$$
\begin{aligned}
&= F_{a+1}F_{b+1}F_{c+1} - F_{a+1}F_{b+1}F_{c-1} + F_aF_bF_c + F_{a+1}F_bF_{c-1} \\
&\qquad -F_{a-1}F_{b-1}F_{c-1} + F_{a+1}F_{b-1}F_{c-1} \\
&= F_{a+1}F_{b+1}F_{c+1} + F_aF_bF_c - F_{a-1}F_{b-1}F_{c-1} \\
&\qquad -F_{a+1}\left(F_{b+1} - F_b\right)F_{c-1} + F_{a+1}F_{b-1}F_{c-1} \\
&= F_{a+1}F_{b+1}F_{c+1} + F_aF_bF_c - F_{a-1}F_{b-1}F_{c-1} \\
&\qquad -F_{a+1}F_{b-1}F_{c-1} + F_{a+1}F_{b-1}F_{c-1} \\
&= F_{a+1}F_{b+1}F_{c+1} + F_aF_bF_c - F_{a-1}F_{b-1}F_{c-1},
\end{aligned}
$$

as required.

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101.45 A note on bounds for the expected value of a random variable

The following result appears in [1]. It confirms an interesting conjecture made in [2] and provides some generalisations.

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Theorem: Let $f \geqslant 0$ be defined on $[0, A]$ with $\int_0^A f(x)\,dx = 1$. If $f \leqslant M$ on $[0, A]$, then

$$\frac{1}{2M} \leqslant \int_0^A xf(x)\,dx.$$

This gives a lower bound on the expected value of the random variable corresponding to f. We begin by pointing out that the theorem is in fact a consequence of the following (see [3]).

Steffensen's inequality: Let ϕ be increasing and g integrable on $[a, b]$, with $0 \leqslant g \leqslant 1$ and $\lambda = \int_a^b g(x)\,dx$. Then

$$\int_a^{a+\lambda} \phi(x)\,dx \leqslant \int_a^b \phi(x)\,g(x)\,dx$$

Indeed, as the reader may verify, taking $[a, b] = [0, A]$, $\phi(x) = x$, and $g(x) = f(x)/M$ in Steffensen's inequality yields the theorem. (In fact the proof given in [1] is in essence D. Mitrinović's proof [4] of Steffensen's inequality. See also Chapter XI of [5].)

Replacing g with $1 - g$ in Steffensen's inequality provides the upper bound

$$\int_a^b \phi(x)\,g(x)\,dx \leqslant \int_{b-\lambda}^b \phi(x)\,dx.$$

And similarly, this provides an upper bound in the theorem:

$$\int_0^A xf(x)\,dx \leqslant A - \frac{1}{2M}.$$

Intuitively, Steffensen's inequality is almost obvious: Thinking of ϕ (increasing) as fixed, the smallest that $\int_a^b \phi(x)\,g(x)\,dx$ can possibly be, for $0 \leqslant g \leqslant 1$ and $\lambda = \int_a^b g(x)\,dx$, is when

$$g(x) = \begin{cases} 1 & \text{if } x \in [a, a + \lambda] \\ 0 & \text{if } x \in (a + \lambda, b]. \end{cases}$$

Next we provide a more direct proof of the theorem, which might also seem better suited to its context.

Proof: (Lower Bound)

Define the function g by $g(x) = \dfrac{f(x)}{M}$. Let $G(x) = \int_0^x g(u)\,du$. We observe that,

$$\int_0^A \left(\int_0^x [g(u) - 1]\,du \right) g(x)\,dx = \int_0^A (G(x) - x)\,g(x)\,dx \leqslant 0,$$

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because $0 \leqslant g(x) \leqslant 1$. By using integration by parts, with $u = G(x)$ and $du = g(x)\,dx$, we can see that,

$$\int_0^A \big(G(x)\,g(x) - xg(x)\big)\,dx = \frac{G(A)^2}{2} - \frac{\int_0^A xf(x)\,dx}{M}.$$

After a few substitutions it is evident that,

$$\frac{\left(\int_0^A g(x)\,dx\right)^2}{2} - \frac{\int_0^A xf(x)\,dx}{M} = \frac{\left(\int_0^A f(x)\,dx\right)^2}{2M^2} - \frac{\int_0^A xf(x)\,dx}{M} \leqslant 0.$$

Therefore,

$$\int_0^A xf(x)\,dx \geqslant \frac{1}{2M}.$$

(Upper Bound)

Define the function g by $g(x) = \dfrac{M - f(x)}{AM - 1}$. We observe that $0 \leqslant f(x) \leqslant M$, $0 \leqslant g(x) \leqslant \dfrac{M}{AM - 1}$, and $\int_0^A g(x)\,dx = 1$. Applying the original theorem for the lower bound to $g(x)$ yields,

$$\int_0^A xg(x)\,dx \geqslant \frac{AM - 1}{2M}.$$

By using substitution it is evident that,

$$\int_0^A x\left(\frac{M - f(x)}{AM - 1}\right)dx \geqslant \frac{AM - 1}{2M}.$$

It follows that,

$$\int_0^A xM\,dx - \int_0^A xf(x)\,dx \geqslant \frac{(AM - 1)^2}{2M}.$$

So by integration,

$$\int_0^A xf(x)\,dx \leqslant \frac{A^2M}{2} - \frac{(AM - 1)^2}{2M}.$$

Therefore,

$$\int_0^A xf(x)\,dx \leqslant A - \frac{1}{2M}.$$

The proof is complete.

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101.46 Improved simple pendulum results

Introduction

The period of a simple pendulum making small oscillations (usually quoted as less than 10 degrees) is given by the standard result

$$T = 2\pi\sqrt{\frac{l}{g}}. \tag{1}$$

For larger oscillation of amplitude δ, Quadling and Ramsey [1] give the result

$$T = 4\sqrt{\frac{l}{g}}\int_0^{\frac{1}{2}\pi}\frac{d\phi}{\sqrt{1 - \sin^2\tfrac{1}{2}\delta\,\sin^2\phi}}. \tag{2}$$

In [2], Murphy found that by approximating this integral to the area of a single trapezium gives the much more accurate result

$$T_1 = \pi\left(1 + \sec\frac{\delta}{2}\right)\sqrt{\frac{l}{g}}. \tag{3}$$

Here we define $s = \sin\frac{1}{2}\delta$ and $I = \dfrac{2}{\pi}\displaystyle\int_0^{\pi/2}\left(1 - s^2\sin^2 x\right)^{-1/2}dx$ and find the solution to the integral as a power series in s which will provide an estimate for the error in (3) and lead to other better approximations.

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