J. Aust. Math. Soc. **110** (2021), 71–80
doi:10.1017/S1446788719000296

ON COMPACT HOMOGENEOUS $G_{2(2)}$-MANIFOLDS

WOLFGANG GLOBKE

(Received 4 September 2018; accepted 2 July 2019; first published online 4 September 2019)

Communicated by M. Murray

Abstract

We prove that among all compact homogeneous spaces for an effective transitive action of a Lie group whose Levi subgroup has no compact simple factors, the seven-dimensional flat torus is the only one that admits an invariant torsion-free $G_{2(2)}$-structure.

2010 *Mathematics subject classification*: primary 53C50; secondary 53C30, 53C29.

Keywords and phrases: $G_{2(2)}$ holonomy, compact homogeneous spaces, pseudo-Riemannian manifolds.

1. Introduction and main results

Let G be a connected Lie group whose Levi subgroup has no compact simple factors. Let M be a seven-dimensional connected compact homogeneous space for G, by which we mean that $M = G/H$ for some closed uniform subgroup H of G and G acts effectively on M. Assume now that there exists a G-invariant torsion-free $G_{2(2)}$-structure $(\mathsf{g}_M, \varphi_M)$ on M, where $G_{2(2)}$ denotes the split real form of the complex exceptional Lie group $G_2^{\mathbb{C}}$. This means that g_M is a pseudo-Riemannian metric of signature $(4,3)$ and φ_M is a certain parallel three-form whose stabilizer on each tangent space is the group $G_{2(2)}$.

In this article we will show the following result.

THEOREM A. *Let M be a compact seven-dimensional homogeneous space for a Lie group G whose Levi subgroup does not have compact simple factors. If M has a G-invariant torsion-free $G_{2(2)}$-structure, then $G = M = \mathbb{R}^7_3/\mathbb{Z}^7$ is a flat torus. In particular, the holonomy group of M is trivial.*

The proof of this theorem makes use of results by Baues *et al.* [1] to reduce it to the case of a Lie group with bi-invariant metric. Then it follows from the next theorem.

Wolfgang Globke is supported by the Austrian Science Fund FWF grant I 3248.
© 2019 Australian Mathematical Publishing Association Inc.

https://doi.org/10.1017/S1446788719000296 Published online by Cambridge University Press

THEOREM B. *Let G be a seven-dimensional Lie group and Γ a uniform lattice in G. If $M = G/\Gamma$ has a G-invariant torsion-free $\mathrm{G}_{2(2)}$-structure whose associated metric pulls back to a bi-invariant metric on G, then $G = M = \mathbb{R}^7_3/\mathbb{Z}^7$ is a flat torus. In particular, the holonomy group of M is trivial.*

In Section 2 we use a result by Globke and Nikolayevsky [5] to reduce the problem to nilpotent Lie groups. The nilpotence of G then implies that φ_M induces an $\mathrm{Ad}(G)$-invariant three-form on the Lie algebra of G. We then investigate in Section 3 the obstructions for nilpotent Lie algebras with invariant scalar products to be contained in $\mathfrak{g}_{2(2)}$. Combining these results, we prove Theorems A and B in Section 4.

Contrasting our theorems here, there are many examples of homogeneous spaces for compact Lie groups that admit nontrivial invariant G_2- or $\mathrm{G}_{2(2)}$-structures. Lê and Munir [8] established a classification of these spaces. In the Riemannian G_2-case, this is a complete classification of compact homogeneous G_2-manifolds. See the references in [8] for further related results.

Other existence results in a setting similar to ours were obtained, for example, by Fino and Luján [4], who determined $\mathrm{G}_{2(2)}$-structures on compact nilmanifolds $\Gamma\backslash G$ or, equivalently, left-invariant $\mathrm{G}_{2(2)}$-structures on seven-dimensional nilpotent Lie groups G. Note that, despite the authors calling these structures 'invariant', the metrics on these quotients are not G-invariant. In fact, the additional assumption made in [4] that the metric is definite on the center of G prohibits the pulled-back metric on G from being bi-invariant, which is necessary for the metric on $\Gamma\backslash G$ to be G-invariant (unless G is abelian to begin with).

Another related result is the classification of indecomposable indefinite symmetric spaces with $\mathrm{G}_{2(2)}$-structures by Kath [7]. All of these turn out to be noncompact quotients of nilpotent Lie groups with a bi-invariant metric, and their holonomy groups are three dimensional and abelian.

Notation and conventions. A Lie group or Lie algebra is called *k-step nilpotent* if the kth term in its descending central series is trivial, but the $(k-1)$st term is not.

For a Lie algebra $\mathfrak{g}$ with subalgebra $\mathfrak{h}$, we let $\mathrm{ad}(\mathfrak{g})$ denote the adjoint representation of $\mathfrak{g}$ and, when clarity requires it, we write $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{h})$ for the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$ to distinguish it from the adjoint representation of $\mathfrak{h}$ on itself. A similar notation is used for Lie groups.

A Lie algebra is called *metric* if it has an invariant scalar product. The vector space $\mathbb{R}^n$, equipped with a scalar product of signature $(n-s, s)$, is denoted by $\mathbb{R}^n_s$.

2. Reduction to nilpotent groups

In this section we will show that for our purposes it is sufficient to consider nilpotent Lie groups.

LEMMA 2.1. *Let M be a compact homogeneous space for a Lie group G whose Levi subgroup does not have compact simple factors. Suppose that G acts effectively on M. If M has a G-invariant pseudo-Riemannian metric, then $M = G/\Gamma$ for some uniform*

https://doi.org/10.1017/S1446788719000296 Published online by Cambridge University Press

lattice Γ *in* G *and in particular* $\dim G = \dim M$. *Moreover, the pseudo-Riemannian metric on* M *is induced by a bi-invariant metric on* G.

Proof. Write $M = G/H$ for a closed uniform subgroup H of G. Via pullback to G, g_M induces a symmetric bilinear form on the Lie algebra of G whose kernel is the Lie algebra of H. Baues *et al.* [1, Theorem A] showed that this bilinear form is $\mathrm{Ad}(G)$-invariant, so that its kernel is an ideal. By the effectivity of the G-action, this ideal, and hence the identity component of H, must be trivial. This means that H is a uniform lattice in G, so that $\dim G = \dim M$. By the invariance of the bilinear form on the Lie algebra, the metric on M is induced by a bi-invariant metric on G. □

The existence of a torsion-free $G_{2(2)}$-structure $(\mathsf{g}_M, \varphi_M)$ on a manifold M means that the three-form φ_M is parallel and is thus preserved by the holonomy group at every point $p \in M$. The stabilizer of $(\varphi_M)_p$ in $\mathrm{GL}(T_pM)$ is $G_{2(2)}$, which means that $\mathrm{Hol}(\mathsf{g}_M) \subseteq G_{2(2)}$. Moreover, the metric g_M of the $G_{2(2)}$-structure has vanishing Ricci curvature (see Bonan [3]).

Lemma 2.2. *Let* G *be a seven-dimensional connected Lie group with a bi-invariant metric* g_G *and* Γ *a uniform lattice in* G. *Let* $M = G/\Gamma$ *and let* g_M *be the metric induced on* M *by* g_G. *If* g_M *is Ricci-flat, then:*

(1) *G is nilpotent;*

(2) *the connected holonomy group of* g_M *is* $\mathrm{Hol}(\mathsf{g}_M)^\circ \cong \mathrm{Ad}_{\mathfrak{g}}([G, G])$.

In particular, this holds if g_M *is the metric of a torsion-free* $G_{2(2)}$*-structure* $(\mathsf{g}_M, \varphi_M)$ *on* M *and in this case* $\mathrm{Ad}_{\mathfrak{g}}(G)$ *is a subgroup of* $G_{2(2)}$.

Proof. The bi-invariance of g_G implies that the Ricci tensor, restricted to the left-invariant vector fields on G, is proportional to the Killing form of G. Since g_M, and hence g_G, is Ricci-flat, the Killing form of G is zero, which means that G is solvable. Moreover, the Ricci-flatness implies that M is an Einstein manifold. It was shown by Globke and Nikolayevsky [5, proof of Theorem 1.3] that a solvable Lie group acting transitively on a compact pseudo-Riemannian Einstein manifold of dimension less than or equal to seven is nilpotent. So, G is nilpotent.

As a Lie group with bi-invariant metric, the curvature tensor on G is given by $\mathsf{R}(X, Y)Z = \frac{1}{4}[[X, Y], Z]$ for any left-invariant vector fields X, Y, Z on G. The Ambrose–Singer theorem (cf. Besse [2, Theorem 10.58]) now implies that the connected holonomy group of G is $\mathrm{Hol}(\mathsf{g}_G)^\circ \cong \mathrm{Ad}_{\mathfrak{g}}([G, G])$. Since G is a covering space of M, $\mathrm{Hol}(\mathsf{g}_G)^\circ \cong \mathrm{Hol}(\mathsf{g}_M)^\circ$ (cf. Besse [2, 10.16]).

Suppose that $(\mathsf{g}_M, \varphi_M)$ is a torsion-free $G_{2(2)}$-structure on M. Then g_M and g_G are Ricci-flat, so all of the above applies. The three-form φ_M pulls back to a left-invariant three-form φ_G on G that is right-invariant under Γ. This means that the induced three-form $\varphi_{\mathfrak{g}}$ on $\mathfrak{g}$ is $\mathrm{Ad}_{\mathfrak{g}}(\Gamma)$-invariant. But the lattice $\mathrm{Ad}_{\mathfrak{g}}(\Gamma)$ in the nilpotent Lie group $\mathrm{Ad}_{\mathfrak{g}}(G)$ is Zariski-dense and $\varphi_{\mathfrak{g}}$ is a polynomial expression; hence, $\varphi_{\mathfrak{g}}$ is $\mathrm{Ad}_{\mathfrak{g}}(G)$-invariant. Since the real algebraic group $G_{2(2)}$ is the stabilizer of $\varphi_{\mathfrak{g}}$, this means that $\mathrm{Ad}_{\mathfrak{g}}(G)$ is a subgroup of $G_{2(2)}$. □

https://doi.org/10.1017/S1446788719000296 Published online by Cambridge University Press

3. Nilpotent metric Lie algebras

3.1. Nilpotent metric Lie algebras in low dimensions. Let $\mathfrak{n}$ be a nilpotent Lie algebra with invariant scalar product $\langle\cdot,\cdot\rangle$ of signature (p,q). Let $\mathfrak{z}(\mathfrak{n})$ denote the center of $\mathfrak{n}$. Set

$$\mathfrak{j} = \mathfrak{z}(\mathfrak{n}) \cap [\mathfrak{n},\mathfrak{n}].$$

This is a totally isotropic ideal in $\mathfrak{n}$. Also, $\mathfrak{j}^\perp$ is an ideal in $\mathfrak{n}$ that contains $[\mathfrak{n},\mathfrak{n}]$ and $\mathfrak{z}(\mathfrak{n})$. We have vector space decompositions

$$\mathfrak{n} = \mathfrak{j}^* \oplus \mathfrak{j}^\perp, \quad \mathfrak{j}^\perp = \mathfrak{w} \oplus \mathfrak{j}, \tag{3.1}$$

where $\mathfrak{j}^*$ is totally isotropic, dually paired with $\mathfrak{j}$ via $\langle\cdot,\cdot\rangle$ and orthogonal to $\mathfrak{w}$, and $\langle\cdot,\cdot\rangle$ has signature $(p-\dim\mathfrak{j}, q-\dim\mathfrak{j})$ on $\mathfrak{w}$.

The decomposition (3.1) is helpful in understanding the algebraic structure of Lie algebras with invariant scalar products. The following is a simple yet very useful property of the ideal $\mathfrak{j}$.

Lemma 3.1. *$\mathfrak{n}$ is abelian if and only if $\mathfrak{j} = \mathbf{0}$.*

Proof. If $\mathfrak{n}$ is abelian, then clearly $\mathfrak{j} = \mathbf{0}$. If $\mathfrak{n}$ is not abelian, its descending central series eventually intersects $\mathfrak{z}(\mathfrak{n})$ nontrivially, which means that $\mathfrak{j} \neq \mathbf{0}$. □

Lemma 3.2. *If $\dim\mathfrak{n} \leq 4$, then $\mathfrak{n}$ is abelian.*

Proof. The claim is trivial for $\dim\mathfrak{n} \leq 2$. Suppose first that $\dim\mathfrak{n} = 3$. By (3.1) and Lemma 3.1, if $\mathfrak{n}$ was not abelian, then $\dim\mathfrak{j} = \dim\mathfrak{j}^* = \dim\mathfrak{w} = 1$. With the invariance of $\langle\cdot,\cdot\rangle$, these dimensions imply that $\mathfrak{j} = [\mathfrak{j}^*,\mathfrak{w}] \perp \mathfrak{j}^*$, contradicting the definition of $\mathfrak{j}^*$.

Now assume that $\dim\mathfrak{n} = 4$ and $\mathfrak{n}$ is not abelian. If $\dim\mathfrak{j} = 2$, then $\mathfrak{n} = \mathfrak{j}^* \oplus \mathfrak{j}$ and invariance of $\langle\cdot,\cdot\rangle$ implies that $[\mathfrak{n},\mathfrak{n}] = [\mathfrak{j}^*,\mathfrak{j}^*] \perp \mathfrak{j}^*$, but also $[\mathfrak{j}^*,\mathfrak{j}^*] \subset \mathfrak{j}$, which is a contradiction. This leaves the case $\dim\mathfrak{j} = 1$. Here, invariance and $\dim\mathfrak{j}^* = 1$ imply that $[\mathfrak{j}^*,\mathfrak{w}] \subseteq \mathfrak{w}$. By nilpotence of $\mathfrak{n}$, $[\mathfrak{w},\mathfrak{w}] \subseteq \mathfrak{j}$ and, since $\mathfrak{j}^*$ is dually paired with $\mathfrak{j}$, it follows that $\mathfrak{j}^*$ acts nontrivially on $\mathfrak{w}$. But invariance then requires $[\mathfrak{j}^*,\mathfrak{w}] = \mathfrak{w}$, contradicting the nilpotence of $\mathfrak{n}$. □

With regard to Lemma 2.2, we are interested in the question of when for nonabelian $\mathfrak{n}$ of dimension seven, $\mathrm{ad}_\mathfrak{n}(\mathfrak{n})$ is contained in $\mathfrak{g}_{2(2)}$. So, let us now further assume that $\dim\mathfrak{n} = 7$ and that $\mathfrak{n}$ is k-step nilpotent. Since it is possible that $\mathfrak{n}$ is *decomposable*, meaning that it can be further decomposed into an orthogonal direct sum $\mathfrak{n} = \mathfrak{n}_1 \times \mathfrak{n}_0$ of ideals, we are also interested in nonabelian nilpotent Lie algebras $\mathfrak{n}_1$ of dimension $\dim\mathfrak{n}_1 \leq 7$ with an invariant scalar product.

As is evident from Kath's classification [6, Theorem 4.7], the Examples 3.3 to 3.5 below are the only nonabelian nilpotent Lie algebras $\mathfrak{n}_1$ of dimension less than eight with an invariant scalar product that cannot be further decomposed into orthogonal direct sums of ideals. We use the following notation: if $d = \dim\mathfrak{j}$, let $a_1,\dots,a_d \in \mathfrak{j}^*$, $w_1,\dots,w_{n-2d} \in \mathfrak{w}$ and $z_1,\dots,z_d \in \mathfrak{j}$ be bases of the respective subspaces in (3.1) such that the a_i and z_j are dual bases to each other with respect to $\langle\cdot,\cdot\rangle$. The number $\varepsilon = \pm 1$ depends on whether $p \geq q$ (then $\varepsilon = 1$) or $p < q$ (then $\varepsilon = -1$).

https://doi.org/10.1017/S1446788719000296 Published online by Cambridge University Press

EXAMPLE 3.3. Let $\dim \mathfrak{n}_1 = 7$, $\dim \mathfrak{j} = 2$, with bracket relations

$$[a_1, a_2] = w_1, \quad [a_1, w_1] = w_2, \quad [a_1, w_2] = -\varepsilon w_3, \quad [a_1, w_3] = -z_2,$$
$$[a_2, w_1] = 0, \quad [a_2, w_2] = 0, \quad [a_2, w_3] = z_1,$$
$$[w_1, w_2] = \varepsilon z_1, \quad [w_1, w_3] = 0, \quad [w_2, w_3] = 0,$$

where the scalar product on $\mathfrak{w}$ is given by

$$\langle w_1, w_1\rangle = \langle w_1, w_2\rangle = \langle w_2, w_3\rangle = \langle w_3, w_3\rangle = 0, \quad \langle w_1, w_3\rangle = 1, \quad \langle w_2, w_2\rangle = \varepsilon.$$

We denote this Lie algebra by $\mathfrak{n}_{\mathrm{I}}$. Here, $\mathfrak{n}_{\mathrm{I}}$ is three-step nilpotent, $\dim \mathfrak{w} = 3$ and the algebra $\mathrm{ad}_{\mathfrak{n}}(\mathfrak{w})$ is three dimensional and abelian.

EXAMPLE 3.4. Let $\dim \mathfrak{n}_1 = 6$, $\dim \mathfrak{j} = 3$, with nonzero bracket relations

$$[a_1, a_2] = z_3, \quad [a_2, a_3] = z_1, \quad [a_3, a_1] = z_2.$$

We denote this Lie algebra by $\mathfrak{n}_{\mathrm{II}}$. Here, $\mathfrak{n}_{\mathrm{II}}$ is two-step nilpotent and $\mathfrak{w} = \mathbf{0}$. The invariant scalar product has split signature $(3, 3)$.

EXAMPLE 3.5. Let $\dim \mathfrak{n}_1 = 5$, $\dim \mathfrak{j} = 2$, with nonzero bracket relations

$$[a_1, a_2] = w, \quad [a_1, w] = -\varepsilon z_2, \quad [a_2, w] = \varepsilon z_1.$$

We denote this Lie algebra by $\mathfrak{n}_{\mathrm{III}}$. Here, $\mathfrak{n}_{\mathrm{III}}$ is three-step nilpotent and $\dim \mathfrak{w} = 1$. The signature of $\mathfrak{n}_{\mathrm{III}}$ is $(3, 2)$ or $(2, 3)$, depending on ε.

3.2. Nilpotent subalgebras of $\mathfrak{g}_{2(2)}$. The Lie algebra $\mathfrak{so}(\mathfrak{n}, \langle\cdot,\cdot\rangle)$ of skew-symmetric linear maps for $\langle\cdot,\cdot\rangle$ can be identified with $\mathfrak{so}_{4,3}$. Since $\langle\cdot,\cdot\rangle$ is invariant, we have $\mathrm{ad}(\mathfrak{n}) \subset \mathfrak{der}(\mathfrak{n}) \cap \mathfrak{so}_{4,3}$.

The exceptional simple Lie algebra $\mathfrak{g}_{2(2)}$ is also a subalgebra of $\mathfrak{so}_{4,3}$. It is the stabilizer subalgebra in $\mathfrak{so}_{4,3}$ of a certain three-form φ on $\mathfrak{n}$. In a suitable Witt basis of $\mathfrak{n}$ with respect to $\langle\cdot,\cdot\rangle$, we identify $\mathfrak{g}_{2(2)}$ with the following Lie subalgebra of $\mathfrak{so}_{4,3}$:

$$\mathfrak{g}_{2(2)} = \left\{ \begin{pmatrix} u_7 & u_9 & u_{10} & u_{12} & u_{13} & u_{14} & 0 \\ u_1 & u_8 & u_{11} & \frac{u_{10}}{2} & -\frac{u_{12}}{4} & 0 & -u_{14} \\ u_2 & u_3 & u_7 - u_8 & -\frac{u_9}{2} & 0 & \frac{u_{12}}{4} & -u_{13} \\ u_4 & 4u_2 & -4u_1 & 0 & \frac{u_9}{2} & -\frac{u_{10}}{2} & -u_{12} \\ u_5 & -2u_4 & 0 & 4u_1 & u_8 - u_7 & -u_{11} & -u_9 \\ u_6 & 0 & 2u_4 & -4u_2 & -u_3 & -u_8 & -u_9 \\ 0 & -u_6 & -u_5 & -u_4 & -u_2 & -u_1 & -u_7 \end{pmatrix} \middle| \; u_1, \ldots, u_{14} \in \mathbb{R} \right\}$$

(this matrix representation is borrowed from Leistner *et al.* [9], with some adjustments to the parameter labelling).

Any strictly triangular subalgebra of $\mathfrak{g}_{2(2)}$ is contained in a maximal triangular subalgebra of $\mathfrak{g}_{2(2)}$. All maximal triangular subalgebras of $\mathfrak{g}_{2(2)}$ are conjugate to each other by $G_{2(2)}$, so we may assume that the strictly triangular subalgebra $\mathrm{ad}(\mathfrak{n})$

https://doi.org/10.1017/S1446788719000296 Published online by Cambridge University Press

is conjugate to a subalgebra of

$$\mathfrak{m} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ u_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ u_2 & u_3 & 0 & 0 & 0 & 0 & 0 \\ u_4 & 4u_2 & -4u_1 & 0 & 0 & 0 & 0 \\ u_5 & -2u_4 & 0 & 4u_1 & 0 & 0 & 0 \\ u_6 & 0 & 2u_4 & -4u_2 & -u_3 & 0 & 0 \\ 0 & -u_6 & -u_5 & -u_4 & -u_2 & -u_1 & 0 \end{pmatrix} \middle| \, u_1, \dots, u_6 \in \mathbb{R} \right\}. \tag{3.2}$$

In fact, this subalgebra $\mathfrak{m}$ is a maximal strictly triangular subalgebra of a Borel subalgebra of $\mathfrak{g}_{2(2)}$ and hence a conjugate of $\mathrm{ad}(\mathfrak{n})$ is contained in $\mathfrak{m}$.

LEMMA 3.6. *Let A be the matrix in (3.2). Suppose that $\operatorname{rk} A \leq 2$. Then $u_1 = 0$ and in addition one of the following holds:*

(a) *either $u_2 = u_4 = 0$;*
(b) *or $u_2 \neq 0, u_4 \neq 0$ and $u_3 = 4u_2^2/u_4$, $u_5 = -u_4^2/2u_2$.*

PROOF. If $u_1 \neq 0$, then columns one, three, four and six are linearly independent. So, assume henceforth that $u_1 = 0$, that is,

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ u_2 & u_3 & 0 & 0 & 0 & 0 & 0 \\ u_4 & 4u_2 & 0 & 0 & 0 & 0 & 0 \\ u_5 & -2u_4 & 0 & 0 & 0 & 0 & 0 \\ u_6 & 0 & 2u_4 & -4u_2 & -u_3 & 0 & 0 \\ 0 & -u_6 & -u_5 & -u_4 & -u_2 & 0 & 0 \end{pmatrix}.$$

First, assume that $u_2 = 0$. Then columns one, two, three and four are linearly independent, unless also $u_4 = 0$. This is case (a).

Next, assume that $u_2 \neq 0$. If $u_4 = 0$, we see that columns one, two and four are linearly independent. So, assume also that $u_4 \neq 0$. Given $\operatorname{rk} A = 2$, the subcolumns $\begin{pmatrix} u_2 \\ u_4 \\ u_5 \end{pmatrix}$ and $\begin{pmatrix} u_3 \\ 4u_2 \\ -2u_4 \end{pmatrix}$ must be linearly dependent. So, add $-u_3/u_2$ times the first column of A to its second column and obtain the identities $4u_2 - u_3u_4/u_2 = 0$ and $-2u_4 - u_3u_5/u_2 = 0$. By solving for u_3 and u_5,

$$u_3 = \frac{4u_2^2}{u_4}, \quad u_5 = -\frac{u_4^2}{2u_2}.$$

Inserting this into A, we can verify directly that $\operatorname{rk} A = 2$. This is case (b). □

REMARK 3.7. Note that condition (a) in Lemma 3.6 is necessary but not sufficient for $\operatorname{rk} A = 2$. Also, the condition $\operatorname{rk} A \leq 2$ already implies that $\operatorname{rk} A = 2$ for the nonzero matrices in case (a) or (b).

Below, we use the notation $A(u_3, u_5, u_6)$ for matrices of case (a) in Lemma 3.6 and $B(u_2, u_4, u_6)$ for matrices of case (b),

https://doi.org/10.1017/S1446788719000296 Published online by Cambridge University Press

$$A(u_3,u_5,u_6)=\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&u_3&0&0&0&0&0\\0&0&0&0&0&0&0\\u_5&0&0&0&0&0&0\\u_6&0&0&0&-u_3&0&0\\0&-u_6&-u_5&0&0&0&0\end{pmatrix},$$

$$B(u_2,u_4,u_6)=\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\u_2&\dfrac{4u_2^2}{u_4}&0&0&0&0&0\\u_4&4u_2&0&0&0&0&0\\-\dfrac{u_4^2}{2u_2}&-2u_4&0&0&0&0&0\\u_6&0&2u_4&-4u_2&-\dfrac{4u_2^2}{u_4}&0&0\\0&-u_6&\dfrac{u_4^2}{2u_2}&-u_4&-u_2&0&0\end{pmatrix},\quad u_2,u_4\neq 0.$$

Lemma 3.8. *The maximal nilpotent subalgebra* $\mathfrak{m}$ *of* $\mathfrak{g}_{2(2)}$ *does not contain a three-dimensional Lie subalgebra* $\mathfrak{b}$ *consisting only of matrices of rank two (and zero).*

Proof. Rank-two matrices in $\mathfrak{m}$ belong to one of the two types described in Lemma 3.6. First, assume that $\mathfrak{b}$ contains only matrices of the form $A(u_3,u_5,u_6)$. In order to be of rank two, one of the parameters u_3,u_5,u_6 must be zero in each element of $\mathfrak{b}$. But the set of matrices with this property is a finite union of two-dimensional subspaces, contradicting $\dim\mathfrak{b}=3$.

So, assume that $\mathfrak{b}$ contains a nonzero matrix $B(u_2,u_4,u_6)$. If $\mathfrak{b}$ contains two elements $B_1=B(u_2,u_4,u_6)$, $B_2=B(v_2,v_4,v_6)$ that are not multiples of each other, then one of $u_2+v_2\neq 0$ or $u_4+v_4\neq 0$ holds. The sum $B_{12}=B_1+B_2$ is

$$B_{12}=\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\u_2+v_2&\dfrac{4u_2^2}{u_4}+\dfrac{4v_2^2}{v_4}&0&0&0&0&0\\u_4+v_4&4u_2+4v_2&0&0&0&0&0\\-\dfrac{u_4^2}{2u_2}-\dfrac{v_4^2}{2v_2}&-2u_4-2v_4&0&0&0&0&0\\u_6+v_6&0&2u_4+2v_4&-4u_2-4v_2&-\dfrac{4u_2^2}{u_4}-\dfrac{4v_2^2}{v_4}&0&0\\0&-u_6-v_6&\dfrac{u_4^2}{2u_2}+\dfrac{v_4^2}{2v_2}&-u_4-v_4&-u_2-v_2&0&0\end{pmatrix}.$$

https://doi.org/10.1017/S1446788719000296 Published online by Cambridge University Press

If $\operatorname{rk} B_{12} = 2$, then, since one of $u_2 + v_2$ and $u_4 + v_4$ is nonzero, both must be nonzero (B_{12} is case (b) of Lemma 3.6). Set $w_2 = u_2 + v_2$, $w_4 = u_2 + v_4$. By Lemma 3.6, entry $(3, 2)$ of B_{12} must be $w_3 = 4w_2^2/w_4 = 4(u_2^2 + 2u_2u_4 + u_4^2)/(u_4 + v_4)$. But this entry is $(4u_2^2/u_4) + (4v_2^2/v_4)$ and the resulting equation $4(u_2^2 + 2u_2u_4 + u_4^2)/(u_4 + v_4) = (4u_2^2/u_4) + (4v_2^2/v_4)$ simplifies to $u_2v_4 - u_4v_2 = 0$. Solving for v_2 and inserting back into $B(v_2, v_4, v_6)$ shows that

$$B_1 - \frac{u_4}{v_4} B_2 = A\Big(0, 0, u_6 - \frac{u_4}{v_4} v_6\Big).$$

If this is zero, then B_1 is a multiple of B_2, in contradiction to our assumption.

So, $\mathfrak{b}$ contains B_1 and $A(0, 0, 1)$ as basis elements, and all other elements $B(v_2, v_4, v_6) \in \mathfrak{b}$ lie in the two-dimensional subspace U spanned by those two elements. Since $\dim \mathfrak{b} = 3$, there exists a third basis vector of $\mathfrak{b}$ of the form $B(u_2, u_4, u_6) + A(w_3, w_5, 0)$ for some $u_2, u_4 \neq 0$, and at least one of w_3 or w_5 is nonzero, for otherwise the element lies in the subspace U. But then the resulting matrix cannot be of rank two, as it is neither of type (a) nor of type (b) in Lemma 3.6. In fact, it is not of type (a) since $u_2 \neq 0$ and, for type (b), the entries u_3, u_5 are functions of u_2, u_4, which prohibits adding a nonzero term w_3 or w_5.

In conclusion, a Lie subalgebra with the properties required for $\mathfrak{b}$ does not exist in $\mathfrak{m}$. □

Lemma 3.9. *Let $\mathfrak{n}$ be a seven-dimensional two-step nilpotent Lie algebra with an invariant scalar product of index three. Then* $\operatorname{ad}(\mathfrak{n})$ *is not contained in* $\mathfrak{g}_{2(2)}$.

Proof. The invariance of $\langle \cdot, \cdot \rangle$ and two-step nilpotence of $\mathfrak{n}$ imply that $\mathfrak{z}(\mathfrak{n}) = \mathfrak{w} \oplus \mathfrak{j}$ and $[\mathfrak{n}, \mathfrak{n}] = [\mathfrak{j}^*, \mathfrak{j}^*] = \mathfrak{j}$. In particular, $\dim \operatorname{ad}(\mathfrak{n}) = \dim \mathfrak{j}^* = \dim \mathfrak{j}$. By assumption, $\mathfrak{j} \neq \mathbf{0}$ and hence $\dim \mathfrak{j} = 3$. In fact, if $\dim \mathfrak{j}^* \leq 2$, then $[\mathfrak{j}^*, \mathfrak{j}^*] = \mathbf{0}$ due to the invariance of $\langle \cdot, \cdot \rangle$, which is a contradiction. Also by invariance, $\operatorname{im} \operatorname{ad}(x) = x^\perp \cap \mathfrak{j}$ for all nonzero $x \in \mathfrak{j}^*$. So, $\operatorname{rk} \operatorname{ad}(x) = 2$ for all nonzero $x \in \mathfrak{j}^*$.

If $\operatorname{ad}(\mathfrak{n})$ is contained in $\mathfrak{g}_{2(2)}$, then it is a nilpotent triangular subalgebra and as such contained in a maximal nilpotent triangular subalgebra. This means that $\operatorname{ad}(\mathfrak{n})$ is conjugate to a three-dimensional abelian subalgebra of $\mathfrak{m}$ in (3.2) consisting of matrices of rank two (and zero). But, by Lemma 3.8, such a subalgebra cannot exist in $\mathfrak{m}$. □

Lemma 3.10. *Let $\mathfrak{n}$ be a seven-dimensional nilpotent but nonabelian Lie algebra with an invariant scalar product of index three. Then* $\operatorname{ad}(\mathfrak{n})$ *is not contained in* $\mathfrak{g}_{2(2)}$.

Proof. If $\operatorname{ad}(\mathfrak{n})$ is contained in $\mathfrak{g}_{2(2)}$, then it is a nilpotent triangular subalgebra and as such contained in a maximal nilpotent triangular subalgebra. This means that $\operatorname{ad}(\mathfrak{n})$ is conjugate to a subalgebra of the maximal nilpotent subalgebra $\mathfrak{m}$ of $\mathfrak{g}_{2(2)}$ given by (3.2). The possibilities for $\mathfrak{n}$ are covered by the examples in Section 3, up to some abelian factor. We use the notation from these examples.

Suppose first that $\mathfrak{n}$ is indecomposable. Then $\mathfrak{n}$ must be the Lie algebra $\mathfrak{n}_i$ from Example 3.3. Now $\operatorname{ad}_{\mathfrak{n}_i}(\mathfrak{w})$ is a three-dimensional abelian subalgebra of $\operatorname{ad}_{\mathfrak{n}_i}(\mathfrak{n}_i)$,

https://doi.org/10.1017/S1446788719000296 Published online by Cambridge University Press

generated by the elements $\mathrm{ad}(w_i)$, $i = 1, 2, 3$. From the commutator relations of $\mathfrak{n}_\iota$, it follows that the image of every nonzero $Q \in \mathrm{ad}_{\mathfrak{n}_\iota}(\mathfrak{w})$ is the span of z_1 and Qa_1. Hence, $\mathrm{rk}\, Q = 2$ for all nonzero Q. Now Lemma 3.8 tells us that such a subalgebra does not exist in $\mathfrak{m}$ (alternatively, this case could be excluded by comparison with Kath's classification result [7, Theorem 6.8]).

Suppose now that $\mathfrak{n}$ is decomposable, that is, an orthogonal direct sum $\mathfrak{g} = \mathfrak{n}_1 \times \mathfrak{n}_0$ of nonzero metric Lie algebras. We may assume that $\dim \mathfrak{n}_1 > \dim \mathfrak{n}_0$. Then $1 \le \dim \mathfrak{n}_0 \le 4$, so that $\mathfrak{n}_0$ is abelian by Lemma 3.2. If $\mathfrak{n}_1$ is not abelian itself, then $5 \le \dim \mathfrak{n}_1 \le 6$. This leaves $\mathfrak{n}_{\mathrm{II}}$ and $\mathfrak{n}_{\mathrm{III}}$ from Examples 3.4 and 3.5 as possibilities for $\mathfrak{n}_1$. Since $\mathfrak{n}_{\mathrm{II}} \times \mathbb{R}$ is two-step nilpotent, its adjoint representation cannot be contained in $\mathfrak{g}_{2(2)}$ by Lemma 3.9.

So, assume that $\mathfrak{n} = \mathfrak{n}_{\mathrm{III}} \times \mathbb{R}^2_1$. By the relations in Example 3.5, the algebra $\mathrm{ad}(\mathfrak{n})$ is three dimensional, spanned by $\mathrm{ad}(a_1)$, $\mathrm{ad}(a_2)$ and $\mathrm{ad}(w)$. Furthermore, it follows that the image of a linear combination $Q = \alpha_1 \mathrm{ad}(a_1) + \alpha_2 \mathrm{ad}(a_2) + \beta \mathrm{ad}(w)$ equals $\mathrm{span}\{\alpha_1 w - \beta\varepsilon z_1, \alpha_2 \varepsilon z_1 - \alpha_1 \varepsilon z_2\}$. So, $\mathrm{rk}\, Q = 2$ for all nonzero $Q \in \mathfrak{b}$. We can now apply Lemma 3.8 to conclude that $\mathrm{ad}(\mathfrak{n})$ is not contained in $\mathfrak{g}_{2(2)}$. □

4. Proofs of the main theorems

Proof (Theorem B). In this situation, Lemma 2.2 applies. This means that G is nilpotent, $\dim G = 7$ and $\mathrm{ad}(\mathfrak{g})$ is a subalgebra of $\mathfrak{g}_{2(2)}$. But, by Lemma 3.10, this is not possible unless $\mathfrak{g}$ and hence G is abelian. Since G acts effectively on $M = G/\Gamma$, the lattice Γ must be trivial, which implies that G itself is compact. This means that $G = M$ is a torus $\mathbb{R}^7_3/\mathbb{Z}^7$. □

Proof (Theorem A). By Lemma 2.1, the group G has dimension seven and $M = G/\Gamma$ for some uniform lattice Γ in G, and the metric $\mathfrak{g}_M$ on M is induced by a bi-invariant metric $\mathfrak{g}_G$ on G. Now we are in the situation of Theorem B, which concludes the proof. □

Acknowledgements

I would like to thank Thomas Leistner for helpful discussions of holonomy groups, and also the anonymous referee for many helpful suggestions that improved the readability of the paper.

References

[1] O. Baues, W. Globke and A. Zeghib, 'Isometry Lie algebras of indefinite homogeneous spaces of finite volume', *Proc. Lond. Math. Soc.* **119**(4) (2019), 1115–1148.
[2] A. L. Besse, *Einstein Manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 10 (Springer, Berlin, 1987).
[3] E. Bonan, 'Sur les variétés Riemanniennes à groupe d'holonomie G_2 ou Spin(7)', *C. R. Math. Acad. Sci. Paris* **262** (1966), 127–129.
[4] A. Fino and I. Luján, 'Torsion-free $G^*_{2(2)}$-structures with full holonomy on nilmanifolds', *Adv. Geom.* **15**(3) (2015), 381–392.

https://doi.org/10.1017/S1446788719000296 Published online by Cambridge University Press

[5] W. Globke and Y. Nikolayevsky, 'Compact pseudo-Riemannian homogeneous Einstein manifolds in low dimension', *Differential Geom. Appl.* **54**(Part B) (2017), 475–489.
[6] I. Kath, 'Nilpotent metric Lie algebras of small dimension', *J. Lie Theory* **17** (2007), 41–61.
[7] I. Kath, 'Indefinite symmetric spaces with $G_{2(2)}$-structure', *J. Lond. Math. Soc.* **87**(2) (2013), 853–876.
[8] H. V. Lê and M. Munir, 'Classification of compact homogeneous spaces with invariant G_2-structures', *Adv. Geom.* **12**(2) (2012), 303–328.
[9] T. Leistner, P. Nurowski and K. Sagerschnig, 'New relations between G_2-geometries in dimensions 5 and 7', *Int. J. Math.* **28**(13) (2017), 1750094.

WOLFGANG GLOBKE, Faculty of Mathematics,
Oskar-Morgenstern-Platz 1, Universität Wien,
1090 Vienna, Austria
e-mail: wolfgang.globke@univie.ac.at

https://doi.org/10.1017/S1446788719000296 Published online by Cambridge University Press