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SPARSITY OF ITERATIONS OF RATIONAL FUNCTIONS ON THE UNIT CIRCLE

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Abstract

We use a recent result of Orevkov and Pakovich ['On intersection of lemniscates of rational functions', *Arnold J. Math.* **11** (2025), 7–26] to obtain a bound on the frequency with which elements in an orbit of polynomial iterations can fall on the unit circle.

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1. Introduction

Given a rational function $f \in \mathbb{C}(X)$ and an initial point $\alpha \in \mathbb{C}$, we consider the orbit

$$\mathcal{O}_f(\alpha) = (f^n(\alpha) : n = 0, 1, \ldots),$$

where we set $f^0(X) = X$ and then define recursively

$$f^n(X) = f(f^{n-1}(X)), \quad n = 1, 2 \ldots.$$

We also assume that the orbit $\mathcal{O}_f(\alpha)$ terminates if α is a pole of $f^n(X)$ for some n.

We emphasise that we treat orbits as naturally ordered sequences, rather than as sets. If this sequence becomes eventually periodic, we call α *preperiodic*.

There is a vast amount of work studying the distribution of orbits $\mathcal{O}_f(\alpha)$ in various subsets $\mathcal{S} \subseteq \mathbb{C}$, under various algebraic and arithmetic constraints on the function f, the initial point α, and the set $\mathcal{S}$ (see [1, 2, 4, 5, 7–10] and the references therein).

Here, we use a recent result of Orevkov and Pakovich [6] to study an apparently new question and estimate the frequency of the elements in an orbit $\mathcal{O}_f(\alpha)$, which fall on the unit circle

$$\mathbb{U} = \{z \in \mathbb{C} : \ |z| = 1\}.$$

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Some arithmetic analogues of this question have been considered by Ostafe [7], where a rational function $f \in \mathbb{K}(X)$ is defined over a number field $\mathbb{K}$ and the unit circle is replaced by the set $\mathcal{U}_{\mathbb{K}}$ of roots of unity of $\mathbb{K}$. In particular, in this setting, using [7, Theorem 1.2], one can derive the finiteness of the sets $\mathcal{O}_f(\alpha) \cap \mathcal{U}_{\mathbb{K}}$ (under some natural conditions on α and f).

2. Main result

Given an integer N, we define

$$T_f(\alpha, N) = \#\{n : f^n(\alpha) \in \mathbb{U}, 1 \leqslant n \leqslant N\}.$$

We say that $f(z) \in \mathbb{C}(z)$ is *special* if $f^n(\mathbb{U}) \subseteq \mathbb{U}$ for some n, that is, f^n maps $\mathbb{U}$ to $\mathbb{U}$.

For example, if $f(z)$ is a polynomial, than as follows from the discussion in [6, Section 1], in particular, [6, (3)], this is possible only if $f^n(z)$ is a monomial, that is, $f^n(z) = z^k$ for some integer $k \geqslant 0$ and thus $f(z)$ is also a monomial.

THEOREM 2.1. *For any nonspecial rational function $f(z) \in \mathbb{C}(z)$ and an initial point $\alpha \in \mathbb{C}$, which is not preperiodic, we have*

$$T_f(\alpha, N) = O(N/\log N),$$

where the implied constant depends only on $\deg f$.

As we have mentioned, nonmonomial polynomials are always nonspecial. However, it is not immediately clear how to characterise nonspecial rational functions, which we pose as an open question.

3. Preliminaries

The following result is a special case of [6, Theorem 1.1], which improves a similar bound from [11] on the number of intersections of lemniscates.

LEMMA 3.1. *Let $F(z) \in \mathbb{C}(z)$ be a rational function of degree D such that $F(\mathbb{U}) \not\subseteq \mathbb{U}$. Then, $\#\{z \in \mathbb{U} : F(z) \in \mathbb{U}\} \leqslant 2D$.*

We also need the following simple combinatorial statement given by [3, Lemma 5.7].

LEMMA 3.2. *Let $2 \leqslant M < N/2$. For any sequence $0 \leqslant n_1 < \cdots < n_M \leqslant N$, there exists $r \leqslant 2N/(M-1)$ such that $n_{i+1} - n_i = r$ for at least $(M-1)^2/4N$ values of $i \in \{1, \ldots, M-1\}$.*

4. Proof of Theorem 2.1

We fix some $\alpha \in \mathbb{C}$, which is not preperiodic, and define $z_n = f^n(\alpha)$, $n = 0, 1, \ldots$, which are obviously pairwise distinct.

If $T_f(\alpha, N) = 0$, there is nothing to prove. Thus, we now assume that $T_f(\alpha, N) \geqslant 1$.

For $t = T_f(\alpha, N) - 1$, let $1 \leqslant n_1 < \ldots < n_{t+1} \leqslant N$ be such that $z_{n_i} \in \mathbb{U}$. Consider the gaps $d_i = n_{i+1} - n_i$, $i = 1, \ldots, t$.

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By Lemma 3.2, there is some $r \leqslant 2N/t$ such that $d_i = r$ for at least $I \geqslant t^2/(4N)$ values of $i \in \{1, \ldots, t\}$. Hence, there are at least I solutions to the equation

$$|f^r(z)| = |z| = 1$$

given by $z_i = f^{n_i}(\alpha)$ for each i such that $d_i = r$. Note that since $\alpha \in \mathbb{C}$ is not preperiodic, all these points are distinct.

Since $\deg f^r \leqslant (\deg f)^r$ and f is nonspecial, by Lemma 3.1,

$$\frac{t^2}{4N} \leqslant I \leqslant 2(\deg f)^r \leqslant 2(\deg f)^{2N/t}. \tag{4.1}$$

If

$$2(\deg f)^{2N/t} \leqslant N^{1/2}, \tag{4.2}$$

then from (4.1), we have $t \leqslant 2N^{3/4}$, which is much stronger that the desired bound. Otherwise, that is, if (4.2) is false,

$$\frac{2N}{t} \log(\deg f) + \log 2 > \frac{1}{2} \log N,$$

which implies the result.

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