and $z^1_\eta \in S_1$ such that, for all $i < 2$, we have $y_\eta \prec z^i_\eta \prec x$. Then let $y_{\eta+1} = z^0_\eta \cup z^1_\eta$. If $\xi < \kappa_x$ is a limit ordinal, let $y_\xi = \bigcup \{y_\eta \mid \eta < \xi\}$. The set of $\eta < \kappa_x$ for which $\kappa_{y_\eta} = \eta$ is a club in κ_x, so, since κ_x is weakly Mahlo, we can fix some regular cardinal $\eta < \kappa_x$ such that $\kappa_{y_\eta} = \eta$. A now-familiar argument then shows that S_0 and S_1 are both $\prec$-cofinal in y_η, and hence $y_\eta \in T$.

Since A is 1-strongly stationary in $P_{\kappa_x} x$, we can find $w \in A$ such that T is $\prec$-cofinal in $P_{\kappa_w} w$. It follows immediately that S_0 and S_1 are both $\prec$-cofinal in $P_{\kappa_w} w$; therefore, A is 1-s-strongly stationary in $P_{\kappa_x} x$.

For the "moreover" clause, assume that κ_x is strongly inaccessible and A is 1-strongly stationary in $P_{\kappa_x} x$. The fact that κ_x is Mahlo follows from the previous paragraphs. To show that A is strongly stationary in $P_{\kappa_x} x$, suppose C is a weak club subset of $P_{\kappa_x} x$. Since A is 1-strongly stationary there is some $y \in A$ such that C is $\prec$-cofinal in $P_{\kappa_y} y$. Since C is weakly closed we have $y \in A \cap C$.

Finally, suppose κ_x is Mahlo and A is strongly stationary in $P_{\kappa_x} x$. To show that A is 1-strongly stationary in $P_{\kappa_x} x$, fix a set S which is $\prec$-cofinal in $P_{\kappa_x} x$. Since κ_x is Mahlo, it follows from Proposition 3.3 that $d_0(S)$ is a weak club in $P_{\kappa_x} x$. Thus, by Fact 2.1, $d_0(S) \cap A$ is strongly stationary in $P_{\kappa_x} x$ and hence there is a $y \in A \cap P_{\kappa_x} x$ such that S is $\prec$-cofinal in $P_{\kappa_y} y$. ⊣

3.3. The τ_1 topology on $P_\kappa X$. We now discuss the *first derived topology* τ_1 *on* $P_\kappa X$. Recall that this is the topology generated by

$$\mathcal{B}_1 = \mathcal{B}_0 \cup \{d_0(A) \mid A \subseteq P_\kappa X\}.$$

REMARK 3.11. Recall that the subbase for the first derived topology on an ordinal δ is always a base for that topology (see [2]). By definition, $\mathcal{B}_1$ is a subbase for the first derived topology τ_1 on $P_\kappa X$, but it is not clear whether it is a base for τ_1 for the

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following reason. Suppose $x \in I \cap d_0(A_0) \cap \cdots \cap d_0(A_{n-1})$ where $n \geq 1$. Then each A_i, for $i < n$, is $\prec$-cofinal in $P_{\kappa_x} x$ and hence κ_x is a limit cardinal. If κ_x is a Mahlo cardinal, then it follows by Proposition 3.3 that each $d_0(A_i)$, for $i < n$, is a weak club in $P_{\kappa_x} x$, and hence $I \cap d_0(A_0) \cap \cdots \cap d_0(A_{n-1})$ is a weak club in $P_{\kappa_x} x$. Thus, $x \in d_0(I \cap d_0(A_0) \cap \cdots \cap d_0(A_{n-1})) \subseteq I \cap d_0(A_0) \cap \cdots \cap d_0(A_{n-1})$. However, if κ_x is not Mahlo, then it is not clear whether $d_0(A_i)$ is a weak club in $P_{\kappa_x} x$ for $i < n$, and furthermore, it is not clear how to proceed. This difference seems not to create too much difficulty so we proceed with our definition as is, but in Section 3.6 we show that, if we pass to a certain club subset C of $P_\kappa X$, then the natural restriction of $\mathcal{B}_1$ to C is a base for the subspace topology on C induced by τ_1.

We will need the following lemma.

LEMMA 3.12. *Fix $x \in P_\kappa X$, and suppose that A is 1-s-strongly stationary in $P_{\kappa_x} x$ and $A_0, \ldots, A_{n-1}$ are all 0-s-strongly stationary (i.e., $\prec$-cofinal) in $P_{\kappa_x} x$, where $n \geq 2$. Then $d_0(A_0) \cap \cdots \cap d_0(A_{n-1}) \cap A$ is 1-s-strongly stationary in $P_{\kappa_x} x$.*

PROOF. First let us use a straightforward inductive argument on $n \geq 2$ to show that whenever $A_0, \ldots, A_{n-1}$ are 0-strongly stationary in $P_{\kappa_x} x$, the set $d_0(A_0) \cap \cdots \cap d_0(A_{n-1}) \cap A$ is 0-strongly stationary in $P_{\kappa_x} x$. Suppose A_0 and A_1 are $\prec$-cofinal in $P_{\kappa_x} x$ and note that κ_x must be a limit cardinal. To show that $d_0(A_0) \cap d_0(A_1) \cap A$ is $\prec$-cofinal in $P_{\kappa_x} x$, fix $y \in P_{\kappa_x} x$. Then $A_0 \cap (y, x)$ and $A_1 \cap (y, x)$ are $\prec$-cofinal in $P_{\kappa_x} x$. Since A is 1-s-strongly stationary in $P_{\kappa_x} x$ there is an $a \in A \cap P_{\kappa_x} x$ such that $A_0 \cap (y, x)$ and $A_1 \cap (y, x)$ are both $\prec$-cofinal in $P_{\kappa_a} a$, and hence $y < a$. Therefore $a \in d_0(A_0) \cap d_0(A_1) \cap A \cap (y, x)$. Now suppose the result holds for n, and suppose $A_0, \ldots, A_{n-1}, A_n$ are all $\prec$-cofinal in $P_{\kappa_x} x$. By our inductive hypothesis, $d_0(A_0) \cap \cdots \cap d_0(A_{n-1})$ is $\prec$-cofinal in $P_{\kappa_x} x$, and by the base case the set $d_0(d_0(A_0) \cap \cdots \cap d_0(A_{n-1})) \cap d_0(A_n) \cap A$ is $\prec$-cofinal in $P_{\kappa_x} x$. But

$$\begin{aligned} d_0(d_0(A_0) \cap \cdots \cap d_0(A_{n-1})) \cap d_0(A_n) &\subseteq d_0(d_0(A_0)) \cap \cdots \cap d_0(d_0(A_{n-1})) \cap d_0(A_n) \\ &\subseteq d_0(A_0) \cap \cdots \cap d_0(A_{n-1}) \cap d_0(A_n). \end{aligned}$$

Now we prove the statement of the lemma. Fix sets $A_0, \ldots, A_{n-1} \subseteq P_\kappa X$ that are $\prec$-cofinal in $P_{\kappa_x} x$. To show that $d_0(A_0) \cap \cdots \cap d_0(A_{n-1}) \cap A$ is 1-s-strongly stationary in $P_{\kappa_x} x$, fix sets S and T that are $\prec$-cofinal in $P_{\kappa_x} x$. By the previous paragraph, it follows that the set

$$d_0(S) \cap d_0(T) \cap d_0(A_0) \cap \cdots \cap d_0(A_{n-1}) \cap A$$

is $\prec$-cofinal in $P_{\kappa_x} x$ and hence there is some

$$y \in d_0(S) \cap d_0(T) \cap d_0(A_0) \cap \cdots \cap d_0(A_{n-1}) \cap A,$$

which establishes that $d_0(A_0) \cap \cdots \cap d_0(A_{n-1}) \cap A$ is 1-s-strongly stationary. ⊣

COROLLARY 3.13. *Suppose $P_{\kappa_x} x$ is 1-s-strongly stationary. Then a set A is 1-s-strongly stationary in $P_{\kappa_x} x$ if and only if for every set C which is a 0-s-weak club in $P_{\kappa_x} x$ we have $A \cap C \cap P_{\kappa_x} x \neq \emptyset$.*

PROOF. Suppose A is 1-s-strongly stationary in $P_{\kappa_x} x$ and C is a 0-s-weak club in $P_{\kappa_x} x$. Then $d_0(C) \cap P_{\kappa_x} x \subseteq C \cap P_{\kappa_x} x$ and by Lemma 3.12, $d_0(C) \cap A$ is 1-s-strongly stationary in $P_{\kappa_x} x$. Thus $A \cap C \cap P_{\kappa_x} x \neq \emptyset$. Conversely, assume that

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$A \cap C \cap P_{\kappa_x} x \neq \emptyset$ whenever C is a 0-s-weak club in $P_{\kappa_x} x$. Fix sets S and T that are 0-s-strongly stationary in $P_{\kappa_x} x$. Then $d_0(S) \cap d_0(T)$ is a 0-s-weak club in $P_{\kappa_x} x$ because $d_0(S) \cap d_0(T) \cap P_{\kappa_x} x$ is 1-s-strongly stationary and hence 0-s-strongly stationary in $P_{\kappa_x} x$ by Lemma 3.12, and $d_0(S) \cap d_0(T)$ is 0-s-closed in $P_{\kappa_x} x$ since

$$d_0(d_0(S) \cap d_0(T)) \subseteq d_0(S) \cap d_0(T)$$

as a consequence of the fact that d_0 is the limit point operator of the space $(P_\kappa X, \tau_0)$. ⊣

PROPOSITION 3.14. *If $A \subseteq P_\kappa X$ then*

$$d_1(A) = \{x \in P_\kappa X \mid A \text{ is 1-s-strongly stationary in } P_{\kappa_x} x\}.$$

PROOF. Suppose A is not 1-s-strongly stationary in $P_{\kappa_x} x$. If κ_x is a successor cardinal then x is isolated in $(P_\kappa X, \tau_1)$ by Corollary 3.2 and hence $x \notin d_1(A)$. Suppose κ_x is a limit cardinal. Then there are sets S and T which are 0-strongly stationary in $P_{\kappa_x} x$ such that $d_0(S) \cap d_0(T) \cap A \cap P_{\kappa_x} x = \emptyset$. Then it follows that $d_0(S) \cap d_0(T) \cap (0, x]$ is an open neighborhood of x in the τ_1 topology that does not intersect A in some point other than x. Hence $x \notin d_1(A)$.

Conversely, suppose A is 1-s-strongly stationary in $P_{\kappa_x} x$. We must show that x is a limit point of A in the τ_1 topology. Suppose $x \in I \cap d_0(A_0) \cap \cdots \cap d_0(A_{n-1})$, where $I \in \mathcal{B}_0$ and $A_0, \ldots, A_{n-1} \subseteq P_\kappa X$. Then the sets $A_0, \ldots, A_{n-1}$ are all $\prec$-cofinal in $P_{\kappa_x} x$, and by Lemma 3.12, the set $I \cap d_0(A_0) \cap \cdots d_0(A_{n-1}) \cap A$ is $\prec$-cofinal in $P_{\kappa_x} x$, which implies that the open neighborhood $I \cap d_0(A_0) \cap \cdots \cap d_0(A_{n-1})$ of x intersects A in some point other than x. ⊣

COROLLARY 3.15. *A point $x \in P_\kappa X$ is not isolated in $(P_\kappa X, \tau_1)$ if and only if $P_{\kappa_x} x$ is 1-s-strongly stationary in $P_{\kappa_x} x$.*

3.4. The τ_ξ topology on $P_\kappa X$ for $\xi \geq 2$. We now move to the general setting. Let us first characterize limit points of sets in the spaces $(P_\kappa X, \tau_\xi)$ in terms of ξ-s-strong stationarity.

THEOREM 3.16. *For all $\xi < \kappa$ the following hold.*

$(1)_\xi$ *We have*

$$d_\xi(A) = \{x \in P_\kappa X \mid A \text{ is } \xi\text{-s-strongly stationary in } P_{\kappa_x} x\}.$$

$(2)_\xi$ *For all $x \in P_\kappa X$, a set A is $\xi + 1$-s-strongly stationary in $P_{\kappa_x} x$ if and only if for all $\zeta \leq \xi$ and every pair S, T of subsets of $P_{\kappa_x} x$ that are ζ-s-strongly stationary in $P_{\kappa_x} x$, we have $A \cap d_\zeta(S) \cap d_\zeta(T) \neq \emptyset$ (equivalently $A \cap d_\zeta(S) \cap d_\zeta(T)$ is ζ-s-strongly stationary in $P_{\kappa_x} x$).*

$(3)_\xi$ *For all $x \in P_\kappa X$, if A is ξ-s-strongly stationary in $P_{\kappa_x} x$ and A_i is ζ_i-s-strongly stationary in $P_{\kappa_x} x$ for some $\zeta_i < \xi$ and all $i < n$, then $A \cap d_{\zeta_0}(A_0) \cap \cdots \cap d_{\zeta_{n-1}}(A_{n-1})$ is ξ-s-strongly stationary in $P_{\kappa_x} x$.*

PROOF. We have already established that $(1)_\xi$ and $(3)_\xi$ hold for $\xi \leq 1$ and $(2)_0$ holds. Given these base cases, the fact that (1), (2), and (3) hold for all $\xi < \kappa$ can be established by simultaneous induction using an argument which is essentially identical to that of [2, Proposition 2.10]. For the reader's convenience, we include the argument here.

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First, suppose $(1)_\zeta$, $(2)_\zeta$, and $(3)_\zeta$ hold for all ζ less than some limit ordinal $\xi < \kappa$. It is clear that $(1)_\xi$ and $(3)_\xi$ also must hold. Let us prove that $(2)_\xi$ holds. Notice that the backward direction of $(2)_\xi$ easily follows from the definition of $\xi+1$-s-strong stationarity and the fact that $(1)_\zeta$ holds for $\zeta \leq \xi$. For the forward direction of $(2)_\xi$, suppose A is $\xi+1$-s-strongly stationary in $P_{\kappa_x}x$. Fix $\zeta \leq \xi$ and a pair S, T of ζ-s-strongly stationary subsets of $P_{\kappa_x}x$. To show that $A \cap d_\zeta(S) \cap d_\zeta(T)$ is ζ-s-strongly stationary in $P_{\kappa_x}x$, fix sets A, B that are η-s-strongly stationary in $P_{\kappa_x}x$ where $\eta < \zeta$. Using the fact that (3) holds for ζ, we see that $S \cap d_\eta(A) \cap d_\eta(B)$ is ζ-s-strongly stationary in $P_{\kappa_x}x$. Since A is $\xi+1$-s-strongly stationary, and applying the fact that $(1)_\zeta$ holds, we have

$$\emptyset \neq d_\zeta(d_\eta(A) \cap d_\eta(B) \cap S) \cap d_\zeta(T) \cap A.$$

But, by Lemma 3.5,

$$d_\zeta(d_\eta(A) \cap d_\eta(B) \cap S) \cap d_\zeta(T) \cap A = d_\eta(A) \cap d_\eta(B) \cap d_\zeta(S) \cap d_\zeta(T) \cap A.$$

Thus, $d_\zeta(S) \cap d_\zeta(T) \cap A$ is ζ-s-strongly stationary in $P_{\kappa_x}x$.

One can show that if $(1)_{\leq\xi}$, $(2)_{\leq\xi}$, and $(3)_{\leq\xi}$ hold then, by induction on n, $(3)_{\xi+1}$ must also hold. For the reader's convenience we provide a proof that $(3)_{\xi+1}$ holds for $n=1$, the remaining argument is the same as [2, Proposition 2.10]. Suppose $n=1$. To prove that $A \cap d_{\zeta_0}(A_0)$ is $\xi+1$-s-strongly stationary in $P_{\kappa_x}x$, fix sets S and T that are η-s-strongly stationary in $P_{\kappa_x}x$ for some $\eta \leq \xi$. By $(1)_{\leq\xi}$, it will suffice to show that $A \cap d_{\zeta_0}(A_0) \cap d_\eta(S) \cap d_\eta(T) \neq \emptyset$. If $\zeta_0 = \eta$, then by $(2)_{\leq\xi}$, it follows that the set $A \cap d_{\zeta_0}(A_0) \cap d_\eta(d_\eta(S) \cap d_\eta(T))$, which is contained in $A \cap d_{\zeta_0}(A_0) \cap d_\eta(S) \cap d_\eta(T)$, is ζ_0-s-strongly stationary in $P_{\kappa_x}x$, and thus $A \cap d_{\zeta_0}(A_0) \cap d_\eta(S) \cap d_\eta(T) \neq \emptyset$. If $\zeta_0 < \eta$, then by $(3)_\eta$, if follows that $d_{\zeta_0}(A_0)$ is η-s-strongly stationary in $P_{\kappa_x}x$, and by $(2)_\xi$, the set $A \cap d_\eta(d_{\zeta_0}(A_0)) \cap d_\eta(d_\eta(S) \cap d_\eta(T))$, which is contained in $A \cap d_{\zeta_0}(A_0) \cap d_\eta(S) \cap d_\eta(T)$, is η-s-strongly stationary in $P_{\kappa_x}x$. If $\zeta_0 > \eta$ then by $(2)_\xi$ the set $A \cap d_{\zeta_0}(A_0)$ is ζ_0-s-strongly stationary in $P_{\kappa_x}x$ and thus $A \cap d_{\zeta_0}(A_0) \cap d_\eta(S) \cap d_\eta(T) \neq \emptyset$.

Let us prove that if $(1)_{\leq\xi}$, $(2)_{\leq\xi}$, and $(3)_{\leq\xi+1}$ hold then $(1)_{\xi+1}$ holds (this argument is similar to that of Proposition 3.14). Suppose A is not $\xi+1$-s-strongly stationary in $P_{\kappa_x}x$. Then by $(1)_{\leq\xi}$, for some $\zeta \leq \xi$ there are sets S and T which are ζ-s-strongly stationary in $P_{\kappa_x}x$ such that $A \cap d_\zeta(S) \cap d_\zeta(T) = \emptyset$. Thus $d_\zeta(S) \cap d_\zeta(T) \cap (0, x]$ is an open neighborhood of x in the $\tau_{\xi+1}$ topology that does not intersect A in some point other than x. Conversely, suppose A is $\xi+1$-s-strongly stationary in $P_{\kappa_x}x$. To show that $x \in d_{\xi+1}(A)$, let U be an arbitrary basic open neighborhood of x in the $\tau_{\xi+1}$ topology. By Lemma 3.6, we can assume that U is of the form

$$U = I \cap d_\zeta(A_0) \cap \cdots \cap d_\zeta(A_{n-1}),$$

where $I \in \mathcal{B}_0$, $n < \omega$, $\zeta < \xi+1$ and $A_i \subseteq P_\kappa X$ for $i < n$. Since $x \in U$ it follows from $(1)_\zeta$ that each A_i is ζ-s-strongly stationary in $P_{\kappa_x}x$, and thus by $(3)_{\xi+1}$ we see that $A \cap d_\zeta(A_0) \cap \cdots \cap d_\zeta(A_{n-1})$ is $\xi+1$-s-strongly stationary in $P_{\kappa_x}x$, and thus U intersects A in some point other than x.

Finally, we prove that if $(1)_{\leq\xi+1}$, $(2)_{\leq\xi}$, and $(3)_{\leq\xi+1}$ hold, then $(2)_{\xi+1}$ must also hold. Suppose A is $\xi+2$-s-strongly stationary in $P_{\kappa_x}x$. By $(2)_{\leq\xi}$, it suffices to show that whenever S and T are $\xi+1$-s-strongly stationary in $P_{\kappa_x}x$, the set $A \cap d_{\xi+1}(S) \cap d_{\xi+1}(T)$ is $\xi+1$-s-strongly stationary in $P_{\kappa_x}x$. So, fix Y and Z

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which are ζ-s-strongly stationary in $P_{\kappa_x}x$ for some $\zeta \leq \xi$. By $(3)_{\xi+1}$, it follows that $S \cap d_\zeta(Y)$ and $T \cap d_\zeta(Z)$ are $\xi+1$-s-strongly stationary in $P_{\kappa_x}x$, and hence by the $\xi+1$-s-strong stationarity of A and $(1)_{\xi+1}$ we have $A \cap d_{\xi+1}(S \cap d_\zeta(Y)) \cap d_{\xi+1}(T \cap d_\zeta(Z)) \neq \emptyset$. But

$$d_{\xi+1}(S \cap d_\zeta(Y)) \cap d_{\xi+1}(T \cap d_\zeta(Z)) = d_{\xi+1}(S) \cap d_{\xi+1}(T) \cap d_\zeta(Y) \cap d_\zeta(Z)$$

by Lemma 3.5, and thus $A \cap d_{\xi+1}(S) \cap d_{\xi+1}(T)$ is $\xi+1$-s-strongly stationary in $P_{\kappa_x}x$. The backward direction of $(2)_{\xi+1}$ follows easily from $(1)_{\leq\xi}$. ⊣

COROLLARY 3.17. *Suppose $P_{\kappa_x}x$ is ξ-s-strongly stationary where $\xi \leq \kappa_x$ and A is ζ-s-strongly stationary in $P_{\kappa_x}x$ for some $\zeta < \xi$. Then, for all $\zeta \leq \zeta' \leq \xi$, $d_\zeta(A)$ is a ζ'-s-weak club in $P_{\kappa_x}x$.*

PROOF. Fix ζ' with $\zeta \leq \zeta' \leq \xi$. It follows from Theorem 3.16(3) that $d_\zeta(A)$ is ξ-s-strongly stationary and hence ζ'-s-strongly stationary in $P_{\kappa_x}x$. Furthermore, $d_\zeta(A)$ is ζ'-s-closed below $P_{\kappa_x}x$ since $d_{\zeta'}(d_\zeta(A)) \subseteq d_\zeta(d_\zeta(A)) \subseteq d_\zeta(A)$. ⊣

COROLLARY 3.18. *Suppose that $x \in P_\kappa X$ and $\xi \leq \kappa_x$. Then x is not isolated in $(P_\kappa X, \tau_\xi)$ if and only if $P_{\kappa_x}x$ is ξ-s-strongly stationary.*

PROOF. For the forward direction, suppose that $P_{\kappa_x}x$ is not ξ-s-strongly stationary. Then there is $\zeta < \xi$ and sets $S, T \subseteq P_{\kappa_x}x$ such that S and T are both ζ-s-strongly stationary in $P_{\kappa_x}x$ but there is no $y \prec x$ such that S and T are both ζ-s-strongly stationary in $P_{\kappa_y}y$. Then, by Theorem 3.16(1), we have $d_\zeta(S) \cap d_\zeta(T) = \{x\}$, so x is isolated in $(P_\kappa X, \tau_\xi)$.

For the converse, suppose that $P_{\kappa_x}x$ is ξ-s-strongly stationary, and fix an interval $I \in \mathcal{B}_0$, an $n < \omega$, ordinals $\xi_0, \ldots, \xi_{n-1} < \xi$, and sets $A_0, \ldots, A_{n-1} \subseteq P_{\kappa_x}x$ such that

$$x \in U := I \cap d_{\xi_0}(A_0) \cap \cdots \cap d_{\xi_{n-1}}(A_{n-1}).$$

Let $\zeta := \max\{\zeta_i \mid i < n\} < \xi$. By Corollary 3.17, each of $I, d_{\xi_0}(A_0), \ldots, d_{\xi_{n-1}}(A_{n-1})$ is a ζ-s-weak club in $P_{\kappa_x}x$. By Corollary 3.19, U is also ζ-s-weak club in $P_{\kappa_x}x$. In particular, $U \neq \{x\}$; hence, x is not isolated in $P_{\kappa_x}x$. ⊣

COROLLARY 3.19. *Suppose $P_{\kappa_x}x$ is ξ-s-strongly stationary where $0 < \xi \leq \kappa_x$.*

(1) *A set A is ξ-s-strongly stationary in $P_{\kappa_x}x$ if and only if for all $\zeta < \xi$ we have $A \cap C \neq \emptyset$ for every set $C \subseteq P_{\kappa_x}x$ which is a ζ-s-weak club in $P_{\kappa_x}x$. Thus, the filter on $P_{\kappa_x}x$ generated by the collection of all sets which are ζ-s-weak clubs in $P_{\kappa_x}x$ for some $\zeta < \xi$ is the filter dual to $\mathrm{NS}^\xi_{\kappa_x,x}$.*

(2) *A set A is $\xi+1$-s-strongly stationary in $P_{\kappa_x}x$ if and only if $A \cap C \neq \emptyset$ for every set $C \subseteq P_{\kappa_x}x$ which is a ξ-s-weak club in $P_{\kappa_x}x$. Thus the filter generated by the ξ-s-weak club subsets of $P_{\kappa_x}x$ is the filter dual to $\mathrm{NS}^{\xi+1}_{\kappa_x,x}$.*

PROOF. We only provide a proof of (1) since the proof of (2) is essentially identical. Suppose A is ξ-s-strongly stationary in $P_{\kappa_x}x$. Fix $\zeta < \xi$ and assume that $C \subseteq P_{\kappa_x}x$ is a ζ-s-weak club in $P_{\kappa_x}x$. Since C is ζ-s-strongly stationary in $P_{\kappa_x}x$ there is some $y \in d_\zeta(C) \cap A$, but since $d_\zeta(C) \subseteq C$ we have $y \in C \cap A$. Conversely, suppose that for all $\zeta < \xi$ and every $C \subseteq P_{\kappa_x}x$ that is a ζ-s-weak club in $P_{\kappa_x}x$ we have $A \cap C \neq \emptyset$. To show that A is ξ-s-strongly stationary in $P_{\kappa_x}x$, suppose S and T are ζ-s-strongly stationary in $P_{\kappa_x}x$ for some $\zeta < \xi$. Then, since we are assuming that

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$P_{\kappa_x} x$ is ξ-s-strongly stationary, it follows by Theorem 3.16(3) that $d_\zeta(S) \cap d_\zeta(T)$ is ξ-s-strongly stationary in $P_{\kappa_x} x$. Furthermore,

$$d_\zeta(d_\zeta(S) \cap d_\zeta(T)) \subseteq d_\zeta(d_\zeta(S)) \cap d_\zeta(d_\zeta(T)) \subseteq d_\zeta(S) \cap d_\zeta(T),$$

which implies that $d_\zeta(S) \cap d_\zeta(T)$ is a ζ-s-weak club in $P_{\kappa_x} x$. Thus $A \cap d_\zeta(S) \cap d_\zeta(T) \cap P_{\kappa_x} x \neq \emptyset$, and hence A is ξ-s-strongly stationary in $P_{\kappa_x} x$ as desired. ⊣

PROPOSITION 3.20. *For $x \in P_\kappa X$ and $\xi \leq \kappa_x$, the set $P_{\kappa_x} x$ is ξ-s-strongly stationary if and only if $\mathrm{NS}^\xi_{\kappa_x,x}$ is a nontrivial ideal.*

PROOF. Suppose $P_{\kappa_x} x$ is 0-s-strongly stationary. Then $\mathrm{NS}^0_{\kappa_x,x}$ is the ideal $I_{\kappa_x,x}$ consisting of all subsets A of $P_{\kappa_x} x$ such that there is some $y \in P_{\kappa_x} x$ with $A \cap (y, x) = \emptyset$. Clearly this is a nontrivial ideal since $P_{\kappa_x} x \notin I_{\kappa_x,x}$.

Now suppose $\xi > 0$. Let us show that $\mathrm{NS}^\xi_{\kappa_x,x}$ is an ideal. Suppose A and B are both not ξ-s-strongly stationary in $P_{\kappa_x} x$. By Corollary 3.19, there are sets $C_A, C_B \subseteq P_{\kappa_x} x$ such that C_A is a ζ_A-s-weak club in $P_{\kappa_x} x$ for some $\zeta_A < \xi$, C_B is a ζ_B-s-weak club in $P_{\kappa_x} x$ for some $\zeta_B < \xi$, such that $C_A \cap A = \emptyset$ and $C_B \cap B = \emptyset$. Then $d_{\zeta_A}(C_A) \cap d_{\zeta_B}(C_B) \cap (A \cup B) = \emptyset$ where $d_{\zeta_A}(C_A) \cap d_{\zeta_B}(C_B)$ is a ζ-s-weak club in $P_{\kappa_x} x$ for $\zeta = \max\{\zeta_A, \zeta_B\}$ because $d_{\zeta_A}(C_A) \cap d_{\zeta_B}(C_B) \cap P_{\kappa_x} x$ is ζ-s-strongly stationary in $P_{\kappa_x} x$ by Theorem 3.16(3) and furthermore

$$d_\zeta(d_{\zeta_A}(C_A) \cap d_{\zeta_B}(C_B)) \subseteq d_\zeta(C_A) \cap d_\zeta(C_B).$$ ⊣

THEOREM 3.21. *Suppose that $0 < \xi < \kappa$. Then the following are equivalent*:

(1) *$\mathcal{B}_\xi$ is a base for τ_ξ*;
(2) *for every $\zeta \leq \xi$, every $x \in P_\kappa X$, and every $A \subseteq P_\kappa X$, if A is ζ-strongly stationary in $P_{\kappa_x} x$, then A is ζ-s-strongly stationary in $P_{\kappa_x} x$.*

PROOF. For the forward direction, suppose that (2) fails, and let ζ, x, and A form a counterexample, with ζ minimal among all such counterexamples. Note that we must have $\zeta > 0$. ⊣

CLAIM 3.22. *$P_{\kappa_x} x$ is not ζ-s-strongly stationary.*

PROOF. Suppose otherwise. We will show that A is in fact ζ-s-strongly stationary, contradicting our choice of A. By Corollary 3.19, it suffices to show that, for all $\eta < \zeta$ and every η-s-weak club C in $P_{\kappa_x} x$, we have $A \cap C \neq \emptyset$. Fix such η and C. Then C is η-s-strongly stationary in $P_{\kappa_x} x$ and hence, by the minimality of ζ, η-strongly stationary in $P_{\kappa_x} x$. Thus, since A is ζ-strongly stationary, there is $y \in A$ such that C is η-strongly stationary in $P_{\kappa_y} y$ and hence, again by the minimality of ζ, η-s-strongly stationary in $P_{\kappa_y} y$. But then, since C is an η-s-weak club in $P_{\kappa_x} x$, we have $y \in C \cap A$, as desired. ⊣

We can therefore fix an $\eta < \zeta$ and sets $S, T \subseteq P_{\kappa_x} x$ such that S and T are both η-s-strongly stationary in $P_{\kappa_x} x$ but there is no $y \in P_{\kappa_x} x$ such that S and T are both η-s-strongly stationary in $P_{\kappa_x} x$. Then we have $d_\eta(S) \cap d_\eta(T) = \{x\}$, and hence $\{x\} \in \tau_\xi$. To show that (1) fails, it thus suffices to show that $\{x\} \notin \mathcal{B}_\xi$.

Since $P_{\kappa_x} x$ is 1-strongly stationary, it follows that κ_x is a limit cardinal, and hence $\{x\} \notin \mathcal{B}_0$. Now suppose that $B \subseteq P_{\kappa_x} x$, $\xi_0 < \xi$, and $x \in d_{\xi_0}(B)$. Since $P_{\kappa_x} x$ is not ζ-s-strongly stationary, it follows that $\xi_0 < \zeta$ and B is ξ_0-s-stationary in $P_{\kappa_x} x$.

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By minimality of ζ, B is ξ_0-stationary in $P_{\kappa_x}x$, so, since $P_{\kappa_x}x$ is ζ-strongly stationary, there is $y \in P_{\kappa_x}x$ such that B is ξ_0-strongly stationary in $P_{\kappa_y}y$. Again by minimality of ζ, B is ξ_0-s-strongly stationary in $P_{\kappa_y}y$, so $y \in d_{\xi_0}(B)$. It follows that $\{x\} \notin \mathcal{B}_\xi$.

For the backward direction, suppose that (2) holds, and fix $x \in P_\kappa X$, $I \in \mathcal{B}_0$, $0 < n < \omega$, ordinals $\xi_0, \dots, \xi_{n-1} < \xi$, and sets $A_0, \dots, A_{n-1} \subseteq P_{\kappa_x}x$ such that

$$x \in I \cap d_{\xi_0}(A_0) \cap \cdots \cap d_{\xi_{n-1}}(A_{n-1}).$$

Let $\zeta := \max\{\xi_0, \dots, \xi_{n-1}\} < \xi$. It follows that P_{κ_x} is ζ-s-strongly stationary. If $P_{\kappa_x}x$ is not $(\zeta+1)$-strongly stationary, then there is $A \subseteq P_{\kappa_x}x$ such that $d_\zeta(A) = \{x\}$. We can therefore assume that $P_{\kappa_x}x$ is $(\zeta+1)$-strongly stationary and hence, by (2), $(\zeta+1)$-s-strongly stationary. But then it follows that $I \cap d_{\xi_0}(A_0) \cap \cdots \cap d_{\xi_{n-1}}(A_{n-1})$ is a ζ-s-weak club in $P_{\kappa_x}x$. In particular, it is ζ-s-strongly stationary in $P_{\kappa_x}x$, so

$$x \in d_\zeta(I \cap d_{\xi_0}(A_0) \cap \cdots \cap d_{\xi_{n-1}}(A_{n-1})) \subseteq I \cap d_{\xi_0}(A_0) \cap \cdots \cap d_{\xi_{n-1}}(A_{n-1}),$$

and $d_\zeta(I \cap d_{\xi_0}(A_0) \cap \cdots \cap d_{\xi_{n-1}}(A_{n-1})) \in \mathcal{B}_\xi$. Therefore, $\mathcal{B}_\xi$ is a base for τ_ξ.

3.5. Consequences of Π^1_ξ-indescribability. In this section we establish the consistency of the ξ-s-strong stationarity of $P_{\kappa_x}x$, for $\xi \leq \kappa_x$, using a two-cardinal version of transfinite indescribability.

The classical notion of Π^m_n-indescribability studied by Levy [24] was generalized to the two-cardinal setting in a set of handwritten notes by Baumgartner (see [9, Section 4]). More recently, various transfinite generalizations of classical Π^1_n-indescribability, involving certain infinitary formulas have been studied in the cardinal context [2–4, 12, 13, 15] and in the two-cardinal context [11].

Let us review the definition of Π^1_ξ-indescribability in the two-cardinal context used in [11]. For the reader's convenience, we review the notion of Π^1_ξ formula introduced in [2]. Recall that a formula of second-order logic is Π^1_0, or equivalently Σ^1_0, if it does not have any second-order quantifiers, but it may have finitely many first-order quantifiers and finitely many first- and second-order free variables. We use the standard convention that uppercase letters denote second-order variables, unless other specification is given. Bagaria inductively defined the notion of Π^1_ξ formula for any ordinal ξ as follows. A formula is $\Sigma^1_{\xi+1}$ if it is of the form

$$\exists X_0 \dots \exists X_k \varphi(X_0, \dots, X_k),$$

where φ is Π^1_ξ, and a formula is $\Pi^1_{\xi+1}$ if it is of the form

$$\forall X_0 \dots \forall X_k \varphi(X_0, \dots, X_k),$$

where φ is Σ^1_ξ. If ξ is a limit ordinal, we say that a formula is Π^1_ξ if it is of the form

$$\bigwedge_{\zeta<\xi} \varphi_\zeta,$$

where φ_ζ is Π^1_ζ for all $\zeta < \xi$ and the infinite conjunction has only finitely many free second-order variables. We say that a formula is Σ^1_ξ if it is of the form

$$\bigvee_{\zeta<\xi} \varphi_\zeta,$$

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where φ_ζ is Σ^1_ζ for all $\zeta < \xi$ and the infinite disjunction has only finitely many free second-order variables.

The two-cardinal definition of Π^1_ξ-indescribability below uses the following two-cardinal version of the usual V_α-hierarchy below a fixed cardinal κ. Suppose κ is an uncountable regular cardinal and X is a set of ordinals with $|X| \geq \kappa$. For $\alpha \leq \kappa$ we define

$$\begin{aligned} V_0(\kappa, X) &= X, \\ V_{\alpha+1}(\kappa, X) &= P_\kappa(V_\alpha(\kappa, X)) \cup V_\alpha(\kappa, X), \text{ and} \\ V_\alpha(\kappa, X) &= \bigcup_{\eta<\alpha} V_\alpha(\kappa, X) \text{ if } \alpha \text{ is a limit.} \end{aligned}$$

Clearly $V_\kappa \subseteq V_\kappa(\kappa, X)$ and if X is transitive then so is $V_\alpha(\kappa, X)$ for $\alpha \leq \kappa$. Furthermore, both $P_\kappa X$ and $P_\kappa X \times P_\kappa X$ are subsets of $V_\kappa(\kappa, X)$. For more regarding the expressive power of Π^1_ξ formulas over structures of the form $(V_\kappa(\kappa, X), \in, R_0, \dots, R_{n-1})$, where $R_0, \dots, R_{n-1} \subseteq V_\kappa(\kappa, X)$, one may consult [1, Section 3] or [11].

DEFINITION 3.23 [11]. For $\xi < \kappa$ we say that $S \subseteq P_\kappa X$ is Π^1_ξ-*indescribable in* $P_\kappa X$ if for any $R_0, \dots, R_{n-1} \subseteq V_\kappa(\kappa, X)$ and any Π^1_ξ sentence φ such that

$$(V_\kappa(\kappa, X), \in, R_0, \dots, R_{n-1}) \models \varphi,$$

there is an $x \in S$ such that $x \cap \kappa = \kappa_x$ and

$$(V_{\kappa_x}(\kappa_x, x), \in, R_0 \cap V_{\kappa_x}(\kappa_x, x), \dots, R_{n-1} \cap V_{\kappa_x}(\kappa_x, x)) \models \varphi.$$

The collection

$$\Pi^1_\xi(\kappa, X) = \{A \subseteq P_\kappa X \mid A \text{ is not } \Pi^1_\xi\text{-indescribable in } P_\kappa X\}$$

is called the Π^1_ξ-*indescribability ideal on* $P_\kappa X$.

Standard arguments, which we omit, establish the consistency of two-cardinal indescribability from supercompactness (see, for example, [1, Theorem D], [11, Corollary 5.5] and [15, Proposition 3.11]).

PROPOSITION 3.24. *Suppose κ is λ-supercompact where $\kappa \leq \lambda$ and $\lambda^{<\kappa} = \lambda$. Then $P_\kappa\lambda$ is Π^1_ξ-indescribable for all $\xi < \kappa$. Furthermore, the set*

$$\{x \in P_\kappa\lambda \mid \kappa_x = x \cap \kappa \text{ and } P_{\kappa_x} x \text{ is } \Pi^1_\xi\text{-indescribable for all} \xi < \kappa_x\}$$

is in any normal measure U on $P_\kappa\lambda$.

Abe [1, Lemma 4.1] showed that if $P_\kappa X$ is Π^1_n-indescribable then $\Pi^1_n(\kappa, X)$ is a strongly normal ideal on $P_\kappa X$. As pointed out in [16], a straightforward application of the arguments for [1, Lemma 4.1] and [2, Proposition 4.4], which is left to the reader, establishes the following.

PROPOSITION 3.25. *For $\xi < \kappa$, if $P_\kappa X$ is Π^1_ξ-indescribable then $\Pi^1_\xi(\kappa, X)$ is a strongly normal ideal on $P_\kappa X$.*

Next we show that the ξ-s-strong stationarity of a set S in $P_{\kappa_x} x$ can be expressed by a Π^1_ξ formula.

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LEMMA 3.26. *For all $\xi < \kappa$ there is a Π^1_ξ formula $\Phi_\xi(R, S, T)$ with three free second-order variables such that for $x \in P_\kappa X$, a set $A \subseteq P_{\kappa_x} x$ is ξ-s-strongly stationary in $P_{\kappa_x} x$ if and only if*

$$(V_{\kappa_x}(\kappa_x, x), \in, A, P_{\kappa_x} x, \prec_x) \models \Phi_\xi[A, P_{\kappa_x} x, \prec_x],$$

where $\prec_x$ denotes the usual strong subset ordering $\prec$ restricted to $P_{\kappa_x} x$.

PROOF. We proceed by induction on ξ. We let $\Phi_0(R, S, T)$ be the Π^1_0 formula

$$(\forall y \in S)(\exists x \in R)\ (y, x) \in T$$

so that $\Phi_0[A, P_{\kappa_x} x, \prec]$ expresses that A is 0-s-strongly stationary (i.e., $\prec$-cofinal) in $P_{\kappa_x} x$ over the structure $(V_{\kappa_x}(\kappa_x, x), \in, A, P_{\kappa_x} x, \prec)$.

Suppose ξ is a limit ordinal. It is easy to see that $\Phi_\xi = \bigwedge_{\zeta<\xi} \Phi_\zeta$ is as desired.

Suppose $\xi = \zeta + 1$. Let Φ_ζ be the Π^1_ζ-formula obtained from the induction hypothesis. Then for all $x \in P_\kappa X$ a set $A \subseteq P_{\kappa_x} x$ is ζ-s-strongly stationary in $P_{\kappa_x} x$ if and only if

$$(V_{\kappa_x}(\kappa_x, x), \in, A, P_{\kappa_x} x, \prec_x) \models \Phi_\zeta[A, P_{\kappa_x} x, \prec_x].$$

For $x \in P_\kappa X$ we see that $A \subseteq P_{\kappa_x} x$ is ξ-s-strongly stationary in $P_{\kappa_x} x$ if and only if

$$\Phi_\zeta[A, P_{\kappa_x} x, \prec_x] \wedge (\forall S \subseteq P_{\kappa_x} x)(\forall T \subseteq P_{\kappa_x} x)[\Phi_\zeta[S, P_{\kappa_x} x, \prec_x] \wedge \Phi_\zeta[T, P_{\kappa_x} x, \prec_x] \longrightarrow$$

$$(\exists y \in A)\Phi_\zeta[S \cap P_{\kappa_y} y, P_{\kappa_y} y, \prec_y] \wedge \Phi_\zeta[T \cap P_{\kappa_y} y, P_{\kappa_y} y, \prec_y]]$$

holds in $(V_{\kappa_x}(\kappa_x, x), \in, A, P_{\kappa_x} x, \prec_x)$. It is easy to check that the previous formula is equivalent to a Π^1_ξ formula, hence the desired formula $\Phi_\xi(R, S, T)$ exists. ⊣

COROLLARY 3.27. *For $x \in P_\kappa X$ with $\kappa_x = x \cap \kappa$, if $A \subseteq P_{\kappa_x} x$ is Π^1_ξ-indescribable in $P_{\kappa_x} x$ then A is $\xi + 1$-s-strongly stationary in $P_{\kappa_x} x$.*

PROOF. To show that A is $\xi + 1$-s-strongly stationary in $P_{\kappa_x} x$, fix sets $S, T \subseteq P_{\kappa_x} x$ that are ζ-s-strongly stationary where $\zeta \leq \xi$ and let Φ_ζ be the Π^1_ζ-formula obtained from Lemma 3.26. Since A is Π^1_ξ-indescribable, it is Π^1_ζ-indescribable and the fact that

$$(V_{\kappa_x}(\kappa_x, x), \in, S, T, P_{\kappa_x} x, \prec_x) \models \Phi_\zeta[S, P_{\kappa_x} x, \prec_x] \wedge \Phi_\zeta[T, P_{\kappa_x} x, \prec_x]$$

implies that there is some $y \in A \cap P_{\kappa_x} x$ with $\kappa_y = y \cap \kappa$ such that the structure

$$(V_{\kappa_y}(\kappa_y, y), \in, S \cap P_{\kappa_y} y, T \cap P_{\kappa_y} y, P_{\kappa_y} y, \prec_y)$$

satisfies

$$\Phi_\zeta[S \cap P_{\kappa_y} y, P_{\kappa_y} y, \prec_y] \wedge \Phi_\zeta[T \cap P_{\kappa_y} y, P_{\kappa_y} y, \prec_y],$$

and hence S and T are ζ-s-strongly stationary in y. Therefore A is $\xi + 1$-s-strongly stationary in $P_{\kappa_x} x$. ⊣

COROLLARY 3.28. *For $\xi < \kappa$, if there is an $x \in P_\kappa X$ such that $P_{\kappa_x} x$ is Π^1_ξ-indescribable then the $\tau_{\xi+1}$-topology on $P_\kappa X$ is not discrete.*

PROPOSITION 3.29. *Suppose $P_\kappa X$ is Π^1_1-indescribable. Then a set $A \subseteq P_\kappa X$ is 2-s-strongly stationary in $P_\kappa X$ if and only if for every pair S, T of strongly stationary*

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subsets of $P_\kappa X$ there is an $x \in A$ such that $x \cap \kappa = \kappa_x$ is a Mahlo cardinal and the sets S and T are both strongly stationary in $P_{\kappa_x} x$.

PROOF. Suppose A is 2-s-strongly stationary in $P_\kappa X$. Fix sets S and T that are strongly stationary in $P_\kappa X$. The fact that κ is Mahlo and the sets S and T are strongly stationary in $P_\kappa X$ can be expressed by a Π^1_1 sentence:

$$(V_\kappa(\kappa, X), \in, P_\kappa X, S, T) \models \varphi.$$

The set

$$C = \{x \in P_\kappa X \mid (V_{\kappa_x}(\kappa_x, x), \in, P_{\kappa_x} x, S \cap V_{\kappa_x}(\kappa_x, x), T \cap V_{\kappa_x}(\kappa_x, x)) \models \varphi\}$$

is in the filter $\Pi^1_1(\kappa, X)^*$. Thus C is, in particular, strongly stationary in $P_\kappa X$ and so by Lemma 3.10 we see that C is 1-s-strongly stationary in $P_\kappa X$. Since A is 2-s-strongly stationary in $P_\kappa X$, there is an $x \in A \cap C$ and it follows that κ_x is Mahlo and the sets S and T are strongly stationary in $P_{\kappa_x} x$.

Conversely, to show that A is 2-s-strongly stationary in $P_\kappa X$, fix sets Q and R that are 1-s-strongly stationary in $P_\kappa X$. By Lemma 3.10, Q and R are strongly stationary in $P_\kappa X$. Thus, by assumption, there is an $x \in A$ such that $x \cap \kappa = \kappa_x$ is Mahlo and the sets Q and R are both strongly stationary in $P_{\kappa_x} x$. By Lemma 3.10, Q and R are both 1-s-strongly stationary in $P_{\kappa_x} x$. Hence A is 2-s-strongly stationary in $P_\kappa X$. ⊣

PROPOSITION 3.30. *For $x \in P_\kappa X$ with $x \cap \kappa = \kappa_x$, if $P_{\kappa_x} x$ is Π^1_ξ-indescribable where $\xi < \kappa_x$, then the ideal $\mathrm{NS}^{\xi+1}_{\kappa_x, x}$ (see Definition 3.9) is strongly normal.*

PROOF. For each $z \in P_{\kappa_x} X$ choose $C_z \in (\mathrm{NS}^{\xi+1}_{\kappa_x, x})^*$. Without loss of generality, by Corollary 3.19, we may assume that each C_z is a ξ-s-weak club in $P_{\kappa_x} x$.

Since each C_z is in the filter $\Pi^1_\xi(\kappa_x, x)^*$ and $\Pi^1_\xi(\kappa_x, x)$ is strongly normal, it follows that the set $C = \triangle_\prec\{C_z \mid z \in P_{\kappa_x} x\}$ is in the filter $\Pi^1_\xi(\kappa_x, x)^*$ and thus C is $\xi + 1$-s-strongly stationary in $P_{\kappa_x} x$ by Corollary 3.27. By Theorem 3.16(2), it follows that $d_\xi(C)$ is ξ-s-strongly stationary in $P_{\kappa_x} x$, and since d_ξ is the Cantor derivative of the space $(P_\kappa X, \tau_\xi)$, it follows that $d_\xi(d_\xi(C)) \subseteq d_\xi(C)$ and hence $d_\xi(C)$ is a ξ-s-weak club in $P_\kappa X$. Thus it will suffice to show that $d_\xi(C) \subseteq C$.

Let us verify that $d_\xi(C) \subseteq \triangle_\prec\{d_\xi(C_z) \mid z \in P_{\kappa_x} x\}$. Suppose $y \in d_\xi(C)$, then the set $\triangle_\prec\{C_z \mid z \in P_{\kappa_x} x\}$ is ξ-s-strongly stationary in $P_{\kappa_y} y$. To show that $y \in \triangle_\prec\{d_\xi(C_z) \mid z \in P_{\kappa_x} x\}$ we must verify that $y \in \bigcap_{z \prec y} d_\xi(C_z)$. Fix $z_0 \prec y$, then $(z_0, y) \cap \triangle_\prec\{C_z \mid z \in P_{\kappa_x} x\} \subseteq C_{z_0}$ and since $(z_0, y) \cap \triangle_\prec\{C_z \mid z \in P_{\kappa_x} x\}$ is ξ-s-strongly stationary in $P_{\kappa_y} y$ we see that $y \in d_\xi(C_{z_0})$. Thus $d_\xi(C) \subseteq \triangle_\prec\{d_\xi(C_z) \mid z \in P_{\kappa_x} x\}$.

Since each C_z is a ξ-s-weak club in $P_{\kappa_x} x$, it follows that $d_\xi(C_z) \subseteq C_z$ and thus

$$d_\xi(C) \subseteq \triangle_\prec\{d_\xi(C_z) \mid z \in P_{\kappa_x} x\} \subseteq \triangle_\prec\{C_z \mid z \in P_{\kappa_x} x\} = C.$$ ⊣

3.6. Variations. In this subsection, we investigate a couple of variations on the sequence of derived topologies considered above. First, we show that by restricting our attention to a certain natural club subset of $P_\kappa X$, certain questions about the resulting spaces become more tractable.

Let $P'_\kappa X$ be the set of $x \in P_\kappa X$ for which $\kappa_x = x \cap \kappa$. Similarly, if $x \in P'_\kappa X$, then $P'_{\kappa_x} x = P'_\kappa X \cap P_{\kappa_x} x$. If κ is weakly inaccessible, then $P'_\kappa X$ is evidently a club, and

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hence a weak club, in $P_\kappa X$. It follows that, if $\xi < \kappa$, $x \in P_\kappa X$, and κ_x is weakly inaccessible, then

$$(P_{\kappa_x} x \text{ is } \xi\text{-s-stationary}) \iff (P'_{\kappa_x} x \text{ is } \xi\text{-s-stationary in } P_{\kappa_x} x). \tag{1}$$

For each $\xi < \kappa$, let τ'_ξ be the subspace topology on $P'_\kappa X$ induced by τ_ξ, and let $\mathcal{B}'_\xi = \{U \cap P'_\kappa X \mid U \in \mathcal{B}_\xi\}$; it follows that τ'_ξ is the topology on $P'_\kappa X$ generated by $\mathcal{B}'_\xi$.

PROPOSITION 3.31. *Suppose that $x \in P'_\kappa X$. Then the following are equivalent*:

(1) *κ_x is weakly inaccessible*;
(2) *x is not isolated in $(P'_\kappa X, \tau'_0)$.*

PROOF. If κ_x is weakly inaccessible and $y \prec x$, with $y \in P_\kappa X$, then, letting λ be the least cardinal with $|y| < \lambda$, we have $y \cup \lambda \in (y, x] \cap P'_\kappa X$. The implication (1) $\implies$ (2) follows immediately.

For the converse, suppose first that $\kappa_x = \lambda^+$ is a successor cardinal, and let $y \prec x$ be such that $|y| = \lambda$. Then $(y, x] = \{x\}$, so x is isolated in τ_0, and hence also in τ'_0. Suppose next that κ_x is singular, and let $y \subseteq \kappa_x$ be a cofinal subset such that $|y| = \mathrm{cf}(\kappa_x)$. Then $(y, x] \cap P'_\kappa X = \{x\}$, so x is isolated in τ'_0. ⊣

Using this proposition, we can establish the following characterization of when $\mathcal{B}'_\xi$ forms a base for τ'_ξ. Since the proof is essentially the same as that of Theorem 3.21, we leave it to the reader.

THEOREM 3.32. *Suppose that $0 < \xi < \kappa$. Then the following are equivalent*:

(1) *$\mathcal{B}'_\xi$ is a base for τ'_ξ*;
(2) *for every $\zeta \leq \xi$, every $x \in P'_\kappa X$ for which κ_x is weakly inaccessible, and every $A \subseteq P_\kappa X$, if A is ζ-strongly stationary in $P_{\kappa_x} x$, then A is ζ-s-strongly stationary in $P_{\kappa_x} x$.*

COROLLARY 3.33. *$\mathcal{B}'_1$ is a base for τ'_1.*

PROOF. This is immediate from Proposition 3.10 and Theorem 3.32. ⊣

We saw above that the topology $(P_\kappa X, \tau_1)$ can be characterized by specifying that, if $x \in P_\kappa X$ and $A \subseteq P_\kappa X$, then x is a limit point of A if and only if A is strongly 1-s-stationary in $P_{\kappa_x} x$. By Proposition 3.10, if κ_x is regular, then this is equivalent to A being 1-strongly stationary in $P_{\kappa_x} x$, and if κ_x is Mahlo, it is in turn equivalent to A being strongly stationary in $P_{\kappa_x} x$. One can ask if there is a variant on this topology in which limit points are characterized by *stationarity* in the sense of [23] (recall the discussion at the end of Section 2). We now show that the answer is positive as long as κ is weakly inaccessible and one only requires this of $x \in P_\kappa X$ for which κ_x is weakly inaccessible. We first establish the following proposition.

PROPOSITION 3.34. *Suppose that κ is weakly inaccessible, $A \subseteq P_\kappa X$ and the set*

$$\{x \in P_\kappa X \mid \kappa_x \text{ is regular and } A \cap P_{\kappa_x} x \text{ is stationary in } P_{\kappa_x} x\}$$

is stationary in $P_\kappa X$. Then A is stationary.

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PROOF. Fix a club C in $P_\kappa X$. Since κ is regular and uncountable, by [27, Theorem 1.5], we can find a function $f : [X]^2 \to P_\kappa X$ such that $B_f \subseteq C$, where

$$B_f := \{x \in P_\kappa X \mid f``[x]^2 \subseteq P(x)\}.$$

We actually get slightly more. Namely, let C_f be the set of $x \in P_\kappa X$ for which $f``[x]^2 \subseteq P_{\kappa_x} x$. Then clearly $C_f \subseteq B_f \subseteq C$, and, moreover, C_f is a club in $P_\kappa X$. To see this, simply note that C_f is clearly closed and, if $\langle y_n \mid n < \omega\rangle$ is a $\prec$-increasing sequence of elements of B_f, then $\bigcup\{y_n \mid n < \omega\} \in C_f$, so C_f is cofinal in $P_\kappa X$. (This is where we use the fact that κ is weakly inaccessible, and hence a limit cardinal.)

By assumption, we can find $x \in P_\kappa X$ such that:

(1) κ_x is regular;
(2) $A \cap P_{\kappa_x} x$ is stationary in $P_{\kappa_x} x$;
(3) $x \in C_f$.

Since $x \in C_f$, we know that $g := f \restriction [x]^2$ satisfies $g : [x]^2 \to P_{\kappa_x} x$. Since κ_x is regular, it follows that B_g is a club in $P_{\kappa_x} x$. Then item (2) above implies that $B_g \cap A \cap P_{\kappa_x} x \neq 0$. Since $B_g \subseteq B_f \subseteq C$, it follows that $C \cap A \neq 0$. The choice of C was arbitrary, and hence A is stationary in $P_\kappa X$. ⊣

Note that Proposition 3.34 fails if $\kappa > \aleph_1$ is a successor cardinal. Indeed, if $\kappa = \nu^+ > \aleph_1$, then $A = P_\nu X$ satisfies the hypothesis of Proposition 3.34 but is not stationary in $P_\kappa X$.

Now, if κ is weakly inaccessible, define a function $c : P(P_\kappa X) \to P(P_\kappa X)$ by letting

$$c(A) = A \cup \{x \in P_\kappa X \mid \kappa_x \text{ is weakly inaccessible and } A \cap P_{\kappa_x} x \text{ is stationary in } P_{\kappa_x} x\}.$$

Proposition 3.34 implies that c is a closure operator. If τ is the topology

$$\{U \subseteq P_\kappa X \mid c(P_\kappa X \setminus U) = P_\kappa X \setminus U\}$$

on $P_\kappa X$ generated by c, then, clearly τ is a witness to the following.

COROLLARY 3.35. *If κ is weakly inaccessible and X is a set of ordinals with $\kappa \subseteq X$, then there is a topology τ on $P_\kappa X$ such that for $A \subseteq P_\kappa X$, x is a limit point of A if and only if κ_x is weakly inaccessible and $A \cap P_{\kappa_x} x$ is stationary in $P_{\kappa_x} x$. In particular, x is a nonisolated point of the space $(P_\kappa X, \tau)$ if and only if κ_x is weakly inaccessible.*

§4. On Ramseyness and indescribability. In this section we answer questions concerning the relationship between Ramseyness and indescribability, which were raised by the first author and Holy [15] in the context of cardinals, and by the first author and White [16] in the two-cardinal context. We provide detailed arguments in the cardinal context and simply state definitions and results in the two-cardinal context since the proofs are similar.

Let us review the definition and some basic properties of canonical functions. We follow the definitions and notation given in [18]. The sequence of canonical functions $\langle f_\alpha \mid \alpha < \lambda^+\rangle$ is a sequence of canonical representatives of the ordinals less than λ^+ in the generic ultrapower obtained by forcing with any normal ideal I

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on $Z \subseteq P(\lambda)$. We recursively define $\langle f_\alpha \mid \alpha < \lambda^+ \rangle$ as follows. For $\alpha < \lambda$ we let

$$f_\alpha(z) = \mathrm{ot}(z \cap \alpha)$$

for all $z \in Z$. For $\lambda < \alpha < \lambda^+$ define

$$f_\alpha(z) = \sup\{f_{b_{\lambda,\alpha}(\eta)}(z) + 1 \mid \eta \in z\},$$

where $b_{\lambda,\alpha} : \lambda \to \alpha$ is a bijection. Let us note that if we take $Z = \lambda$, then each f_α represents the ordinal α in any generic ultrapower obtained by forcing with a normal ideal on λ. Whereas, in the two-cardinal setting, if we take $Z = P_\kappa\lambda$, the function f_α represents α in any generic ultrapower obtained by forcing with a normal ideal on $P_\kappa\lambda$.

Let us review some basic definitions concerning ineffable and Ramsey operators on cardinals. For $S \subseteq \kappa$, we say that $\vec{S} = \langle S_\alpha \mid \alpha \in S \rangle$ is an *S-list* if $S_\alpha \subseteq \alpha$ for all $\alpha \in S$. Given an S-list $\vec{S}$, a set $H \subseteq S$ is said to be *homogeneous for* $\vec{S}$ if whenever $\alpha, \beta \in H$ with $\alpha < \beta$ we have $S_\alpha = S_\beta \cap \alpha$. If I is an ideal on κ, we define another ideal $\mathcal{I}(I)$ such that for $S \subseteq \kappa$ we have $S \in \mathcal{I}(I)^+$ if and only if for every S-list $\vec{S} = \langle S_\alpha \mid \alpha \in S \rangle$ there is a set $H \in P(S) \cap I^+$ which is homogeneous for $\vec{S}$. We say that κ is *almost ineffable* if $\kappa \in \mathcal{I}([\kappa]^{<\kappa})^+$ and κ is *ineffable* if $\kappa \in \mathcal{I}(\mathrm{NS}_\kappa)^+$. The function $\mathcal{I}$ is referred to as the *ineffable operator on* κ.

Recall that for a cardinal κ and a set $S \subseteq \kappa$, a function $f : [\kappa]^{<\omega} \to \kappa$ is called *regressive on* S if $f(x) < \min(x)$ for all $x \in [S]^{<\omega}$. Given a function $f : [\kappa]^{<\omega} \to \kappa$, a set $H \subseteq \kappa$ is said to be *homogeneous for* f if $f \restriction [H]^n$ is constant for every $n < \omega$. If I is an ideal on a cardinal κ, we define another ideal $\mathcal{R}(I)$ such that for $S \subseteq \kappa$ we have $S \in \mathcal{R}(I)^+$ if and only if for every function $f : [\kappa]^{<\omega} \to \kappa$ that is regressive on S, there is a set $H \in P(S) \cap I^+$ which is homogeneous for f. We say that a set $S \subseteq \kappa$ is *Ramsey in* κ if $S \in \mathcal{R}([\kappa]^{<\kappa})^+$. Let us note that the definition of Ramsey set and, more generally, the definition of $\mathcal{R}(I)$ given above are standard and have many equivalent formulations (see [12, Proposition 2.8 and Theorem 2.10] for details). The function $\mathcal{R}$ is called the *Ramsey operator on* κ.

The ineffable operator $\mathcal{I}$ and the Ramsey operator $\mathcal{R}$ on κ are examples of what are called *ideal operators*, which have been studied in a broader context by several authors [15, 21, 22], and which are discussed in slightly more detail below. For a given ideal I and ideal operator $\mathcal{O}$, such as $\mathcal{O} \in \{\mathcal{I}, \mathcal{R}\}$, we inductively define new ideals by letting

$$\begin{aligned} \mathcal{O}^0(I) &= I, \\ \mathcal{O}^{\alpha+1}(I) &= \mathcal{O}(\mathcal{O}^\alpha(I)), \text{ and} \\ \mathcal{O}^\alpha(I) &= \bigcup_{\beta<\alpha} \mathcal{O}^\beta(I). \end{aligned}$$

So, for example, $S \in \mathcal{R}([\kappa]^{<\kappa})^2$ if and only if for every function $f : [\kappa]^{<\omega} \to \kappa$ that is regressive on S there is a set H that is Ramsey in κ and homogeneous for f.

Recall that a set $S \subseteq \kappa$ is *Π^1_n-indescribable in* κ if $(V_\kappa, \in, R) \models \varphi$ implies there is an $\alpha \in S$ with $(V_\alpha, \in, R \cap V_\alpha) \models \varphi$ whenever $S \subseteq V_\kappa$ and φ is a Π^1_n sentence. Recall that φ is Π^1_0 if it is first order with finitely many second-order free variables. When κ is Π^1_n-indescribable, the collection $\Pi^1_n(\kappa)$ of all subsets of κ which are not

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Π^1_n-indescribable in κ forms a normal ideal on κ [24]. Baumgartner studied ideals on κ of the form $\mathcal{I}^\gamma(\Pi^1_n(\kappa))$ for $\gamma < \kappa^+$ and $n \in \omega \cup \{-1\}$ where for notational convenience we take $\Pi^1_{-1}(\kappa) = [\kappa]^{<\kappa}$ (see [5, Section 7] and [6]). Ideals of the form $\mathcal{R}^\gamma([\kappa]^{<\kappa})$ and $\mathcal{R}^\gamma(\mathrm{NS}_\kappa)$ were introduced by Feng [17]; note that if κ is inaccessible then $\Pi^1_0(\kappa) = \mathrm{NS}_\kappa$.

Bagaria [2] introduced a notion of Π^1_ξ-indescribability of a cardinal κ for $\xi < \kappa$. The first author [13] extended Bagaria's definition and introduced a notion of Π^1_ξ-indescribability of κ for $\xi < \kappa^+$.[2] Instead of reviewing the rather lengthy definition, we refer the reader to [13] for the definition of the Π^1_ξ-indescribability of a subset S of κ for $\xi < \kappa^+$. The Π^1_ξ-indescribability ideal on κ is then

$$\Pi^1_\xi(\kappa) = \{S \subseteq \kappa \mid S \text{ is not } \Pi^1_\xi\text{-indescribable in } \kappa\}.$$

Let us note that, in some sense, the definition of Π^1_ξ-indescribability does not play a large role in what follows because it is being "black boxed" by Lemma 4.5 and Theorem 4.2 (see the proof of Corollary 4.6).

Ideals of the form $\mathcal{R}^\gamma(\Pi^1_\xi(\kappa))$ were studied by the first author [12], and more generally, ideals of the form $\mathcal{I}^\gamma(\Pi^1_\xi(\kappa))$ and $\mathcal{R}^\gamma(\Pi^1_\xi(\kappa))$ for $\gamma < \kappa^+$ and $\xi \in \kappa \cup \{-1\}$ were studied by the first author and Holy [15] (in fact the framework presented in [15] and [21] handles many ideal operators other than $\mathcal{I}$ and $\mathcal{R}$).

Notice that for a cardinal κ, to each ideal of the form $\mathcal{O}^\gamma(\Pi^1_\xi(\kappa))$ where $\mathcal{O} \in \{\mathcal{I}, \mathcal{R}\}$, $\gamma < \kappa^+$ and $\xi \in \kappa^+ \cup \{-1\}$, there is a corresponding large cardinal hypothesis, namely $\kappa \in \mathcal{O}^\gamma(\Pi^1_\xi(\kappa))^+$.

DEFINITION 4.1. Suppose κ is a cardinal, $\gamma < \kappa^+$ and $\xi \in \kappa^+ \cup \{-1\}$. Let $\Pi^1_{-1}(\kappa) = [\kappa]^{<\kappa}$. We say that κ is γ-Π^1_ξ-*ineffable* if $\kappa \in \mathcal{I}^\gamma(\Pi^1_\xi(\kappa))^+$, and κ is γ-Π^1_ξ-*Ramsey* if $\kappa \in \mathcal{R}^\gamma(\Pi^1_\xi(\kappa))^+$.

For example, κ is 1-Π^1_1-Ramsey if and only if every regressive function $f : [\kappa]^{<\kappa} \to \kappa$ has a homogeneous set which is Π^1_1-indescribable in κ.

Recall that Baumgartner proved [5, Theorem 4.1] that when κ is a subtle cardinal the set

$$\{\alpha < \kappa \mid \alpha \text{ is } \Pi^1_n\text{-indescribable for all } n < \omega\}$$

is in the subtle filter on κ. More generally, the first author and Holy [15, Corollary 3.5] proved that when κ is subtle the set

$$\{\alpha < \kappa \mid \alpha \text{ is } \Pi^1_\xi\text{-indescribable for all } \xi < \alpha^+\}$$

is in the subtle filter on κ. Since whenever a cardinal κ is almost ineffable it must also be subtle, it follows that the existence of an almost ineffable cardinal is strictly stronger in consistency strength than the existence of a cardinal κ such that κ is Π^1_ξ-indescribable for all $\xi < \kappa^+$. Furthermore, as shown in [15], this result can be pushed up the almost ineffability hierarchy. For example, the existence of an

[2] Let us note that previously, Sharpe and Welch [29] had used games to define a notion of Π^1_ξ-indescribability of a cardinal κ for all $\kappa < \xi^+$, but the relationship between their notion and that of [13] is still not known.

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uncountable cardinal such that $\kappa \in \mathcal{I}^2([\kappa]^{<\kappa})^+$ is strictly stronger than the existence of an uncountable cardinal κ such that for all $\xi < \kappa^+$ we have $\kappa \in \mathcal{I}(\Pi^1_1(\kappa))^+$.

THEOREM 4.2 [15, Theorem 3.8]. *Suppose* $\gamma < \kappa^+$, $S \in \mathcal{I}^{\gamma+1}([\kappa]^{<\kappa})^+$ *and* $\vec{S} = \langle S_\alpha \mid \alpha \in S \rangle$ *is an S-list. Let A be the set of all ordinals* α *such that*

$$\exists X \subseteq S \cap \alpha \left[(\forall \xi < \alpha^+ \; X \in \mathcal{I}^{f^\kappa_\gamma(\alpha)}(\Pi^1_\xi(\alpha))^+) \wedge (X \cup \{\alpha\} \text{ is hom. for } \vec{S}) \right].$$

Then, $S \setminus A \in \mathcal{I}^{\gamma+1}([\kappa]^{<\kappa})$.

COROLLARY 4.3 [15, Corollary 3.9]. *Suppose* $\kappa \in \mathcal{I}^{\gamma+1}([\kappa]^{<\kappa})^+$ *where* $\gamma < \kappa^+$. *Then the set*

$$\{\alpha < \kappa \mid (\forall \xi < \alpha^+) \; \alpha \in \mathcal{I}^{f^\kappa_\gamma(\alpha)}(\Pi^1_\xi(\alpha))^+\}$$

is in the filter $\mathcal{I}^{\gamma+1}([\kappa]^{<\kappa})^*$. *In other words, if* κ *is* $\gamma+1$*-almost ineffable then the set of* $\alpha < \kappa$ *which are* $f^\kappa_\gamma(\alpha)$*-*Π^1_ξ*-ineffable for all* $\xi < \alpha$ *is in the filter* $\mathcal{I}^{\gamma+1}([\kappa]^{<\kappa})^*$.

4.1. New results on Ramseyness and indescribability. Now let us address the following question, and its generalizations, which were originally posed in [15].

QUESTION 4.4. *Is the existence of an uncountable cardinal* κ *with* $\kappa \in \mathcal{R}^2([\kappa]^{<\kappa})^+$ *strictly stronger than the existence of a cardinal* κ *such that* $\kappa \in \mathcal{R}(\Pi^1_\xi(\kappa))^+$ *for all* $\xi < \kappa^+$?

The following lemma is standard and is an easy consequence of Feng's characterization of Ramsey sets in terms of (ω, S)-sequences [17, Theorem 2.3].

LEMMA 4.5. *Suppose* κ *is a Ramsey cardinal. Then*

$$\mathcal{I}([\kappa]^{<\kappa}) \subseteq \mathcal{R}([\kappa]^{<\kappa}).$$

COROLLARY 4.6 [15, Theorem 10.3]. *Suppose* $S \in \mathcal{R}([\kappa]^{<\kappa})^+$ *and let*

$$T = \{\alpha \in S \mid (\forall \xi < \alpha^+) \; S \cap \alpha \in \Pi^1_\xi(\alpha)^+\}.$$

Then $S \setminus T \in \mathcal{R}([\kappa]^{<\kappa})$.

PROOF. Suppose $S \in \mathcal{R}([\kappa]^{<\kappa})^+$ and let T be as in the statement of the corollary. By Lemma 4.5 we see that $S \in \mathcal{I}([\kappa]^{<\kappa})^+$ and by Theorem 4.2 we have $S \setminus A \in \mathcal{I}([\kappa]^{<\kappa}) \subseteq \mathcal{R}([\kappa]^{<\kappa})$. But $A \subseteq T$ so $S \setminus T \subseteq S \setminus A$ and hence $S \setminus T \in \mathcal{R}([\kappa]^{<\kappa})$. ⊣

The next result shows that Corollary 4.6 can, in a sense, be pushed up the Ramsey hierarchy, and provides an affirmative answer to Questions 10.4–10.6 and 10.9 in [15]; it is at present the best known generalization of Theorem 4.2 from the context of the ineffable operator to that of the Ramsey operator.

THEOREM 4.7. *Suppose* $\gamma < \kappa^+$, $S \in \mathcal{R}^{\gamma+1}([\kappa]^{<\kappa})^+$ *and let*

$$T = \{\alpha \in S \mid (\forall \xi < \alpha^+) \; S \cap \alpha \in \mathcal{R}^{f^\kappa_\gamma(\alpha)}(\Pi^1_\xi(\alpha))^+\}.$$

Then $S \setminus T \in \mathcal{R}^{\gamma+1}([\kappa]^{<\kappa})$.

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PROOF. If $\gamma = 0$ the result follows directly from Corollary 4.6.

Suppose $\gamma = \delta + 1 < \kappa^+$ is a successor ordinal, and suppose for a contradiction that $S \setminus T \in \mathcal{R}^{\gamma+1}([\kappa]^{<\kappa})^+$. Recall that the set $C = \{\alpha < \kappa \mid f^\kappa_{\delta+1}(\alpha) = f^\kappa_\delta(\alpha) + 1\}$ is a club in κ and thus the set

$$E = (S \setminus T) \cap C$$

is in $\mathcal{R}^{\gamma+1}([\kappa]^{<\kappa})^+$. For each $\alpha \in E$ fix $\xi_\alpha < \alpha^+$ such that

$$S \cap \alpha \in \mathcal{R}^{f^\kappa_\delta(\alpha)+1}(\Pi^1_{\xi_\alpha}(\alpha)) = \mathcal{R}(\mathcal{R}^{f^\kappa_\delta(\alpha)}(\Pi^1_{\xi_\alpha}(\alpha)),$$

and fix a regressive function $g_\alpha : [S \cap \alpha]^{<\omega} \to \alpha$ which has no homogeneous set in $\mathcal{R}^{f^\kappa_\delta(\alpha)}(\Pi^1_{\xi_\alpha}(\alpha))^+$. Let $f : [E]^{<\omega} \to \kappa$ be a regressive function such that

$$f(\alpha_0, \dots, \alpha_n) = g_{\alpha_n}(\alpha_0, \dots, \alpha_{n-1})$$

for $n < \omega$ and $(\alpha_0, \dots, \alpha_n) \in [E]^{n+1}$. Since $E \in \mathcal{R}^{\gamma+1}([\kappa]^{<\kappa})^+$, there is a set $H \in P(E) \cap \mathcal{R}^{\delta+1}([\kappa]^{<\kappa})^+$ homogeneous for f. By the inductive hypothesis it follows that if we let

$$T_H = \{\alpha \in H \mid (\forall \xi < \alpha^+)\, H \cap \alpha \in \mathcal{R}^{f^\kappa_\delta(\alpha)}(\Pi^1_\xi(\alpha))^+\}$$

then $H \setminus T_H \in \mathcal{R}^{\delta+1}([\kappa]^{<\kappa})$. Thus we can fix an $\alpha \in T_H$. It follows that $H \cap \alpha \in \mathcal{R}^{f^\kappa_\delta(\alpha)}(\Pi^1_{\xi_\alpha}(\alpha))^+$ and $H \cap \alpha \subseteq E \cap \alpha \subseteq S \cap \alpha$ is homogeneous for g_α, a contradiction.

Now suppose $\gamma < \kappa^+$ is a limit ordinal, and suppose again for a contradiction that $S \setminus T \in \mathcal{R}^{\gamma+1}([\kappa]^{<\kappa})^+$. Recall that $C = \{\alpha < \kappa \mid f^\kappa_\gamma(\alpha) \text{ is a limit ordinal}\}$ is a club subset of κ and thus

$$E = (S \setminus T) \cap C$$

is in $R^{\gamma+1}([\kappa]^{<\kappa})^+$. For each $\alpha \in E$, using the fact that $\alpha \notin T$, let $\xi_\alpha < \alpha^+$ be such that

$$S \cap \alpha \in \mathcal{R}^{f^\kappa_\gamma(\alpha)}(\Pi^1_{\xi_\alpha}(\alpha)).$$

Since $f^\kappa_\gamma(\alpha) = \sup\{f^\kappa_{b_{\kappa,\gamma}(\eta)}(\alpha) + 1 \mid \eta \in \alpha\} < \alpha^+$ is a limit ordinal, we can choose an ordinal $r(\alpha) < \alpha$ such that

$$S \cap \alpha \in \mathcal{R}^{f^\kappa_{b_{\kappa,\gamma}(r(\alpha))}(\alpha)+1}(\Pi^1_{\xi_\alpha}(\alpha)).$$

This defines a regressive function $r : E \to \kappa$, and by normality of $\mathcal{R}^{\gamma+1}([\kappa]^{<\kappa})$ (see [17, Theorem 2.1]), there is an $E^* \in P(E) \cap \mathcal{R}^{\gamma+1}([\kappa]^{<\kappa})^+$ and some $\beta_0 < \kappa$ such that $g(\alpha) = \beta_0$ for all $\alpha \in E^*$. Let $\nu = b_{\kappa,\gamma}(\beta_0)$ and notice that for all $\alpha \in E^*$,

$$S \cap \alpha \in \mathcal{R}^{f^\kappa_\nu(\alpha)+1}(\Pi^1_{\xi_\alpha}(\alpha)).$$

For each $\alpha \in E^*$, we fix a regressive function $g_\alpha : [S \cap \alpha]^{<\omega} \to \kappa$ which has no homogeneous set in $\mathcal{R}^{f^\kappa_\nu(\alpha)}(\Pi^1_{\xi_\alpha}(\alpha))^+$. Let $f : [E^*]^{<\omega} \to \kappa$ be a regressive function such that

$$f(\alpha_0, \dots, \alpha_n) = g_{\alpha_n}(\alpha_0, \dots, \alpha_{n-1})$$

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for $n < \omega$ and $(\alpha_0, \dots, \alpha_n) \in [E]^{n+1}$. Since $E^* \in \mathcal{R}^{\gamma+1}([\kappa]^{<\kappa})^+$ there is a set $H \in P(E^*) \cap \mathcal{R}^{\gamma}([\kappa]^{<\kappa})^+$ homogeneous for f. Since $\nu < \gamma$ we have

$$H \in \mathcal{R}^{\gamma}([\kappa]^{<\kappa})^+ \subseteq \mathcal{R}^{\nu+1}([\kappa]^{<\kappa})^+,$$

and we may apply the inductive hypothesis to see that the set

$$T_H = \{\alpha \in H \mid (\forall \xi < \alpha^+)\ H \cap \alpha \in \mathcal{R}^{f^{\kappa}_{\nu}(\alpha)}(\Pi^1_{\xi}(\alpha))^+\}$$

satisfies $H \setminus T_H \in \mathcal{R}^{\nu+1}([\kappa]^{<\kappa})$. Thus we can fix an $\alpha \in T_H$. But then the set

$$H \cap \alpha \subseteq E^* \cap \alpha \subseteq E \cap \alpha \subseteq S \cap \alpha$$

is in $\mathcal{R}^{f^{\kappa}_{\nu}(\alpha)}(\Pi^1_{\xi_\alpha}(\alpha))^+$ and is homogeneous for g_α, which is a contradiction. ⊣

COROLLARY 4.8. *Suppose $\gamma < \kappa^+$. If $\kappa \in \mathcal{R}^{\gamma+1}([\kappa]^{<\kappa})^+$ then the set of $\alpha < \kappa$ which are $f^{\kappa}_{\gamma}(\alpha)$-$\Pi^1_{\xi}$-Ramsey for all $\xi < \alpha^+$ is in the filter $\mathcal{R}^{\gamma+1}([\kappa]^{<\kappa})^*$.*

4.2. New results on two-cardinal Ramseyness. Let us now discuss two-cardinal versions of the ineffable and Ramsey operator, which are defined using the strong subset ordering $\prec$. Suppose κ is a cardinal and X is a set of ordinals with $\kappa \subseteq X$. For $S \subseteq P_\kappa X$, we say that $\vec{S} = \langle S_x \mid x \in P_\kappa X\rangle$ is an $(S, \prec)$-list if $S_x \subseteq P_{\kappa_x} x$ for all $x \in S$. Given an $(S, \prec)$-list, a set $H \subseteq S$ is said to be *homogeneous for* $\vec{S}$ if whenever $x, y \in H$ with $x \prec y$ we have $S_x = S_y \cap P_{\kappa_x} x$. If I is an ideal on $P_\kappa X$, we define another ideal $\mathcal{I}_{\prec}(I)$ such that for $S \subseteq P_\kappa X$ we have $S \in \mathcal{I}_{\prec}(I)^+$ if and only if for every $(S, \prec)$-list $\vec{S}$ there is a set $H \in P(S) \cap I^+$ which is homogeneous for $\vec{S}$. We say that $P_\kappa X$ is *strongly ineffable* if $P_\kappa X \in \mathcal{I}_{\prec}(\mathrm{NSS}_{\kappa,X})^+$ and *almost strongly ineffable* if $P_\kappa X \in \mathcal{I}_{\prec}(I_{\kappa,X})^+$. Here $I_{\kappa,X}$ is the ideal on $P_\kappa X$ consisting of all subsets of $P_\kappa X$ which are not $\prec$-cofinal in $P_\kappa X$.

Let $[S]^{\leq\omega}_{\prec}$ be the collection of all tuples $\vec{x} = (x_0, \dots, x_{n-1}) \in S^n$ such that $n < \omega$ and $x_0 \prec \cdots \prec x_{n-1}$. A function $f : [P_\kappa X]^{\leq\omega}_{\prec} \to P_\kappa X$ is called *$\prec$-regressive on S* if $f(x_0, \dots, x_{n-1}) \prec x_0$ for all $(x_0, \dots, x_{n-1}) \in [S]^{\leq\omega}_{\prec}$. Given a function $f : [P_\kappa X]^{\leq\omega}_{\prec} \to P_\kappa X$, a set $H \subseteq P_\kappa X$ is said to be *homogeneous for f* if $f \restriction [H]^n$ is constant for all $n < \omega$. For $S \subseteq P_\kappa X$, let $S \in \mathcal{R}_{\prec}(I)^+$ if and only if for every function $f : [P_\kappa X]^{<\omega} \to P_\kappa X$ that is $\prec$-regressive on S, there is a set $H \in P(S) \cap I^+$ which is homogeneous for f. We say that $P_\kappa X$ is *strongly Ramsey* if $P_\kappa X \in \mathcal{R}_{\prec}(I_{\kappa,X})^+$.

The first author and White [16] showed that many results from the literature [5, 6, 12, 15, 17] on the ineffable operator $\mathcal{I}$ and the Ramsey operator $\mathcal{R}$, and their relationship with indescribability, can be extended to $\mathcal{I}_{\prec}$ and $\mathcal{R}_{\prec}$. For example, by iterating the ideal operators $\mathcal{I}_{\prec}$ and $\mathcal{R}_{\prec}$, one obtains hierarchies in the two-cardinal setting which are analogous to the classical ineffable and Ramsey hierarchies. One question left open by [16] is that which is analogous to Question 4.4 for the two-cardinal context. For example, if $P_\kappa X \in \mathcal{R}^2_{\prec}(I_{\kappa,X})^+$, does it follow that the set

$$\{x \in P_\kappa X \mid (\forall \xi < \kappa_x)\ x \in \mathcal{R}_{\prec}(\Pi^1_{\xi}(\kappa_x, x))^+\}$$

is in the filter $\mathcal{R}_{\prec}(I_{\kappa,X})^*$?

The proof of Theorem 4.7 generalizes in a straight-forward way to establish the following.

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THEOREM 4.9. *Suppose* $\gamma < |X|^+$, $S \in \mathcal{R}^{\gamma+1}_{\prec}(I_{\kappa,X})^+$ *and let*

$$T = \{x \in S \mid (\forall \xi < \kappa_x)\ S \cap P_{\kappa_x} x \in \mathcal{R}^{f_\gamma(x)}_{\prec}(\Pi^1_\xi(\kappa_x, x))^+\}.$$

Then $S \setminus T \in \mathcal{R}_{\prec}(I_{\kappa,X})$.

COROLLARY 4.10. *Suppose* $\gamma < |X|^+$. *If* $P_\kappa X \in \mathcal{R}_{\prec}(I_{\kappa,X})^+$, *then the set*

$$\{x \in P_\kappa X \mid (\forall \xi < \kappa_x)\ P_{\kappa_x} x \in \mathcal{R}^{f_\gamma(x)}_{\prec}(\Pi^1_\xi(\kappa_x, x))^+\}$$

is in the filter $\mathcal{R}^{\gamma+1}_{\prec}(I_{\kappa,X})^*$.

§5. Questions and ideas. Let us formulate a few open questions relevant to the topics of this article. For this section, let us assume κ is some regular uncountable cardinal and $X \supseteq \kappa$ is a set of ordinals. First, we consider the following questions regarding the consistency strength of various principles considered above.

QUESTION 5.1. *What is the consistency strength of "whenever* $S \subseteq P_\kappa X$ *is strongly stationary there is some* $x \in P_\kappa X$ *for which* $S \cap P_{\kappa_x} x$ *is strongly stationary in* $P_{\kappa_x} x$*"? Is this similar to the situation for cardinals? Is the strength of this kind of reflection of strong stationary sets strictly between the "great Mahloness" of* $P_\kappa X$ *and the* Π^1_1*-indescribability of* $P_\kappa X$?

QUESTION 5.2. *What is the consistency strength of the* 2*-s-strong stationarity of* $P_\kappa X$? *What is the consistency strength of the hypothesis that whenever* S *and* T *are strongly stationary in* $P_\kappa X$ *there is some* $x \in P_\kappa X$ *such that* S *and* T *are both strongly stationary in* $P_{\kappa_x} x$?

The following questions regarding separation of various properties considered in this article remain open.

QUESTION 5.3. *Can we separate reflection of strongly stationary sets from pairwise simultaneous reflection of strongly stationary sets? In other words, is it consistent that whenever* S *is strongly stationary in* $P_\kappa X$ *there is some* $x \in P_\kappa X$ *such that* S *is strongly stationary in* $P_{\kappa_x} x$, *but at the same time, pairwise reflection fails in the sense that there exists a pair* S, T *of strongly stationary subsets of* $P_\kappa X$ *such that for every* $x \in P_\kappa X$ *both* S *and* T *are not strongly stationary in* $P_{\kappa_x} x$?

It is conceivable that some two-cardinal $\square(\kappa)$-like principle could be used to address Questions 5.3. For example, $\square(\kappa)$ implies that every stationary subset of κ can be partitioned into two disjoint stationary sets that do not simultaneously reflect (see [20, Theorem 2.1] as well as [8, Theorem 3.50] and [14, Theorem 7.1] for generalizations).

QUESTION 5.4. *Is some two-cardinal* $\square(\kappa)$*-like principle formulated using weak clubs* (*defined in Section* 2) *consistent? Does it deny pairwise simultaneous reflection of strongly stationary subsets of* $P_\kappa X$?

It is also natural to ask whether the various reflection properties introduced here can be separated from the large cardinal notions that imply them.

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QUESTION 5.5. *Can we separate $\xi+1$-strong stationarity or $\xi+1$-s-strong stationarity in $P_\kappa X$ from*:

(1) *Π^1_ξ-indescribability in $P_\kappa X$ similar to what was done in* [3]; *or*
(2) *Π^1_1-indescribability in $P_\kappa X$ similar to what was done in* [7]?

In [3], it was shown that consistently $\mathrm{NS}^{\xi+1}_\kappa$ can be non-trivial while κ is not Π^1_ξ-indescribable. In [7, Definition 0.7], a normal version of the ideal NS^ξ_κ was introduced, $\mathrm{NS}^{\xi,d}_\kappa$. It was shown that consistently, $\mathrm{NS}^{\xi,d}_\kappa$ can be non-trivial for all $\xi<\omega$ while κ is not even Π^1_1-indescribable.

QUESTION 5.6. *Is it consistent that $\kappa\in\mathcal{I}(\Pi^1_\xi(\kappa))$ and $\kappa\notin\mathcal{I}(\mathrm{NS}^{\xi+1}_\kappa)$. Is it consistent that $\kappa\in\mathcal{I}(\Pi^1_1(\kappa))$ and $\kappa\notin\mathcal{I}(\mathrm{NS}^{\xi,d}_\kappa)$ for all $\xi<\omega$*?

Finally, let us consider some questions that arise by considering Proposition 3.8 and [13]. Bagaria noticed that, using the definitions of [2], no ordinal α is $\alpha+1$-stationary (see the discussion after Definition 2.6 in [2]) and no cardinal κ is Π^1_κ-indescribable (see the discussion after Definition 4.2 in [2]). The first author showed that Bagaria's definitions of ξ-s-stationarity and derived topologies $\langle\tau_\xi\mid\xi<\delta\rangle$ on an ordinal δ, can be modified in a natural way so that a regular cardinal μ can carry a longer sequence of derived topologies $\langle\tau_\xi\mid\xi<\mu^+\rangle$, such that, for each $\xi<\mu$ there is a club C_ξ in δ such that $\alpha\in C_\xi$ is not isolated in the τ_ξ topology if and only if α is $f^\mu_\xi(\alpha)$-s-stationary (see [13, Theorem 6.15]). The first author also generalized Bagaria's notion of Π^1_ξ-indescribability so that a cardinal κ can be Π^1_ξ-indescribable for all $\xi<\kappa^+$, and that the Π^1_ξ-indescribability of κ implies the $\xi+1$-s-stationarity of κ for all $\xi<\kappa^+$ (see [13, Proposition 6.18]). It is natural to ask whether similar techniques can be used to generalize the results in Section 3.2 of the present article. For example, can one modify the definition of ξ-strong stationarity so that Proposition 3.8 can fail for the modified notion?

QUESTION 5.7. *Can one use canonical functions to modify the definition of ξ-s-strong stationarity so that it is possible for $x\in P_\kappa X$ to be ξ-strongly stationary or ξ-s-strongly stationary for some $\xi>\kappa_x$*?

QUESTION 5.8. *Can the definitions of two-cardinal Π^1_ξ-indescribability (Definition 3.23), $\xi+1$-s-strong stationarity (Definition 3.7), and the two-cardinal derived topologies (see Section 3.2) be modified using canonical functions so that Corollary 3.27 might generalize to values of ξ for which $\kappa_x<\xi<|x|^+$ and Theorem 3.16 might generalize to values of ξ for which $\kappa<\xi<|X|^+$*?

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