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# Eddy diffusivities of inertial particles under gravity

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The large-scale/long-time transport of inertial particles of arbitrary mass density under gravity is investigated by means of a formal multiple-scale perturbative expansion in the scale-separation parameter between the carrier flow and the particle concentration field. The resulting large-scale equation for the particle concentration is determined, and is found to be diffusive with a positive definite eddy diffusivity. The calculation of the latter tensor is reduced to the resolution of an auxiliary differential problem, consisting of a coupled set of two differential equations in a $(6 + 1)$-dimensional coordinate system (three space coordinates plus three velocity coordinates plus time). Although expensive, numerical methods can be exploited to obtain the eddy diffusivity, for any desirable non-perturbative limit (e.g. arbitrary Stokes and Froude numbers). The aforementioned large-scale equation is then specialized to deal with two different relevant perturbative limits: (i) vanishing of both Stokes time and sedimenting particle velocity; (ii) vanishing Stokes time and finite sedimenting particle velocity. Both asymptotics lead to a greatly simplified auxiliary differential problem, now involving only space coordinates and thus easily tackled by standard numerical techniques. Explicit, exact expressions for the eddy diffusivities have been calculated, for both asymptotics, for the class of parallel flows, both static and time-dependent. This allows us to investigate analytically the role of gravity and inertia on the diffusion process by varying relevant features of the carrier flow, such as the form of its temporal correlation function. Our results exclude a universal role played by gravity and inertia on the diffusive behaviour: regimes of both enhanced and reduced diffusion may exist, depending on the detailed structure of the carrier flow.

**Key words:** mixing

## 1. Introduction

Transport of passive particles is a problem of major importance in a variety of domains ranging from astrophysics and geophysics to technological applications and

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biology (Ruiz, Macías & Peters 2004). A quantity of particular interest is the rate at which particles are transported by the flow.

For inertialess particles (i.e. fluid particles), for times large compared to those characteristic of the given velocity field, transport is usually diffusive and characterized by effective (enhanced) diffusion coefficients (Kraichnan 1987), the so-called *eddy diffusivities*. These coefficients incorporate all the dynamical effects played by the velocity field on the particle transport.

Although the diffusive scenario with eddy diffusivities is the typical one, there exist cases where superdiffusion is already observed for simple incompressible laminar flows (Castiglione *et al.* 1999) and synthetic flows (Andersen *et al.* 2000).

From an applied point of view, eddy diffusivity has long been a fruitful concept in turbulent-transport theory, and its use has made the computation of turbulent-transport problems possible at Péclet numbers too high for full numerical simulations (see e.g. Lesieur 1997; Celani, Martins Afonso & Mazzino 2006).

In an asymptotic case of interaction between modes whose space and time scales are strongly separated, eddy diffusivities have been calculated by Biferale *et al.* (1995), exploiting a multiple-scale expansion in the scale-separation parameter. Conversely, for realistic flows, active on all space–time scales, eddy diffusivities are generally dependent on all flow characteristics and no general expression for them is known (Kraichnan 1987).

The situation is even more complex in the case of transport of particles with inertia (Gatignol 1983; Maxey & Riley 1983; Balkovsky, Falkovich & Fouxon 2001; Wilkinson & Mehlig 2003; Cencini *et al.* 2006*a*; Volk *et al.* 2008). The main difficulty arises from the fact that in this case the fluid velocity does not coincide with the particle velocity, with the result that a larger phase space has to be considered (i.e. involving both particle position and velocity). The standard Fokker–Planck equation (which only involves space variables), holding for the concentration field in the case of fluid particles, is replaced, for particles with inertia, by a Kramers equation for the phase-space density now involving both space and velocity coordinates.

In many cases of interest, the dependence of the phase-space particle density on the velocity coordinates is not of particular interest (a relevant example is in the realm of pollutant dispersion, where the crucial objective is to predict the space–time behaviour of the pollutant concentration rather than its velocity distribution), and can be averaged out in such a way that only space–time variables are involved.

Closed equations for such an observable (the so-called marginal density) have been obtained by Pavliotis & Stuart (2005) exploiting a formal multiple-scale expansion in the scale-separation parameter. There, attention was focused only on the case of heavy particles and in the absence of gravity. The latter is, however, relevant in many situations of interest, including those where the particle acceleration is (at least) of the same order of magnitude as the gravitational acceleration (see e.g. Maxey 1987). This latter case corresponds to finite terminal sedimenting velocity even in the presence of particles with small inertia. A question which naturally arises is whether a constant (gravity-induced) falling/rising terminal velocity is accompanied by a correction to the diffusion process occurring in the absence of gravity. Answering this question is one of the main contributions of the present paper.

To achieve our aim we have exploited a formal multiple-scale expansion in the scale-separation parameter on the Kramers equation for the particle probability density function. Different particle densities have been considered, ranging from the limit of heavy particles to that of light particles (i.e. bubbles). The spirit of this work is similar to that which motivated Biferale *et al.* (1995). Unfortunately, for the reasons

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already mentioned, the analytical treatment is here much more cumbersome, a fact which imposes a more formal treatment of the multiple-scale expansion. With the hope of making the present manuscript more readable, we decided to postpone the most technical issues to dedicated appendices. Here is a short summary of our main achievements.

We found that the large-scale/long-time behaviour of the particle concentration field is diffusive and the eddy diffusivity is positive definite. This holds for all Stokes times and particle densities. To obtain the eddy-diffusivity tensor one has to solve (numerically) a system of two, one-way coupled, partial differential equations (the auxiliary problem) in a phase space involving both the particle coordinates and their velocity (plus time). In a three-dimensional space, the differential problem thus involves seven coordinates. Consequently, the numerical solution requires considerable computational effort.

In order to simplify the problem, and at the same time motivated by situations of interest where the Stokes time is small, a further perturbative analysis has been carried out in the limit of small inertia. Two asymptotics have been considered: one corresponding to small Stokes number and vanishing terminal sedimenting velocity, the other corresponding to small Stokes number and finite terminal sedimenting velocity as in Maxey (1987). In both cases, the resulting expressions for the eddy-diffusivity tensor greatly simplify: they still involve an auxiliary problem, but now in three-dimensional space. Also in this case, in general, only numerical approaches are capable of obtaining the eddy-diffusivity tensor. At the perturbative order considered here, gravity does not affect the diffusion process in the first asymptotics, while it does in the second one.

As we mentioned above, eddy diffusivity is a very old idea in turbulence (see e.g. Frisch 1995), permitting simulation of high Reynolds/Péclet number flows. Motivated by this consideration, we have isolated from both our asymptotics (vanishing or finite sedimenting velocity) a class of relevant flow fields from which the eddy diffusivity can be computed analytically. This is indeed the case for the class of parallel flows. The aim was to understand whether gravity acts to enhance or reduce the rate of transport. Answering this question might help researchers involved in closure problems to try to parametrize the role of gravity in the diffusion process in terms of relevant properties of the flow field. Our results are not encouraging from this point of view: we performed a detailed sensitivity study, from which there clearly emerges dependence of the effect played by gravity and inertia on the details of the carrying flow. A simple, universal, way to parametrize the role of gravity and inertia on the diffusion process seems to be very difficult to achieve.

The paper is organized as follows. In § 2 we sketch the problem under consideration by recalling the significant equations, and we perform a multiscale expansion to obtain the eddy-diffusivity tensor and the auxiliary equation. In § 3 we analyse our small-inertia expansion order by order. Section 4 shows explicit results for parallel flows and, namely, for a specific case, the well-known Kolmogorov flow (steady or random in time). To test our analytical findings and to extend them beyond the aforementioned perturbative limits, in § 5 we perform Monte Carlo simulations of heavy particles in a two-dimensional steady cellular flow. Conclusions and perspectives follow in § 6. Appendices A, B and C are devoted to proving the positive definite character of the eddy diffusivity, to giving the mathematical details of the calculation and to recalling some basic notions about the Ornstein–Uhlenbeck process, respectively.

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## 2. Large-scale dynamics and multiscale expansion

The model of transport we consider here refers to point-like inertial particles subject to a constant (gravitational) acceleration $\boldsymbol{g}$ (for the sake of generality, in $d$ spatial dimensions) and carried by an *incompressible* velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$. We suppose that the velocity field is steady or periodic in time (with period $\mathscr{T}$), and periodic in space with unit cell $\mathbb{B}$ of linear size $\ell$. From the analysis point of view, it is not a restriction to focus on velocity fields the average of which vanishes over $\mathbb{B}$:

$$\int_{\mathbb{B}} \mathrm{d}\boldsymbol{x}\, \boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{0}. \tag{2.1}$$

The same technique can be extended to handle the case of a random, homogeneous, and stationary velocity field with some non-trivial modifications in the rigorous proofs of convergence (Avellaneda & Majda 1991). For an interesting investigation of the role played by mean currents on the eddy diffusivity (of tracers, in that case), see e.g. Mazzino & Vergassola (1997) and Ferrari & Nikurashin (2010).

Neglecting any feedback effect on the transporting fluid and taking into account the added-mass effect in a simplified way as in Martins Afonso (2008), the Lagrangian dynamics reduces to the following set of stochastic differential equations for particle position ($\boldsymbol{\mathcal{X}}(t)$) and co-velocity ($\boldsymbol{\mathcal{V}}(t)$) (Maxey & Riley 1983; Gatignol 1983):

$$\left.\begin{aligned} \dot{\boldsymbol{\mathcal{X}}}(t) &= \boldsymbol{\mathcal{V}}(t) + \beta \boldsymbol{u}(\boldsymbol{\mathcal{X}}(t), t), \\ \dot{\boldsymbol{\mathcal{V}}}(t) &= -\frac{\boldsymbol{\mathcal{V}}(t) - (1-\beta)\boldsymbol{u}[\boldsymbol{\mathcal{X}}(t), t]}{\tau_S} + (1-\beta)\boldsymbol{g} + \frac{\sqrt{2\kappa}}{\tau_S}\boldsymbol{\eta}(t). \end{aligned}\right\} \tag{2.2}$$

In (2.2), $\boldsymbol{\eta}(t)$ is a white-noise process coupled by a constant diffusivity $\kappa$(Reeks 1988), and the pure number $\beta$ is the added-mass factor $\beta := 3\rho_f/(\rho_f + 2\rho_p) \in [0, 3]$ built from the constant fluid ($\rho_f$) and particle ($\rho_p$) mass densities. Non-vanishing values of $\beta$, i.e. particles not much heavier than the fluid, induce a discrepancy between the particle velocity, $\dot{\boldsymbol{\mathcal{X}}}(t)$, and co-velocity, $\boldsymbol{\mathcal{V}}(t)$ (Bec 2003). Finally, the Stokes time $\tau_S$ expresses the typical response delay of particles to flow variations. For spherical inertial particles of radius $R$, the Stokes time is related to the kinematic viscosity $\nu$ of the carrier fluid by $\tau_S = R^2/(3\nu\beta)$.

A standard application of stochastic calculus (see e.g. Chandrasekhar 1943; Van Kampen 2007) yields the Kramers equation for $\rho(\boldsymbol{x}, \boldsymbol{v}, t)$, the phase-space density:

$$\left\{\frac{\partial}{\partial t} + \frac{\partial}{\partial \boldsymbol{x}} \cdot (\boldsymbol{v} + \beta \boldsymbol{u}) + \frac{\partial}{\partial \boldsymbol{v}} \cdot \left[\frac{(1-\beta)\boldsymbol{u} - \boldsymbol{v}}{\tau_S} + (1-\beta)\boldsymbol{g}\right] - \frac{\kappa}{\tau_S^2}\Delta_v\right\}\rho = 0. \tag{2.3}$$

Our aim is to derive the large-scale asymptotics of the marginal density $P$,

$$P(\boldsymbol{x}, t) := \int_{\mathbb{R}^d} \mathrm{d}\boldsymbol{v}\, \rho(\boldsymbol{x}, \boldsymbol{v}, t), \tag{2.4}$$

describing variations of the configuration-space marginal distribution over spatial and temporal scales $O(x) \approx L \gg \ell$ and $O(t) \approx T \gg \mathscr{T}$, respectively. The technique we employ to extricate this large-scale asymptotics from the full dynamics is known as *homogenization* or *multiple-scale expansion* (Bender & Orszag 1978; Bensoussan, Lions & Papanicolaou 1978; Pavliotis & Stuart 2007). The idea is to use the scale-separation parameter

$$\epsilon := \ell/L \ll 1 \tag{2.5}$$

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to introduce a new set of ('slow' or 'large-scale') variables

$$\left.\begin{aligned} \boldsymbol{X} &:= \epsilon \boldsymbol{x}, \\ T &:= \epsilon^2 t, \\ T^{\ddagger} &:= \epsilon t, \end{aligned}\right\} \tag{2.6}$$

which we shall treat as independent from the corresponding 'fast' variables $\boldsymbol{x}$ and $t$. We do not introduce any slow variable $\boldsymbol{V}$ associated to the co-velocity $\boldsymbol{v}$, since we are interested only in the asymptotics of the marginal configuration-space distribution (2.4). We define instead the large-scale time variables $T$ and $T^{\ddagger}$ to decouple diffusive from ballistic degrees of freedom (Cencini *et al.* 2006*b*). As our goal is to find a diffusion tensor, we define $T$ to be proportional to $\epsilon^2$ ('diffusive rescaling'), and indeed we will obtain a diffusion equation in such a variable. It is also necessary, however, to introduce another time variable $T^{\ddagger} \propto \epsilon$ with a 'ballistic rescaling', in order to take sedimentation into account. This is due to the fact that we chose a frame of reference with vanishing mean fluid velocity, and not with vanishing effective settling velocity, as was done in Pavliotis & Stuart (2005), for instance (see their discussion of the 'centring condition'). We also hypothesize that the expression of the initial conditions for (2.3) in terms of fast and slow variables takes the form

$$\rho(\boldsymbol{x}, \boldsymbol{X}, \boldsymbol{v}, 0, 0, 0) = \bar{\rho}_x(\boldsymbol{x})\, \bar{\rho}_v(\boldsymbol{v})\, \bar{P}(\boldsymbol{X}), \tag{2.7}$$

where $\bar{\rho}_x$ has the same periodicity as the velocity field $\boldsymbol{u}$. The product form (2.7) is non-restrictive owing to the linearity of the dynamics (2.3). This latter is couched by the change of variables (2.6),

$$\frac{\partial}{\partial \boldsymbol{x}} \mapsto \frac{\partial}{\partial \boldsymbol{x}} + \epsilon \frac{\partial}{\partial \boldsymbol{X}}, \quad \frac{\partial}{\partial t} \mapsto \frac{\partial}{\partial t} + \epsilon \frac{\partial}{\partial T^{\ddagger}} + \epsilon^2 \frac{\partial}{\partial T}, \quad \frac{\partial}{\partial \boldsymbol{v}} \mapsto \frac{\partial}{\partial \boldsymbol{v}}, \tag{2.8}$$

into the form

$$(\mathscr{L}^{(0)} + \epsilon \mathscr{L}^{(1)} + \epsilon^2 \mathscr{L}^{(2)})\rho = 0, \tag{2.9}$$

with

$$\mathscr{L}^{(0)} := \frac{\partial}{\partial t} + \frac{\partial}{\partial \boldsymbol{x}} \cdot (\boldsymbol{v} + \beta \boldsymbol{u}) + \frac{\partial}{\partial \boldsymbol{v}} \cdot \left[ \frac{(1-\beta)\boldsymbol{u} - \boldsymbol{v}}{\tau_S} + (1-\beta)\boldsymbol{g} \right] - \frac{\kappa}{\tau_S^2} \Delta_v, \tag{2.10a}$$

$$\mathscr{L}^{(1)} := (\boldsymbol{v} + \beta \boldsymbol{u}) \cdot \frac{\partial}{\partial \boldsymbol{X}} + \frac{\partial}{\partial T^{\ddagger}}, \tag{2.10b}$$

$$\mathscr{L}^{(2)} := \frac{\partial}{\partial T}. \tag{2.10c}$$

Our programme is then to solve (2.9) in a power-series expansion in the scale-separation parameter,

$$\rho(\boldsymbol{x}, \boldsymbol{X}, \boldsymbol{v}, t, T, T^{\ddagger}) = \sum_{n=0}^{\infty} \epsilon^n \rho^{(n)}(\boldsymbol{x}, \boldsymbol{X}, \boldsymbol{v}, t, T, T^{\ddagger}), \tag{2.11a}$$

with initial conditions

$$\rho^{(n)}(\boldsymbol{x}, \boldsymbol{X}, \boldsymbol{v}, 0, 0, 0) = \delta_{n0} \bar{\rho}_x(\boldsymbol{x}) \bar{\rho}_v(\boldsymbol{v}) \bar{P}(\boldsymbol{X}), \tag{2.11b}$$

based on the 'unperturbed' Cauchy problem in the fast variables,

$$\mathscr{L}^{(0)} \rho^{(0)} = 0, \tag{2.12a}$$

$$\rho^{(0)}(\boldsymbol{x}, \boldsymbol{X}, \boldsymbol{v}, 0, T, T^{\ddagger}) = \bar{\rho}_x(\boldsymbol{x}) \bar{\rho}_v(\boldsymbol{v}) P^{(0)}(\boldsymbol{X}, T, T^{\ddagger}). \tag{2.12b}$$

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The final objective is to attain closed equations for the large-scale dynamics, at the leading order described by $P^{(0)}$, in the form of solvability conditions for the perturbative hierarchy. As a result, the details of the small-scale dynamics are summarized in the definition of renormalized transport coefficients, such as the *effective terminal velocity* (see also Maxey 1987; Martins Afonso 2008, and references therein) and the *eddy diffusivity*, for the advective and diffusive rescalings respectively.

For the fast variables, here the aim is to find square-integrable solutions which enjoy the same space–time periodicity as the flow field $\boldsymbol{u}$, and which decay sufficiently fast at infinity in the co-velocity variable. Under physically reasonable regularity assumptions on $\boldsymbol{u}$, the results of chapter 3 of Bensoussan *et al.* (1978) carry over to the operator $\mathscr{L}^{(0)}$. Its adjoint on the space $\mathbb{L}^2(\mathbb{B}\times\mathbb{R}^d\times[0,\mathscr{T}])$,

$$\mathscr{L}^{(0)\dagger} := -\frac{\partial}{\partial t} - (\boldsymbol{v}+\beta\boldsymbol{u})\cdot\frac{\partial}{\partial \boldsymbol{x}} - \left[\frac{(1-\beta)\boldsymbol{u}-\boldsymbol{v}}{\tau_S} + (1-\beta)\boldsymbol{g}\right]\cdot\frac{\partial}{\partial \boldsymbol{v}} - \frac{\kappa}{\tau_S^2}\Delta_{\boldsymbol{v}}, \tag{2.13}$$

has as zero modes only the constants with respect to the fast variables. By the Fredholm alternative, the existence and uniqueness of the inverse of $\mathscr{L}^{(0)}$ (Bensoussan *et al.* 1978; Pavliotis & Stuart 2007) is guaranteed by imposing that the kernel of $\mathscr{L}^{(0)\dagger}$ be orthogonal to the perturbations spawned by (2.10*b*) and (2.10*c*), or in other words that the integrals of the right-hand sides of (2.19) and (2.28) be vanishing. This will allow us to determine the dependence on the slow variables.

### 2.1. *Derivation of the large-scale evolution*

Since the fundamental solution of (2.3), acting on a normalizable initial phase-space density, preserves its normalization, it is not restrictive to posit

$$\int_0^{\mathscr{T}}\frac{\mathrm{d}t}{\mathscr{T}}\int_{\mathbb{B}}\mathrm{d}\boldsymbol{x}\int_{\mathbb{R}^d}\mathrm{d}\boldsymbol{v}\int_{\mathbb{R}^d}\mathrm{d}\boldsymbol{X}\,\rho(\boldsymbol{x},\boldsymbol{X},\boldsymbol{v},t,T,T^{\ddagger}) = 1. \tag{2.14}$$

Notice that the appearance of the double spatial integral on both $\boldsymbol{x}$ and $\boldsymbol{X}$ has been rigorously proved by Nguetseng (1989) and Allaire (1992) (see also Pavliotis & Stuart 2007, and references therein). Combined with (2.11*b*) and (2.14), it allows us to fix the normalization of the terms of the multiscale expansion to

$$\int_0^{\mathscr{T}}\frac{\mathrm{d}t}{\mathscr{T}}\int_{\mathbb{B}}\mathrm{d}\boldsymbol{x}\int_{\mathbb{R}^d}\mathrm{d}\boldsymbol{v}\int_{\mathbb{R}^d}\mathrm{d}\boldsymbol{X}\,\rho^{(n)} = \delta_{n0}. \tag{2.15}$$

Since the solution of the unperturbed Cauchy problem (2.12*a*)–(2.12*b*) takes the form

$$\rho^{(0)}(\boldsymbol{x},\boldsymbol{X},\boldsymbol{v},t,T,T^{\ddagger}) = p^{(0)}(\boldsymbol{x},\boldsymbol{v},t)P^{(0)}(\boldsymbol{X},T,T^{\ddagger}), \tag{2.16}$$

with

$$\mathscr{L}^{(0)}p^{(0)}(\boldsymbol{x},\boldsymbol{v},t) = 0, \tag{2.17}$$

from (2.15) we obtain

$$\int_0^{\mathscr{T}}\frac{\mathrm{d}t}{\mathscr{T}}\int_{\mathbb{B}}\mathrm{d}\boldsymbol{x}\int_{\mathbb{R}^d}\mathrm{d}\boldsymbol{v}\,p^{(0)}(\boldsymbol{x},\boldsymbol{v},t) = 1 = \int_{\mathbb{R}^d}\mathrm{d}\boldsymbol{X}\,P^{(0)}(\boldsymbol{X},T,T^{\ddagger}). \tag{2.18}$$

We will show below that this condition is mathematically self-consistent because the solvability conditions are consistent with the interpretation of $P$ as a probability distribution in configuration space. Physically, this is desirable as it corresponds to the statement that, when focusing only on the large scales, $P$ itself can be interpreted as

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a mass density. It is also a necessary condition to correctly reproduce the well-known small-inertia limit for the eddy diffusivity of tracer particles.

2.1.1. *Solvability condition for $\rho^{(1)}$: large-scale ballistic motion*

At $O(\epsilon^1)$ we have

$$\mathscr{L}^{(0)}\rho^{(1)} = -\mathscr{L}^{(1)}\rho^{(0)} = -\left[(\boldsymbol{v}+\beta\boldsymbol{u})\cdot\frac{\partial}{\partial \boldsymbol{X}} + \frac{\partial}{\partial T^{\ddagger}}\right]\rho^{(0)}. \tag{2.19}$$

By the Fredholm alternative the problem admits a unique solution if and only if

$$\int_0^{\mathscr{T}}\frac{\mathrm{d}t}{\mathscr{T}}\int_{\mathbb{B}}\mathrm{d}\boldsymbol{x}\int_{\mathbb{R}^d}\mathrm{d}\boldsymbol{v}\,\mathscr{L}^{(1)}\rho^{(0)} = -\int_0^{\mathscr{T}}\frac{\mathrm{d}t}{\mathscr{T}}\int_{\mathbb{B}}\mathrm{d}\boldsymbol{x}\int_{\mathbb{R}^d}\mathrm{d}\boldsymbol{v}\mathscr{L}^{(0)}\rho^{(1)} = 0. \tag{2.20}$$

Inserting (2.16) into (2.20) we get the solvability condition

$$\frac{\partial}{\partial T^{\ddagger}}P^{(0)}(\boldsymbol{X},T,T^{\ddagger}) + \boldsymbol{w}\cdot\frac{\partial}{\partial \boldsymbol{X}}P^{(0)}(\boldsymbol{X},T,T^{\ddagger}) = 0, \tag{2.21}$$

with $\boldsymbol{w}$ the *terminal velocity* (Maxey 1987; Martins Afonso 2008) specified by

$$\boldsymbol{w} := \int_0^{\mathscr{T}}\frac{\mathrm{d}t}{\mathscr{T}}\int_{\mathbb{B}}\mathrm{d}\boldsymbol{x}\int_{\mathbb{R}^d}\mathrm{d}\boldsymbol{v}\,[\boldsymbol{v}+\beta\boldsymbol{u}(\boldsymbol{x},t)]p^{(0)}(\boldsymbol{x},\boldsymbol{v},t). \tag{2.22}$$

The physical interpretation of (2.21) is to associate the terminal velocity to ballistic large-scale motion. The solution of (2.21) requires $P^{(0)}$ to satisfy

$$P^{(0)}(\boldsymbol{X},T,T^{\ddagger}) = P^{(0)}(\boldsymbol{X}-\boldsymbol{w}\,T^{\ddagger},T,0) \tag{2.23}$$

without fixing the functional dependence with respect to the first two arguments. As in Mazzino, Musacchio & Vulpiani (2005), we can leverage this fact to find the function $P^{(0)}_*(\boldsymbol{X},T)$ satisfying the Fredholm alternative condition to second order in $\epsilon$. We can then reconstruct the full solution by the Galilean transformation

$$P^{(0)}(\boldsymbol{X},T,T^{\ddagger}) := P^{(0)}_*(\boldsymbol{X}-\boldsymbol{w}\,T^{\ddagger},T). \tag{2.24}$$

2.1.2. *Solution of the equation for the leading-order correction $\rho^{(1)}$*

Exploiting (2.21), the first-order equation (2.19) reduces to

$$\mathscr{L}^{(0)}\rho^{(1)} = -p^{(0)}(\boldsymbol{v}+\beta\boldsymbol{u}-\boldsymbol{w})\cdot\frac{\partial P^{(0)}}{\partial \boldsymbol{X}}. \tag{2.25}$$

We look for a solution of (2.25) in the form

$$\rho^{(1)}(\boldsymbol{x},\boldsymbol{X},\boldsymbol{v},t,T,T^{\ddagger}) = \boldsymbol{p}^{(1)}(\boldsymbol{x},\boldsymbol{v},t)\cdot\frac{\partial P^{(0)}}{\partial \boldsymbol{X}}(\boldsymbol{X},T,T^{\ddagger}), \tag{2.26}$$

with $\boldsymbol{p}^{(1)}$ a $d$-dimensional vector field satisfying

$$\mathscr{L}^{(0)}\boldsymbol{p}^{(1)}(\boldsymbol{x},\boldsymbol{v},t) = -[\boldsymbol{v}+\beta\boldsymbol{u}(\boldsymbol{x},t)-\boldsymbol{w}]p^{(0)}(\boldsymbol{x},\boldsymbol{v},t), \tag{2.27a}$$

$$\boldsymbol{p}^{(1)}(\boldsymbol{x},\boldsymbol{v},0) = \boldsymbol{0}. \tag{2.27b}$$

Equation (2.27*a*) is usually called an 'auxiliary equation' or 'cell problem'. The advantage of (2.26) is that it satisfies at a glance the normalization condition (2.15) with $n=1$, in consequence of the gradient form in the slow variable $\boldsymbol{X}$.

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2.1.3. *Solvability condition for $\rho^{(2)}$: large-scale diffusive motion*

Large-scale diffusive effects set in only at order $O(\epsilon^2)$. Namely,

$$\begin{aligned}\mathscr{L}^{(0)}\rho^{(2)} &= -\mathscr{L}^{(1)}\rho^{(1)} - \mathscr{L}^{(2)}\rho^{(0)} \\ &= -\left[(\boldsymbol{v}+\beta\boldsymbol{u})\cdot\frac{\partial}{\partial \boldsymbol{X}} + \frac{\partial}{\partial T^{\ddagger}}\right]\rho^{(1)} - \frac{\partial}{\partial T}\rho^{(0)}\end{aligned} \tag{2.28}$$

admits a unique solution if and only if

$$\begin{aligned}&\int_0^{\mathscr{T}}\frac{\mathrm{d}t}{\mathscr{T}}\int_{\mathbb{B}}\mathrm{d}\boldsymbol{x}\int_{\mathbb{R}^d}\mathrm{d}\boldsymbol{v}\,(\mathscr{L}^{(1)}\rho^{(1)} + \mathscr{L}^{(2)}\rho^{(0)}) \\ &\quad = -\int_0^{\mathscr{T}}\frac{\mathrm{d}t}{\mathscr{T}}\int_{\mathbb{B}}\mathrm{d}\boldsymbol{x}\int_{\mathbb{R}^d}\mathrm{d}\boldsymbol{v}\,\mathscr{L}^{(0)}\rho^{(2)} = 0.\end{aligned} \tag{2.29}$$

Inserting on the right-hand side of (2.29) the explicit expressions (2.10*b*) of $\mathscr{L}^{(1)}$ and (2.26) of $\rho^{(1)}$, after straightforward algebra we obtain

$$\frac{\partial}{\partial T}P^{(0)}(\boldsymbol{X},T,T^{\ddagger}) = K_{\mu\nu}\frac{\partial^2}{\partial X_\mu \partial X_\nu}P^{(0)}(\boldsymbol{X},T,T^{\ddagger}), \tag{2.30}$$

where we have introduced the *eddy-diffusivity* tensor

$$K_{\mu\nu} := -\mathscr{S}\int_0^{\mathscr{T}}\frac{\mathrm{d}t}{\mathscr{T}}\int_{\mathbb{B}}\mathrm{d}\boldsymbol{x}\int_{\mathbb{R}^d}\mathrm{d}\boldsymbol{v}\,[v+\beta u(\boldsymbol{x},t)-w]_\mu\, p_v^{(1)}(\boldsymbol{x},\boldsymbol{v},t), \tag{2.31}$$

with $\mathscr{S}$ the trace-preserving index symmetrization operator: $\mathscr{S}A_{\mu\nu} := (A_{\mu\nu}+A_{\nu\mu})/2$.

Finding explicit expressions for (2.31), however, requires knowledge of the terminal velocity $\boldsymbol{w}$ (2.22) and of the vector field $\boldsymbol{p}^{(1)}$ (2.27*a*), and thus also of the density $p^{(0)}$ (2.17). The partial differential equations describing these two latter quantities are defined in a $(2d+1)$-dimensional space according to the definition of $\mathscr{L}^{(0)}$ (2.10*a*), and are thus prohibitive to solve numerically. In § 3 we will turn to the derivation of systematic approximations for these quantities in the limit of small inertia, which allows us to decouple the co-velocity degree of freedom and therefore to recast the problem in terms of 'innocent' forced advection–diffusion equations, which are $(d+1)$-dimensional and can be solved relatively easily for specific flows (§ 4) or in general from the numerical point of view.

In order to prove the diffusive nature of the large-scale motion described by (2.30), it remains to show that the eddy-diffusivity tensor (2.31) is positive definite, which is done in appendix A.

The particular solution $P_*^{(0)}$ of (2.30) satisfying the initial condition

$$P_*^{(0)}(\boldsymbol{X},0) = \bar{P}(\boldsymbol{X}) \tag{2.32}$$

yields the large-scale asymptotics we set out to derive. Namely, using (2.6), (2.11*a*) and (2.16) and what follows after (2.22), we obtain

$$\rho(\boldsymbol{x},\boldsymbol{v},t) = p^{(0)}(\boldsymbol{x},\boldsymbol{v},t)P_*^{(0)}(\epsilon\boldsymbol{x}-\epsilon\boldsymbol{w}t,\epsilon^2 t) + O(\epsilon), \tag{2.33}$$

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which – applying (2.4) – finally gives

$$P(\boldsymbol{x},t)=P^{(0)}_{*}(\epsilon\boldsymbol{x}-\epsilon\boldsymbol{w}t,\epsilon^2 t)\int_{\mathbb{R}^d}\mathrm{d}\boldsymbol{v}\,p^{(0)}(\boldsymbol{x},\boldsymbol{v},t)+O(\epsilon). \tag{2.34}$$

## 3. Small-inertia expansion

It is expedient to define the $(d+1)$-dimensional vector fields $\boldsymbol{\psi}$ and $\boldsymbol{W}$ with components

$$\left.\begin{array}{ll}\psi_\mu:=p^{(0)} \text{ and } W_\mu:=0, & \text{for } \mu=0,\\ \psi_\mu:=p^{(1)}_\mu \text{ and } W_\mu:=v_\mu+\beta u_\mu(\boldsymbol{x},t)-w_\mu, & \text{for } 1\leqslant\mu\leqslant d.\end{array}\right\} \tag{3.1}$$

(Notice that $\psi_0$ thus turns out to have dimensions (length$^{-d}\times$ velocity$^{-d}$) different from those of the remaining components (length$^{1-d}\times$ velocity$^{-d}$). One could easily fix this point by setting $\psi_0:=\ell p^{(0)}$, but we prefer to avoid such an operation for the sake of notational simplicity.) From (2.17) and (2.27*a*) it follows immediately that

$$\mathscr{L}^{(0)}\psi_\mu(\boldsymbol{x},\boldsymbol{v},t)=-W_\mu(\boldsymbol{x},\boldsymbol{v},t)\psi_0(\boldsymbol{x},\boldsymbol{v},t), \tag{3.2a}$$

$$\psi_\mu(\boldsymbol{x},\boldsymbol{v},0)=\delta_{\mu 0}\bar{\rho}_x(\boldsymbol{x})\bar{\rho}_v(\boldsymbol{v}). \tag{3.2b}$$

The problem of deriving systematic approximations for the auxiliary fields $p^{(0)}$, $\boldsymbol{p}^{(1)}$ reduces to that of solving the spectral properties of $\mathscr{L}^{(0)}$ in some suitable limits. The *small-inertia* limit recovering the over-damped Fokker–Planck equation from the Kramers dynamics (see e.g. Gardiner 1985; Risken 1989; Van Kampen 2007) is a natural candidate. To set the scene, let us denote by $U$ the typical magnitude of the advecting velocity field ($U$ may be defined, for example, as the space–time average of $\|\boldsymbol{u}\|(\boldsymbol{x},t)$ over $\mathbb{B}\times[0,\mathscr{T}]$). From $U$, the unit cell linear size $\ell$, the Stokes time $\tau_S$, the Brownian diffusivity $\kappa$ and the gravitational acceleration $g$, we can construct the hydrodynamical numbers

$$\text{Stokes, } St:=\frac{\tau_S U}{\ell},\quad \text{Froude, } Fr:=\frac{U}{\sqrt{g\ell}},\quad \text{Péclet, } Pe:=\frac{\ell U}{\kappa}. \tag{3.3}$$

The small-inertia limit corresponds to letting $St$ tend to zero for arbitrary but finite values of $Fr$ and $Pe$. In such a limit the Stokes time sets the shortest time scale in the problem. The corresponding typical co-velocity variation is diffusion-dominated. The root mean square along any Cartesian axis,

$$v_\tau=\sqrt{\frac{2\kappa}{\tau_S}}=\sqrt{\frac{2}{St\,Pe}}\,U, \tag{3.4}$$

suggests rescaling the co-velocities as

$$\boldsymbol{v}':=\sqrt{\frac{St\,Pe}{2}}\,\boldsymbol{v} \tag{3.5}$$

and decoupling the time-dependence of solutions of (3.2*a*) into a fast ($t^\ddagger$) and a slow ($t'$) component. It then becomes expedient to introduce a new vector field $\boldsymbol{\psi}^*(\boldsymbol{x},\boldsymbol{v}',t',t^\ddagger)$, obtained from $\boldsymbol{\psi}$ by multiplication by the Jacobian for the change of variables (3.5).

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On inserting

$$\frac{\partial}{\partial t} = \frac{1}{St}\frac{\partial}{\partial t^{\ddagger}} + \frac{\partial}{\partial t'} \tag{3.6}$$

into (3.2*a*), the variables (3.5) yield for $\mathscr{L}^{(0)}$ the expansion

$$\mathscr{L}^{(0)} = \sum_{n=0}^{3} St^{n/2-1}\mathscr{L}^{(0:n)}, \tag{3.7}$$

where (omitting from now on the prime superscript in $\boldsymbol{v}'$ and $t'$ to simplify the notation):

$$\mathscr{L}^{(0:0)} := \frac{\partial}{\partial t^{\ddagger}} - \frac{\sigma_\ell^2}{\tau_\ell}\Delta_{\boldsymbol{v}} - \frac{1}{\tau_\ell}\frac{\partial}{\partial \boldsymbol{v}}\cdot\boldsymbol{v}, \tag{3.8a}$$

$$\mathscr{L}^{(0:1)} := \frac{1-\beta}{\tau_\ell}\sqrt{\frac{Pe}{2}}\,\boldsymbol{u}\cdot\frac{\partial}{\partial \boldsymbol{v}} + \sqrt{\frac{2}{Pe}}\,\boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{x}}, \tag{3.8b}$$

$$\mathscr{L}^{(0:2)} := \frac{\partial}{\partial t} + \beta\boldsymbol{u}\cdot\frac{\partial}{\partial \boldsymbol{x}}, \tag{3.8c}$$

$$\mathscr{L}^{(0:3)} := \frac{\sigma_\ell(1-\beta)}{\tau_\ell Fr^2}\sqrt{Pe}\,\boldsymbol{G}\cdot\frac{\partial}{\partial \boldsymbol{v}}. \tag{3.8d}$$

Henceforth, the digit after the colon in the superscripts represents the order of the expansion in $St$, bearing in mind that the latter is in half-integer powers and for some quantities might not start from the zeroth order; this notation was used in order to distinguish such an expansion from the multiscale one shown in the previous section. In (3.8*a*) we defined

$$\sigma_\ell^2 := \frac{U^2}{2}, \quad \tau_\ell := \frac{\ell}{U} \tag{3.9}$$

to identify the variance of the co-velocity steady-state Maxwell–Boltzmann distribution (see § B.1 below) and the advective time scale, and in (3.8*d*) we defined

$$\boldsymbol{G} := \frac{\boldsymbol{g}}{\|\boldsymbol{g}\|} \tag{3.10}$$

to exhibit the Froude number dependence.

Two remarks are in order. First, the effect of the rescaling (3.5) is to balance drift and diffusion terms in the co-velocity dynamics for vanishing $St$. The solution of the auxiliary problem is thus turned into a multiscale expansion in powers of $St$ based on the well-known Ornstein–Uhlenbeck operator (3.8*a*) (see appendix C for further details). Second, the unique element of the kernel of $\mathscr{L}^{(0:0)}$ defines a Maxwell–Boltzmann co-velocity distribution to which any integrable initial density relaxes for time scales much larger than the Stokes time $\tau_S$, i.e. for $t^{\ddagger} \gg 1$. In this limit, we have

$$\boldsymbol{\psi}^*(\boldsymbol{x},\boldsymbol{v},t,t^{\ddagger}) \overset{t^{\ddagger}\nearrow\infty}{\longrightarrow} \sum_{n=0}^{\infty} St^{n/2}\boldsymbol{\psi}^{(:n)}(\boldsymbol{x},\boldsymbol{v},t), \tag{3.11}$$

so we can focus on the evaluation of $\boldsymbol{\psi}^{(:n)}$. The expansion of the terminal velocity (2.22) in powers of $St$ yields

$$\boldsymbol{w} = \frac{\boldsymbol{w}^{(:-1)}}{St^{1/2}} + \sum_{n=0}^{\infty} St^{n/2}\boldsymbol{w}^{(:n)}, \tag{3.12}$$

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where

$$\boldsymbol{w}^{(:n)} = \begin{cases} \displaystyle\int_0^{\mathscr{T}} \frac{\mathrm{d}t}{\mathscr{T}} \int_{\mathbb{B}} \mathrm{d}\boldsymbol{x} \int_{\mathbb{R}^d} \mathrm{d}\boldsymbol{v} \sqrt{\frac{2}{Pe}}\, \boldsymbol{v}\psi_0^{(:0)} & n=-1, \\ \displaystyle\int_0^{\mathscr{T}} \frac{\mathrm{d}t}{\mathscr{T}} \int_{\mathbb{B}} \mathrm{d}\boldsymbol{x} \int_{\mathbb{R}^d} \mathrm{d}\boldsymbol{v} \left( \sqrt{\frac{2}{Pe}}\, \boldsymbol{v}\psi_0^{(:n+1)} + \beta \boldsymbol{u}\psi_0^{(:n)} \right) & n \geqslant 0, \end{cases} \tag{3.13}$$

so that the auxiliary equation (2.27*a*) spawns a perturbative hierarchy, the first terms of which are as follows (no sum on $\mu$ is implied here). $O(St^{-1})$:

$$\mathscr{L}^{(0:0)}\psi_\mu^{(:0)} = 0. \tag{3.14}$$

$O(St^{-1/2})$:

$$\mathscr{L}^{(0:0)}\psi_\mu^{(:1)} = -\mathscr{L}^{(0:1)}\psi_\mu^{(:0)} - (1-\delta_{\mu 0}) \left( \sqrt{\frac{2}{Pe}}\, v_\mu - w_\mu^{(:-1)} \right) \psi_0^{(:0)}. \tag{3.15}$$

$O(St^{0})$:

$$\begin{aligned} \mathscr{L}^{(0:0)}\psi_\mu^{(:2)} = &- \sum_{n=0}^{1} \mathscr{L}^{(0:2-n)}\psi_\mu^{(:n)} - (1-\delta_{\mu 0}) \\ &\times \left[ \left( \sqrt{\frac{2}{Pe}}\, v_\mu - w_\mu^{(:-1)} \right) \psi_0^{(:1)} + (\beta u_\mu - w_\mu^{(:0)})\psi_0^{(:0)} \right]. \end{aligned} \tag{3.16}$$

$O(St^{1/2})$:

$$\begin{aligned} \mathscr{L}^{(0:0)}\psi_\mu^{(:3)} = &- \sum_{n=0}^{2} \mathscr{L}^{(0:3-n)}\psi_\mu^{(:n)} - (1-\delta_{\mu 0}) \\ &\times \left[ \left( \sqrt{\frac{2}{Pe}}\, v_\mu - w_\mu^{(:-1)} \right) \psi_0^{(:2)} + (\beta u_\mu - w_\mu^{(:0)})\psi_0^{(:1)} - w_\mu^{(:1)}\psi_0^{(:0)} \right]. \end{aligned} \tag{3.17}$$

We solved (3.14)–(3.17) by first deriving the coefficients of the expansion of $\psi_0 \equiv p^{(0)}$ (the phase-space density) to use them thereafter for the evaluation of the remaining components of the auxiliary vector field $\boldsymbol{\psi}_{\geqslant 1} \equiv \boldsymbol{p}^{(1)}$. The details of the calculation can be found in appendix B, more precisely in § B.1 for the first step and § B.2 for the second one.

### 3.1. *Expression of the eddy diffusivity*

Within third-order accuracy, the co-velocity dependence of both the phase-space density $p^{(0)}$ and the auxiliary vector field $\boldsymbol{p}^{(1)}$ is parity defined (see appendix B). As a consequence, the Stokes-number expansion of the eddy diffusivity (2.31) tensor reduces to

$$\begin{aligned} K_{\mu\nu} = &-\mathscr{S} \int_0^{\mathscr{T}} \frac{\mathrm{d}t}{\mathscr{T}} \int_{\mathbb{B}} \mathrm{d}\boldsymbol{x} \int_{\mathbb{R}^d} \mathrm{d}\boldsymbol{v} \left( \sqrt{\frac{2}{Pe}}\, v_\mu \psi_\nu^{(:1)} + \beta u_\mu \psi_\nu^{(:0)} \right) \\ &- St\, \mathscr{S} \int_0^{\mathscr{T}} \frac{\mathrm{d}t}{\mathscr{T}} \int_{\mathbb{B}} \mathrm{d}\boldsymbol{x} \int_{\mathbb{R}^d} \mathrm{d}\boldsymbol{v} \left( \sqrt{\frac{2}{Pe}}\, v_\mu \psi_\nu^{(:3)} + \beta u_\mu \psi_\nu^{(:2)} \right) + O(St^2). \end{aligned} \tag{3.18}$$

Inserting the terms of the expansion of $\boldsymbol{p}^{(1)}$ leads to further simplifications (see § B.3).

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3.1.1. *Evaluation of the $O(St^0)$ term*

The tracer limit of the particle eddy diffusivity (first line of (3.18)) reduces to

$$K_{\mu\nu}^{(:0)} = \frac{2\tau_\ell \sigma_\ell^2}{Pe}\delta_{\mu\nu} - \mathscr{S}\int_0^{\mathscr{T}} \frac{\mathrm{d}t}{\mathscr{T}} \int_{\mathbb{B}} \mathrm{d}\boldsymbol{x}\, u_\mu(\boldsymbol{x},t)\xi_\nu^{(1:0)}(\boldsymbol{x},t), \tag{3.19}$$

with the auxiliary vector $\boldsymbol{\xi}^{(1:0)}$ obeying an advection–diffusion equation with zero initial condition:

$$\left[\frac{\partial}{\partial t} + \boldsymbol{u}(\boldsymbol{x},t)\cdot\frac{\partial}{\partial \boldsymbol{x}} - \kappa\Delta_{\boldsymbol{x}}\right]\boldsymbol{\xi}^{(1:0)}(\boldsymbol{x},t) = -\frac{\boldsymbol{u}(\boldsymbol{x},t)}{\ell^d}, \tag{3.20a}$$

$$\boldsymbol{\xi}^{(1:0)}(\boldsymbol{x},0) = \boldsymbol{0}. \tag{3.20b}$$

An alternative rewriting of (3.19),

$$K_{\mu\nu}^{(:0)} = \kappa\left[\delta_{\mu\nu} + \int_0^{\mathscr{T}} \frac{\mathrm{d}t}{\mathscr{T}} \int_{\mathbb{B}} \mathrm{d}\boldsymbol{x}\, \ell^d \frac{\partial \xi_\mu^{(1:0)}}{\partial x_{\alpha_1}}(\boldsymbol{x},t)\frac{\partial \xi_\nu^{(1:0)}}{\partial x_{\alpha_1}}(\boldsymbol{x},t)\right], \tag{3.21}$$

shows that this quantity is positive definite and recovers the result of Biferale *et al.* (1995). This indicates that the limits of small inertia and of large-scale separation commute.

3.1.2. *Evaluation of the $O(St^1)$ term*

The leading-order correction in (3.18) turns out to be

$$K_{\mu\nu}^{(:2)} = -\mathscr{S}\int_0^{\mathscr{T}} \frac{\mathrm{d}t}{\mathscr{T}} \int_{\mathbb{B}} \mathrm{d}\boldsymbol{x}\left[u_\nu(\boldsymbol{x},t)\xi_\mu^{(1:2)}(\boldsymbol{x},t) - \frac{\tau_\ell(1-\beta)}{\ell^d}u_\mu(\boldsymbol{x},t)u_\nu(\boldsymbol{x},t)\right], \tag{3.22}$$

where the auxiliary vector $\boldsymbol{\xi}^{(1:2)}$ satisfies another (more complicated) advection–diffusion equation with vanishing initial condition:

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\frac{\partial}{\partial \boldsymbol{x}} - \kappa\Delta_x\right)\xi_\mu^{(1:2)} \\
&\quad = -\tau_\ell(1-\beta)g_{\alpha_1}\frac{\partial}{\partial x_{\alpha_1}}\xi_\mu^{(1:0)} + \tau_\ell^2\frac{\partial}{\partial x_{\alpha_1}} \\
&\qquad \times\left(\sigma_\ell\sqrt{\frac{2}{Pe}}\,\mathscr{L}^{(0:2)}\mathscr{M}_{\alpha_1} + \kappa\frac{\partial}{\partial x_{\alpha_2}}\mathscr{M}_{\alpha_1}\mathscr{M}_{\alpha_2}\right)\xi_\mu^{(1:0)} \\
&\qquad + \kappa\frac{\partial}{\partial x_\mu}\xi^{(0:2)} + \frac{\tau_\ell(1-\beta)}{\ell^d}u_{\alpha_1}\frac{\partial}{\partial x_{\alpha_1}}u_\mu - \frac{\tau_\ell(1-\beta)\kappa}{2\ell^d}\Delta_{\boldsymbol{x}}u_\mu,
\end{aligned} \tag{3.23a}$$

$$\boldsymbol{\xi}^{(1:2)}(\boldsymbol{x},0) = 0. \tag{3.23b}$$

Here, $\mathscr{L}^{(0:2)}$ was defined in (3.8*c*) and

$$\boldsymbol{\mathcal{M}} := \frac{1-\beta}{\tau_\ell\sigma_\ell}\sqrt{\frac{Pe}{2}}\,\boldsymbol{u}(\boldsymbol{x},t) - \sigma_\ell\sqrt{\frac{2}{Pe}}\,\frac{\partial}{\partial \boldsymbol{x}}. \tag{3.24}$$

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3.1.3. *Perturbative expression of the eddy-diffusivity tensor*

Gleaning from (3.21) and (3.22), we get the final result for the Stokes-number expansion of the eddy-diffusivity tensor (2.31):

$$K_{\mu\nu} = \kappa \left( \delta_{\mu\nu} + \int_0^{\mathscr{T}} \frac{\mathrm{d}t}{\mathscr{T}} \int_{\mathbb{B}} \mathrm{d}\boldsymbol{x}\, \ell^d \frac{\partial \xi_\mu^{(1:0)}}{\partial x_{\alpha_1}} \frac{\partial \xi_\nu^{(1:0)}}{\partial x_{\alpha_1}} \right) + St \int_0^{\mathscr{T}} \frac{\mathrm{d}t}{\mathscr{T}} \int_{\mathbb{B}} \mathrm{d}\boldsymbol{x} \left[ \frac{\tau_\ell (1-\beta)}{\ell^d} u_\mu u_\nu - \frac{1}{2} \left( u_\mu \xi_\nu^{(1:2)} + u_\nu \xi_\mu^{(1:2)} \right) \right] + O(St^2), \tag{3.25}$$

with $\boldsymbol{\xi}^{(1:0)}$ and $\boldsymbol{\xi}^{(1:2)}$ given by (3.20*a*) and (3.23*a*), respectively. An important physical implication of (3.25) is that, within $O(St)$-accuracy, the eddy diffusivity depends only implicitly on gravity through (3.23*a*).

A more compact rewriting is

$$K_{\mu\nu} = \frac{U\ell}{Pe}\delta_{\mu\nu} + \int_0^{\mathscr{T}} \frac{\mathrm{d}t}{\mathscr{T}} \int_{\mathbb{B}} \mathrm{d}\boldsymbol{x} [-\mathscr{S} u_\mu \xi_\nu^{(1)} + St(1-\beta) u_\mu u_\nu] + O(St^2), \tag{3.26}$$

with the auxiliary vector $\boldsymbol{\xi}^{(1)}$ admitting a power-series expansion in $St$, whose zeroth and first orders are $\boldsymbol{\xi}^{(1:0)}$ and $\boldsymbol{\xi}^{(1:2)}$.

## 3.2. *Expansion at small inertia and finite terminal velocity*

The results derived above hold for finite $Fr$, i.e. finite gravity effects, which in the limit of small $St$ make $\boldsymbol{w}$ infinitesimal. An alternative approach (Maxey 1987; Marchioli, Fantoni & Soldati 2007) is to treat the terminal velocity as a finite parameter: we will investigate this in the present subsection, denoting with a tilde the quantities which undergo some change.

A limit of physical interest is thus that of vanishing $St$ for large values of $Fr$, so that

$$S_F := \frac{St}{Fr^2} \equiv \frac{g\tau_S}{U} = O(1). \tag{3.27}$$

The limit implies a reshuffling of the hierarchy (3.14)–(3.17) since the operator $\mathscr{L}^{(0:3)}$ (3.8*d*) becomes the same order as (3.8*b*), so that

$$\mathscr{L}^{(0:1)} \mapsto \tilde{\mathscr{L}}^{(0:1)} := \frac{1-\beta}{\tau_\ell} \sqrt{\frac{Pe}{2}} [\boldsymbol{u}(\boldsymbol{x},t) + S_F U \boldsymbol{G}] \cdot \frac{\partial}{\partial \boldsymbol{v}} + \sqrt{\frac{2}{Pe}}\, \boldsymbol{v} \cdot \frac{\partial}{\partial \boldsymbol{x}}, \tag{3.28}$$

whilst

$$\mathscr{L}^{(0:3)} \mapsto \tilde{\mathscr{L}}^{(0:3)} = 0. \tag{3.29}$$

The redefinitions affect both the phase-space density $p^{(0)}$ and the auxiliary field $\boldsymbol{p}^{(1)}$(see § B.4). Inserting these results into the $St$-expansion of the eddy diffusivity, we obtain

$$K_{\mu\nu} \mapsto \tilde{K}_{\mu\nu} = \kappa \left[ \delta_{\mu\nu} + \int_0^{\mathscr{T}} \frac{\mathrm{d}t}{\mathscr{T}} \int_{\mathbb{B}} \mathrm{d}\boldsymbol{x}\, \ell^d \frac{\partial \tilde{\xi}_\mu^{(1:0)}}{\partial x_{\alpha_1}}(\boldsymbol{x},t) \frac{\partial \tilde{\xi}_\nu^{(1:0)}}{\partial x_{\alpha_1}}(\boldsymbol{x},t) \right] + O(St), \tag{3.30}$$

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which now exhibits an implicit dependence on gravity at $O(St^0)$ (for this reason we do not write down the results for $O(St^1)$) through the following new advection–diffusion equation, again supplemented with zero initial condition:

$$\left\{\frac{\partial}{\partial t}+[\boldsymbol{u}(\boldsymbol{x},t)+(1-\beta)S_F U\boldsymbol{G}]\cdot\frac{\partial}{\partial \boldsymbol{x}}-\kappa\Delta_{\boldsymbol{x}}\right\}\tilde{\boldsymbol{\xi}}^{(1:0)}(\boldsymbol{x},t)=-\frac{\boldsymbol{u}(\boldsymbol{x},t)}{\ell^d}, \quad (3.31a)$$

$$\tilde{\boldsymbol{\xi}}^{(1:0)}(\boldsymbol{x},0)=0. \quad (3.31b)$$

## 4. Specific examples: parallel flows and Kolmogorov flow

A flow is called *parallel* if it points always and everywhere to the same direction (e.g. along $x_1$), i.e. if the velocity field can be written as $u_\mu(\boldsymbol{x},t)=\delta_{\mu 1}u(\boldsymbol{x},t)$. In such a case, incompressibility requires $u$ to be independent of $x_1$, i.e.

$$u_\mu(\boldsymbol{x},t)=\delta_{\mu 1}u(x_2,\ldots,x_d,t). \quad (4.1)$$

(Further simplifications can be obtained for specific instances of flow, e.g. if $u_\mu(\boldsymbol{x},t)=\delta_{\mu 1}u(x_d,t)$, only dependent on one spatial direction along which we orient the $x_d$ axis, as is the case for the well-known Kolmogorov flow.)

Equations (3.20*a*) and (3.23*a*), coupled with vanishing initial conditions for $\boldsymbol{\xi}^{(1:0)}$ and $\boldsymbol{\xi}^{(1:2)}$, make $\xi_1^{(1:0)}(\boldsymbol{x},t)$ and $\xi_1^{(1:2)}(\boldsymbol{x},t)$ the only non-zero vectorial components, as they are the only ones forced. Moreover, the forcing is independent of $x_1$, thus the same also holds for $\xi_1^{(1:0)}(x_2,\ldots,x_d,t)$ and $\xi_1^{(1:2)}(x_2,\ldots,x_d,t)$. This means that advective terms of the type $\boldsymbol{u}\cdot\partial/\partial\boldsymbol{x}$ always vanish. Therefore (3.20*a*) and (3.23*a*) greatly simplify, and can easily be solved in Fourier space thanks to the absence of any convolution. We denote Fourier transforms by a hat (for the sake of simplicity, the cell size is assumed to be $\ell$ in every spatial dimension),

$$\hat{f}(\boldsymbol{n}_{\boldsymbol{q}},n_\omega)=\ell^{-d}\mathscr{T}^{-1}\int_0^{\mathscr{T}}\mathrm{d}t\int_{\mathbb{B}}\mathrm{d}\boldsymbol{x}\,\mathrm{e}^{-\mathrm{i}(\boldsymbol{q}\cdot\boldsymbol{x}+\omega t)}f(\boldsymbol{x},t), \quad (4.2)$$

with $\boldsymbol{q}:=2\pi\boldsymbol{n}_{\boldsymbol{q}}/\ell$ and $\omega:=2\pi n_\omega/\mathscr{T}$, so that

$$f(\boldsymbol{x},t)=\sum_{\boldsymbol{n}_{\boldsymbol{q}}\in\mathbb{Z}^d}\sum_{n_\omega\in\mathbb{Z}}\mathrm{e}^{\mathrm{i}(\boldsymbol{q}\cdot\boldsymbol{x}+\omega t)}\hat{f}(\boldsymbol{n}_{\boldsymbol{q}},n_\omega). \quad (4.3)$$

Therefore, we have

$$\left(\frac{\partial}{\partial t}-\kappa\Delta_{\boldsymbol{x}}\right)\xi_\mu^{(1:0)}=-\frac{\delta_{\mu 1}}{\ell^d}u\Longrightarrow\hat{\xi}_\mu^{(1:0)}(\boldsymbol{n}_{\boldsymbol{q}},n_\omega)=-\frac{\delta_{\mu 1}}{\ell^d}\frac{\hat{u}(\boldsymbol{n}_{\boldsymbol{q}},n_\omega)}{\mathrm{i}\omega+\kappa q^2}, \quad (4.4)$$

$$\left(\frac{\partial}{\partial t}-\kappa\Delta_{\boldsymbol{x}}\right)\xi_\mu^{(1:2)}=\left[\kappa^2\tau_\ell\Delta_{\boldsymbol{x}}^2-\kappa\tau_\ell\frac{\partial}{\partial t}\Delta_{\boldsymbol{x}}-\tau_\ell(1-\beta)\boldsymbol{g}\cdot\frac{\partial}{\partial\boldsymbol{x}}\right]\xi_\mu^{(1:0)}-\frac{\tau_\ell(1-\beta)\kappa}{2\ell^d}\Delta_{\boldsymbol{x}}u_\mu$$

$$\Longrightarrow\hat{\xi}_\mu^{(1:2)}(\boldsymbol{n}_{\boldsymbol{q}},n_\omega)=-\frac{\delta_{\mu 1}}{\ell^d}\hat{u}(\boldsymbol{n}_{\boldsymbol{q}},n_\omega)\frac{(1+\beta)(\tau_\ell\kappa^2q^4+\mathrm{i}\tau_\ell\kappa\omega q^2)-2\mathrm{i}\tau_\ell(1-\beta)\boldsymbol{g}\cdot\boldsymbol{q}}{2\,(\mathrm{i}\omega+\kappa q^2)^2}. \quad (4.5)$$

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By Fourier antitransforming and plugging into (3.21) and (3.22), we obtain

$$K^{(:0)}_{\mu\nu} = \kappa\delta_{\mu\nu} - \kappa\ell^d \int_0^{\mathscr{T}} \frac{\mathrm{d}t}{\mathscr{T}} \int_{\mathbb{B}} \mathrm{d}\boldsymbol{x}\, \frac{\delta_{\mu 1}\delta_{\nu 1}}{\ell^{2d}} \times \sum_{\boldsymbol{n_p},\boldsymbol{n_q}\in\mathbb{Z}^d} \sum_{n_\Omega,n_\omega\in\mathbb{Z}} \hat{u}(\boldsymbol{n_p}, n_\Omega)\hat{u}(\boldsymbol{n_q}, n_\omega) \frac{\boldsymbol{p}\cdot\boldsymbol{q}}{(\mathrm{i}\Omega + \kappa p^2)(\mathrm{i}\omega + \kappa q^2)} \mathrm{e}^{\mathrm{i}[(\boldsymbol{p}+\boldsymbol{q})\cdot\boldsymbol{x}+(\Omega+\omega)t]}$$

$$= \kappa \left[ \delta_{\mu\nu} + \delta_{\mu 1}\delta_{\nu 1} \sum_{\boldsymbol{n_q}\in\mathbb{Z}^d} \sum_{n_\omega\in\mathbb{Z}} |\hat{u}(\boldsymbol{n_q}, n_\omega)|^2 \frac{q^2}{\omega^2 + \kappa^2 q^4} \right], \tag{4.6}$$

$$K^{(:2)}_{\mu\nu} = \frac{\delta_{\mu 1}\delta_{\nu 1}}{\ell^d} \int_0^{\mathscr{T}} \frac{\mathrm{d}t}{\mathscr{T}} \int_{\mathbb{B}} \mathrm{d}\boldsymbol{x} \left\{ \tau_\ell(1-\beta)u^2(\boldsymbol{x}, t) + \sum_{\boldsymbol{n_p},\boldsymbol{n_q}\in\mathbb{Z}^d} \sum_{n_\Omega,n_\omega\in\mathbb{Z}} \hat{u}(\boldsymbol{p}, \Omega)\hat{u}(\boldsymbol{q}, \omega) \times \frac{(1+\beta)(\tau_\ell\kappa^2 q^4 + \mathrm{i}\tau_\ell\kappa\omega q^2) - 2\mathrm{i}\tau_\ell(1-\beta)\boldsymbol{g}\cdot\boldsymbol{q}}{2\,(\mathrm{i}\omega + \kappa q^2)^2} \mathrm{e}^{\mathrm{i}[(\boldsymbol{p}+\boldsymbol{q})\cdot\boldsymbol{x}+(\Omega+\omega)t]} \right\}$$

$$= \delta_{\mu 1}\delta_{\nu 1}\tau_\ell \sum_{\boldsymbol{n_q}\in\mathbb{Z}^d} \sum_{n_\omega\in\mathbb{Z}} |\hat{u}(\boldsymbol{n_q}, n_\omega)|^2 \left[ (1-\beta) + \frac{1+\beta}{2} \frac{\kappa^2 q^4}{\omega^2 + \kappa^2 q^4} \right]. \tag{4.7}$$

For generic parallel flows, gravity thus does not appear in the $O(St)$ correction (4.7) to the eddy diffusivity, because of the reality condition of the flow field ($|\hat{u}(\boldsymbol{n_q}, n_\omega)|^2 = |\hat{u}(-\boldsymbol{n_q}, n_\omega)|^2$), which by parity symmetry cancels any contribution linear in $\boldsymbol{q}$ (as would have been the one involving $\boldsymbol{g}$). The tracer limit of the effective diffusivity, (4.6), obviously coincides with the results of Biferale *et al.* (1995), and always shows a positive correction (Pavliotis & Stuart 2005) with respect to the Brownian isotropic counterpart in the absence of any flow.

#### 4.0.1. *Steady Kolmogorov flow*

These results can be further substantiated by focusing on the steady Kolmogorov flow (Obukhov 1983)

$$u_\mu(\boldsymbol{x}, t) = \delta_{\mu 1} U \cos(q_\ell x_d), \tag{4.8}$$

with $q_\ell := 2\pi/\ell$, such that $\hat{u}(\boldsymbol{n_q}, n_\omega) = U\delta_{n_\omega,0}\delta_{n_{q_1},0}\ldots\delta_{n_{q_{d-1}},0}(\delta_{n_{q_d},1} + \delta_{n_{q_d},-1})/2$. In such a case, we have

$$K_{\mu\nu} = \kappa\delta_{\mu\nu} + \frac{U^2}{2q_\ell^2\kappa}\delta_{\mu 1}\delta_{\nu 1} + St\,\frac{U\ell(3-\beta)}{4}\delta_{\mu 1}\delta_{\nu 1} + O(St^2). \tag{4.9}$$

More explicitly, at the leading order we find an enhancement of the eddy diffusivity only along the component parallel to the flow,

$$K_\parallel := K_{11} = \kappa \left[ \left( 1 + \frac{Pe^2}{8\pi^2} \right) + \frac{3-\beta}{4} Pe\, St + O(St^2) \right], \tag{4.10}$$

with gravity eventually contributing to higher orders in $St$, independently of the cell orientation. This diffusivity enhancement grows with $Pe$, and is maximal for very heavy particles ($\beta = 0$). The further enhancement due to inertia can easily be explained by recalling that, according to the Taylor formula, the eddy diffusivity is loosely proportional to the particle Lagrangian autocorrelation time, which is

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definitely longer for inertial particles than for tracers because of ballistic 'coherence' or 'memory'. This latter mechanism can be invoked for the present steady flow, but in general need not hold in unsteady cases. On the other hand, no further enhancement is present for a vanishing Péclet number, nor for bubbles.

The perpendicular component is not modified within accuracy,

$$K_\perp := \frac{K_{\mu\mu} - K_\parallel}{d-1} = \kappa[1 + O(St^2)], \tag{4.11}$$

and it is worth noticing that the $x_d$ axis does not introduce any preferential direction in the $(d-1)$-dimensional subspace orthogonal to the flow, in which the eddy diffusivity is isotropic.

### 4.0.2. *Random-in-time Kolmogorov flow*

As a further example, let us consider a random-in-time Kolmogorov flow,

$$u_\mu(\boldsymbol{x}, t) = \delta_{\mu 1} U \cos(q_\ell x_d) F(t), \tag{4.12}$$

such that the random temporal part is statistically stationary with correlation function (Monin & Yaglom 1975)

$$C(t_\Delta) := \langle F(t + t_\Delta) F(t) \rangle = \mathrm{e}^{-\alpha|t_\Delta|} \cos(\gamma t_\Delta). \tag{4.13}$$

This expression requires the introduction of two additional non-dimensional numbers,

$$\text{Kubo, } Ku := \frac{1}{\alpha \tau_\ell}, \quad \text{Strouhal, } Sr := \frac{\gamma \tau_\ell}{2\pi}, \tag{4.14}$$

measuring, in advective-time units, the vortex lifetime and the recirculation degree, respectively (notice that, in the spirit of a comparison with the time-periodic case, $\gamma$ can roughly be identified with $2\pi/\mathscr{T}$). We can still apply the formalism of the previous sections, if we replace the temporal Fourier series with integrals, and also substitute time averages over a period with statistical averages $\langle\cdot\rangle$. With these provisos, we find

$$\begin{aligned} &\int_0^{\mathscr{T}} \frac{\mathrm{d}t}{\mathscr{T}} \sum_{n_\Omega, n_\omega \in \mathbb{Z}} \hat{u}(\boldsymbol{n_p}, n_\Omega) \hat{u}(\boldsymbol{n_q}, n_\omega) \mathrm{e}^{\mathrm{i}(\Omega+\omega)t} f(\omega, \Omega) \\ &\qquad \mapsto \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{d}\omega \; |\hat{u}_*(\boldsymbol{n_q}, \omega)|^2 f(\omega, -\omega), \end{aligned} \tag{4.15}$$

with $|\hat{u}_*(\boldsymbol{n_q}, \omega)|^2 = U^2 \hat{F}(\omega) \delta_{n_{q_1},0} \ldots \delta_{n_{q_{d-1}},0} (\delta_{n_{q_d},1} + \delta_{n_{q_d},-1})/4$ and

$$\hat{F}(\omega) = \frac{\alpha}{\sqrt{2\pi}} \left[ \frac{1}{\alpha^2 + (\omega+\gamma)^2} + \frac{1}{\alpha^2 + (\omega-\gamma)^2} \right]. \tag{4.16}$$

From (4.6)–(4.7), the upshot is a modification only of the parallel eddy diffusivity:

$$K_\parallel = \kappa \left\{ \left( 1 + \frac{Pe^2}{8\pi^2} \frac{P_K}{Pe} \right) + \left[ \frac{1-\beta}{2} + \frac{1+\beta}{4} \frac{P_K}{Pe} \right] Pe\, St + O(St^2) \right\}. \tag{4.17}$$

Here,

$$P_K := \frac{4\pi^2}{\tau_\ell} \frac{\alpha + q_\ell^2 \kappa}{(\alpha + q_\ell^2 \kappa)^2 + \gamma^2} = \frac{Pe^{-1} + Ku^{-1}/4\pi^2}{(Pe^{-1} + Ku^{-1}/4\pi^2)^2 + Sr^2/4\pi^2} \tag{4.18}$$

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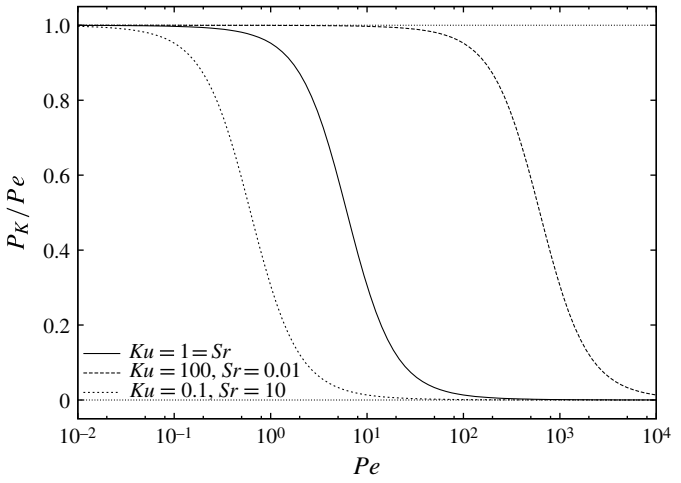



FIGURE 1. Ratio $P_K/Pe$ as a function of $Pe$ for different values of $Ku$ and $Sr$. Notice that the ratio is ever closer to 1 for large $Ku$ and small $Sr$, in which case (4.17) tends to (4.10).

can be identified as a 'modified' Péclet number, which gets back its 'original' value $Pe$ in the steady case, recovered by setting to zero both $\alpha$ and $\gamma$, i.e. for infinite $Ku$ and vanishing $Sr$ (in which case (4.17) coincides with (4.10)). In general, a non-vanishing decay rate $\alpha$ (finite $Ku$) helps increase the role of the molecular diffusivity in the expression of the eddy diffusivity, while a finite recirculation degree $\gamma$ (non-vanishing $Sr$) contributes to reducing the parameter $P_K$.

The outcome is that, at the tracer level ($St \to 0$ in (4.17)), the correction with respect to the Brownian value is always positive. Its magnitude is, however, smaller with respect to the steady-Kolmogorov correction (4.10), because of a multiplicative factor $P_K/Pe$ ($\leqslant 1$ and investigated in figures 1 and 2).

The leading order of the Stokes-number expansion is smaller with respect to the steady-Kolmogorov case (note that $(3-\beta)/4 = (1-\beta)/2 + (1+\beta)/4$, for comparison with (4.10)), and even exhibits a diffusivity depletion for sufficiently light particles, as the coefficient in square brackets in (4.17) is negative for $\beta > \beta^*$. Here, the critical value

$$\beta^* := \frac{3}{2 - P_K/Pe} \in [1.5, 3] \tag{4.19}$$

is a growing function of $P_K$, and thus gets back its 'steady' value 3 for infinite $Ku$ and vanishing $Sr$. Once again, as already shown for generic parallel flows, there is no effect of gravity within leading order in Stokes.

The distinct physical mechanisms in play act on different intrinsic time scales (advective, diffusive, inertial, $1/\alpha$, $1/\gamma$) and can thus interfere constructively – to give a resonance – or destructively. Following Mazzino & Vergassola (1997), Castiglione *et al.* (1998) and Mazzino & Castiglione (1999), we use as a measure of *interference* the quantity

$$\mathscr{K} := K_\parallel - \kappa - \lim_{Pe\to\infty} K_\parallel = U\ell \left[ \frac{1}{8\pi^2}(P_K - P_K^*) + \frac{1+\beta}{4}\frac{P_K}{Pe} St + O(St^2) \right], \tag{4.20}$$

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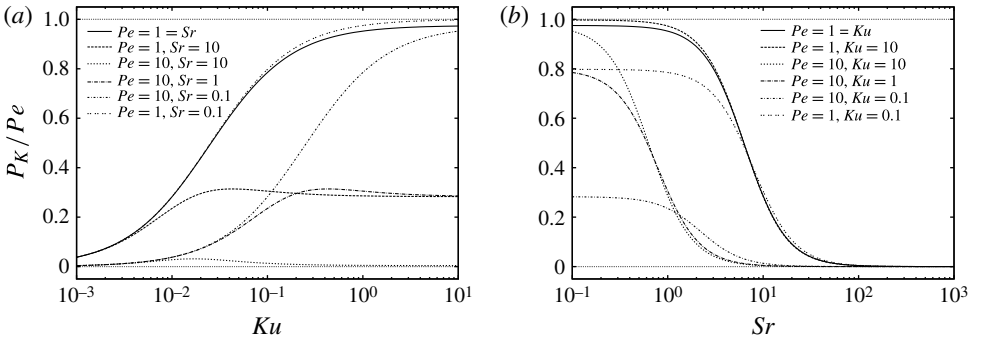



FIGURE 2. (*a*) Ratio $P_K/Pe$ as a function of $Ku$ for different values of $Pe$ and $Sr$. (*b*) Ratio $P_K/Pe$ as a function of $Sr$ for different values of $Pe$ and $Ku$.

| $St < 3.05\,10^{-4}$ | $St = 10^{-3.5}$ | $St = 10^{-3}$ | $St > 1.22 \times 10^{-3}$ |
|---|---|---|---|
| $\nexists\beta$ (no type) | $\beta > 2.86$ (only bubbles) | $\beta > 0.22$ (also tracers) | $\forall\beta$ (every particle) |

TABLE 1. Focusing on the case $Pe = Ku = Sr = 1$, i.e. a situation where the interference is destructive in the tracer approximation $St \to 0$, the table reports (for different, small values of $St$) the ranges of $\beta$ which make the interference constructive on taking into account inertia at its leading order.

where

$$P_K^* := \lim_{Pe\to\infty} P_K = \frac{Ku^{-1}}{Ku^{-2}/4\pi^2 + Sr^2}. \tag{4.21}$$

A positive or negative sign of $\mathscr{K}$ then indicates a constructive or destructive interference of the Brownian diffusivity with the other mechanisms.

At the tracer level, this corresponds to investigating the sign of $(P_K - P_K^*)$, which is done in figure 3, and shows the possible presence of both alternatives. Analytically, one finds a critical value of the Péclet number,

$$Pe^* := \frac{Ku^{-1}}{Sr^2 - Ku^{-2}/4\pi^2}, \tag{4.22}$$

such that the interference is constructive for $Pe > Pe^*$ and destructive otherwise (the interference is always destructive if $Pe^*$ turns out to be negative, e.g. for vanishing $Sr$).

We observe that, within accuracy, inertia always carries a positive contribution to $\mathscr{K}$, linearly increasing in both $\beta$ and $St$. Starting from situations where $\mathscr{K} < 0$ in the tracer approximation $St \to 0$, it is thus interesting to investigate for what ranges of $\beta$ and $St$ the interference becomes positive: an example is sketched in table 1.

### 4.1. *Expansion at small inertia and finite terminal velocity*

Let us now investigate the case of finite terminal velocity (3.27) in the framework of parallel flows, i.e. let us consider (3.31*a*) instead of (3.20*a*). We then obtain a modified

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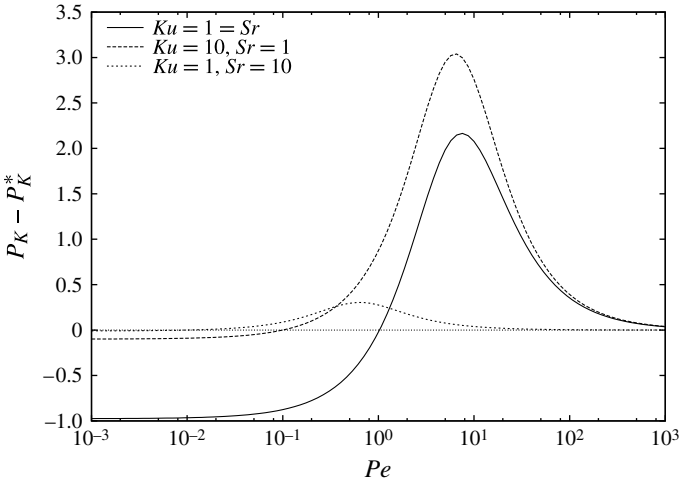



FIGURE 3. Sign of the interference (4.20) as a function of $Pe$ for different values of $Ku$ and $Sr$. Note that the two dashed lines correspond to the same value of the product $Ku\,Sr$ but give different results, which means that the ratio $\gamma/\alpha$ from (4.13) alone in this case is not sufficient to determine the nature of the interference.

version of (4.4):

$$\left[\frac{\partial}{\partial t} + (1-\beta)S_F U\boldsymbol{G}\cdot\frac{\partial}{\partial \boldsymbol{x}} - \kappa\Delta_{\boldsymbol{x}}\right]\tilde{\xi}_\mu^{(1:0)} = -\frac{\delta_{\mu 1}}{\ell^d}u$$
$$\Longrightarrow \hat{\tilde{\xi}}_\mu^{(1:0)}(\boldsymbol{n}_{\boldsymbol{q}}, n_\omega) = -\frac{\delta_{\mu 1}}{\ell^d}\frac{\hat{u}(\boldsymbol{n}_{\boldsymbol{q}}, n_\omega)}{\mathrm{i}\omega + \mathrm{i}(1-\beta)S_F U\boldsymbol{G}\cdot\boldsymbol{q} + \kappa q^2}. \tag{4.23}$$

From (3.30), we deduce that (4.6) is to be modified to

$$K_{\mu\nu}^{(:0)} \mapsto \tilde{K}_{\mu\nu}^{(:0)} = \kappa\delta_{\mu\nu} - \kappa\ell^d \int_0^{\mathscr{T}} \frac{\mathrm{d}t}{\mathscr{T}} \int_{\mathbb{B}} \mathrm{d}\boldsymbol{x} \frac{\delta_{\mu 1}\delta_{\nu 1}}{\ell^{2d}} \sum_{\boldsymbol{n}_{\boldsymbol{p}},\boldsymbol{n}_{\boldsymbol{q}}\in\mathbb{Z}^d} \sum_{n_\Omega, n_\omega\in\mathbb{Z}} \hat{u}(\boldsymbol{n}_{\boldsymbol{p}}, n_\Omega)\hat{u}(\boldsymbol{n}_{\boldsymbol{q}}, n_\omega)$$
$$\times \frac{(\boldsymbol{p}\cdot\boldsymbol{q})\mathrm{e}^{\mathrm{i}[(\boldsymbol{p}+\boldsymbol{q})\cdot\boldsymbol{x}+(\Omega+\omega)t]}}{[\mathrm{i}\Omega + \mathrm{i}(1-\beta)S_F U\boldsymbol{G}\cdot\boldsymbol{p} + \kappa p^2][\mathrm{i}\omega + \mathrm{i}(1-\beta)S_F U\boldsymbol{G}\cdot\boldsymbol{q} + \kappa q^2]}$$
$$= \kappa\left[\delta_{\mu\nu} + \delta_{\mu 1}\delta_{\nu 1}\sum_{\boldsymbol{n}_{\boldsymbol{q}}\in\mathbb{Z}^d}\sum_{n_\omega\in\mathbb{Z}} \frac{|\hat{u}(\boldsymbol{n}_{\boldsymbol{q}}, n_\omega)|^2 q^2}{[\omega + (1-\beta)S_F U\boldsymbol{G}\cdot\boldsymbol{q}]^2 + \kappa^2 q^4}\right]. \tag{4.24}$$

4.1.1. *Steady Kolmogorov flow*

For the steady Kolmogorov flow (4.8), instead of (4.9) we get

$$K_{\mu\nu} \mapsto \tilde{K}_{\mu\nu} = \kappa\left[\delta_{\mu\nu} + \delta_{\mu 1}\delta_{\nu 1}\frac{U^2}{2U^2(1-\beta)^2 S_F^2\cos^2\theta + 2q_\ell^2\kappa^2}\right] + O(St), \tag{4.25}$$

where $\theta$ is the angle between gravity and the $x_d$ direction. In other words, (4.10) and (4.11) now become

$$K_\parallel \mapsto \tilde{K}_\parallel = \kappa\left[1 + \frac{Pe^2}{8\pi^2 + 2Pe^2(1-\beta)^2 S_F^2\cos^2\theta} + O(St)\right], \tag{4.26}$$
$$K_\perp \mapsto \tilde{K}_\perp = \kappa[1 + O(St)]. \tag{4.27}$$

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Comparing (4.26) with the corresponding result for vanishing terminal velocity, (4.10), we deduce that the effect of gravity at $O(St^0)$ is to reduce the enhancement of the parallel eddy diffusivity caused by the presence of the flow with respect to the Brownian value. This reduction is most pronounced when the $x_d$ direction – on which the Kolmogorov flow depends sinusoidally – is vertically aligned, independently whether upwards or downwards. Expression (4.10) is recovered if the flow is vertical ($x_1 \parallel \boldsymbol{G} \Rightarrow x_d \perp \boldsymbol{G} \Rightarrow \cos\theta = 0$) or for neutrally buoyant particles ($\beta = 1$). An analogous mechanism has been discussed for passive scalars, e.g. in Horntrop & Majda (1994), showing that the diffusivity enhancement is less pronounced in the presence of sweeping across a shear flow. For the eddy diffusivity perpendicular to the flow, comparison between (4.11) and (4.27) shows no change at $O(St^0)$, but a finite terminal velocity can in general move the effect of gravity from the original $O(St^2)$ down to $O(St^1)$.

Alternatively, one may study the eddy diffusivity in the vertical and in any horizontal direction; their expressions coincide with (4.26) but with the fraction in square brackets multiplied by $\cos^2\Theta$ and $(d-1)^{-1}\sin^2\Theta$ respectively, where $\Theta$ is the angle between the vertical and the flow direction.

4.1.2. *Random-in-time Kolmogorov flow*

A random Kolmogorov flow (4.12) and (4.13) only affects the parallel component of the diffusivity tensor, with (4.17) replaced by

$$K_\parallel \mapsto \tilde{K}_\parallel = \kappa\left[1 + \frac{Pe^2}{8\pi^2}\frac{\tilde{P}_K}{Pe} + O(St)\right]. \tag{4.28}$$

Here, (4.18) becomes

$$P_K \mapsto \tilde{P}_K := \frac{4\pi^2}{\tau_\ell}\left[\frac{1}{(\alpha + q_\ell^2\kappa)^2 + \Gamma_+^2} + \frac{1}{(\alpha + q_\ell^2\kappa)^2 + \Gamma_-^2}\right]\frac{\alpha + q_\ell^2\kappa}{2}, \tag{4.29}$$

with

$$\Gamma_\pm := \gamma \pm U q_\ell (1-\beta) S_F \cos\theta. \tag{4.30}$$

This latter expression shows that sedimentation contributes to changing the flow characteristic frequency $\gamma$ as viewed (or 'sampled') by the particles, but in a manner symmetric for up/down reflections ($\theta \mapsto \pi - \theta$) because of the symmetrization present in $\tilde{P}_K$ (clearly no modification occurs for neutral particles $\beta = 1$, nor for the orthogonal case $\theta = \pi/2$). In the steady case $\alpha = 0 = \gamma$, expression (4.28) obviously coincides with (4.26).

Another interesting comparison for (4.28) is with the corresponding result (4.17) for vanishing terminal velocity (limit $S_F \to 0$); then the ratio $\tilde{P}_K/P_K$ has to be investigated, which is done in figures 4, 5 and 6: the upshot is that, in the framework of the random-in-time Kolmogorov flow, gravity can either increase or decrease the enhancement of the parallel eddy diffusivity, according to the values of the other parameters.

Let us now analyse the problem of interference again. From the usual point of view, this implies investigating not (4.20) but rather

$$\mathscr{K} \mapsto \tilde{\mathscr{K}} := \tilde{K}_\parallel - \kappa - \lim_{Pe\to\infty}\tilde{K}_\parallel = U\ell\left[\frac{1}{8\pi^2}(\tilde{P}_K - \tilde{P}_K^*) + O(St)\right], \tag{4.31}$$

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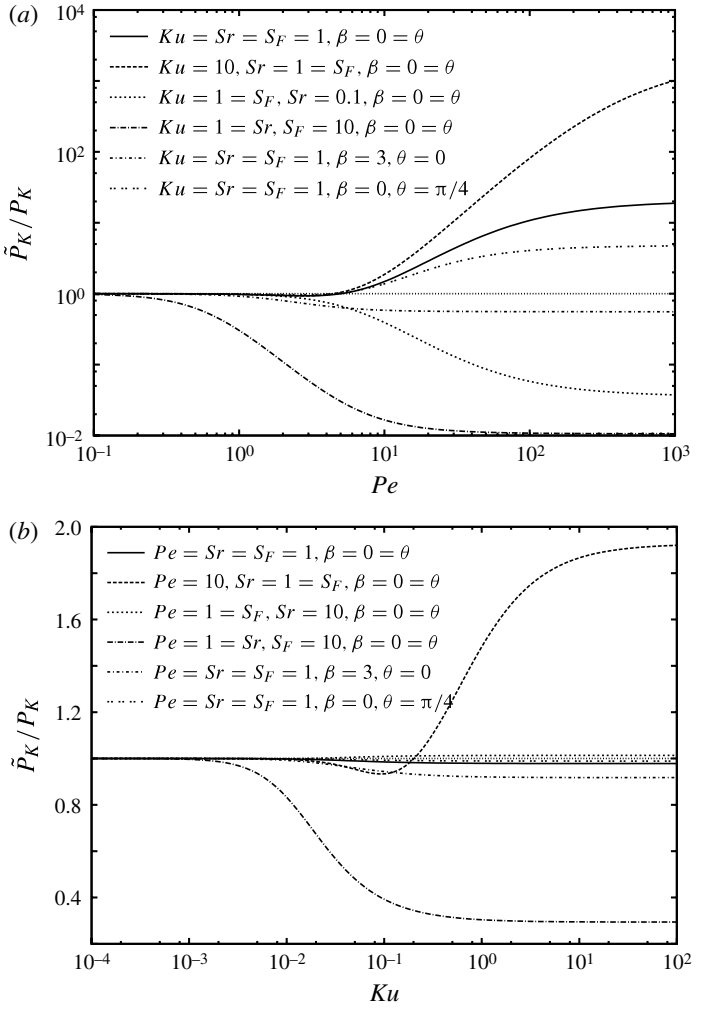



FIGURE 4. (*a*) Ratio $\tilde{P}_K/P_K$ as a function of $Pe$ for different values of $Ku$, $Sr$, $S_F$, $\beta$ and $\theta$; the ratio is ever closer to 1 for small $S_F$, $(1-\beta)$ or $(\pi/2-\theta)$, in which case (4.28) tends to (4.17). (*b*) Ratio $\tilde{P}_K/P_K$ as a function of $Ku$ for different values of $Pe$, $Sr$, $S_F$, $\beta$ and $\theta$.

with (4.21) now becoming

$$P_K^* \mapsto \tilde{P}_K^* := \lim_{Pe\to\infty} \tilde{P}_K = \frac{1}{2}\left\{ \frac{Ku^{-1}}{Ku^{-2}/4\pi^2 + [Sr + (1-\beta)S_F\cos\theta]^2} + \frac{Ku^{-1}}{Ku^{-2}/4\pi^2 + [Sr - (1-\beta)S_F\cos\theta]^2} \right\}. \tag{4.32}$$

Positive or negative sign of $\tilde{\mathscr{K}}$ then indicates a constructive or destructive interference of the *Brownian diffusivity* with the other mechanisms: this corresponds to investigating the sign of $(\tilde{P}_K - \tilde{P}_K^*)$ as a function of $Pe$ ($\propto\kappa^{-1}$), which is done in figure 7 and shows the possible presence of both alternatives.

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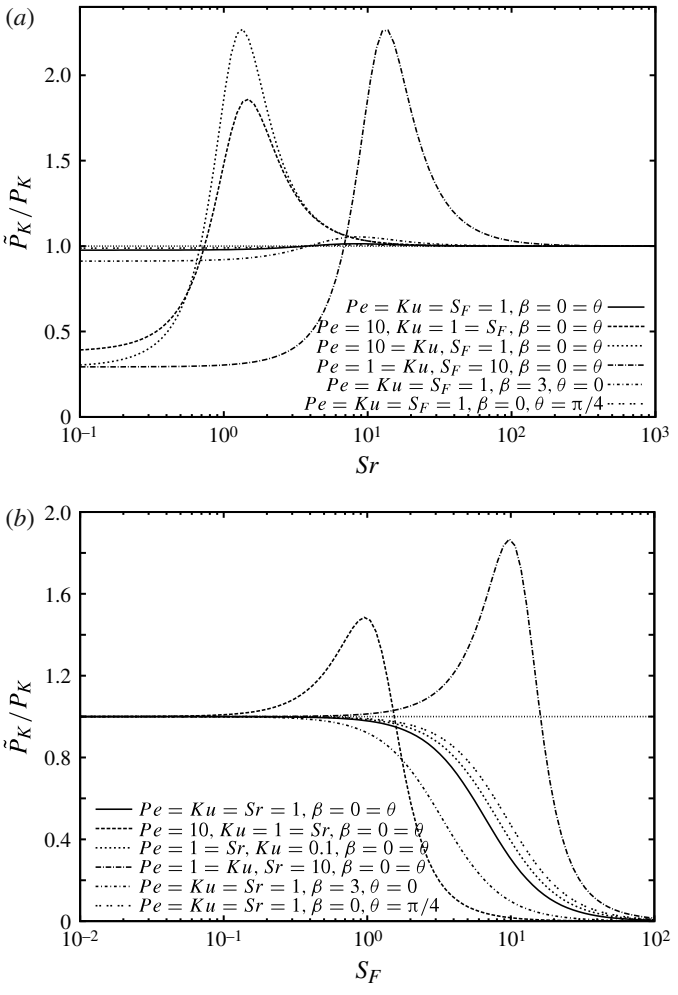



FIGURE 5. (*a*) Ratio $\tilde{P}_K/P_K$ as a function of $Sr$ for different values of $Pe$, $Ku$, $S_F$, $\beta$ and $\theta$. (*b*) Ratio $\tilde{P}_K/P_K$ as a function of $S_F$ for different values of $Pe$, $Ku$, $Sr$, $\beta$ and $\theta$.

However, one could also wonder how *gravity* interferes with the other mechanisms. Therefore, one can study the sign of the quantity

$$\mathscr{K}_* := \left(\tilde{K}_\parallel - \kappa - \tilde{K}_\parallel|_{g=0}\right)\Big|_{\kappa=0} = \lim_{Pe\to\infty}(\tilde{K}_\parallel - K_\parallel)$$
$$= U\ell\left[\frac{1}{8\pi^2}(\tilde{P}_K^* - P_K^*) + O(St)\right], \tag{4.33}$$

i.e. of $(\tilde{P}_K^* - P_K^*)$ as a function of $S_F \propto \|\boldsymbol{g}\|$ (where the Brownian diffusivity must be neglected for sake of consistency). This is done in figure 8 and again shows the possibility of having either destructive or constructive interference. The latter can now be very pronounced near well-defined resonance peaks at a critical value

$$S_F^* := \frac{Sr}{|1-\beta|\cos\theta}, \tag{4.34}$$

except for small values of $Ku$ (which make the denominators in $\tilde{P}_K^*$ large in any case).

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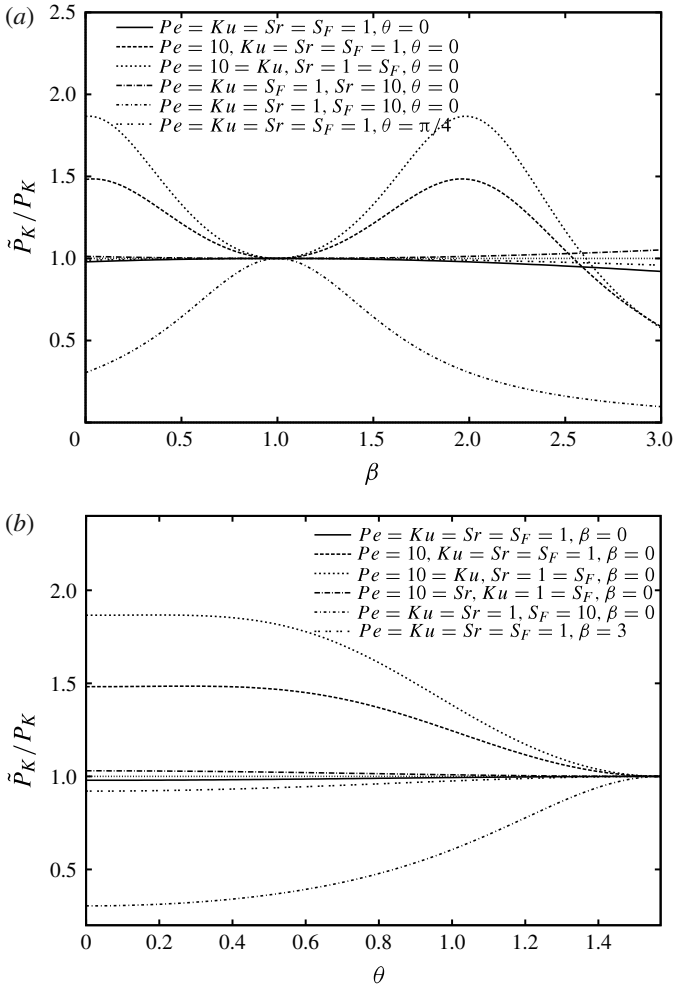



FIGURE 6. (*a*) Ratio $\tilde{P}_K/P_K$ as a function of $\beta$ for different values of $Pe$, $Ku$, $Sr$, $S_F$ and $\theta$. (*b*) Ratio $\tilde{P}_K/P_K$ as a function of $\theta$ for different values of $Pe$, $Ku$, $Sr$, $S_F$ and $\beta$.

Notice that the situation analysed in the present subsection is a generalization of the one described in Mazzino (1997) and Mazzino & Vergassola (1997), with sedimentation here playing the role of a mean large-scale drift.

## 5. Eddy diffusivity for cellular flows: numerical results

As seen in the previous sections, the analytical calculation of eddy diffusivities is possible only under particular circumstances, such as in the presence of simple geometrical properties of the carrier flow. This is the case of the parallel flows discussed in § 4.

The result of these analytical calculations for parallel flows (both static and time-dependent) in the limit of small inertia, and simultaneously of vanishing terminal velocity, is that gravity does not appear at $O(St)$. It is thus impossible, on the basis of our analytical results, to infer the role of gravity on the renormalized transport. To obtain some insights into this, we turned to Monte Carlo simulations of the

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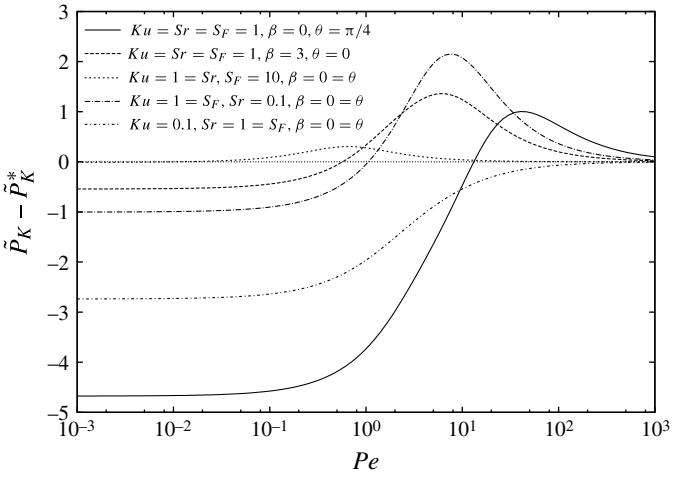



FIGURE 7. Sign of the interference (4.31) as a function of $Pe$ for different values of $Ku$, $Sr$, $S_F$, $\beta$ and $\theta$.

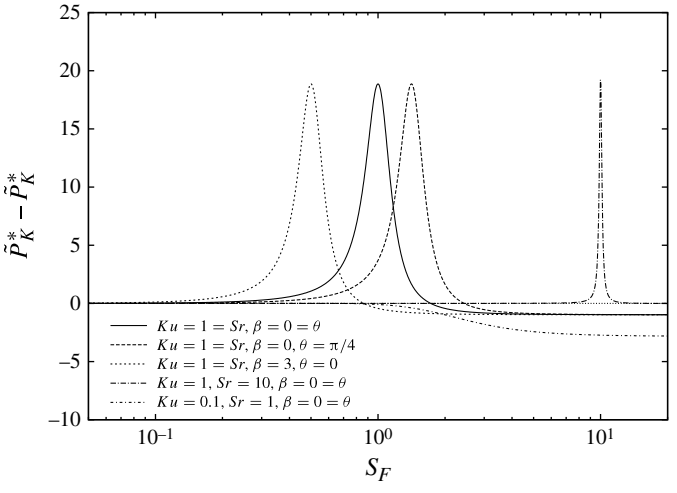



FIGURE 8. Sign of the interference (4.33) as a function of $S_F$ for different values of $Ku$, $Sr$, $\beta$ and $\theta$.

Lagrangian equations (2.2), in order to compute the eddy diffusivity simply by seeding the space with particles, calculating their time evolution and finally evaluating the mean square displacement versus time to extract the effective diffusivity. For the sake of simplicity, we decided to adopt this numerical procedure, rather than performing a full direct numerical simulation of the Eulerian equations (3.20*a*) and (3.23*a*), and to focus our attention on two spatial dimensions. As a complement to the parallel-flow case, we considered a static cellular flow: the latter is indeed characterized by closed streamlines, instead of the open ones typical of parallel flows.

The aims of this study are twofold. First, we investigate whether or not gravity does appear at $O(St)$, possibly unlike what happens for the parallel-flow case. Secondly, we investigate the role of finite inertia on transport for different values of gravity.

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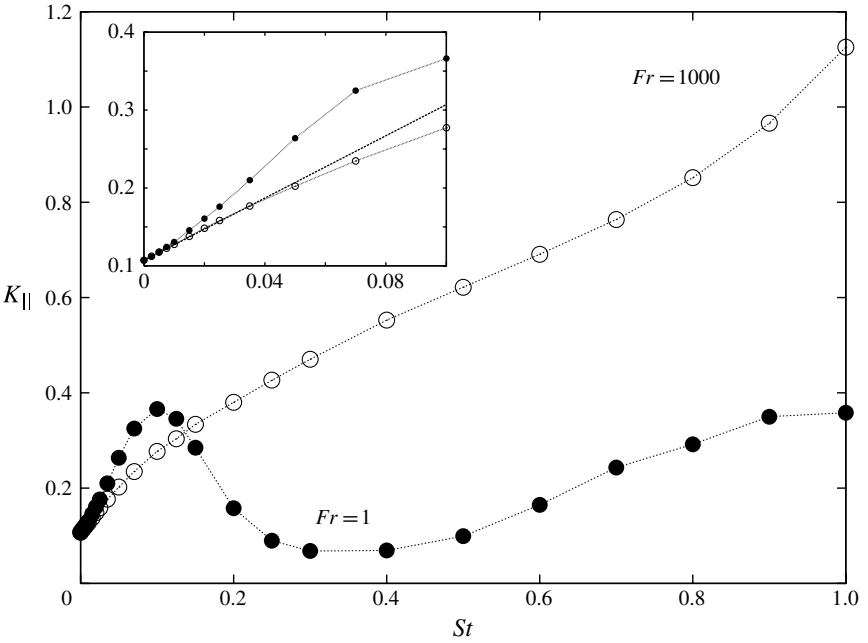



FIGURE 9. Eddy diffusivity behaviour (component parallel to gravity) as a function of Stokes number ($St$) for different Froude numbers ($Fr$). The inset shows a zoom of the same quantity up to $St = 0.1$, together with a linear fit (solid line) drawn for the sake of reference.

Let us start from a more precise definition of the advecting flow. The latter is defined here in terms of the stream function $\phi$ as (Castiglione *et al.* 1999)

$$\phi(\boldsymbol{x}) = \sin(x_1)\sin(x_2), \tag{5.1}$$

where the space and time variables are here made dimensionless by $\ell/(2\pi)$ ($\ell$ being the cell periodicity box) and $\ell/(2\pi U)$ ($U$ being the maximum cell velocity), respectively. The expressions for the Stokes and Froude numbers immediately follow from such non-dimensionalization.

To evaluate the eddy diffusivity we have integrated (2.2) for the space coordinates $\boldsymbol{\mathcal{X}}(t)$ and velocities $\boldsymbol{\mathcal{V}}(t)$ (note that here we only deal with very heavy particles, for which $\beta = 0$ and $\dot{\boldsymbol{\mathcal{X}}} = \boldsymbol{\mathcal{V}}$) of the Lagrangian particles. This is made by using a second-order Runge–Kutta scheme and then performing a linear fit of $\langle[\boldsymbol{\mathcal{X}}(t) - \langle\boldsymbol{\mathcal{X}}(t)\rangle]^2\rangle$ versus $t$, to obtain the (renormalized) diffusion coefficient. Averages are made over different realizations and are performed by uniformly distributing $5 \times 10^5$ particles in the basic periodic cell. The particle evolutions have been computed up to $10^3$ times the typical cell recirculation time $\ell/(2\pi U)$. The results are reported in figures 9 and 10, where the eddy diffusivity (parallel and normal to the gravity direction, respectively) is made dimensionless by $U\ell/(2\pi)$. A zoom up to $St = 0.1$ is reported in the insets of both figures.

Some observations are in order. From the insets one can clearly detect: (i) a linear behaviour with $St$ for both the eddy diffusivity components (which means that we are capturing the $O(St)$ contribution); (ii) a data collapse for two widely separated Froude numbers. This is a robust indication that, as for the class of parallel flows, also in the present case gravity does not enter into play at $O(St)$. Only higher orders are affected by gravity. This is clearly detectable from both figures: the curves corresponding to

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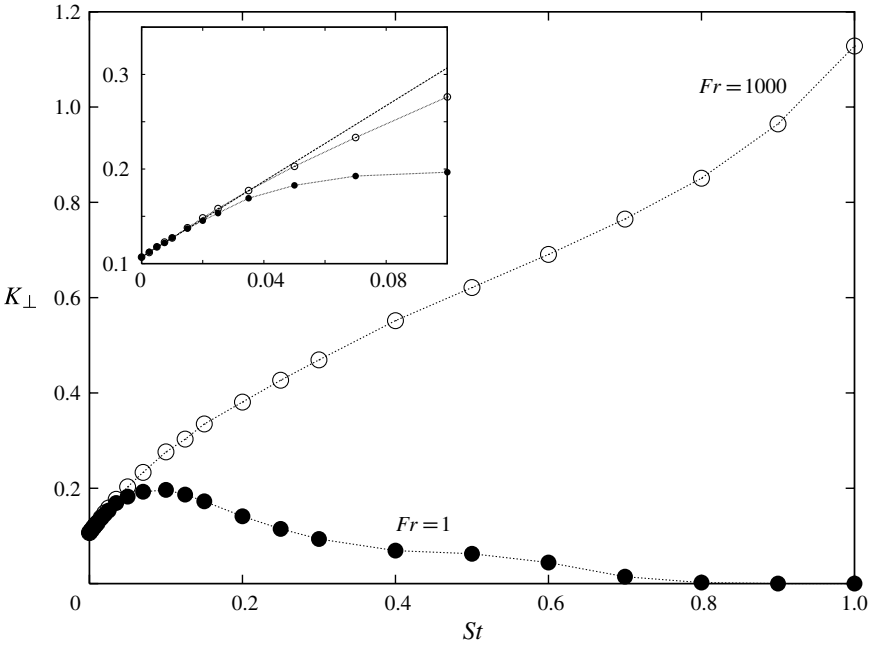



FIGURE 10. As in figure 9 but for the normal component of the eddy diffusivity.

different Froude numbers indeed have completely different behaviour for finite Stokes numbers.

Let us analyse more deeply the different behaviour of eddy diffusivities for different Froude numbers. In all cases, for small but finite Stokes number, $K$ is an increasing function of $St$. As already observed for the parallel-flow case, this fact can be explained as an effect of the increase (with respect to the tracer case) of the Lagrangian autocorrelation time caused by inertia. The increase with $St$, however, stops at some finite Stokes number ($\simeq$0.1 in figures 9 and 10), where gravity dramatically changes the behaviour of the effective diffusivity with the Stokes number. For small Froude numbers (i.e. large gravity) the eddy diffusivity starts to decrease (from $St \simeq 0.1$), while the opposite happens for large Froude numbers. At least in its initial stage (i.e. for sufficiently small Stokes numbers) this effect can again be explained in terms of the Taylor formula (to calculate eddy diffusivities in the inertialess case). Indeed, when gravity becomes sufficiently strong, inertial particles are sedimenting, and in doing that they rapidly cross the periodic cell of the carrier flow. This implies the reduction of the autocorrelation time for increasing Stokes number, thus implying a reduction of eddy diffusivities.

Interestingly, for small Froude numbers, lateral diffusion reduces to zero (see figure 10): sedimentation thus becomes a quasi-one-dimensional process, a phenomenon with possible interesting applications in the framework of falling granular jets, where an unstable broadening of the mean jet section is observed both experimentally and numerically (Möbius 2006).

## 6. Conclusions and perspectives

We have investigated the asymptotic (in space and time) behaviour of the concentration field of inertial particles of arbitrary mass density carried by an incompressible flow field (static or time-dependent) under the action of gravity.

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Exploiting the multiple-scale perturbative expansion in a scale-separation parameter, we found a diffusive equation for the asymptotic dynamics of the particle concentration field, involving a positive definite eddy-diffusivity tensor. The method reduces the computation of this tensor to the solution of an auxiliary differential problem valid for any Stokes, Froude and Péclet numbers. The general results greatly simplify when we focus on two relevant limits of vanishing inertia: one with vanishing sedimenting velocity, the other maintaining a finite value of the latter quantity. Providing estimates on the radius of convergence of the small-Stokes expansion is a challenging task, which might be undertaken along the lines of Babiano *et al.* (2000) and Sapsis & Haller (2008).

As also happens in the simpler problem of inertialess-particle transport, the determination of the eddy-diffusivity tensor is in general possible only by exploiting numerical strategies. Here the problem is particularly evident owing to the larger (compared to the inertialess case) phase space. Nevertheless, we were able to select a relevant class of carrier flows (parallel flows, both static and time-dependent) for which we can derive exact, explicit expressions for the eddy diffusivities. These expressions allow us to evince the highly non-trivial role played by gravity and inertia on the diffusive behaviour: both enhanced and reduced diffusion can be observed depending on the detailed structure of the flow.

Here is a summary of our findings for the parallel Kolmogorov flows, highlighting the non-trivial role of inertia and gravity on the diffusion process in the direction parallel to the flow.

(*a*) An enhancement of the diffusivity is always found, and the Péclet number always plays a positive role in this respect (more precisely, analysing the ratio $K_\parallel/\kappa$). However, if one compares the situations in which other mechanisms – such as flow time dependence or sedimentation or both – are present with the corresponding situations in which they are absent, one can find a weaker enhancement, which is more or less pronounced at different $Pe$. By introducing the concept of interference, one concludes that Brownian diffusivity may interfere constructively or destructively with the other mechanisms depending on the ratio of the Péclet number with a critical value $Pe^*$. This critical value can in principle depend in its turn on the Kubo and Strouhal numbers. These conclusions hold in general, irrespective of considering a tracer or an inertial particle at small inertia, endowed with vanishing or finite sedimentation.

(*b*) The role of inertia (Stokes number) should be taken into account together with the coefficient $\beta$ based on the density ratio. The diffusivity enhancement indeed shows a leading correction proportional to $St$, which is always positive (and decreasing with $\beta$) for a steady flow, but which can become negative for a random time-dependent flow if the particles are light enough. While passing from the former kind of flow to the latter, there is thus a diffusivity reduction, which reaches its maximum at infinite Strouhal number – i.e. vortex frequency or recirculation degree – or at vanishing Kubo number – i.e. the lifetime of the structures before decay. (Also notice that $Ku$ always appears in the form of a harmonic sum with $Pe$, as resistances in parallel, which makes its role more subtle.) $St$ and $\beta$ also contribute linearly and positively to the above-mentioned interference.

(*c*) Gravity (Froude number) plays a significant role only through sedimentation, i.e. when the particle terminal velocity ($S_F$, modulo a prefactor) remains finite even for vanishing inertia. In that case, the key quantity is the combination $(1-\beta)S_F\cos\theta$, $\theta$ being the angle between the vertical and the $x_d$ direction upon which the flow depends sinusoidally (and not the $x_1$ direction along which

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the flow points: this distinction is crucial in the three-dimensional case). This key parameter monotonically damps the eddy diffusivity in the steady case. In the random time-dependent case, this damping: (i) coincides with the vanishing-sedimentation value for a vanishing deviation of $\beta$ from 1, or of $S_F$ from 0, or of $\theta$ from $\pi/2$; (ii) is maximum for some intermediate deviation, corresponding to a resonance with the flow frequency $Sr$; (iii) vanishes for very large deviations, i.e. infinite $S_F$. This behaviour is summarized by the presence of strong resonance peaks denoting constructive interference of gravity with the other mechanisms; this interference becomes destructive for $S_F$ large enough.

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## Appendix A. Proof of the positive definite character of the eddy diffusivity

The effective-diffusivity tensor $K_{\mu\nu}$ (2.31) is positive definite if and only if $K_{\mu\nu}a_\mu a_\nu > 0$ for any constant vector $\boldsymbol{a} \in \mathbb{R}^d$. Let us then prove this inequality by proceeding on the same lines as in Pavliotis & Stuart (2005).

First of all, let us rewrite $\boldsymbol{p}^{(1)}$ as

$$\boldsymbol{p}^{(1)}(\boldsymbol{x}, \boldsymbol{v}, t) = p^{(0)}(\boldsymbol{x}, \boldsymbol{v}, t)\boldsymbol{q}(\boldsymbol{x}, \boldsymbol{v}, t) \tag{A 1}$$

in terms of a new vector $\boldsymbol{q}$, which by definition enjoys the same space–time periodicity as the fluid flow $\boldsymbol{u}(\boldsymbol{x}, t)$. It can easily be shown that

$$\mathscr{L}^{(0)}(p^{(0)}\boldsymbol{q}) = -p^{(0)}\mathscr{L}^{(0)\dagger}\boldsymbol{q} - \frac{2\kappa}{\tau_S^2}\left(\frac{\partial}{\partial \boldsymbol{v}}p^{(0)} \cdot \frac{\partial}{\partial \boldsymbol{v}}\boldsymbol{q} + p^{(0)}\Delta_{\boldsymbol{v}}\boldsymbol{q}\right), \tag{A 2}$$

thanks to (2.10*a*), (2.13) and (2.17).

Let us now define

$$\boldsymbol{Q}(\boldsymbol{x}, \boldsymbol{v}, t) := \boldsymbol{v} + \beta\boldsymbol{u}(\boldsymbol{x}, t) - \boldsymbol{w} \tag{A 3}$$

and

$$\int_{\mathbb{B}\times\mathbb{R}^d\times[0,\mathscr{T}]} := \int_0^{\mathscr{T}} \frac{\mathrm{d}t}{\mathscr{T}} \int_{\mathbb{B}} \mathrm{d}\boldsymbol{x} \int_{\mathbb{R}^d} \mathrm{d}\boldsymbol{v} \tag{A 4}$$

for the sake of notational convenience, and let us compute the quantity of our interest by means of (2.31):

$$\begin{aligned} K_{\mu\nu}a_\mu a_\nu &= -\int_0^{\mathscr{T}} \frac{\mathrm{d}t}{\mathscr{T}} \int_{\mathbb{B}} \mathrm{d}\boldsymbol{x} \int_{\mathbb{R}^d} \mathrm{d}\boldsymbol{v}\, [\boldsymbol{a} \cdot (\boldsymbol{v} + \beta\boldsymbol{u} - \boldsymbol{w})](\boldsymbol{a} \cdot \boldsymbol{p}^{(1)}) \\ &= -\int_{\mathbb{B}\times\mathbb{R}^d\times[0,\mathscr{T}]} (\boldsymbol{a} \cdot \boldsymbol{Q})(\boldsymbol{a} \cdot \boldsymbol{q}p^{(0)}) = -\int_{\mathbb{B}\times\mathbb{R}^d\times[0,\mathscr{T}]} (\boldsymbol{a} \cdot \boldsymbol{Q}p^{(0)})(\boldsymbol{a} \cdot \boldsymbol{q}) \\ &= \int_{\mathbb{B}\times\mathbb{R}^d\times[0,\mathscr{T}]} (\boldsymbol{a} \cdot \mathscr{L}^{(0)}\boldsymbol{p}^{(1)})(\boldsymbol{a} \cdot \boldsymbol{q}) \\ &= \int_{\mathbb{B}\times\mathbb{R}^d\times[0,\mathscr{T}]} [\boldsymbol{a} \cdot \mathscr{L}^{(0)}(p^{(0)}\boldsymbol{q})](\boldsymbol{a} \cdot \boldsymbol{q}), \end{aligned} \tag{A 5}$$

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where we exploited (2.27*a*), (A 1) and (A 3). We now make use of (A 2) and the constancy of $\boldsymbol{a}$, and subsequently integrate by parts, to obtain

$$\begin{aligned} K_{\mu\nu}a_\mu a_\nu &= -\int_{\mathbb{B}\times\mathbb{R}^d\times[0,\mathscr{T}]} \left\{\boldsymbol{a}\cdot\left[p^{(0)}\mathscr{L}^{(0)\dagger}\boldsymbol{q} + \frac{2\kappa}{\tau_S^2}\left(\frac{\partial}{\partial\boldsymbol{v}}p^{(0)}\cdot\frac{\partial}{\partial\boldsymbol{v}}\boldsymbol{q} + p^{(0)}\Delta_v\boldsymbol{q}\right)\right]\right\}(\boldsymbol{a}\cdot\boldsymbol{q}) \\ &= -\int_{\mathbb{B}\times\mathbb{R}^d\times[0,\mathscr{T}]} \left[(\boldsymbol{a}\cdot p^{(0)}\mathscr{L}^{(0)\dagger}\boldsymbol{q})(\boldsymbol{a}\cdot\boldsymbol{q}) - \frac{2\kappa}{\tau_S^2}p^{(0)}\left\|\boldsymbol{a}\cdot\frac{\partial}{\partial\boldsymbol{v}}\boldsymbol{q}\right\|^2\right] \\ &= -\int_{\mathbb{B}\times\mathbb{R}^d\times[0,\mathscr{T}]} \left\{[\boldsymbol{a}\cdot\mathscr{L}^{(0)}(p^{(0)}\boldsymbol{q})](\boldsymbol{a}\cdot\boldsymbol{q}) - \frac{2\kappa}{\tau_S^2}p^{(0)}\left\|\boldsymbol{a}\cdot\frac{\partial}{\partial\boldsymbol{v}}\boldsymbol{q}\right\|^2\right\}. \end{aligned} \tag{A 6}$$

Comparing the last expression of (A 5) with the last one of (A 6), we conclude

$$K_{\mu\nu}a_\mu a_\nu = \frac{\kappa}{\tau_S^2}\int_0^{\mathscr{T}}\frac{\mathrm{d}t}{\mathscr{T}}\int_{\mathbb{B}}\mathrm{d}\boldsymbol{x}\int_{\mathbb{R}^d}\mathrm{d}\boldsymbol{v}\,p^{(0)}(\boldsymbol{x},\boldsymbol{v},t)\left\|\boldsymbol{a}\cdot\frac{\partial}{\partial\boldsymbol{v}}\boldsymbol{q}(\boldsymbol{x},\boldsymbol{v},t)\right\|^2 > 0. \tag{A 7}$$

## Appendix B. Details of the resolution of (3.14)–(3.17) for $\psi$

### B.1. *Phase-space density* $\psi_0 \equiv p^{(0)}$

The Stokes-number expansion of the phase-space density $p^{(0)}$ was studied in detail in Martins Afonso (2008), the main results of which we briefly revisit by couching them in our notation. The results for the Ornstein–Uhlenbeck process in appendix C imply that any integrable initial co-velocity distribution $\bar{\rho}_v$ evolving in $t^\ddagger$ according to (3.8*a*) converges exponentially to the Maxwell–Boltzmann distribution

$$p^{(0:0)}(\boldsymbol{x},\boldsymbol{v},t) = \frac{\mathrm{e}^{-v^2/2\sigma_\ell^2}}{(2\pi\sigma_\ell^2)^{d/2}}\xi^{(0:0)}(\boldsymbol{x},t) \equiv \chi^{0;\sigma_\ell}(\boldsymbol{v})\xi^{(0:0)}(\boldsymbol{x},t). \tag{B 1}$$

The function $\xi^{(0:0)}$ is a constant with respect to the fast variables $(\boldsymbol{v},t^\ddagger)$. We use it to impose the solvability conditions of the perturbative hierarchy starting at (B 1). Higher-order corrections are most conveniently evaluated, observing that the grading (C 10) and recursion (C 11) relations, enjoyed by the Hermite polynomials $H^{n;\sigma_\ell}_{\cdots}$ and their $\mathbb{L}^2(\mathbb{R}^d)$-orthogonal duals $\chi^{n;\sigma_\ell}_{\cdots}$, imply that for any $f(\boldsymbol{x})$

$$\begin{aligned} &\mathscr{L}^{(0:1)}\chi^{n;\sigma_\ell}_{\alpha_1\ldots\alpha_n}(\boldsymbol{v})f(\boldsymbol{x}) \\ &\quad= -\chi^{n+1;\sigma_\ell}_{\alpha_1\ldots\alpha_n\alpha_{n+1}}(\boldsymbol{v})\mathscr{M}_{\alpha_{n+1}}f(\boldsymbol{x}) + \sigma_\ell\sqrt{\frac{2}{Pe}}\sum_{k=1}^{n}\chi^{n-1;\sigma_\ell}_{\alpha_1\ldots\alpha_{k-1}\alpha_{k+1}\ldots\alpha_n}(\boldsymbol{v})\frac{\partial}{\partial x_{\alpha_k}}f(\boldsymbol{x}) \end{aligned} \tag{B 2}$$

and

$$\mathscr{L}^{(0:3)}\chi^{n;\sigma_\ell}_{\alpha_1\ldots\alpha_n}(\boldsymbol{v})f(\boldsymbol{x}) = -\sqrt{\frac{Pe}{2}}\,\frac{(1-\beta)g_{\alpha_{n+1}}}{\sigma_\ell}\chi^{n+1;\sigma_\ell}_{\alpha_1\ldots\alpha_n\alpha_{n+1}}(\boldsymbol{v})f(\boldsymbol{x}). \tag{B 3}$$

For $\chi$ we used the conventions fixed in appendix C, and we defined $\boldsymbol{\mathcal{M}}$ through (3.24).

The leading-order correction does not impose any solvability condition. Since, by (C 13), $\left(\mathscr{L}^{(0:0)}\right)^{-1}$ acts on $\chi^{n;\sigma_\ell}_{\cdots}$ as a multiplication by $\tau_\ell/n$ (asymptotically in $t^\ddagger$), we find

$$p^{(0:1)}(\boldsymbol{x},\boldsymbol{v},t) = \tau_\ell\chi^{1;\sigma_\ell}_{\alpha_1}(\boldsymbol{v})\mathscr{M}_{\alpha_1}\xi^{(0:0)}(\boldsymbol{x},t). \tag{B 4}$$

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The function $\xi^{(0:0)}$ in (B 1) and (B 4) is specified by the solvability condition brought about by (3.16), the solution of which is

$$p^{(0:2)}(\boldsymbol{x},\boldsymbol{v},t)=\chi^{0;\sigma_\ell}(\boldsymbol{v})\xi^{(0:2)}(\boldsymbol{x},t)+\frac{\tau_\ell^2}{2}\chi^{2;\sigma_\ell}_{\alpha_1\alpha_2}(\boldsymbol{v})\mathscr{M}_{\alpha_1}\mathscr{M}_{\alpha_2}\xi^{(0:0)}(\boldsymbol{x},t) \tag{B 5}$$

if and only if $\xi^{(0:0)}$ satisfies the advection–diffusion equation

$$\mathscr{N}\xi^{(0:0)}(\boldsymbol{x},t)=0, \tag{B 6a}$$

$$\xi^{(0:0)}(\boldsymbol{x},0)=\bar{\rho}(\boldsymbol{x}), \tag{B 6b}$$

with

$$\mathscr{N}:=\frac{\partial}{\partial t}+\boldsymbol{u}(\boldsymbol{x},t)\cdot\frac{\partial}{\partial\boldsymbol{x}}-\kappa\Delta_{\boldsymbol{x}}\equiv\frac{\partial}{\partial t}+\boldsymbol{u}(\boldsymbol{x},t)\cdot\frac{\partial}{\partial\boldsymbol{x}}-\frac{2\tau_\ell\sigma_\ell^2}{Pe}\Delta_{\boldsymbol{x}}. \tag{B 7}$$

The solvability condition (B 6*a*) is the Fokker–Planck limit for *incompressible* advection of the Kramers dynamics $\mathscr{L}^{(0)}$ for vanishing Stokes number. As usual in the non-periodic case (Pavliotis & Stuart 2005), the solvability condition arises from the requirement that the non-homogeneous term in (3.16) be orthogonal to the kernel of the zero-frequency component of $\mathscr{L}^{(0:0)\dagger}$, i.e. the generator of the Ornstein–Uhlenbeck process (C 1), with respect to the scalar product on $\mathbb{L}^2(\mathbb{R}^d)$ (C 14). The fulfilment of this condition prevents the occurrence of marginal (polynomial in $t^\ddagger$) instabilities in the perturbative expansion. By (B 6*a*), any integrable initial condition $\bar{\rho}_x$ on the periodicity cell $\mathbb{B}$ converges to a uniform distribution. Since we are ultimately interested in degrees of freedom slower than the $t^\ddagger$-dynamics, it is not restrictive to choose as initial condition the invariant measure

$$\xi^{(0:0)}(\boldsymbol{x},t)=\bar{\rho}_x(\boldsymbol{x})=\frac{1}{\ell^d}. \tag{B 8}$$

Finally, the presence in (B 5) of the homogeneous term $\chi^{0;\sigma_\ell}\xi^{(0:2)}$ takes into account the solvability conditions intervening in the equation for $p^{(0:4)}$ (similar to (3.17) with each superscript to $\boldsymbol{\psi}$ augmented by one). Namely, parity considerations on the co-velocity distribution evince that (3.17) in its original form does not require any solvability condition, and yields

$$\begin{aligned}p^{(0:3)}=&\frac{1-\beta}{\ell^d Fr^2}\sqrt{Pe}\,G_{\alpha_1}\chi^{1;\sigma_\ell}_{\alpha_1}+\tau_\ell\chi^{1;\sigma_\ell}_{\alpha_1}\mathscr{M}_{\alpha_1}\xi^{(0:2)}-\frac{\tau_\ell(1-\beta)}{\sigma_\ell\ell^d}\sqrt{\frac{Pe}{2}}\,\chi^{1;\sigma_\ell}_{\alpha_1}\mathscr{L}_{(0:2)}u_{\alpha_1}\\&+\frac{\tau_\ell^2(1-\beta)}{2\ell^d}\left[\sqrt{\frac{Pe}{2}}\,\frac{\chi^{3;\sigma_\ell}_{\alpha_1\alpha_2\alpha_3}}{3\sigma_\ell}\mathscr{M}_{\alpha_3}-\left(\chi^{1;\sigma_\ell}_{\alpha_1}\frac{\partial}{\partial x_{\alpha_2}}+\chi^{1;\sigma_\ell}_{\alpha_2}\frac{\partial}{\partial x_{\alpha_1}}\right)\right]\mathscr{M}_{\alpha_2}u_{\alpha_1},\end{aligned} \tag{B 9}$$

having used (B 8) and

$$\mathscr{M}_\alpha\xi^{(0:0)}(\boldsymbol{x},t)=\frac{1-\beta}{\tau_\ell\sigma_\ell\ell^d}\sqrt{\frac{Pe}{2}}\,u_\alpha(\boldsymbol{x},t). \tag{B 10}$$

The complete expression of $p^{(0)}$ within third-order accuracy in $St$ is then attained by applying (B 2)–(B 9), which shows that $\xi^{(0:2)}$ satisfies the following advection–diffusion

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equation in $\mathbb{B}$ (Martins Afonso 2008):

$$\mathscr{N}\xi^{(0:2)}(\boldsymbol{x},t)=\frac{\tau_\ell(1-\beta)}{\ell^d}\frac{\partial u_{\alpha_2}}{\partial x_{\alpha_1}}(\boldsymbol{x},t)\frac{\partial u_{\alpha_1}}{\partial x_{\alpha_2}}(\boldsymbol{x},t), \tag{B 11a}$$

$$\xi^{(0:2)}(\boldsymbol{x},0)=0. \tag{B 11b}$$

Owing to periodicity and incompressibility, the forcing sustaining (B 11*a*) has zero average,

$$\int_{\mathbb{B}}\mathrm{d}\boldsymbol{x}\,\frac{\partial u_{\alpha_2}}{\partial x_{\alpha_1}}(\boldsymbol{x},t)\frac{\partial u_{\alpha_1}}{\partial x_{\alpha_2}}(\boldsymbol{x},t)=-\int_{\mathbb{B}}\mathrm{d}\boldsymbol{x}\,u_{\alpha_2}(\boldsymbol{x},t)\frac{\partial^2 u_{\alpha_1}}{\partial x_{\alpha_1}\partial x_{\alpha_2}}(\boldsymbol{x},t)=0, \tag{B 12}$$

so ensuring probability conservation to appropriate accuracy:

$$\int_{\mathbb{B}}\mathrm{d}\boldsymbol{x}\,\xi^{(0:2)}(\boldsymbol{x},t)=\int_0^{\mathscr{T}}\frac{\mathrm{d}t}{\mathscr{T}}\int_{\mathbb{B}}\mathrm{d}\boldsymbol{x}\int_{\mathbb{R}^d}\mathrm{d}\boldsymbol{v}\,p^{(0:2)}(\boldsymbol{x},\boldsymbol{v},t)=0. \tag{B 13}$$

### B.2. *Auxiliary vector field* $\psi_\mu\equiv p^{(1)}_\mu,\ \mu=1,\dots,d$

The Stokes-number expansion of the auxiliary vector field $\boldsymbol{p}^{(1)}$ differs from that of the phase-space density $p^{(0)}$ only by the presence of extra non-homogeneous terms. These latter simplify when the coefficients (3.13) of the Stokes expansion of the terminal velocity are evaluated using the results of the previous subsection for the phase-space density. In particular we have

$$\boldsymbol{w}^{(:-1)}=\boldsymbol{w}^{(:1)}=0, \tag{B 14}$$

since the co-velocity dependence of the coefficients of the phase-space density expansion is defined under parity,

$$p^{(0:n)}(\boldsymbol{x},-\boldsymbol{v},t)=(-1)^n p^{(0:n)}(\boldsymbol{x},\boldsymbol{v},t), \tag{B 15}$$

and by (C 9). By (2.1) we also have

$$\boldsymbol{w}^{(:0)}=\int_0^{\mathscr{T}}\frac{\mathrm{d}t}{\mathscr{T}}\int_{\mathbb{B}}\mathrm{d}\boldsymbol{x}\int_{\mathbb{R}^d}\mathrm{d}\boldsymbol{v}\left\{\boldsymbol{v}\frac{(1-\beta)\chi^{1;\sigma_\ell}_{\alpha_1}(\boldsymbol{v})u_{\alpha_1}(\boldsymbol{x},t)}{\sigma_\ell\ell^d}+\beta\boldsymbol{u}(\boldsymbol{x},t)\frac{\chi^{0;\sigma_\ell}(\boldsymbol{v})}{\ell^d}\right\}=0. \tag{B 16}$$

The first non-vanishing contribution is then

$$\boldsymbol{w}^{(:2)}=\int_0^{\mathscr{T}}\frac{\mathrm{d}t}{\mathscr{T}}\int_{\mathbb{B}}\mathrm{d}\boldsymbol{x}\,\boldsymbol{u}(\boldsymbol{x},t)\xi^{(0:2)}(\boldsymbol{x},t)+\tau_\ell(1-\beta)\boldsymbol{g}, \tag{B 17}$$

using (C 9) again and the vanishing of integrals (over the period) of derivatives of periodic functions.

#### B.2.1. *Leading orders of the Stokes expansion*

Proceeding as for the phase-space density, we look for a solution of (3.14) in the form

$$p^{(1:0)}_\mu(\boldsymbol{x},\boldsymbol{v},t)=\chi^{0;\sigma_\ell}(\boldsymbol{v})\xi^{(1:0)}_\mu(\boldsymbol{x},t). \tag{B 18}$$

In full analogy to § B.1, the vector field $\boldsymbol{\xi}^{(1:0)}$ is determined by the first solvability condition brought about by the perturbative hierarchy. The analogy with the phase-space density extends inasmuch as $(\mathscr{L}^{(0:0)})^{-1}$ remains bounded when acting on the

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non-homogeneous terms sustaining the first correction term in Stokes (3.15), whence

$$p_\mu^{(1:1)}(\boldsymbol{x},\boldsymbol{v},t)=\tau_\ell\chi_{\alpha_1}^{1;\sigma_\ell}(\boldsymbol{v})\mathscr{M}_{\alpha_1}\xi_\mu^{(1:0)}(\boldsymbol{x},t)-\frac{\tau_\ell\sigma_\ell}{\ell^d}\sqrt{\frac{2}{Pe}}\,\chi_\mu^{1;\sigma_\ell}(\boldsymbol{v}). \tag{B 19}$$

Thus, $\boldsymbol{\xi}^{(1:0)}$ is specified by the solvability condition associated to the second-order equation (3.16). This latter differs from (B 6*a*) by the occurrence of a non-homogeneous term on the right-hand side, and gives (3.20*a*) complemented by the initial condition (3.20*b*). An important consequence of the very definition of terminal velocity (2.22) preserved by its Stokes-number expansion (3.12) is the conservation law

$$\int_{\mathbb{B}}\mathrm{d}\boldsymbol{x}\,\boldsymbol{\xi}^{(1:0)}(\boldsymbol{x},t)=\int_0^{\mathscr{T}}\frac{\mathrm{d}t}{\mathscr{T}}\int_{\mathbb{B}}\mathrm{d}\boldsymbol{x}\int_{\mathbb{R}^d}\mathrm{d}\boldsymbol{v}\,\boldsymbol{p}^{(1:0)}(\boldsymbol{x},\boldsymbol{v},t)=\boldsymbol{0}. \tag{B 20}$$

Inserting (3.20*a*) into (3.16) yields

$$\begin{aligned}p_\mu^{(1:2)}(\boldsymbol{x},\boldsymbol{v},t)&=\chi^{0;\sigma_\ell}(\boldsymbol{v})\xi_\mu^{(1:2)}(\boldsymbol{x},t)\\&\quad+\frac{\tau_\ell}{2}\chi_{\alpha_1\alpha_2}^{2;\sigma_\ell}(\boldsymbol{v})\left[\tau_\ell\mathscr{M}_{\alpha_1}\mathscr{M}_{\alpha_2}\xi_\mu^{(1:0)}(\boldsymbol{x},t)-\frac{1-\beta}{\ell^d}\delta_{\alpha_1\mu}u_{\alpha_2}(\boldsymbol{x},t)\right].\end{aligned} \tag{B 21}$$

Like (B 5), (B 21) comprises a term $\chi^{0;\sigma_\ell}\,\boldsymbol{\xi}^{(1:2)}$ solution of the associated homogeneous equation. The introduction of this term is required to impose the solvability condition in the equation for $\boldsymbol{p}^{(1:4)}$ (in which, with respect to (3.17), beyond the aforementioned augmentation of the superscripts, an additional term $-w_\mu^{(:2)}\psi_0^{(:0)}$ appears within square brackets). Again, parity considerations (B 15) as in § B.1 entail that equation (3.17) does not engender any solvability condition. Tedious but straightforward calculations using (B 2), (B 3) and (C 11) yield

$$\begin{aligned}p_\mu^{(1:3)}&=\frac{1-\beta}{Fr^2}\sqrt{Pe}\,G_{\alpha_1}\chi_{\alpha_1}^{1;\sigma_\ell}\xi_\mu^{(1:0)}+\tau_\ell\chi_{\alpha_1}^{1;\sigma_\ell}\mathscr{M}_{\alpha_1}\xi_\mu^{(1:2)}\\&\quad-\tau_\ell^2\chi_{\alpha_1}^{1;\sigma_\ell}\mathscr{L}^{(0:2)}\mathscr{M}_{\alpha_1}\xi_\mu^{(1:0)}+\frac{\tau_\ell^3}{6}\chi_{\alpha_1\alpha_2\alpha_3}^{3;\sigma_\ell}\mathscr{M}_{\alpha_1}\mathscr{M}_{\alpha_2}\mathscr{M}_{\alpha_3}\xi_\mu^{(1:0)}\\&\quad-\frac{\tau_\ell^3\sigma_\ell}{2}\sqrt{\frac{2}{Pe}}\left(\chi_{\alpha_1}^{1;\sigma_\ell}\frac{\partial}{\partial x_{\alpha_2}}+\chi_{\alpha_2}^{1;\sigma_\ell}\frac{\partial}{\partial x_{\alpha_1}}\right)\mathscr{M}_{\alpha_1}\mathscr{M}_{\alpha_2}\xi_\mu^{(1:0)}-\tau_\ell\sigma_\ell\sqrt{\frac{2}{Pe}}\,\chi_\mu^{1;\sigma_\ell}\xi^{(0:2)}\\&\quad-\frac{\tau_\ell^2(1-\beta)}{2\ell^d}\chi_{\alpha_1}^{1;\sigma_\ell}\left(\mathscr{M}_{\alpha_1}u_\mu+\mathscr{M}_\mu u_{\alpha_1}\right)\\&\quad-\frac{\tau_\ell\beta(1-\beta)}{\sigma_\ell\ell^d}\sqrt{\frac{Pe}{2}}\,\chi_{\alpha_1}^{1;\sigma_\ell}u_\mu u_{\alpha_1}+\frac{\tau_\ell^2\sigma_\ell}{2\ell^d}\sqrt{\frac{2}{Pe}}\,\chi_{\alpha_1}^{1;\sigma_\ell}\frac{\partial}{\partial x_\mu}u_{\alpha_1}.\end{aligned} \tag{B 22}$$

The expression within third-order accuracy of $\boldsymbol{p}^{(1)}$ is complete observing that $\boldsymbol{\xi}^{(1:2)}$ satisfies the solvability condition (3.23*a*) with initial condition (3.23*b*), and the conservation law

$$\int_{\mathbb{B}}\mathrm{d}\boldsymbol{x}\,\boldsymbol{\xi}^{(1:2)}(\boldsymbol{x},t)=\int_0^{\mathscr{T}}\frac{\mathrm{d}t}{\mathscr{T}}\int_{\mathbb{B}}\mathrm{d}\boldsymbol{x}\int_{\mathbb{R}^d}\mathrm{d}\boldsymbol{v}\,\boldsymbol{p}^{(1:2)}(\boldsymbol{x},\boldsymbol{v},t)=\boldsymbol{0}. \tag{B 23}$$

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### B.3. *Expression of the eddy diffusivity*

#### B.3.1. *Evaluation of the $O(St^0)$ term*

Using (B 18) and (B 19) and expressing the co-velocity dependence in the integrands as the $\mathbb{L}^2(\mathbb{R}^d)$-scalar product between the Hermite polynomials $H^{n;\sigma_\ell}_{\cdots}$ and their duals $\chi^{n;\sigma_\ell}_{\cdots}$, we have

$$\begin{aligned} K^{(:0)}_{\mu\nu} &= -\mathscr{S}\int_0^{\mathscr{T}}\frac{\mathrm{d}t}{\mathscr{T}}\int_{\mathbb{B}}\mathrm{d}\boldsymbol{x}\int_{\mathbb{R}^d}\mathrm{d}\boldsymbol{v}\left[\sigma_\ell\sqrt{\frac{2}{Pe}}\,H^{1;\sigma_\ell}_\mu(\boldsymbol{v})p^{(1:1)}_\nu+\beta u_\mu H^{0;\sigma_\ell}(\boldsymbol{v})p^{(1:0)}_\nu\right]\\ &= -\mathscr{S}\int_0^{\mathscr{T}}\frac{\mathrm{d}t}{\mathscr{T}}\int_{\mathbb{B}}\mathrm{d}\boldsymbol{x}\int_{\mathbb{R}^d}\mathrm{d}\boldsymbol{v}\,H^{1;\sigma_\ell}_\mu(\boldsymbol{v})\\ &\quad\times\left[\tau_\ell\sigma_\ell\sqrt{\frac{2}{Pe}}\,\chi^{1;\sigma_\ell}_{\alpha_1}(\boldsymbol{v})\mathscr{M}_{\alpha_1}\xi^{(1:0)}_\nu-\frac{2\tau_\ell\sigma_\ell^2}{\ell^d Pe}\chi^{1;\sigma_\ell}_\nu(\boldsymbol{v})\right]\\ &\quad-\mathscr{S}\int_0^{\mathscr{T}}\frac{\mathrm{d}t}{\mathscr{T}}\int_{\mathbb{B}}\mathrm{d}\boldsymbol{x}\int_{\mathbb{R}^d}\mathrm{d}\boldsymbol{v}\,H^{0;\sigma_\ell}(\boldsymbol{v})\beta\chi^{0;\sigma_\ell}(\boldsymbol{v})u_\mu\xi^{(1:0)}_\nu. \end{aligned}\tag{B 24}$$

The orthogonality relations (C 9), together with

$$\int_0^{\mathscr{T}}\frac{\mathrm{d}t}{\mathscr{T}}\int_{\mathbb{B}}\mathrm{d}\boldsymbol{x}\,\frac{\partial\xi^{(1:0)}_\nu}{\partial x_\alpha}(\boldsymbol{x},t)=0,\tag{B 25}$$

yield (3.19). The positive definiteness of $K^{(:0)}_{\mu\nu}$ can be proved through (3.20*a*) and

$$\int_0^{\mathscr{T}}\frac{\mathrm{d}t}{\mathscr{T}}\int_{\mathbb{B}}\mathrm{d}\boldsymbol{x}\left[\frac{\partial}{\partial t}+\boldsymbol{u}(\boldsymbol{x},t)\cdot\frac{\partial}{\partial\boldsymbol{x}}\right]\xi^{(1:0)}_\mu(\boldsymbol{x},t)\xi^{(1:0)}_\nu(\boldsymbol{x},t)=0.\tag{B 26}$$

#### B.3.2. *Evaluation of the $O(St^1)$ term*

The leading-order correction

$$K^{(:2)}_{\mu\nu}=-\mathscr{S}\int_0^{\mathscr{T}}\frac{\mathrm{d}t}{\mathscr{T}}\int_{\mathbb{B}}\mathrm{d}\boldsymbol{x}\int_{\mathbb{R}^d}\mathrm{d}\boldsymbol{v}\left[\sigma_\ell\sqrt{\frac{2}{Pe}}\,H^{1;\sigma_\ell}_\mu(\boldsymbol{v})p^{(1:3)}_\nu+\beta u_\mu H^{0;\sigma_\ell}(\boldsymbol{v})p^{(1:2)}_\nu\right]\tag{B 27}$$

can be evaluated as the sum of two distinct integrals.

The first involves the projection of $p^{(1:3)}_\nu$ over $H^{1;\sigma_\ell}_\mu$. Using (C 9) to integrate out the co-velocity dependence, we are left with

$$\begin{aligned} &-\mathscr{S}\int_0^{\mathscr{T}}\frac{\mathrm{d}t}{\mathscr{T}}\int_{\mathbb{B}}\mathrm{d}\boldsymbol{x}\int_{\mathbb{R}^d}\mathrm{d}\boldsymbol{v}\,\sigma_\ell\sqrt{\frac{2}{Pe}}\,H^{1;\sigma_\ell}_\mu(\boldsymbol{v})p^{(1:3)}_\nu\\ &\quad=-\mathscr{S}\int_0^{\mathscr{T}}\frac{\mathrm{d}t}{\mathscr{T}}\int_{\mathbb{B}}\mathrm{d}\boldsymbol{x}\left[\sqrt{2}\,\frac{\sigma_\ell(1-\beta)}{Fr^2}G_\mu\xi^{(1:0)}_\nu\right.\\ &\qquad\left.+\;\tau_\ell\sigma_\ell\sqrt{\frac{2}{Pe}}\left(\mathscr{M}_\mu\xi^{(1:2)}_\nu-\tau_\ell\mathscr{L}^{(0:2)}\mathscr{M}_\mu\xi^{(1:0)}_\nu\right)\right]\\ &\qquad+\mathscr{S}\int_0^{\mathscr{T}}\frac{\mathrm{d}t}{\mathscr{T}}\int_{\mathbb{B}}\mathrm{d}\boldsymbol{x}\left[\frac{\tau_\ell^3\sigma_\ell^2}{Pe}\,\frac{\partial}{\partial x_\alpha}\left(\mathscr{M}_\mu\mathscr{M}_\alpha+\mathscr{M}_\alpha\mathscr{M}_\mu\right)\xi^{(1:0)}_\nu\right. \end{aligned}$$

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$$+\sqrt{\frac{2}{Pe}}\,\frac{\tau_\ell^2\sigma_\ell(1-\beta)}{2\ell^d}\mathscr{M}_\nu u_\mu\Bigg]$$

$$+\int_0^{\mathscr{T}}\frac{\mathrm{d}t}{\mathscr{T}}\int_{\mathbb{B}}\mathrm{d}\boldsymbol{x}\left[\frac{2\tau_\ell\sigma_\ell^2}{Pe}\delta_{\mu\nu}\xi^{(0:2)}+\frac{\tau_\ell\beta(1-\beta)}{\ell^d}u_\mu u_\nu-\frac{\tau_\ell^2\sigma_\ell^2}{\ell^d Pe}\mathscr{S}\frac{\partial}{\partial x_\nu}u_\mu\right]$$

$$=-\mathscr{S}\int_0^{\mathscr{T}}\frac{\mathrm{d}t}{\mathscr{T}}\int_{\mathbb{B}}\mathrm{d}\boldsymbol{x}\left[(1-\beta)u_\mu\xi_\nu^{(1:2)}-\frac{\tau_\ell(1-\beta)}{\ell^d}u_\mu u_\nu\right], \tag{B 28}$$

where the latter simplifications are due to the conservation laws (B 13) and (B 20), combined with the vanishing of integrals (over the period) of derivatives of periodic functions.

The second term involves (B 21) and is more straightforward to evaluate:

$$-\mathscr{S}\int_0^{\mathscr{T}}\frac{\mathrm{d}t}{\mathscr{T}}\int_{\mathbb{B}}\mathrm{d}\boldsymbol{x}\int_{\mathbb{R}^d}\mathrm{d}\boldsymbol{v}\,\beta u_\mu H^{0;\sigma_\ell}(\boldsymbol{v})p_\nu^{(1:2)}$$

$$=-\int_0^{\mathscr{T}}\frac{\mathrm{d}t}{\mathscr{T}}\int_{\mathbb{B}}\mathrm{d}\boldsymbol{x}\,\frac{\beta}{2}\left(u_\mu\xi_\nu^{(1:2)}+u_\nu\xi_\mu^{(1:2)}\right). \tag{B 29}$$

Expressions (B 28) and (B 29) combine into (3.22), which, together with (3.21), gives the final result (3.25).

### B.4. *Expansion at small inertia and finite terminal velocity*

The leading-order correction in Stokes to the phase-space density becomes

$$p^{(0:1)}\mapsto\tilde{p}^{(0:1)}(\boldsymbol{x},\boldsymbol{v},t)=\tau_\ell\chi_{\alpha_1}^{1;\sigma_\ell}(\boldsymbol{v})\tilde{\mathscr{M}}_{\alpha_1}\tilde{\xi}^{(0:0)}(\boldsymbol{x},t), \tag{B 30}$$

where $\tilde{\xi}^{(0:0)}$ now satisfies

$$\tilde{\mathscr{N}}\tilde{\xi}^{(0:0)}(\boldsymbol{x},t)=0, \tag{B 31}$$

with

$$\mathscr{N}\mapsto\tilde{\mathscr{N}}:=\frac{\partial}{\partial t}+[\boldsymbol{u}+(1-\beta)\boldsymbol{g}\tau_S]\cdot\frac{\partial}{\partial\boldsymbol{x}}-\kappa\Delta_{\boldsymbol{x}}$$

$$\equiv\frac{\partial}{\partial t}+[\boldsymbol{u}+(1-\beta)S_F U\boldsymbol{G}]\cdot\frac{\partial}{\partial\boldsymbol{x}}-\frac{U\ell}{Pe}\Delta_{\boldsymbol{x}}. \tag{B 32}$$

The new solvability condition (B 31), however, remains compatible with the steady-state solution (B 8), which we therefore assume. A further consequence of (B 30) is that the first non-vanishing contribution to the terminal velocity occurs at $O(St^0)$:

$$\boldsymbol{w}^{(:0)}\mapsto\tilde{\boldsymbol{w}}^{(:0)}=(1-\beta)S_F U\boldsymbol{G}. \tag{B 33}$$

The leading-order correction to $\boldsymbol{p}^{(1)}$ in its turn becomes

$$\boldsymbol{p}^{(1:1)}\mapsto\tilde{\boldsymbol{p}}^{(1:1)}(\boldsymbol{x},\boldsymbol{v},t)=\tau_\ell\chi_{\alpha_1}^{1;\sigma_\ell}(\boldsymbol{v})\tilde{\mathscr{M}}_{\alpha_1}\tilde{\boldsymbol{\xi}}^{(1:0)}(\boldsymbol{x},t)-\frac{\tau_\ell\sigma_\ell}{\ell^d}\sqrt{\frac{2}{Pe}}\,\boldsymbol{\chi}^{1;\sigma_\ell}, \tag{B 34}$$

where now $\tilde{\boldsymbol{\xi}}^{(1:0)}$ obeys (3.31*a*) supplemented with (3.31*b*). Accordingly, throughout this section (3.24) becomes

$$\boldsymbol{\mathcal{M}}\mapsto\tilde{\boldsymbol{\mathcal{M}}}:=\frac{(1-\beta)}{\tau_\ell\sigma_\ell}\sqrt{\frac{Pe}{2}}\,[\boldsymbol{u}(\boldsymbol{x},t)+S_F U\boldsymbol{G}]-\sqrt{\frac{2}{Pe}}\,\sigma_\ell\frac{\partial}{\partial\boldsymbol{x}}. \tag{B 35}$$

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## Appendix C. The Ornstein–Uhlenbeck process

We briefly recall some properties of the well-known Ornstein–Uhlenbeck process (see e.g. Van Kampen 2007, and references therein) used in the main text. The generator of the Ornstein–Uhlenbeck process on $\mathbb{R}^d$ with parameters $\tau, \sigma > 0$, defined by

$$\mathscr{G}^{\dagger}_{\tau,\sigma} := -\frac{\boldsymbol{y}}{\tau}\cdot\frac{\partial}{\partial \boldsymbol{y}} + \frac{\sigma^2}{\tau}\Delta_{\boldsymbol{y}} \tag{C 1}$$

is diagonal on the basis of the multidimensional Hermite polynomials $H^{n;\sigma}_{\alpha_1\ldots\alpha_n}$,

$$\mathscr{G}^{\dagger}_{\tau,\sigma} H^{n;\sigma}_{\alpha_1\ldots\alpha_n} = -\frac{n}{\tau} H^{n;\sigma}_{\alpha_1\ldots\alpha_n}, \tag{C 2}$$

with the $1 \leqslant \alpha_i \leqslant d$ $(i = 1, \ldots, n)$ indices ranging over the $\boldsymbol{y}$-vector components. Hermite (multi-)polynomials are most conveniently defined by the Rodrigues formula

$$H^{n;\sigma}_{\alpha_1\ldots\alpha_n}(\boldsymbol{y}) = (-1)^n \frac{\sigma^n}{\chi^{0;\sigma}(\boldsymbol{y})}\left(\prod_{i=1}^{n}\frac{\partial}{\partial y_{\alpha_i}}\right)\chi^{0;\sigma}(\boldsymbol{y}), \tag{C 3}$$

where

$$\chi^{0;\sigma}(\boldsymbol{y}) = \frac{\mathrm{e}^{-y^2/2\sigma^2}}{(2\pi\sigma^2)^{d/2}} \tag{C 4}$$

($y^2 \equiv \|\boldsymbol{y}\|^2$) is the unique element of the kernel (steady-state measure) of the adjoint $\mathscr{G}_{\tau,\sigma}$ operator on $\mathbb{L}^2(\mathbb{R}^d)$:

$$\mathscr{G}_{\tau,\sigma} = \frac{\partial}{\partial \boldsymbol{y}}\cdot\frac{\boldsymbol{y}}{\tau} + \frac{\sigma^2}{\tau}\Delta_{\boldsymbol{y}} = \chi^{0;\sigma}\mathscr{G}^{\dagger}_{\tau,\sigma}\frac{1}{\chi^{0;\sigma}}. \tag{C 5}$$

The last equality in (C 5) implies that $\mathscr{G}_{\tau,\sigma}$ is diagonal,

$$\mathscr{G}_{\tau,\sigma}\chi^{n;\sigma}_{\alpha_1\ldots\alpha_n} = -\frac{n}{\tau}\chi^{n;\sigma}_{\alpha_1\ldots\alpha_n}, \tag{C 6}$$

on the $\mathbb{L}^2(\mathbb{R}^d)$ complete set

$$\chi^{n;\sigma}_{\alpha_1\ldots\alpha_n} = H^{n;\sigma}_{\alpha_1\ldots\alpha_n}\chi^{0;\sigma}. \tag{C 7}$$

We also use the convention

$$\chi^{n;\sigma}_{\alpha_1\ldots\alpha_n} = H^{n;\sigma}_{\alpha_1\ldots\alpha_n} = 0, \quad \text{if } n < 0. \tag{C 8}$$

The Rodrigues formula implies the $\mathbb{L}^2(\mathbb{R}^d)$-orthogonality relations between Hermite polynomials and the basis elements (C 7). Namely, a multiple integration by parts readily shows that the integral is non-vanishing only for $n_1 = n_2 = n$, and if there exists at least one permutation $\mathscr{P}$ of the $n$-tuple $(\beta_1, \ldots, \beta_n)$ such that $\alpha_i = \mathscr{P}_i(\beta_1, \ldots, \beta_n)$

$$\begin{aligned}\int_{\mathbb{R}^d} \mathrm{d}\boldsymbol{y}\, H^{n_1;\sigma}_{\alpha_1\ldots\alpha_{n_1}}\chi^{n_2;\sigma}_{\beta_1\ldots\beta_{n_2}} &= \sigma^{n_1+n_2}\int_{\mathbb{R}^d}\mathrm{d}\boldsymbol{y}\,\chi^{0;\sigma}\left(\prod_{i_2=1}^{n_2}\frac{\partial}{\partial y_{\beta_{i_2}}}\right)\left[\frac{(-1)^{n_1}}{\chi^{0;\sigma}}\left(\prod_{i_1=1}^{n_1}\frac{\partial}{\partial y_{\alpha_{i_1}}}\right)\chi^{0;\sigma}\right]\\ &= \delta_{n_1 n_2}\sum_{\{\mathscr{P}\}}\prod_{i=1}^{n}\delta_{\alpha_i,\mathscr{P}_i(\beta_1,\ldots,\beta_n)}\end{aligned} \tag{C 9}$$

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(the sum ranging over all permutations of the $(\beta_1 \dots \beta_n)$). Further consequences are the grading relation

$$\frac{\partial \chi^{n;\sigma}_{\alpha_1\dots\alpha_n}}{\partial y_{\alpha_{n+1}}} = -\frac{\chi^{n+1;\sigma}_{\alpha_1\dots\alpha_n\alpha_{n+1}}}{\sigma} \tag{C 10}$$

and the recursion relation

$$H^{n+1;\sigma}_{\alpha_1\dots\alpha_n\alpha_{n+1}}(\boldsymbol{y}) = \frac{y_{\alpha_{n+1}}}{\sigma} H^{n;\sigma}_{\alpha_1\dots\alpha_n}(\boldsymbol{y}) - \sum_{k=1}^{n} H^{n-1;\sigma}_{\alpha_1\dots\alpha_{k-1}\alpha_{k+1}\dots\alpha_n}(\boldsymbol{y})\delta_{\alpha_k\alpha_{n+1}}, \tag{C 11}$$

which follows by applying (C 9) and (C 10) to

$$\frac{\partial H^{n;\sigma}_{\alpha_1\dots\alpha_n}}{\partial x_{\alpha_{n+1}}}(\boldsymbol{y}) = -\frac{H^{n+1;\sigma}_{\alpha_1\dots\alpha_n\alpha_{n+1}}(\boldsymbol{y})}{\sigma} + \frac{y_{\alpha_{n+1}}}{\sigma^2} H^{n;\sigma}_{\alpha_1\dots\alpha_n}(\boldsymbol{y}). \tag{C 12}$$

Finally we observe that

$$\begin{aligned}
&\left[(\mathscr{L}^{(0:0)})^{-1}\,\chi^{n;\sigma_\ell}_{\alpha_1\dots\alpha_n}\right](\boldsymbol{y}, t^\star; t_o) \\
&\quad= \int_{t_o}^{t^\star} \mathrm{d}s \int_{\mathbb{R}^d} \mathrm{d}\boldsymbol{z}\, \mathrm{e}^{(t^\star - s)\mathscr{G}_{\tau_\ell,\sigma_\ell}}(\boldsymbol{y}, \boldsymbol{z})\chi^{n;\sigma_\ell}_{\alpha_1\dots\alpha_n}(\boldsymbol{z}) = \chi^{n;\sigma_\ell}_{\alpha_1\dots\alpha_n}(\boldsymbol{y}) \int_{t_o}^{t^\star} \mathrm{d}s\, \mathrm{e}^{-(t^\star - s)n/\tau_\ell} \\
&\quad\xrightarrow{t^\star\to\infty}\ \frac{\tau_\ell}{n}\chi^{n;\sigma_\ell}_{\alpha_1\dots\alpha_n}(\boldsymbol{y}),
\end{aligned} \tag{C 13}$$

which entails that $[(\mathscr{L}^{(0:0)})^{-1} f](\boldsymbol{y}, t; t_o)$ is bounded in $t$ for any integrable $f(\boldsymbol{y})$ if and only if

$$\int_{\mathbb{R}^d} \mathrm{d}\boldsymbol{y}\, H^{0;\sigma}(\boldsymbol{y}) f(\boldsymbol{y}) \equiv \int_{\mathbb{R}^d} \mathrm{d}\boldsymbol{y}\, f(\boldsymbol{y}) = 0. \tag{C 14}$$

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