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SUSLIN TREE PRESERVATION AND CLUB ISOMORPHISMS

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Abstract. We construct a model of set theory in which there exists a Suslin tree and satisfies that any two normal Aronszajn trees, neither of which contains a Suslin subtree, are club isomorphic. We also show that if S is a free normal Suslin tree, then for any positive integer n there is a c.c.c. forcing extension in which S is n-free but all of its derived trees of dimension greater than n are special.

§1. Introduction. Baumgartner [3] proved that Martin's axiom implies that all Aronszajn trees are special. This consequence of Martin's axiom in turn implies Suslin's hypothesis, since any Suslin tree is a nonspecial Aronszajn tree. Later Abraham and Shelah [1] constructed a model in which there exists a Suslin tree and any Aronszajn tree which does not contain a Suslin subtree is special. Abraham and Shelah [1] also introduced the property that any two normal Aronszajn trees are club isomorphic, which implies that all Aronszajn trees are special, and proved its consistency from ZFC and that it follows from the proper forcing axiom.

In light of these results, a natural question is whether there is a variation of the property that all normal Aronszajn trees are club isomorphic which is consistent with the existence of a Suslin tree. In this article we answer this question by constructing a model in which there exists a Suslin tree and any two normal Aronszajn trees, neither of which contains a Suslin subtree, are club isomorphic. We also prove that this statement follows from Todorcevic's forcing axiom $\mathsf{PFA}(S)$.

The main method which we use is that of preserving a Suslin tree S after forcing something about an Aronszajn tree T which is in some sense not near it. More specifically, we are concerned with the relation that forcing with S below some element $x \in S$ adds an uncountable branch to the Aronszajn tree T, or equivalently, that there exists a strictly increasing and height preserving map from a club set of levels of S_x into T. It turns out that if there does not exist such a map, then we can force things about T while preserving S being Suslin.

In addition to our main result discussed above, we give another application of the idea of Suslin tree preservation to the topic of free trees. We show that if S is a free Suslin tree, then for any positive integer n there exists a c.c.c. forcing which forces that S is n-free, but all of the derived trees of S of dimension greater than n are special. This shows that in contrast to the property of homogeneity of Suslin trees, which is upwards absolute, freeness is highly malleable by forcing.

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§2. Preliminaries. We assume that the reader is familiar with Aronszajn and Suslin trees, as well as the basics of forcing and forcing axioms. In this section we go over our notation, review some standard background results, and prove some elementary lemmas which we need later.

All of the trees we discuss in this article have height ω_1. An ω_1-*tree* is a tree of height ω_1 which has countable levels. We write $\mathrm{ht}_T(x)$ for the height of an element x of a tree T, T_α for level α of T consisting of all x with $\mathrm{ht}_T(x) = \alpha$, $T \restriction \alpha = \bigcup\{T_\beta : \beta < \alpha\}$, and more generally, $T \restriction C$ for the set of x in T with $\mathrm{ht}_T(x) \in C$. If $\alpha < \mathrm{ht}_T(x)$ we write $x \restriction \alpha$ for the unique $y <_T x$ with height α. For incomparable elements x and y of T, $\Delta_T(x, y)$ is the order type of the set of z below both x and y.

A *branch* of a tree T is a maximal chain. If B is a branch and α is an ordinal less than its order type, $B(\alpha)$ denotes the element of B of height α. An *antichain* of T is a set of pairwise incomparable elements of T. A tree T is *normal* if it has a root, every element of T has at least two immediate successors, every element of T has some element above it at any higher level, and there is at most one upper bound to any chain of T whose order type is a limit ordinal. A *subtree* of T is any subset of T with the order inherited from T.

An *Aronszajn tree* is an ω_1-tree with no cofinal branch, and a *Suslin tree* is a tree with no uncountable chain or antichain. A tree T of height ω_1 is *special* if it has a *specializing function*, which is a function $f : T \to \omega$ such that $x <_T y$ implies $f(x) \neq f(y)$. Every special ω_1-tree is Aronszajn, and any Suslin tree is a nonspecial Aronszajn tree. A function $f : T \to U$ between trees is *strictly increasing* if $x <_T y$ implies $f(x) <_U f(y)$, and is *height preserving* if $\mathrm{ht}_T(x) = \mathrm{ht}_U(f(x))$ for all $x \in T$. For trees T and U of height ω_1, we say that T and U are *club isomorphic* if there exists a club $C \subseteq \omega_1$ such that $T \restriction C$ and $U \restriction C$ are isomorphic.

The next result is an essential tool for forcings involving Aronszajn trees.

THEOREM 2.1 (Baumgartner [3, Chapter 4]). *Suppose that T is an Aronszajn tree and $\{x_\alpha : \alpha < \omega_1\}$ is a collection of pairwise disjoint finite subsets of T. Then there exist $\alpha < \beta$ such that every element of x_α is incomparable in T with every element of x_β.*

In this article we are interested in preserving the Suslin property of a given Suslin tree after forcing. When iterating forcing, we only need to verify Suslin tree preservation at successor stages.

THEOREM 2.2 (Abraham and Shelah [2, Theorem 3.1], Miyamoto [6, Lemma 1.2]). *Let S be a Suslin tree. Then the property of a forcing poset being proper and forcing that S is Suslin is preserved by any countable support forcing iteration.*

We also want to preserve a Suslin tree after a finite support forcing iteration of c.c.c. forcings.

THEOREM 2.3. *Let S be a Suslin tree. Then the property of a forcing poset being c.c.c. and forcing that S is Suslin is preserved by any finite support forcing iteration.*

This theorem was known previously, but since we do not have a reference for a proof we provide a brief sketch.

PROOF (SKETCH). The result is immediate for iterations of length a successor ordinal, so let δ be a limit ordinal and suppose that we have a finite support forcing

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iteration $\langle \mathbb{P}_i, \dot{\mathbb{Q}}_j : i \leq \delta,\ j < \delta \rangle$ of c.c.c. forcings such that for all $i < \delta$, $\mathbb{P}_i$ forces that S is Suslin. Then $\mathbb{P}_\delta$ is c.c.c. Let $p \in \mathbb{P}_\delta$ and assume for a contradiction that

$$p \Vdash_\delta \text{“}\{\dot{x}_\alpha : \alpha < \omega_1\} \text{ is an uncountable antichain of } S\text{.”}$$

Then for each α we can choose $p_\alpha \leq p$ in $\mathbb{P}_\delta$ and $y_\alpha \in S$ such that $p_\alpha \Vdash_{\mathbb{P}_\delta}$ “$\dot{x}_\alpha = y_\alpha$.” By a standard Δ-system argument on the domains of the p_α's, find $\beta < \delta$ and an uncountable set $X \subseteq \omega_1$ such that for all $i < j$ in X, if $p_i \restriction \beta$ and $p_j \restriction \beta$ are compatible in $\mathbb{P}_\beta$, then p_i and p_j are compatible in $\mathbb{P}_\delta$.

For all $i \in X$, $p_i \restriction \beta \leq p \restriction \beta$ in $\mathbb{P}_\beta$. By a standard fact about c.c.c. forcings, there exists $u \leq p \restriction \beta$ in $\mathbb{P}_\beta$ such that

$$u \Vdash_\beta \text{“}\{i \in X : p_i \restriction \beta \in \dot{G}_{\mathbb{P}_\beta}\} \text{ is uncountable.”}$$

Let G be a generic filter on $\mathbb{P}_\beta$ which contains u. Let $Y := \{i \in X : p_i \restriction \beta \in G\}$, which is uncountable. Then for all $i < j$ in Y, $p_i \restriction \beta$ and $p_j \restriction \beta$ are in G and hence are compatible in $\mathbb{P}_\beta$. As $Y \subseteq X$, p_i and p_j are compatible in $\mathbb{P}_\delta$, which in turn easily implies that y_i and y_j are incomparable in S. Hence, in $V[G]$ the set $\{y_i : i \in Y\}$ is an uncountable antichain of S, which contradicts our assumption that S is Suslin in $V^{\mathbb{P}_\beta}$. ⊣

We sometimes consider a normal tree S as a forcing poset with the reversed order, which we also write as S. Then elements a and b of S are compatible in this forcing poset iff they are comparable in the tree S. So an antichain of the tree S is the same as an antichain of the forcing poset S. Hence, the tree S is Suslin iff the forcing poset S is c.c.c. If S is Suslin, then forcing with S yields a generic filter which is a cofinal branch of S.

Suppose that S is a Suslin tree. Then for any dense open set $D \subseteq S$ (in the forcing poset), there exists some $\alpha < \omega_1$ such that $S_\alpha \subseteq D$. Namely, pick a maximal antichain $A \subseteq D$. Since A is countable, we can fix $\alpha < \omega_1$ such that $A \subseteq S \restriction \alpha$. As A is maximal and D is open, it easily follows that $S_\alpha \subseteq D$.

Given finitely many ω_1-trees $T_0, \dots, T_{n-1}$, the product $T_0 \times \cdots \times T_{n-1}$ ordered componentwise by $<_T$ is a strict partial order. The suborder $T_0 \otimes \cdots \otimes T_{n-1}$ consists of all n-tuples in the product whose elements all have the same height. Since this suborder is dense in the product assuming the trees are normal, the suborder is c.c.c. iff the product is c.c.c. In particular, for $n > 1$, the tree $T_0 \otimes \cdots \otimes T_{n-1}$ is Suslin iff the tree $T_0 \otimes \cdots \otimes T_{n-2}$ is Suslin and $\Vdash_{T_0 \otimes \cdots \otimes T_{n-2}}$ “T_{n-1} is Suslin.” This follows from the basic fact about c.c.c. forcings that $\mathbb{P} \times \mathbb{Q}$ is c.c.c. iff $\mathbb{P}$ is c.c.c. and $\Vdash_{\mathbb{P}}$ “$\mathbb{Q}$ is c.c.c.” Note that the height of a tuple in the tree $T_0 \otimes \cdots \otimes T_{n-1}$ is equal to the height of the elements of that tuple in the trees $T_0, \dots, T_{n-1}$. It follows that $T_0 \otimes \cdots \otimes T_{n-2} \otimes T_{n-1}$ is isomorphic to $(T_0 \otimes \cdots \otimes T_{n-2}) \otimes T_{n-1}$.

Let T be an ω_1-tree. For every $a \in T$, define T_a as the subtree consisting of all $b \in T$ such that either $b \leq_T a$ or $a \leq_T b$. For any positive integer n and n-tuple $\vec{a} = (a_0, \dots, a_{n-1})$ of elements of T of the same height, define $T_{\vec{a}} := T_{a_0} \otimes \cdots \otimes T_{a_{n-1}}$, which is called a *derived tree of T of dimension n*. The tree T is said to be *n-free* if all of its derived trees of dimension n are Suslin, and is *free* if it is n-free for all positive integers n. Jensen [4] proved that $\diamondsuit$ implies the existence of a free tree.

In the remainder of this section we prove some easy facts about products of trees which will be helpful to refer to later.

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LEMMA 2.4. *Let T be an ω_1-tree. Then for any derived tree $U = T_{a_0} \otimes \cdots \otimes T_{a_{n-1}}$ of T, U is Aronszajn iff for some $i < n$, T_{a_i} is Aronszajn.*

PROOF. Suppose that U is not Aronszajn and let B be a cofinal branch of U. For each $\alpha < \omega_1$, write $B(\alpha) = (B_0(\alpha), \dots, B_{n-1}(\alpha))$. Then for each $i < n$, B_i is a cofinal branch of T_{a_i}, so T_{a_i} is not Aronszajn. Conversely, suppose that for each $i < n$, B_i is a cofinal branch of T_{a_i}. Then B defined by $B(\alpha) := (B_0(\alpha), \dots, B_{n-1}(\alpha))$ is a cofinal branch of U. ⊣

LEMMA 2.5. *Suppose that S and T are Suslin trees and $S \otimes T$ is special. Then $\Vdash_S$ "T is special."*

PROOF. Let $f : S \otimes T \to \omega$ be a specializing function. In V^S, let B be a cofinal branch of S, and we define a specializing function $g : T \to \omega$. For any $x \in T$, define $g(x) := f(B(\mathrm{ht}_T(x)), x)$. If $x <_T y$, then $(B(\mathrm{ht}_T(x)), x) <_{S\otimes T} (B(\mathrm{ht}_T(y)), y)$, so $g(x) = f(B(\mathrm{ht}_T(x)), x)) \neq f(B(\mathrm{ht}_T(y)), y) = g(y)$. ⊣

LEMMA 2.6. *Let S be an ω_1-tree and $b_0, \dots, b_{n-1}$ distinct elements of T of the same height α. Suppose that $\alpha < \xi < \omega_1$. If for all $c_0, \dots, c_{n-1}$ above $b_0, \dots, b_{n-1}$ respectively of heights ξ, $S_{c_0} \otimes \cdots \otimes S_{c_{n-1}}$ is Suslin, then $S_{b_0} \otimes \cdots \otimes S_{b_{n-1}}$ is Suslin.*

PROOF. Suppose that $\{(d_{i,0}, \dots, d_{i,n-1}) : i < \omega_1\}$ is an uncountable antichain of $S_{b_0} \otimes \cdots \otimes S_{b_{n-1}}$. Since level ξ of S is countable, we can find an uncountable set $X \subseteq \omega_1$ and some $c_0, \dots, c_{n-1}$ of height ξ such that for all $i \in X$, $d_{i,0} \restriction \xi = c_0, \dots, d_{i,n-1} \restriction \xi = c_{n-1}$. Then $\{(d_{i,0}, \dots, d_{i,n-1}) : i \in X\}$ is an uncountable antichain of $S_{c_0} \otimes \cdots \otimes S_{c_{n-1}}$. ⊣

LEMMA 2.7. *Let T be an ω_1-tree, and $U = T_{a_0} \otimes \cdots \otimes T_{a_{n-1}}$ a derived tree of T. Suppose that $m \leq n$, $i_0 < \cdots < i_{m-1} \leq n - 1$, and $W := T_{a_{i_0}} \otimes \cdots \otimes T_{a_{i_{m-1}}}$ is special. Then U is special.*

PROOF. Let $f : W \to \omega$ be a specializing function. Define $g : U \to \omega$ by letting $g(c_0, \dots, c_{n-1}) := f(c_{i_0}, \dots, c_{i_{m-1}})$. Then easily g is a specializing function for U, so U is special. ⊣

§3. Preserving a Suslin tree. In this section we discuss the topic of forcing a property of some Aronszajn tree while preserving another tree being Suslin. Specifically, we consider forcings to make an Aronszajn tree special or to make two Aronszajn trees club isomorphic. This is not always possible; if there exists a strictly increasing function from a Suslin tree S into an Aronszajn tree T, then specializing T also specializes S.

The relation between a Suslin tree S and an Aronszajn tree T which we are interested in is whether adding a cofinal branch to S also adds a cofinal branch to T. We use the following characterization of this relation.

PROPOSITION 3.1 (Lindström [5]). *Let S be a normal Suslin tree, and let T be a normal Aronszajn tree. Then $\Vdash_S$ "T has a cofinal branch" iff there exists a club $C \subseteq \omega_1$ and a strictly increasing and height preserving function $f : S \restriction C \to T \restriction C$.*

In particular, for all $x \in S$, $x \Vdash_S$ "T has a cofinal branch" iff there exists a club $C \subseteq \omega_1$ and a strictly increasing and height preserving function

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$f : S_x \restriction C \to T \restriction C$. It is easy to check that in this case the range $f[S_x \restriction C]$ has no uncountable antichain, and hence T contains a Suslin subtree.

In Section 4.1 of [2], Abraham and Shelah proved that if S is a Suslin tree and T is an Aronszajn tree, and S forces that T is Aronszajn, then there is a forcing poset $\mathbb{P}$ which specializes T while preserving S being Suslin. The forcing $\mathbb{P}$ consists of countably infinite conditions and does not add new countable sets of ordinals. It is natural to ask whether the same property is true for Baumgartner's c.c.c. forcing for making T special using finite conditions.

DEFINITION 3.2 (Baumgartner [3]). Let T be a tree of height ω_1. Define $\mathbb{Q}(T)$ to be the forcing poset whose conditions are finite functions $p : \mathrm{dom}(p) \subseteq T \to \omega$ such that $x <_T y$ in $\mathrm{dom}(p)$ implies $p(x) \neq p(y)$, ordered by reverse inclusion.

THEOREM 3.3. *Let T be a tree of height ω_1. Then T has no cofinal branch iff $\mathbb{Q}(T)$ is c.c.c.*

PROOF. See Chapter 4 of [3] for the forward direction. Conversely, suppose that B is a cofinal branch of T. For each $\alpha < \omega_1$ define $p_\alpha := \{(B(\alpha), 0)\}$. Then $\{p_\alpha : \alpha < \omega_1\}$ is an uncountable antichain of $\mathbb{Q}(T)$. ⊣

So assuming that T is an Aronszajn tree, $\mathbb{Q}(T)$ is c.c.c. and forces that T is special.

THEOREM 3.4. *Let S be a Suslin tree and T an Aronszajn tree. Then*

$$\Vdash_S \text{“}T \text{ is Aronszajn”} \iff \Vdash_{\mathbb{Q}(T)} \text{“}S \text{ is Suslin.”}$$

PROOF. We use the fact that for c.c.c. forcings $\mathbb{P}$ and $\mathbb{Q}$, $\mathbb{P} \times \mathbb{Q}$ is c.c.c. iff $\Vdash_{\mathbb{P}}$ “$\mathbb{Q}$ is c.c.c.” Recall that the tree S is Suslin iff the forcing poset S (with the reversed order) is c.c.c. So both forcings S and $\mathbb{Q}(T)$ are c.c.c. Therefore,

$$\Vdash_{\mathbb{Q}(T)} \text{“}S \text{ is Suslin”} \iff \Vdash_{\mathbb{Q}(T)} \text{“}S \text{ is c.c.c.”} \iff$$
$$\mathbb{Q}(T) \times S \text{ is c.c.c.} \iff S \times \mathbb{Q}(T) \text{ is c.c.c.} \iff \Vdash_S \text{“}\mathbb{Q}(T) \text{ is c.c.c.”}$$

Now since $\mathbb{Q}(T)$ is defined by finite conditions, by absoluteness $\mathbb{Q}(T)^V = \mathbb{Q}(T)^{V^S}$. Thus, $\Vdash_S$ “$\mathbb{Q}(T)$ is c.c.c.” iff $\Vdash_S$ “$\mathbb{Q}(T)^{V^S}$ is c.c.c.,” which by Theorem 3.3 is equivalent to $\Vdash_S$ “T is Aronszajn.” ⊣

Now we move on to the topic of making two normal Aronszajn trees club isomorphic while preserving some Suslin tree. We begin by reviewing the definition of a forcing poset $\mathbb{Q}(T, U)$ for making T and U club isomorphic. This forcing is due to Abraham and Shelah; their definition is slightly different but their poset is isomorphic to a dense subset of $\mathbb{Q}(T, U)$. See Section 5 of [1] for their definition and the proof of Theorem 3.6.

DEFINITION 3.5. Let T and U be normal Aronszajn trees. Define the forcing poset $\mathbb{Q}(T, U)$ to consist of all pairs (x, f), where x is a finite set of countable limit ordinals, f is an injective function whose domain is a finite downwards closed subset of $T \restriction x$ mapping into U, and f is strictly increasing and height preserving. The ordering of $\mathbb{Q}(T, U)$ is defined by $(y, g) \leq (x, f)$ if $x \subseteq y$ and $f \subseteq g$.

THEOREM 3.6. *For any normal Aronszajn trees T and U, the forcing poset $\mathbb{Q}(T, U)$ is proper.*

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In Theorem 3.10, we prove that if S is a normal Suslin tree, T and U are normal Aronszajn trees, and forcing with S does not add an uncountable branch to either T or U, then forcing with $\mathbb{Q}(T, U)$ preserves S. The proof relies on an analysis about compatibility of conditions in $\mathbb{Q}(T, U)$.

LEMMA 3.7. *Let T and U be normal Aronszajn trees and (x, f) and (y, g) conditions in $\mathbb{Q}(T, U)$. Suppose that $\alpha < \beta < \omega_1$ are limit ordinals and the following statements hold*:

(1) $\alpha \in x$ *and* $\beta \in y$.
(2) $x \subseteq \beta$ *and* $x \cap \alpha = y \cap \beta$.
(3) $f \restriction (T \restriction \alpha) = g \restriction (T \restriction \beta)$.
(4) *For all* $a, b \in \mathrm{dom}(g) \cap T_\beta$, *the ordinals* $\Delta_T(a, b)$ *and* $\Delta_U(g(a), g(b))$ *are less than* α.
(5) *Every member of* $\mathrm{dom}(f) \setminus (T \restriction \alpha)$ *is incomparable in T with every member of* $\mathrm{dom}(g) \setminus (T \restriction \beta)$, *and every member of* $\mathrm{ran}(f) \setminus (U \restriction \alpha)$ *is incomparable in U with every member of* $\mathrm{ran}(g) \setminus (U \restriction \beta)$.

Then (x, f) and (y, g) are compatible in $\mathbb{Q}(T, U)$.

PROOF. Using (5), it is easy to check that the pair $(x \cup y, f \cup g)$ satisfies all of the requirements of being a condition except that the domain of the function $f \cup g$ is not necessarily downwards closed in $T \restriction (x \cup y)$. So we extend $f \cup g$ to a function h whose domain is downwards closed in $T \restriction (x \cup y)$ and then verify that $(x \cup y, h)$ is a condition. Such an extension h is obtained by adding to the domain of $f \cup g$ all elements of T of the form $a \restriction \gamma$, where $a \in \mathrm{dom}(g) \cap T_\beta$ and $\gamma \in x \setminus \alpha$, and defining $h(a \restriction \gamma) := g(a) \restriction \gamma$. Note that by (4), for any new element c of height γ there exists a *unique* element $a \in \mathrm{dom}(g) \cap T_\beta$ such that $c = a \restriction \gamma$.

Obviously h is height preserving and its domain is downwards closed in $T \restriction (x \cup y)$. If h is not injective, then there are distinct c and d of the same height $\gamma \in x \setminus \alpha$, at least one of which is new, such that $h(c) = h(d)$. Suppose that c and d are both new. Then there are distinct a and b in $\mathrm{dom}(g) \cap T_\beta$ such that $c = a \restriction \gamma$ and $d = b \restriction \gamma$. But then $h(c) = h(d) \leq_U g(a), g(b)$, which contradicts that $\Delta_U(g(a), g(b)) < \alpha$ by (4). If just one of them is new, then we may assume $c = a \restriction \gamma$ where $a \in \mathrm{dom}(g) \cap T_\beta$ and $d \in \mathrm{dom}(f)$. Then $f(d) = h(d) = h(c) = g(a) \restriction \gamma <_U g(a)$, contradicting (5).

It remains to prove that for all $c, d \in \mathrm{dom}(h)$, $c <_T d$ implies $h(c) <_U h(d)$. It suffices to verify this in the case where at least one of c or d is new.

Case 1: $\mathrm{ht}_T(c) < \alpha$ and $d = a \restriction \gamma$ is new. Note $c \in \mathrm{dom}(g)$. If $c <_T d$ then $c <_T a$, hence $g(c) <_U g(a)$. Therefore, $h(c) = g(c) <_U g(a) \restriction \gamma = h(a \restriction \gamma) = h(d)$.

Case 2: $c \in \mathrm{dom}(f) \setminus (T \restriction \alpha)$ and $d = a \restriction \gamma$ is new. Then c cannot be below d in T, for otherwise $c <_T a$ which contradicts (5).

Case 3: $c = a \restriction \gamma$ is new and $d \in \mathrm{dom}(f) \setminus \alpha$. If $c <_T d$, then $a \restriction \alpha = c \restriction \alpha <_T d$, and hence $a \restriction \alpha = d \restriction \alpha$. But $\mathrm{dom}(f)$ is downwards closed in $T \restriction x$ and $\alpha \in x$, so $d \restriction \alpha$ is in $\mathrm{dom}(f)$. then $d \restriction \alpha = a \restriction \alpha <_T a$, which contradicts (5).

Case 4: $c = a \restriction \gamma$ and $d = b \restriction \xi$ are both new. Assume $c <_T d$. Then $c <_T a, b$. If $a \neq b$, then $\Delta_T(a, b) < \alpha \leq \gamma = \mathrm{ht}_T(c)$, which contradicts that $c <_T a, b$. So $a = b$. Therefore, $h(c) = g(a) \restriction \gamma <_U g(a) \restriction \xi = h(d)$.

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Case 5: $c = a \restriction \gamma$ is new and $d \in \mathrm{dom}(g) \setminus \beta$. Assume $c <_T d$. Then $a \leq_T d$, for otherwise by (4) $\Delta_T(a,d) = \Delta_T(a, d \restriction \beta) < \alpha \leq \mathrm{ht}_T(c)$, which contradicts that $c <_T a, d$. Hence, $h(c) = g(a) \restriction \gamma <_U g(a) \leq_U g(d) = h(d)$. ⊣

LEMMA 3.8. *Suppose that T and U are normal Aronszajn trees. Let $Y \subseteq \omega_1$ be a stationary set of limit ordinals. Assume that $\{(x_\alpha, f_\alpha) : \alpha \in Y\}$ is a set of conditions in $\mathbb{Q}(T,U)$ such that for all $\alpha \in Y$, $\alpha \in x_\alpha$. Then there exists $\alpha < \beta$ in Y such that (x_α, f_α) and (x_β, f_β) are compatible.*

PROOF. By a straightforward pressing down argument, we can find a stationary set $Y_0 \subseteq Y$, a function f, a set x, and an ordinal $\gamma < \omega_1$ less than $\min(Y_0)$ such that for all $\alpha \in Y_0$:

(1) $x_\alpha \cap \alpha = x$.
(2) $f_\alpha \restriction (T \restriction \alpha) = f$.
(3) For all distinct a and b in $\mathrm{dom}(f_\alpha) \cap T_\alpha$, both of the ordinals $\Delta_T(a,b)$ and $\Delta_U(f_\alpha(a), f_\alpha(b))$ are less than γ.
(4) For all $\beta \in Y_0$ larger than α, $x_\alpha \subseteq \beta$.

Now applying Theorem 2.1 to the disjoint union of the trees T and U, we can find $\alpha < \beta$ in Y_0 such that every member of $\mathrm{dom}(f_\alpha) \setminus (T \restriction \alpha)$ is incomparable in T with every member of $\mathrm{dom}(f_\beta) \setminus (T \restriction \beta)$, and every member of $\mathrm{ran}(f_\alpha) \setminus (U \restriction \alpha)$ is incomparable in U with every member of $\mathrm{ran}(f_\beta) \setminus (U \restriction \beta)$. By Lemma 3.7, the conditions (x_α, f_α) and (x_β, f_β) are compatible. ⊣

We need one more general result about Suslin trees.

LEMMA 3.9. *Let S be a Suslin tree. Consider $\{b_\alpha : \alpha \in Z\} \subseteq S$, where $Z \subseteq \omega_1$ is stationary. Then there exists some $a \in S$ such that for all $d >_S a$, the set $Z_d := \{\alpha \in Z : d \leq_S b_\alpha\}$ is stationary.*

PROOF. Suppose not. Then for all $a \in S$ we can fix $d_a >_S a$ and a club $C_a \subseteq \omega_1$ such that $C_a \cap Z_{d_a} = \emptyset$. Now the set $\{d_a : a \in S\}$ is obviously dense, so its upwards closure is dense open. Since S is Suslin, we can fix some $\gamma < \omega_1$ such that for all $y \in S_\gamma$, there exists some $a \in S$ such that $d_a \leq_S y$.

Let $D := \bigcap\{C_a : a \in S \restriction \gamma\}$, which is a club since $S \restriction \gamma$ is countable. As Z is stationary, $D \cap Z$ is stationary. So we can fix some $\alpha \in D \cap Z$ such that b_α has height greater than or equal to γ. Let $y := b_\alpha \restriction \gamma$. By the choice of γ, there exists some $a \in S$ such that $d_a \leq_S y$. Then $a \in S \restriction \gamma$, so $D \subseteq C_a$, and therefore $\alpha \in C_a$. But $d_a \leq_S y \leq_S b_\alpha$, which means that $\alpha \in Z_{d_a}$. So $\alpha \in C_a \cap Z_{d_a} = \emptyset$, which is a contradiction. ⊣

THEOREM 3.10. *Suppose that S is a normal Suslin tree. Let T and U be normal Aronszajn trees such that*

$$\Vdash_S \text{“}T \text{ and } U \text{ are Aronszajn.”}$$

Then $\Vdash_{\mathbb{Q}(T,U)}$ “S is Suslin.”

PROOF. We prove the contrapositive. Assume that there is a condition $p \in \mathbb{Q}(T,U)$ such that

$$p \Vdash_{\mathbb{Q}(T,U)} \text{“}\dot{A} = \{\dot{a}_\alpha : \alpha < \omega_1\} \text{ is an uncountable antichain of } S.\text{”}$$

We will find some $a \in S$ which forces in S that either T or U is not Aronszajn.

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Write $p = (x, f)$ and let Z be the set of limit ordinals in ω_1 above $\max(x)$. For each $\alpha \in Z$, let $p_\alpha = (x \cup \{\alpha\}, f)$, which is clearly a condition below p. Extend each p_α to some $q_\alpha = (x_\alpha, f_\alpha)$ which forces, for some $b_\alpha \in S$, that $\dot{a}_\alpha$ equals b_α. This gives us a family of conditions $\{(x_\alpha, f_\alpha) : \alpha \in Z\}$ satisfying that for all $\alpha \in Z$, $\alpha \in x_\alpha$.

Consider any $\alpha < \beta$ in Z and suppose that q_α and q_β are compatible in $\mathbb{Q}(T, U)$. Fix $r \leq q_\alpha, q_\beta$. Then r forces that b_α and b_β are both in the antichain $\dot{A}$ and hence are incomparable in S. But then b_α and b_β really are incomparable in S.

Applying Lemma 3.9 to the collection $\{b_\alpha : \alpha \in Z\}$, we can find some $a \in S$ such that for all $d >_S a$, the set

$$Z_d := \{\alpha \in Z : d \leq_S b_\alpha\}$$

is stationary. We claim that

$$a \Vdash_S \text{“}\{\alpha \in Z : b_\alpha \in \dot{G}_S\} \text{ is stationary.”}$$

If not, then there exists some $d >_S a$ and an S-name $\dot{D}$ for a club subset of ω_1 such that

$$d \Vdash_S \text{“for all } \alpha \in \dot{D} \cap Z, b_\alpha \notin \dot{G}_S\text{.”}$$

Since S is c.c.c., we can find a club $E \subseteq \omega_1$ such that d forces that $E \subseteq \dot{D}$. Now Z_d is stationary, so we can fix some $\alpha \in Z_d \cap E$. Then d forces that $\alpha \in \dot{D}$ and hence that $b_\alpha \notin \dot{G}_S$. On the other hand, $\alpha \in Z_d$ so $d \leq_S b_\alpha$. But then b_α extends d in the forcing S, so b_α forces that $b_\alpha \notin \dot{G}_S$, which is impossible. This completes the proof of the claim.

Let G be a generic filter on S such that $a \in G$. We will prove that in $V[G]$, either T or U is not Aronszajn. Suppose for a contradiction that both T and U are Aronszajn in $V[G]$. Note that the definition of $\mathbb{Q}(T, U)$ is absolute between V and $V[G]$ due to the finiteness of the conditions. That is, $\mathbb{Q}(T, U)^V = \mathbb{Q}(T, U)^{V[G]}$. In $V[G]$, define $Y := \{\alpha \in Z : b_\alpha \in G\}$. By the claim, Y is stationary. Moreover, the collection $\{(x_\alpha, f_\alpha) : \alpha \in Y\}$ is a subset of $\mathbb{Q}(T, U)^{V[G]}$ which satisfies that for all $\alpha \in Y$, $\alpha \in x_\alpha$.

Since T and U are normal Aronszajn trees in $V[G]$, we can apply Lemma 3.8 in $V[G]$ to find some $\alpha < \beta$ in Y such that $q_\alpha = (x_\alpha, f_\alpha)$ and $q_\beta = (x_\beta, f_\beta)$ are compatible in $\mathbb{Q}(T, U)^{V[G]}$. By absoluteness, q_α and q_β are compatible in $\mathbb{Q}(T, U)$ in V. As observed above, the compatibility of q_α and q_β in $\mathbb{Q}(T, U)$ implies that b_α and b_β are incomparable in S. But b_α and b_β are both in G, so they are comparable in S. This contradiction completes the proof that either T or U is not an Aronszajn tree in $V[G]$. ⊣

§4. Consistency results. In this section we will apply the theorems of the previous section to prove some consistency results concerning Suslin trees. In our first result, we construct a model in which there exists a Suslin tree and any two normal Aronszajn trees, neither of which contains a Suslin subtree, are club isomorphic. In the second result, we prove that if S is a free Suslin tree, then for any positive integer n there exists a c.c.c. forcing poset which forces that S is n-free but any derived tree of dimension $n + 1$ is special.

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Previously, Abraham and Shelah proved that it is consistent that there exists a Suslin tree and any Aronszajn tree either contains a Suslin tree or is special (see Section 4 of [1]). We strengthen their result by constructing a model with a Suslin tree in which there exists an essentially unique Aronszajn tree with no Suslin subtree (which of course must be special).

THEOREM 4.1. *Suppose that S is a normal Suslin tree, $2^\omega = \omega_1$, and $2^{\omega_1} = \omega_2$. Then there exists a forcing poset which forces*:

- *S is a Suslin tree.*
- *If T and U are normal Aronszajn trees, neither of which contains a Suslin subtree, then T and U are club isomorphic.*

In contrast to the aforementioned model of Abraham and Shelah which satisfies GCH, in our model we have that $2^\omega = 2^{\omega_1}$. This is necessary since by Section 2 of [1], the weak diamond principle $2^\omega < 2^{\omega_1}$ implies the existence of 2^{ω_1} many pairwise non-club isomorphic special Aronszajn trees.

PROOF. Define by recursion a countable support forcing iteration

$$\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \omega_2,\ \beta < \omega_2 \rangle$$

of proper forcings which preserve S. After defining $\mathbb{P}_\alpha$, we consider by some bookkeeping a pair of Aronszajn trees T_α and U_α in $V^{\mathbb{P}_\alpha}$ and ask whether or not S forces over $V^{\mathbb{P}_\alpha}$ that T_α and U_α remain Aronszajn. If so, then by Theorem 3.10 forcing with $\mathbb{Q}(T_\alpha, U_\alpha)$ over $V^{\mathbb{P}_\alpha}$ preserves S being Suslin. In this case, define $\dot{\mathbb{Q}}_\alpha$ as a $\mathbb{P}_\alpha$-name for $\mathbb{Q}(T_\alpha, U_\alpha)$, and otherwise let $\dot{\mathbb{Q}}_\alpha$ be a $\mathbb{P}_\alpha$-name for the trivial forcing.

Consider on the other hand the case that there exists $x \in S$ and $W \in \{T_\alpha, U_\alpha\}$ such that in $V^{\mathbb{P}_\alpha}$, $x \Vdash_S$ "W has a cofinal branch." By Proposition 3.1, there exists a club $C \subseteq \omega_1$ and a strictly increasing and height preserving function $f : S_x \restriction C \to W \restriction C$ in $V^{\mathbb{P}_\alpha}$. By upwards absoluteness, f has the same property in $V^{\mathbb{P}_{\omega_2}}$. Since S, and hence S_x, is Suslin in $V^{\mathbb{P}_{\omega_2}}$, it follows that in $V^{\mathbb{P}_{\omega_2}}$ the tree W contains a Suslin subtree, namely $f[S_x \restriction C]$.

By standard proper forcing iteration theorems and our cardinal arithmetic assumptions, for all $\beta < \omega_2$, $\mathbb{P}_\beta$ is ω_2-c.c. and has cardinality at most ω_2 (see Chapter VIII of [8]). Thus, by a standard bookkeeping argument we can arrange that all pairs of Aronszajn trees in the final model have been handled at some stage less than ω_2. ⊣

Recall that for a coherent normal Suslin tree S, the forcing axiom PFA(S) of Todorcevic [9] states that for any proper forcing $\mathbb{P}$ which preserves S being Suslin, for any collection $\mathcal{D}$ of ω_1 many dense subsets of $\mathbb{P}$, there exists a filter G on $\mathbb{P}$ which meets every dense set in $\mathcal{D}$.

THEOREM 4.2. *The forcing axiom PFA(S) implies that there exists a Suslin tree (namely, S) and any two normal Aronszajn trees, neither of which contains a Suslin subtree, are club isomorphic.*

PROOF. Assume PFA(S) and consider normal Aronszajn trees T and U which have no Suslin subtree. By Proposition 3.1 and the comments which follow it, S forces that T and U are Aronszajn. By Theorem 3.10, $\mathbb{Q}(T, U)$ preserves S being

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Suslin. Also $\mathbb{Q}(T, U)$ is proper. By choosing a filter meeting an appropriate collection of dense subsets of $\mathbb{Q}(T, U)$, it is easy to show that T and U are club isomorphic. ⊣

Note that the consistency result of Theorem 4.1 does not use large cardinals, in contrast to $\mathsf{PFA}(S)$. Also, we are not using the coherence of S and the same conclusion holds for $\mathsf{PFA}(U)$ for any Suslin tree U whether it is coherent or not.

We move on to our second application which concerns the topic of free trees. Free trees were originally introduced by Jensen [4] as a counterpoint to homogeneous Suslin trees, which are trees such that for any distinct a and b of the same height, there exists an automorphism of the tree which maps a to b and b to a. Note that the property of being homogeneous is upwards absolute. In contrast, as the next theorem shows, free trees are highly malleable by forcing.

THEOREM 4.3. *Suppose that S is a free normal Suslin tree. Then for any positive integer n, there exists a c.c.c. forcing poset which forces that S is n-free but all derived trees of S of dimension greater than n are special.*

PROOF. Fix a positive integer n. We define by recursion a finite support forcing iteration

$$\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \omega_1,\ \beta < \omega_1 \rangle$$

of c.c.c. forcings. We will arrange that for each $\beta < \omega_1$, there exists an $(n+1)$-tuple $\vec{a}_\beta = (a_{\beta,0}, \dots, a_{\beta,n})$ of distinct elements of S of the same height such that

$$\Vdash_\beta \text{“}S_{\vec{a}_\beta} \text{ is Suslin and } \dot{\mathbb{Q}}_\beta = \mathbb{Q}(S_{\vec{a}_\beta})\text{.”}$$

By Theorem 3.3, each $\dot{\mathbb{Q}}_\beta$ is forced to be c.c.c. We bookkeep our forcings in such a way that for all $\gamma < \beta < \omega_1$, the height of the elements of $\vec{a}_\gamma$ are less than or equal to the height of the elements of $\vec{a}_\beta$, which is possible since the levels of S are countable.

Let us say that an $(n+1)$-tuple $\vec{a}$ of distinct elements of S of the same height has been *handled* by stage δ if $\Vdash_\delta$ “$S_{\vec{a}}$ is special.” A given $(n+1)$-tuple $\vec{a}$ can be handled either explicitly by forcing with $\mathbb{Q}(S_{\vec{a}})$, or incidentally as a consequence of forcing other trees to be special. Our bookkeeping ensures that all $(n+1)$-tuples of elements of one level of S are handled before we move on and handle the $(n+1)$-tuples of the next level. We need to maintain at each step that every derived tree of S of dimension n remains Suslin. In order to make sure we can handle all $(n+1)$-tuples, we also maintain that any derived tree of dimension $n+1$ which has not been handled by a given stage δ is still Suslin in $V^{\mathbb{P}_\delta}$. The following inductive hypothesis achieves these goals.

Inductive Hypothesis on $\delta < \omega_1$: Let $\vec{b} = (b_0, \dots, b_{m-1})$ be a tuple of distinct elements of S satisfying that for all $\gamma < \delta$:

(1) the height of the elements of $\vec{b}$ are greater than or equal to the height of the elements of $\vec{a}_\gamma$;
(2) for all $\gamma < \delta$ there exists $i \leq n$ such that for all $j < m$, $a_{\gamma,i} \not\leq_S b_j$.

Then $\Vdash_\delta$ “$S_{\vec{b}}$ is Suslin.”

Assume for now that the inductive hypothesis is true for all $\delta < \omega_1$, and we describe how it can be used to prove the theorem. To begin, we show that we can arrange all derived trees of dimension $n+1$ to be special. So consider $\delta < \omega_1$ and assume that

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a particular $(n+1)$-tuple $\vec{a} = (a_0, \dots, a_n)$ has not been handled by stage δ. Since we are specializing derived trees one level of S at a time, it follows that for all $\gamma < \delta$, the height of the elements of $\vec{a}_\gamma$ is less than or equal to the height of the elements of $\vec{a}$.

We claim that $\Vdash_\delta$ "$S_{\vec{a}}$ is Suslin." By the inductive hypothesis, it suffices to show that for all $\gamma < \delta$ there exists $i \leq n$ such that for all $j \leq n$, $a_{\gamma,i} \not\leq_S a_j$. Suppose for a contradiction that $\gamma < \delta$ and for all $i \leq n$ there is some $j_i \leq n$ such that $a_{\gamma,i} \leq_S a_{j_i}$. Since the elements of $\vec{a}_\gamma$ are distinct and S is a tree, it follows that the map $i \mapsto j_i$ from $n+1$ to $n+1$ is an injection, and hence a bijection. Consequently, $S_{a_{j_0}} \otimes \cdots \otimes S_{a_{j_n}}$ is a subtree of $S_{\vec{a}_\gamma}$. But $S_{\vec{a}_\gamma}$ is special in $V^{\mathbb{P}_{\gamma+1}}$, and hence in $V^{\mathbb{P}_\delta}$. So $S_{a_{j_0}} \otimes \cdots \otimes S_{a_{j_n}}$ is also special in $V^{\mathbb{P}_\delta}$, since any subtree of a special tree is special. As $S_{\vec{a}}$ and $S_{a_{j_0}} \otimes \cdots \otimes S_{a_{j_n}}$ are isomorphic, $S_{\vec{a}}$ is special in $V^{\mathbb{P}_\delta}$ as well, which contradicts our assumption that $\vec{a}$ has not been handled by stage δ.

In summary, any derived tree of dimension $n+1$ which has not been handled by stage $\delta < \omega_1$ is still Suslin, and hence Aronszajn, in $V^{\mathbb{P}_\delta}$. Thus, we can easily arrange by bookkeeping that the forcing iteration $\mathbb{P}_{\omega_1}$ eventually handles all derived trees of S of dimension $n+1$. Therefore, $\mathbb{P}_{\omega_1}$ forces that all derived trees of S of dimension $n+1$ are special. By Lemma 2.7, it follows that $\mathbb{P}_{\omega_1}$ forces that all derived trees of S of dimension greater than n are special.

Next let us see that the inductive hypothesis implies that all derived trees of S of dimension n are Suslin in $V^{\mathbb{P}_{\omega_1}}$. So let $\vec{b} = (b_0, \dots, b_{n-1})$ be an n-tuple of distinct elements of S of the same height. To show that $S_{\vec{b}}$ is Suslin in $V^{\mathbb{P}_{\omega_1}}$, it suffices to show that for all $\delta < \omega_1$, $S_{\vec{b}}$ is Suslin in $V^{\mathbb{P}_\delta}$. As δ is countable, we can fix some $\xi < \omega_1$ greater than the height of the elements of $\vec{b}$ such that for all $\gamma < \delta$, ξ is greater than the height of the elements of $\vec{a}_\gamma$.

By Lemma 2.6, in order to prove that $S_{\vec{b}}$ is Suslin in $V^{\mathbb{P}_\delta}$, it suffices to show that for all $\vec{c}$ in $S_{\vec{b}}$ whose elements have height ξ, $S_{\vec{c}}$ is Suslin in $V^{\mathbb{P}_\delta}$. So let such $\vec{c} = (c_0, \dots, c_{n-1})$ be given. By the inductive hypothesis, it suffices to show that for all $\gamma < \delta$ there exists some $i \leq n$ such that for all $j < n$, $a_{\gamma,i} \not\leq_S c_j$. Let $\gamma < \delta$. Let α be the height of the elements of $\vec{a}_\gamma$, which is less than ξ by the choice of ξ. The set $\{c_0 \restriction \alpha, \dots, c_{n-1} \restriction \alpha\}$ has size at most n, whereas $\{a_{\gamma,0}, \dots, a_{\gamma,n}\}$ has size $n+1$. So we can choose $i \leq n$ such that $a_{\gamma,i}$ is not an element of $\{c_0 \restriction \alpha, \dots, c_{n-1} \restriction \alpha\}$. Then for all $j < n$, $a_{\gamma,i} \not\leq_S c_j$ and we are done.

It remains to prove the inductive hypothesis. Let $\delta < \omega_1$ and assume that the inductive hypothesis holds for all $\beta < \delta$. Let $\vec{b} = (b_0, \dots, b_{m-1})$ be a tuple of distinct elements of S of the same height satisfying that for all $\gamma < \delta$: (a) the height of the elements of $\vec{b}$ are greater than or equal to the height of the elements of $\vec{a}_\gamma$, and (b) there exists $i \leq n$ such that for all $j < m$, $a_{\gamma,i} \not\leq_S b_j$. We will prove that $\Vdash_\delta$ "$S_{\vec{b}}$ is Suslin."

Note that for all $\delta_0 < \delta$, $\vec{b}$ satisfies properties (a) and (b) for all $\gamma < \delta_0$. By the inductive hypothesis, for all $\delta_0 < \delta$, $\Vdash_{\delta_0}$ "$S_{\vec{b}}$ is Suslin." If δ is a limit ordinal, then by Theorem 2.3 it follows that $\Vdash_\delta$ "$S_{\vec{b}}$ is Suslin" and we are done.

Suppose that $\delta = \beta + 1$ is a successor ordinal. Then as just observed, $\Vdash_\beta$ "$\vec{b}$ is Suslin." So it suffices to prove that in $V^{\mathbb{P}_\beta}$, $\Vdash_{\mathbb{Q}_\beta}$ "$S_{\vec{b}}$ is Suslin." Recall that $\mathbb{Q}_\beta$ is equal to $\mathbb{Q}(S_{\vec{a}_\beta})$. By our assumptions on $\vec{b}$, we can fix $i^* \leq n$ such that for all $j < m$, $a_{\beta,i^*} \not\leq_S b_j$. Let α be the height of the elements of $\vec{b}$.

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We work in $V^{\mathbb{P}_\beta}$. In order to show that $\Vdash_{\mathbb{Q}(S_{\vec{a}_\beta})}$ "$S_{\vec{b}}$ is Suslin," by Theorem 3.4 it suffices to show that $\Vdash_{S_{\vec{b}}}$ "$S_{\vec{a}_\beta}$ is Aronszajn." By Lemma 2.4, it is enough to show that $\Vdash_{S_{\vec{b}}}$ "$S_{a_{\beta,i^*}}$ is Aronszajn." By a simple argument, it suffices to show that whenever $a \in S_\alpha$ is above a_{β,i^*}, then $\Vdash_{S_{\vec{b}}}$ "S_a is Aronszajn." So let such an a be given. By the choice of i^*, a is not equal to any of $b_0, \dots, b_{m-1}$.

There are two possibilities to consider. First, assume that for all $\gamma < \beta$, there exists some $i \leq n$ such that $a_{\gamma,i}$ is not less than or equal to any of $b_0, \dots, b_{m-1}, a$. Applying the inductive hypothesis for β we get that $S_{b_0} \otimes \cdots \otimes S_{b_{m-1}} \otimes S_a$ is Suslin in $V^{\mathbb{P}_\beta}$. As discussed in Section 2, it follows that $S_{\vec{b}}$ forces that S_a is Suslin, and hence Aronszajn, and we are done.

Secondly, assume that there exists some $\gamma < \beta$ such that for all $i \leq n$, $a_{\gamma,i}$ is less than or equal to one of $b_0, \dots, b_{m-1}, a$. By assumption (b) about $\vec{b}$, there exists $i \leq n$ such that for all $j < m$, $a_{\gamma,i} \not\leq_S b_j$, and therefore $a_{\gamma,i} \leq_S a$. For any $l \leq n$ different from i, $a_{\gamma,i}$ and $a_{\gamma,l}$ are different, so they cannot both be below a. Hence, we can pick $j_l < m$ such that $a_{\gamma,l} \leq_S b_{j_l}$. Then the map $l \mapsto j_l$ is injective. Define $d_i := a$ and for $l \leq n$ different from i, $d_l := b_{j_l}$. Then $S_{d_0} \otimes \cdots \otimes S_{d_n}$ is a subtree of $S_{\vec{a}_\gamma}$. Now in $V^{\mathbb{P}_{\gamma+1}}$, and hence in $V^{\mathbb{P}_\beta}$, $S_{\vec{a}_\gamma}$ is special. Since $S_{d_0} \otimes \cdots \otimes S_{d_n}$ is a subtree of $S_{\vec{a}_\gamma}$, it is special as well. By Lemma 2.7, it follows that $S_{b_0} \otimes \cdots \otimes S_{b_{m-1}} \otimes S_a$ is special. By Lemma 2.5, $S_{\vec{b}}$ forces that S_a is special and hence Aronszajn. ⊣

We mention a related result of Scharfenberger-Fabien [7] that under $\diamondsuit$, for each positive integer n there exists an n-free tree which is not $(n+1)$-free.

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