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Tracking control design for underactuated constrained systems

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SUMMARY

An extension of a model-based tracking control strategy to underactuated systems is presented. Originally, it is designed for actuated systems that can perform tasks specified by equations of algebraic or differential constraints referred to as programmed, which may be nonintegrable. It is demonstrated that for both actuated and underactuated nonholonomically constrained systems one tracking control strategy can be designed. Systems we consider are nonholonomic because of constraints put on their motions as well as unactuated degrees of freedom. We detail an example, which illustrates the theory and demonstrates advantages of application of one tracking control strategy for actuated and underactuated constrained systems.

KEYWORDS: Tracking control; Underactuated systems; Nonholonomic models.

I. INTRODUCTION

In reference [1] we presented a development of a model-based tracking strategy referred to as a model reference tracking control strategy for programmed motion. It is based on two dynamic models. The first is a reference model, which is a dynamic model derived by the generalized programmed motion equations (GPME)[2] and it includes all tasks and motion specifications via the programmed and material constraint equations. The second is a dynamic control model, which is equivalent to models control theory currently uses. The difference is that it is derived with the aid of the GPME, which yield equations in the reduced state form. This tracking strategy is developed for fully actuated systems. However, there are many systems designed as underactuated. Dynamic models for them are classified as second order nonholonomic system models, since the equation for an unactuated degree of freedom is regarded as a second order nonholonomic constraint. It is proved in reference [3] that the equation related to the unactuated degree of freedom is nonintegrable, in general. Underactuated systems are challenging from dynamic and control design perspectives. Reasons are a drift term that is present in a state-space control formulation for them and a second order nonholonomic constraint equation. Based on results in reference [1] we extend the programmed motion tracking strategy to underactuated systems, which can also be subjected to programmed constraints.

II. TRACKING CONTROL FOR UNDERACTUATED SYSTEMS

To extend the programmed motion tracking strategy to underactuated systems, we reformulate the dynamic control model and the reference model remains unchanged[1,2].

Theorem: The dynamic control model

$$\begin{gathered} M(q)\ddot{q} + C(q,\dot{q})\dot{q} + D(q) = T(q)\tau, \\ B_1(q)\dot{q} = 0 \end{gathered} \tag{1}$$

developed by the GPME[1,2] for fully actuated systems can be a unified dynamic control model for both actuated and underactuated systems subjected to material and programmed constraints.

Proof: We present (1) in the form

$$\begin{gathered} M(q)\ddot{q}_1 + C(q,\dot{q}) = T(q)\tau, \\ \ddot{q}_2 = G(q)\ddot{q}_1 + \dot{G}(q)\dot{q}_1. \end{gathered} \tag{2}$$

In (2) the matrix $G(q) = -B_{12}^{-1}(q)B_{11}(q)$. Since the constraint equation $B_1(q)\dot{q} = 0$ must be satisfied, it is enough to choose a control law such that $q_1 \to q_p$ for $q_1 \in R^{n-m}$, $q_2 \in R^m$, where "p" stands for "programmed" and the number of degrees of freedom is $k = (n-m)$. Assume that (2) concern an underactuated system and we have s, $s < n-m$, inputs. To reflect the fewer number of inputs than degrees of freedom, we rearrange (2) as follows. We partition q_1 as $q_1 = (q_{1a}, q_{1f})$, where $q_{1a} \in R^s$, $q_{1f} \in R^{k-s}$, and "a" and "f" stand for actuated and "free". Then, we write (2) as

$$\tilde{M}(q)\ddot{q} + \tilde{C}(q,\dot{q}) = \tilde{T}(q)\tau \tag{3}$$

where $q = (q_{1a}, q_{1f}, q_2)$, $\tilde{M}$ is an $(n \times n)$ inertia matrix, $\tilde{C}$ and $\tilde{T}$ are

$$\tilde{M}(q) = \begin{bmatrix} M_{11}(q) & M_{12}(q) & 0_{s\times k} \\ M_{21}(q) & M_{22}(q) & 0_{(m-s)\times k} \\ G_{3a}(q) & G_{3f}(q) & I_{k\times k} \end{bmatrix},$$

$$\tilde{C}(q,\dot{q}) = \begin{bmatrix} C_1(q,\dot{q}) \\ C_2(q,\dot{q}) \\ 0 \end{bmatrix}, \quad \tilde{T}(q) = \begin{bmatrix} T(q) \\ 0 \\ 0 \end{bmatrix}.$$

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The last row in $\tilde{M}$ consists of terms that result from partition of $G(q)$, i.e.

$$G(q) = \begin{bmatrix} I_{s\times s} & 0 \\ 0 & I_{(m-s)\times(m-s)} \\ G_{3a} & G_{3f} \end{bmatrix},$$

G_{3a} is $(k \times s)$, and G_{3f} is a $k \times (m-s)$ matrix.

The vector $\tilde{C}(q, \dot{q})$ components are $C_1 \in R^s$, $C_2 \in R^{m-s}$, and $T(q)$ is the $(s \times s)$ matrix. Now, assign $q_{1a} = q_1$, $q_1 \in R^s$ and $(q_{1f}, q_2) = q_2$, $q_2 \in R^{n-s}$. Equations (3) with $q = (q_1, q_2)$ are equivalent to the control model in [4]. Indeed, the second equation in (3) can be solved for $\ddot{q}_{1f}$ as $\ddot{q}_{1f} = -M_{22}^{-1}(q)[M_{21}(q)\ddot{q}_{1a} + C_2(q, \dot{q})]$ and substituted into the first and third equations in (3) to obtain

$$\begin{aligned} M^*(q)\ddot{q}_1 + C^*(q, \dot{q}) &= T(q)\tau, \\ M_{33}(q)\ddot{q}_2 + \tilde{M}^*(q)\ddot{q}_1 + \tilde{C}^*(q, \dot{q}) &= 0. \end{aligned} \tag{4}$$

In (4) matrices $M^*(q)$, $C^*(q, \dot{q})$, $\tilde{M}^*(q)$ and $\tilde{C}^*(q, \dot{q})$ are defined as

$$\begin{aligned} M^*(q) &= M_{11}(q) - M_{12}(q)M_{22}^{-1}(q)M_{21}(q), \\ C^*(q, \dot{q}) &= C_1(q, \dot{q}) - M_{12}(q)M_{22}^{-1}(q)C_2(q, \dot{q}), \end{aligned} \tag{5}$$

$$\begin{aligned} \tilde{M}^*(q) &= M_{31}(q) - M_{32}(q)M_{22}^{-1}(q)M_{21}(q), \\ \tilde{C}^*(q, \dot{q}) &= C_3(q, \dot{q}) - M_{31}(q)M_{22}^{-1}(q)C_2(q, \dot{q}). \end{aligned} \tag{6}$$

Now, we can apply the partial feedback linearizing controller

$$\tau = T^{-1}(q)[M^*(q)u + C^*(q, \dot{q})], \tag{7}$$

which transforms (4) to

$$\begin{aligned} \ddot{q}_1 &= u, \\ \ddot{q}_2 &= R(q)\ddot{q}_1 + H(q, \dot{q}), \end{aligned} \tag{8}$$

and $R(q) = -M_{33}^{-1}(q)\tilde{M}^*(q)$, $H(q) = -M_{33}^{-1}(q)\tilde{C}^*(q, \dot{q})$.

The control formulation (8) is equivalent to the one developed in reference [4].

Our result applies to a large class of underactuated systems, which can be subjected to material and programmed constraints.

III. EXAMPLE

Consider the underactuated system (8), which is a two-link planar manipulator. Only its first joint is actuated. The system motion is described by Θ_1, Θ_2. A control objective is to move its end-effector according to a programmed motion, which is motion along a trajectory such that the change of its curvature is specified by some function. In joint coordinates this constraint has the form

$$\ddot{\Theta}_2 = F_1 - F_2\ddot{\Theta}_1. \tag{9}$$

All details about (9), the constrained dynamics and programmed motion tracking for this system when it is fully actuated can be found in references [1,5]. We examine here the same system for several reasons. Firstly, it often illustrates control theoretical developments and it can be

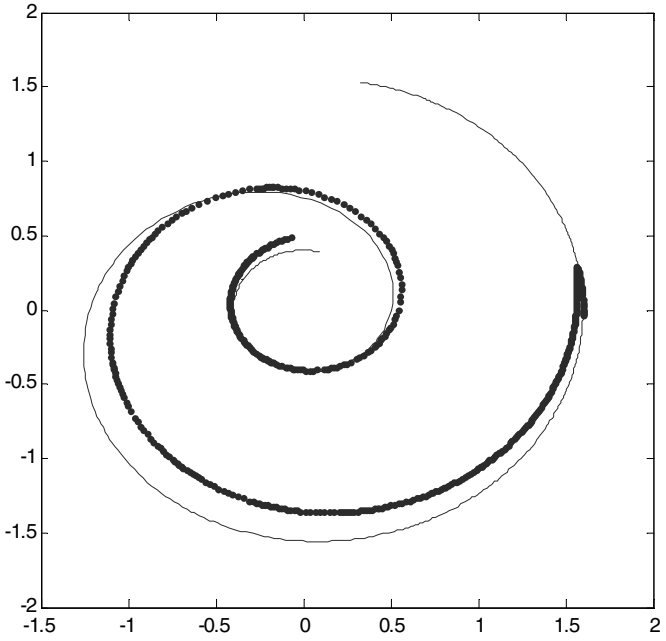


Fig. 1. Programmed motion tracking by the PD controller.

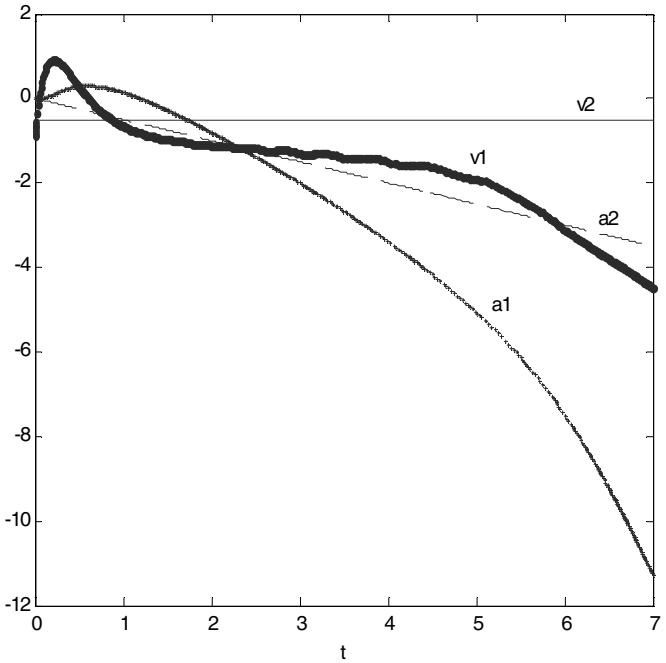


Fig. 2. Controlled joint positions ($a1$, $a2$) and velocities ($v1$, $v2$) vs time.

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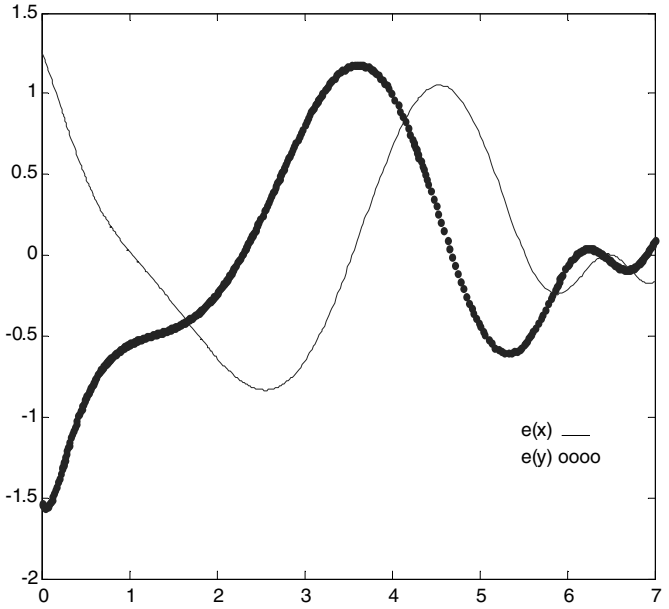


Fig. 3. Joint position errors during tracking vs time.

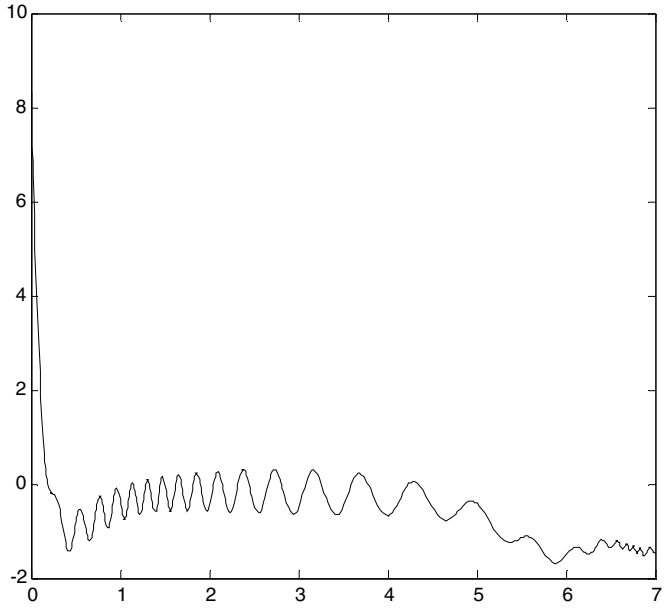


Fig. 4. Control torque based on the PD controller applied to the first joint vs time.

verified that tracking controller designs based on classical dynamics and control results fail for the systems with the constraints (9). Secondly, we obtained the underactuated model by simply removing one of the control torques. In a real situation one control torque may fail and if a controller for an underactuated system is available, we can complete a task and recover from the failure. Also, the constraint (9) may mimic tasks such as writing, painting or scribing. Finally, this underactuated manipulator is completely second order nonholonomic.[3,4] Figures 1–4 present simulation results for the programmed motion tracking.

IV. CONCLUSIONS

We presented the extension of the programmed motion tracking control strategy to underactuated systems that can be subjected to other constraints, for example, programmed. The novelty of the approach consists of the application of the GPME method to develop a unified dynamic control model in the reduced state form and one tracking control design for both actuated and underactuated systems.

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