Proc. Camb. Phil. Soc. (1970), **68**, 173
PCPS 68–23
Printed in Great Britain

Complementary variational principles for $\phi^{iv} = f(\phi)$

BY N. ANDERSON AND A. M. ARTHURS
University of York

(*Received* 20 *September* 1968)

Abstract. Complementary variational principles are developed for boundary-value problems with field equations $\phi^{iv} = f(\phi)$ in $(0, 1)$ subject to $\phi(0) = \phi(1) = 0$, $\phi'(0) = \phi'(1) = 0$. The theory is applied in the non-linear case $f(\phi) = -k e^{\phi}$ and accurate variational solutions, which are simple polynomials in x, are obtained.

1. *Introduction.* In a recent series of papers (**1, 2, 3**), variational principles associated with a wide class of boundary-value problems have been derived which under certain circumstances lead to both maximum and minimum (complementary) principles. These principles provide a systematic method of obtaining approximate analytic solutions to various linear and non-linear problems which have field equations of the form

$$T\phi = \frac{\partial H}{\partial u}, \tag{1}$$

$$T^* u = \frac{\partial H}{\partial \phi}, \tag{2}$$

T being a linear operator and T^* its adjoint. The results already established cover the cases (i) $T = d/dx$, (ii) $T = \text{grad}$, (iii) $T = \text{curl}$, (iv) $T = \text{integral operator}$, (v) $T = \text{matrix}$.

In this paper we consider the case $T = d^2/dx^2$, and in particular we discuss complementary variational principles for boundary-value problems with field equations

$$\phi^{iv} = f(\phi), \quad x \in (0, 1), \tag{3}$$

$$\phi(0) = 0, \quad \phi(1) = 0, \quad \phi'(0) = 0, \quad \phi'(1) = 0. \tag{4}$$

Throughout the paper primes denote differentiation with respect to x. We assume the existence of a solution of (3) and (4). In the case when

$$\frac{f(\phi) - f(\psi)}{\phi - \psi} \leqslant 0 \quad (\text{all } \phi, \psi) \tag{5}$$

it can be shown that the solution is unique.

We remark that inhomogeneous boundary conditions

$$\phi(0) = a, \quad \phi(1) = b, \quad \phi'(0) = c, \quad \phi'(1) = d$$

can always be reduced to those of (4) by means of a suitable change of dependent variable.

https://doi.org/10.1017/S0305004100001201 Published online by Cambridge University Press

2. *Variational principles.* Consider equation (3) in canonical form

$$\frac{d^2\Phi}{dx^2} = U, \tag{6}$$

$$\frac{d^2U}{dx^2} = f(\Phi), \tag{7}$$

where $$\Phi(0) = \Phi(1) = 0, \quad \Phi'(0) = \Phi'(1) = 0. \tag{8}$$

The exact solution of this problem will be denoted by (u, ϕ). Now we introduce the associated generalized action functional

$$I(U, \Phi) = \int_0^1 \left\{U\frac{d^2\Phi}{dx^2} - \tfrac{1}{2}U^2 - F(\Phi)\right\} dx - \left[U\frac{d\Phi}{dx} - \frac{dU}{dx}\Phi\right]_0^1 \tag{9}$$

$$= \int_0^1 \left\{\frac{d^2U}{dx^2}\Phi - \tfrac{1}{2}U^2 - F(\Phi)\right\} dx, \tag{10}$$

where $$F(\Phi) = \int^{\Phi} f(\theta)\, d\theta. \tag{11}$$

Then the following results are readily obtained.

(*a*) *First variational principle.* For arbitrary independent trial functions U, Φ the functional $I(U, \Phi)$ is stationary at (u, ϕ) if equations (6), (7) and (8) hold simultaneously at (u, ϕ).

(*b*) *Second variational principle.* Choose a trial function $\Phi = \phi + \epsilon\xi$ such that $\Phi(0) = \Phi(1) = 0$, $\Phi'(0) = \Phi'(1) = 0$. Determine $U(\Phi)$ so that (6) is satisfied identically. Then if (7) and (8) hold at (u, ϕ) we have from (9)

$$J(\Phi) \equiv I(U(\Phi), \Phi) = \int_0^1 \{\tfrac{1}{2}(\Phi'')^2 - F(\Phi)\}\, dx \tag{12}$$

$$= I(u, \phi) + \delta^2 J(\Phi) + O(\epsilon^3), \tag{13}$$

where $$\delta^2 J(\Phi) = \frac{1}{2}\int_0^1 \left\{(U(\Phi) - u)^2 - (\Phi - \phi)^2 \frac{df(\phi)}{d\phi}\right\} dx \tag{14}$$

is the second variation. Thus $J(\Phi)$ is stationary as Φ varies around ϕ. The expression (12) has been discussed previously in the literature (4, 5, 6). It arises naturally in an Euler–Lagrange treatment.

(*c*) *Third variational principle.* Choose a trial function $U = u + \epsilon\eta$, and determine (if possible) $\Phi(U)$ so that (7) is satisfied identically. Then if (6) and (8) hold at (u, ϕ) we have from (10)

$$G(U) \equiv I(U, \Phi(U)) = \int_0^1 \{U''f^{-1}(U'') - \tfrac{1}{2}U^2 - F[f^{-1}(U'')]\}\, dx, \tag{15}$$

$$= I(u, \phi) + \delta^2 G(U) + O(\epsilon^3), \tag{16}$$

where $$\delta^2 G(U) = -\frac{1}{2}\int_0^1 \left\{(U - u)^2 - (\Phi(U) - \phi)^2 \frac{df(\phi)}{d\phi}\right\} dx \tag{17}$$

is the second variation. Thus $G(U)$ is stationary as U varies round u. The expression (15) does not appear to have been considered previously.

https://doi.org/10.1017/S0305004100001201 Published online by Cambridge University Press

Since the exact function u is related to ϕ by $u = \phi''$, it is desirable to choose the trial function U to have the form

$$U = \Psi'', \tag{18}$$

where Ψ is a trial function intended to be close to ϕ. Then from (15) and (18)

$$G(\Psi'') = \int_0^1 \{\Psi^{iv} f^{-1}(\Psi^{iv}) - \tfrac{1}{2}(\Psi'')^2 - F[f^{-1}(\Psi^{iv})]\}\, dx. \tag{19}$$

If the inverse of f does not exist, the functional G must be modified since an additional constraint on U or Ψ'' will be necessary. For example take the case when

$$f(\Phi) = p(x) \tag{20}$$

so that $f(\Phi)$ does not depend on Φ. Then we cannot determine $\Phi(U)$ from (7) since that equation would read

$$\frac{d^2U}{dx^2} = p(x). \tag{21}$$

We therefore choose U to satisfy the constraint $U'' = p(x)$ for $0 < x < 1$. Then, using the fact that $F(\Phi) = p(x)\,\Phi$ in this case, we have from (10) a modified form of (15) given by

$$G(U) = -\frac{1}{2}\int_0^1 U^2\, dx. \tag{15'}$$

(*d*) *Complementary variational principles.* If terms of higher order than the second in (13) and (16) are negligible (or vanish, as is the case for linear $f(\phi)$), it follows from 2(*b*) and 2(*c*) that

$$G(U) \leqslant I(u, \phi) \leqslant J(\Phi) \quad \text{if} \quad \delta^2 G(U) \leqslant 0, \quad \delta^2 J(\Phi) \geqslant 0, \tag{22}$$

or

$$G(U) \geqslant I(u, \phi) \geqslant J(\Phi) \quad \text{if} \quad \delta^2 G(U) \geqslant 0, \quad \delta^2 J(\Phi) \leqslant 0. \tag{23}$$

Thus we obtain maximum and minimum (complementary) principles, which give upper and lower bounds for $I(u, \phi)$, provided that $\delta^2 G$ and $\delta^2 J$ do not have the same sign. The pair of functions (u, ϕ) furnishes the exact solution of the problem described by (6), (7) and (8). From (14) and (17) it is clear that the bounds in (22) certainly hold for local variations if

$$\frac{df(\phi)}{d\phi} \leqslant 0 \quad (0 < x < 1). \tag{24}$$

When (24) is satisfied, the uniqueness condition (5) also holds.

3. *An application.* To illustrate this theory we consider the case with

$$f(\phi) = -k\, e^{\phi}. \tag{25}$$

Then the boundary-value problem in (3) and (4) becomes

$$\phi^{iv} = -k\, e^{\phi} \quad (0 < x < 1), \tag{26}$$

and

$$\phi(0) = 0, \quad \phi(1) = 0, \quad \phi'(0) = 0, \quad \phi'(1) = 0. \tag{27}$$

The associated variational problem involves the functionals

$$J(\Phi) = \int_0^1 \{\tfrac{1}{2}(\Phi'')^2 + k\, e^{\Phi}\}\, dx, \tag{28}$$

https://doi.org/10.1017/S0305004100001201 Published online by Cambridge University Press

where
$$\Phi(0) = 0, \quad \Phi(1) = 0, \quad \Phi'(0) = 0, \quad \Phi'(1) = 0; \tag{29}$$

$$I(u, \phi) = \int_0^1 \{\tfrac{1}{2}(\phi'')^2 + k\, e^{\phi}\}\, dx; \tag{30}$$

and
$$G(\Psi'') = \int_0^1 \{\Psi^{\mathrm{iv}} \ln [-k^{-1}\Psi^{\mathrm{iv}}] - \tfrac{1}{2}(\Psi'')^2 - \Psi^{\mathrm{iv}}\}\, dx, \tag{31}$$

where there are no essential conditions on Ψ. From (22) and (24) we have

$$G(\Psi'') \leqslant I(u, \phi) \leqslant J(\Phi) \quad \text{if} \quad k > 0. \tag{32}$$

For the trial functions Φ and Ψ we shall choose very simple polynomials in x

$$\Phi = -\alpha k x^2 (1-x)^2/24, \tag{33}$$

$$\Psi = -\beta k x^2 (1-x)^2/24, \tag{34}$$

which each contain one variational parameter (α or β) and which satisfy the boundary conditions (27). In the case of Φ in (28) the boundary conditions are essential, while for Ψ in (31) they are natural conditions. The parameters α and β are determined by minimizing $J(\Phi)$ and maximizing $G(\Psi'')$ respectively.

Calculations have been carried out for three values of k. The optimization of J and G was performed numerically, which is a very simple task for one-parameter trial functions like (33) and (34), and the optimum values of α and β were found to four significant figures. The results are given in Table 1, the values of J and G being the local extremum values, minimum for J and maximum for G. For each value of k the bounds of G and J are very close and so we expect the corresponding trial functions to be accurate analytical variational solutions of the problem. This accuracy can be measured in terms of an error bound derived by Ciarlet, Schultz and Varga (6). For our class of problems in (3) and (4) this error bound is given by

$$\sup_{x \in [0,1]} |\Phi - \phi| \leqslant \frac{K}{\sqrt{\Lambda}} \|\tau(\Phi)\|_{L^2[0,1]} = E(\Phi), \tag{35}$$

where
$$\tau(\Phi) = -\Phi^{\mathrm{iv}} + f(\Phi) \quad (0 < x < 1), \tag{36}$$

Λ is a lower bound for the eigenvalues of the associated eigenvalue problem $\phi^{\mathrm{iv}} = \lambda\phi$, $0 < x < 1$, subject to the boundary conditions (4), and K is a real constant such that

$$\sup_{x \in [0,1]} |y(x)| \leqslant K \left\{ \int_0^1 (y'')^2\, dx \right\}^{\frac{1}{2}} \tag{37}$$

for all functions $y(x)$ satisfying

$$\left.\begin{aligned} &y(0) = y(1) = y'(0) = y'(1) = 0, \\ &y' \text{ continuous in } (0, 1), \quad y'' \in L^2[0, 1]. \end{aligned}\right\} \tag{38}$$

It can be shown that
$$K = \sqrt{(3)}/24, \quad \Lambda = 500. \tag{39}$$

For $f(\Phi) = -k\, e^{\Phi}$ and a given function Φ we can thus readily compute the error bound $E(\Phi)$ in (35). The results corresponding to the optimum forms of (33) and (34) are given in Table 2. For $k = 1$ and $k = 2$ the error bounds for (33) and (34) are identical, while for $k = 5$ the error bound for (34) is better than that for (33). Thus for $k = 5$ the

https://doi.org/10.1017/S0305004100001201 Published online by Cambridge University Press

variational solution arising from the new functional G is more accurate than that from J. Increased accuracy can be achieved by choosing more elaborate trial functions, as in the classical Rayleigh–Ritz procedure, and we refer to (6) and (7) for a discussion of this numerical aspect of the variational method and for a comparison of various kinds of trial functions.

Table 1. *Variational parameters and bounds*

k	β	G	α	J
1	0·9988	0·9993065	0·9988	0·9993069
2	0·9972	1·9972299	0·9972	1·9972332
5	0·9932	4·9827583	0·9904	4·9828088

Table 2. *Error bounds*

k	β	$E(\beta)$	α	$E(\alpha)$
1	0·9988	0·2988(−5)	0·9988	0·2988(−5)
2	0·9972	0·1168(−4)	0·9972	0·1168(−4)
5	0·9932	0·7239(−4)	0·9904	0·8478(−4)

Here $N(-m)$ means $N \times 10^{-m}$.

For similar problems involving other boundary conditions the theory can be modified in some cases by adding extra terms to the basic functional $I(U, \Phi)$. For example, if boundary conditions $\phi(0) = 0, \phi'(0) = 0, \phi''(1) = 0, \phi'''(1) = 0$ are required, the functional $I(U, \Phi)$ is modified by the addition of the term $-[U'\Phi]_0^1$.

We wish to thank Dr B. W. Martin for helpful discussions and the referee for valuable comments on an earlier version of the work.

REFERENCES

(1) Noble, B. University of Wisconsin Mathematics Research Center Report No. 473 (1964, unpublished).
(2) Rall, L. B. *J. Math. Anal. Appl.* **14** (1966), 174–184.
(3) Arthurs, A. M. *Proc. Cambridge Philos. Soc.* **65** (1969), 803–806.
(4) Levinson, N. *J. Math. Mech.* **12** (1963), 567–575.
(5) Mikhlin, S. G. In *Differential equations and their applications* (ed. Babuska, I.), Academic Press, New York (1963), 77–92.
(6) Ciarlet, P. G., Schultz, M. H. and Varga, R. S. *Numer. Math.* **9** (1967), 394–430.
(7) Ciarlet, P. G., Schultz, M. H. and Varga, R. S. *Numer. Math.* **12** (1968), 266–279.

https://doi.org/10.1017/S0305004100001201 Published online by Cambridge University Press