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MODIFIED EXTENDER BASED FORCING

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Abstract. We analyze the modified extender based forcing from Assaf Sharon's PhD thesis. We show there is a bad scale in the extension and therefore weak square fails. We also present two metatheorems which give a rough characterization of when a diagonal Prikry-type forcing forces the failure of weak square.

§1. Introduction. In the 1980s Woodin asked if the failure of the Singular Cardinals Hypothesis (SCH) was compatible with the failure of weak square at a singular cardinal μ. The case when $\mu = \aleph_\omega$ is of particular interest and remains open. In [5], Gitik and Magidor developed a diagonal extender based forcing construction to violate SCH. This forcing was in part motivated by searching for a way to obtain failure of weak square at a large singular cardinal together with not SCH. (Their forcing has since been refined by Gitik, see for example Section 2 of [4].) However, in his PhD Thesis [8] Sharon showed that this forcing actually adds a weak square sequence.

A positive answer to Woodin's question for a large singular cardinal was obtained by Gitik and Sharon [6]. In fact they obtained a stronger result that the approachability property fails. Cummings and Foreman [1] were able to show that there is a PCF theoretic object called a bad scale in Gitik and Sharon's model, which is responsible for the failure of the approachability property. Gitik and Sharon's forcing admits the addition of collapses in a natural way to make the singular cardinal into $\aleph_{\omega^2}$, which gives a positive answer to Woodin's question for $\aleph_{\omega^2}$. Recently, the authors [9] added collapses to the Gitik and Sharon forcing to make κ into $\aleph_\omega$. The result was a model where SCH fails at $\aleph_\omega$, but weak square holds. The argument that weak square holds is a modification of Sharon's argument that weak square holds in Gitik and Magidor's model mentioned above. In a second paper [10], we were able to show that the forcing from our previous paper gives the failure of the intermediate weak square principles $\square_{\aleph_\omega, \aleph_n}$ for $n < \omega$. This is currently the best known partial result towards a positive answer to Woodin's question.

This paper is motivated by a search for other posets which might give a positive answer to Woodin's question at $\aleph_\omega$. It is reasonable to ask whether there is some way to modify Gitik and Magidor's extender based forcing in order to obtain the failure of weak square. Further, if there is such a forcing, then is it possible to add collapses

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to obtain a positive answer to Woodin's question at $\aleph_\omega$. In this paper, we show that the answer to the first question is yes and the answer to the second question is no. In particular, we present a version of extender based Prikry forcing from Sharon's PhD thesis, which we call *modified extender based forcing* and show that it forces the failure of SCH and the existence of a bad scale, which implies the failure of weak square. We give a strong answer to the second question by proving two metatheorems which characterize when a diagonal Prikry forcing forces the failure of weak square. As an application, we show that a reasonable attempt to add collapses to the modified extender based forcing to make κ into $\aleph_\omega$ will add weak square. This suggests that extender based forcing is unlikely to provide a positive answer to Woodin's question for $\aleph_\omega$.

The sections of the paper are organized as follows. In Section 2, we give the definition of the modified extender based forcing and prove some of its properties. The definitions and theorems of this section are all due to Sharon. In Section 3, we show that assuming κ is supercompact, there is a bad scale in the extension by the modified extender based forcing. In Section 4, we prove our metatheorems and show that certain extender based forcings with short extenders add weak square.

§2. Modified extender based forcing. In order to motivate the definition of the modified extender based forcing, we describe a natural failed attempt to modify the extender based forcing with long extenders from Section 2 of Gitik's chapter in the Handbook of Set Theory [4]. This long extender forcing from [4] is a simplification of the forcing construction of Gitik and Magidor mentioned above. The reader is advised to have a good understanding of it before attempting to read this section of the paper.

As a background assumption, we have an increasing sequence of cardinals $\langle \kappa_n \mid n < \omega \rangle$ with $\sup_{n<\omega} \kappa_n = \kappa_\omega$ such that each κ_n carries a $(\kappa_n, \kappa_\omega^{++})$-extender $E_n = \langle E_{n\alpha} \mid \alpha < \kappa_\omega^{++} \rangle$. Recall that we say $\alpha \leq_{E_n} \beta$ if and only if there is a function $f : \kappa \to \kappa$ such that $j_n(f)(\beta) = \alpha$ where j_n is the ultrapower map generated by E_n. For such α and β, we fix a map $\pi_{\beta\alpha} : \kappa \to \kappa$ that witnesses $\alpha \leq_{E_n} \beta$. We also set $\mu = \kappa_\omega^+$.

The long extender based forcing is a way of gluing together ω many cells. Each cell $\mathbb{Q}_n$ has two parts $\mathbb{Q}_{n0}$ and $\mathbb{Q}_{n1}$. $\mathbb{Q}_{n1}$ is a version of Cohen forcing and $\mathbb{Q}_{n0}$ is a constrained version of Cohen forcing where some coordinates are controlled by a measure one set from some measure in the extender E_n. The order on a cell is the order on each of the pieces together with a method for passing from $\mathbb{Q}_{n0}$ to $\mathbb{Q}_{n1}$ by selecting an element from the current measure one set and using the constraint to add information to the Cohen part of the condition.

For Gitik and Magidor, the size of the Cohen parts is at most κ_ω and a careful read of Sharon's thesis reveals that these large Cohen parts are responsible for the addition of weak square. Unfortunately, the size of the Cohen parts also allows for an easy proof of the Prikry lemma, since the Cohen parts are closed enough to become universal.

A natural attempt to avoid adding weak square is to decrease the size of the Cohen parts in each cell. To this end we define $\mathbb{Q}_{n1} = \{f : \kappa_\omega^{++} \to \kappa_n \mid |f| \leq \kappa_n\}$ and $\mathbb{Q}_{n0}$ to be as in Definition 2.6 of [4], but with the modified version of $\mathbb{Q}_{n1}$.

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DEFINITION 2.1. Conditions in $\mathbb{Q}_{n0}$ are triples $\langle a, A, f \rangle$ where

(1) $f \in \mathbb{Q}_{n1}$,
(2) $a \subseteq \kappa_\omega^{++}$ such that $|a| < \kappa_n$ and it has a maximal ordinal $\mathrm{mc}(a)$ in the sense of $\leq_{E_n}$ which is also its largest element,
(3) $A \in E_{n\,\mathrm{mc}(a)}$,
(4) if $\beta < \alpha$ with $\alpha, \beta \in a$, then for all $\nu \in A$, $\pi_{\mathrm{mc}(a)\beta}(\nu) < \pi_{\mathrm{mc}(a)\alpha}(\nu)$, and
(5) if $\alpha, \beta, \gamma \in a$ with $\gamma \leq_{E_n} \beta \leq_{E_n} \alpha$, then for all $\nu \in \pi_{\mathrm{mc}(a)\alpha}$"$A$, $\pi_{\alpha\gamma}(\nu) = \pi_{\beta\gamma}(\pi_{\alpha\beta}(\nu))$.

We define $(a, A, f) \leq (b, B, g)$ if and only if $f \supseteq g$, $a \supseteq b$ and $\pi_{\mathrm{mc}(a)\,\mathrm{mc}(b)}$"$A \subseteq B$.

The combination of $\mathbb{Q}_{n1}$ and $\mathbb{Q}_{n0}$ to form $\mathbb{Q}_n$ is the standard one.

DEFINITION 2.2. Let $\mathbb{Q}_n = \mathbb{Q}_{n0} \cup \mathbb{Q}_{n1}$ where we set $p \leq q$ if

(1) $p, q \in \mathbb{Q}_{ni}$ for $i \in 2$ and $p \leq q$ or
(2) $p \in \mathbb{Q}_{n1}$, $q \in \mathbb{Q}_{n0}$ and $p \supseteq q \frown \nu$ (defined below) for some ν.

For $\langle a, A, f \rangle \in \mathbb{Q}_{n0}$ and $\nu \in A$, let $\langle a, A, f \rangle^\frown \nu = f \cup \{\langle \beta, \pi_{\mathrm{mc}(a),\beta}(\nu)\rangle \mid \beta \in a\}$.

We start with some easy propositions.

PROPOSITION 2.3. (1) *$\mathbb{Q}_n$ is κ_n^+-closed.*
(2) *$\mathbb{Q}_n$ has the κ_n^{++} chain condition.*
(3) *$\mathbb{Q}_{n0}$ is κ_n-closed, preserves κ_n^{++}, and collapses $(\kappa_n^+)^V$ to κ_n.*

One can then define a poset $\hat{\mathbb{P}}$ as in Definitions 2.13 and 2.14 of [4]. More precisely, conditions in $\hat{\mathbb{P}}$ are of the form $p = \langle p_n \mid n < \omega\rangle$, with length $\mathrm{lh}(p)$, such that for $n < \mathrm{lh}(p)$, $p_n \in \mathbb{Q}_{n1}$ and for $n \geq \mathrm{lh}(p)$, $p_n \in \mathbb{Q}_{n0}$. The order is given by $q \leq p$ if for all n, $q_n \leq_{\mathbb{Q}_n} p_n$, and also $q \leq^* p$ if $q \leq p$ and they have the same length. The naive expectation is that this poset will satisfy the Prikry property (and hence preserve cardinals up to κ_ω) and also force the failure of weak square. Unfortunately, it does not satisfy the Prikry property, in fact it collapses κ_ω to be countable. The definition of modified extender based forcing can be seen as a way to fix $\hat{\mathbb{P}}$ in order to recover the Prikry property.

Let us work for the moment with a condition $p \in \hat{\mathbb{P}}$ and a statement φ in the forcing language for $\hat{\mathbb{P}}$. Possible extensions of p are determined by finite sequences of points from the measure one sets in p. Let $\vec{\nu}$ be such a sequence, and let $p^\frown\vec{\nu}$ denote the weakest possible extension with respect to $\vec{\nu}$ (below we give a precise definition). If there is a condition $q \leq^* p^\frown\vec{\nu}$ which decides φ, then we record this condition as $q_{\vec{\nu}}$. To prove the Prikry lemma we would like a direct extension of p which captures each $q_{\vec{\nu}}$. Unfortunately the Cohen parts f are no longer closed enough to do this, since we have reduced their size.

What we are left with is for each $n < \omega$ a function from $\{\vec{\nu} \mid \vec{\nu}$ is a possible extension of p with $\mathrm{lh}(p^\frown\vec{\nu}) > n\}$ to $\mathbb{Q}_{n1}$ given by $\vec{\nu} \mapsto q_{\vec{\nu}}(n)$. Since we cannot capture such a function by a direct extension in $\hat{\mathbb{P}}$, we replace the current f-parts with functions of this form under a suitable ordering. We note that if n is larger than the length of p, then our n^{th} function will not depend on $\vec{\nu} \restriction (n - \mathrm{lh}(p))$.

Let $[\prod_{i \geq l} A_i]^{<\omega}$ denote the set of all sequences $\langle \nu_l, \ldots, \nu_n\rangle$, where each $\nu_i \in A_i$.

DEFINITION 2.4. Conditions in $\mathbb{P}$ are of the form

$$p = \langle f_0, \ldots, f_{l-1}, \langle a_l, A_l, f_l\rangle, \langle a_{l+1}, A_{l+1}, f_{l+1}\rangle, \ldots\rangle,$$

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where:

(1) For $n < l$, $f_n : [\prod_{i \geq l} A_i]^{<\omega} \to \mathbb{Q}_{n1}$.
(2) For $n \geq l$, $\mathrm{dom}(f_n) = [\prod_{i>n} A_i]^{<\omega}$, and for each $\vec{\nu} \in \mathrm{dom}(f_n)$, $\langle a_n, A_n, f_n(\vec{\nu})\rangle \in \mathbb{Q}_{n0}$.
(3) For $l \leq n < m$, we have $a_n \subset a_m$.

For p as above and $\nu \in A_l$, $p^\frown \nu$ is the condition $\langle f'_0, \ldots, f'_{l-1}, f'_l, \langle a'_{l+1}, A'_{l+1}, f'_{l+1}\rangle, \ldots\rangle$, where:

(1) for $n < l = \mathrm{lh}(p)$, for every $h \in \mathrm{dom}(f'_n)$, $f'_n(h) = f_n(\nu^\frown h)$,
(2) for every $h \in \mathrm{dom}(f'_l)$, $f'_l(h) = \langle a_l, A_l, f_l(h)\rangle^\frown \nu$,
(3) for $n > l$, $\langle a'_n, A'_n, f'_n\rangle = \langle a_n, A_n, f_n\rangle$.

Similarly, define $p^\frown \vec{\nu}$.

For p as above with $\mathrm{lh}(p) = l$, we denote $p := \langle p_n \mid n < \omega\rangle$ or $p = \langle f^p_0, \ldots, f^p_{l-1}, \langle a^p_l, A^p_l, f^p_l\rangle, \langle a^p_{l+1}, A^p_{l+1}, f^p_{l+1}\rangle, \ldots\rangle$. Recall that for each $n \geq \mathrm{lh}(p)$, a^p_n has a $\leq_{E_n}$-maximal element which we call $\mathrm{mc}(a^p_n)$.

Definition 2.5. $q \leq^* p$ if

(1) $\mathrm{lh}(q) = \mathrm{lh}(p) = l$,
(2) for all $n \geq l$, $a^q_n \supseteq a^p_n$ and $\pi_{\mathrm{mc}(a^q_n)\,\mathrm{mc}(a^p_n)}$"$A^q_n \subseteq A^p_n$, and
(3) for all $n < \omega$ and all $\vec{\nu} \in [\prod_{i>n} A^q_i]^{<\omega}$, $f^q_n(\vec{\nu}) \leq f^p_n(\pi(\vec{\nu}))$ where π is the map that for each i applies $\pi_{\mathrm{mc}(a^q_i)\,\mathrm{mc}(a^p_i)}$ to the i^{th} coordinate of $\vec{\nu}$.

Definition 2.6. Then $q \leq p$ if for some $\vec{\nu}$, $q \leq^* p^\frown \vec{\nu}$.

It is straightforward to show that the ordering is transitive. The following is standard, but for completeness we sketch the proof.

Lemma 2.7. *$\mathbb{P}$ has the Prikry property: for every sentence in the forcing language ϕ and condition p, there is $q \leq^* p$, such that q decides ϕ.*

Proof. Suppose that $p \in \mathbb{P}$ and φ is statement in the forcing language. By the discussion preceding the definition of the forcing there is a $p_0 \leq^* p$ such that for all sequences $\vec{\nu}$ if there is $q \leq^* p_0^\frown \vec{\nu}$ which decides φ, then $p_0^\frown \vec{\nu}$ decides φ in the same way. Note that at this point in the construction, we have fixed the a-parts of the conditions we consider and hence all measure one sets will come from the measures $E_{m\,\mathrm{mc}(a^{p_0}_m)}$ for $m \geq \mathrm{lh}(p_0)$.

We now reduce the measure one sets as follows. Suppose that $\vec{\nu}$ is a sequence of ordinals. We partition the set of ρ such that $p_0^\frown \vec{\nu}^\frown \rho$ is a condition into three sets. The set of ρ for which $p_0^\frown \vec{\nu}^\frown \rho$ forces φ, forces $\neg\varphi$ or doesn't decide φ. Let $A_{\vec{\nu}}$ be the unique set of the three which is measure one for the appropriate measure. Now for each k, $\bigcap\{A_{\vec{\nu}} \mid \mathrm{lh}(\vec{\nu}) = k\}$ is measure one and hence we can strengthen p_0 to p_1 by restricting its measure one sets to these intersections.

To finish the proof we claim that there is a direct extension of p_1 which decides φ. Otherwise, we take an extension q of minimal length which decides it (without loss of generality assume that it forces φ). By the construction, we can take $q = p_1^\frown \vec{\nu}^\frown \rho^*$ for some $\vec{\nu}$ and ρ^*. Now by the reduction of measure one sets we know that $A_{\vec{\nu}}$ is the set of ρ for which $p_1^\frown \vec{\nu}^\frown \rho$ forces φ. It follows that $p_1^\frown \vec{\nu}$ forces φ, a contradiction to the minimality of the length of q. ⊣

A similar argument gives a standard strengthening of the previous lemma.

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LEMMA 2.8. *For every dense open set D and condition p, there is $q \leq^* p$ and $n < \omega$, such that all $r \leq q$ of length n are in D.*

PROPOSITION 2.9. *$\mathbb{P}$ has the following properties*:

(1) *$\langle \mathbb{P}, \leq^* \rangle$ is κ_0- directed closed.*
(2) *$\langle \mathbb{P} \restriction [n, \omega), \leq^* \rangle$ is κ_n-closed, where $\mathbb{P} \restriction [n, \omega) := \{p \restriction [n, \omega) \mid p \in \mathbb{P}\}$.*
(3) *$\mathbb{P}$ does not add bounded subsets of κ_0.*
(4) *Both $\mathbb{P}$ and $\langle \mathbb{P}, \leq^* \rangle$ preserve μ*

Items 1 through 3 are straightforward. The last item is one of the key properties of this forcing which allows us to argue that there is a bad scale in the extension, so we devote particular attention to its proof. Below we prove that certain restrictions of $\mathbb{P}$ have the chain condition. This shows that $\langle \mathbb{P}, \leq^* \rangle$ preserves μ. The fact that $\mathbb{P}$ preserves μ is due to the Prikry property.

For $n > 0$ and $p \in \mathbb{P}_0 := \{p \in \mathbb{P} \mid \mathrm{lh}(p) = 0\}$, let $\pi_n(p) = \langle p_0, p_1, \ldots, p_{n-1} \rangle$. Set $\mathbb{P}_{0n} := \{\pi_n(p) \mid p \in \mathbb{P}_0\}$ with the natural induced ordering from $\leq^*$.

PROPOSITION 2.10. *For all $n \geq 0$, $\mathbb{P}_{0n+1}$ has the κ_n^{++}-c.c.*

PROOF. By induction on n, suppose for contradiction that $\{\pi_{n+1}(p^\eta) \mid \eta < \kappa_n^{++}\}$ is an antichain in $\mathbb{P}_{0n+1}$, we may assume that the part above n of each p^η is the same, i.e. for some $p \in \mathbb{P}$, for all $m > n$ and $\eta < \kappa_n^{++}$, $p^\eta_m = p_m$. For $m > n$, let $\alpha_m = \mathrm{mc}(a^p_m)$. For $m > n$, set $i_m := j_{E_{n+1,\alpha_{n+1}}} \circ \ldots j_{E_{m,\alpha_m}}$.

We will define functions f^η_m for $n < m, \eta < \kappa_n^{++}$ as follows.

- If $m = n + 1$, for $v \in A^p_{n+1}$, let $f^\eta_{n+1} := [v \mapsto f^{p^\eta}_n(v)]_{E_{n+1,\alpha_{n+1}}}$. Since $|f^{p^\eta}_n(v)| \leq \kappa_n$, which is below the critical point of $E_{n+1,\alpha_{n+1}}$, we have that $|f^\eta_{n+1}| \leq \kappa_n$.
- If $m = n + 2$, for $v \in A^p_{n+1}$, let,
  - $f^\eta_v := [\delta \mapsto f^{p^\eta}_n(v^\frown \delta)]_{E_{n+2,\alpha_{n+2}}}$;
  - $f^\eta_{n+2} := [v \mapsto f^\eta_v]_{E_{n+1,\alpha_{n+1}}}$.

  As before, $|f^\eta_{n+2}| \leq \kappa_n$.
- ... continue in a similar fashion for all $m > n$.

Then each f^η_m is a partial function from $i_m(\mu^+)$ to κ_n of size less than or equal to κ_n. Now define a partial function $F^\eta_m : i_m(\mu^+) \rightharpoonup \{Y\} \cup \kappa_n$ by setting

$$F^\eta_m(\alpha) := \begin{cases} Y & \text{if } \alpha \in i_m(a^{p_\eta}_n), \\ f^\eta_m(\alpha) & \text{if } \alpha \in \mathrm{dom}(f^\eta_m). \end{cases}$$

Let F^η be the function given by $F^\eta(m, \alpha) = F^\eta_m(\alpha)$. This is a function of size less than κ_n^+. So, by applying the Δ-system lemma, we get an unbounded $I \subset \kappa_n^{++}$, such that $\langle F^\eta \mid \eta \in I \rangle$ forms a Δ system, and the functions have the same value on the kernel. Note that this implies that for all η, δ in I and for all $n < m$, $i_m(a^{p_\eta}_n) \cap \mathrm{dom}(f^\delta_m) = \emptyset$.

By the inductive hypothesis, if $n > 0$, $\mathbb{P}_{0n}$ has the κ_{n-1}^{++}-c.c. So let η, δ be distinct points in I, such that if $n > 0$, $\pi_n(p^\eta)$ and $\pi_n(p^\delta)$ are compatible. We will construct $p \in \mathbb{P}_0$, such that $\pi_{n+1}(p)$ is a common extension of $\pi_{n+1}(p^\eta)$ and $\pi_{n+1}(p^\delta)$.

Let $\xi < \mu^+$ be above the supremum of the domains of $f^{p_\eta}_k(\vec{\delta})$ and $f^{p_\delta}_k(\vec{\delta})$, for $k \leq n, \vec{\delta} \in \mathrm{dom}(f^{p_\eta}_k) \cap \mathrm{dom}(f^{p_\delta}_k)$. Also, let r be a common extension of $\pi_n(p_\eta)$ and $\pi_n(p_\delta)$, such that for all $k < n$, $a^r_k = a^{p^\eta}_k \cup a^{p^\delta}_k \cup c$, where $c \subset \mu^+ \setminus \xi$.

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For $i < n$, set $a^p_i = a^r_i, A^p_i = A^r_i$. Then let

$$a^p_n = a^{p^\eta}_n \cup a^{p^\delta}_n \cup \bigcup_{i<n} a^p_i \cup \{\hat{\xi}\},$$

where $\hat{\xi} > \xi$. Then, set

$$A^p_n = \pi^{-1}_{\hat{\xi},\mathrm{mc}(a^{p^\eta}_n)}(A^{p^\eta}_n) \cap \pi^{-1}_{\hat{\xi},\mathrm{mc}(a^{p^\delta}_n)}(A^{p^\delta}_n).$$

Finally, for all $m > n$, let

$$f^m = f^\eta_m \cup f^\delta_m.$$

This is a well-defined function because the values on the kernel of the Δ-system obtained above are the same.

Denote:

- $f^m = [\nu \mapsto f^m_n(\nu)]_{E_{n+1,\alpha_{n+1}}}$;
- $f^m_n(\nu) := [\delta \mapsto f^m_n(\nu)(\delta)]_{E_{n+2,\alpha_{n+2}}}$;
- ... and so on until we reach m.

We use "$\forall^*_E$" to denote "for a measure one set in E." Then we have that:
$\forall^*_{E_{n+1.\alpha_{n+1}}} \nu_{n+1}, \forall^*_{E_{n+2},\alpha_{n+2}} \nu_{n+2}, \ldots, \forall^*_{E_m,\alpha_m} \nu_m$

$$(\dagger) : f^m_n(\nu_{n+1}), \ldots, (\nu_m) = f^{p_\eta}_n(\langle \nu_{n+1}, \ldots, \nu_m \rangle) \cup f^{p_\delta}_n(\langle \nu_{n+1}, \ldots, \nu_m \rangle)$$

and

$$\mathrm{dom}(f^m_n(\nu_{n+1}), \ldots, (\nu_m)) \cap a^p_n = \emptyset.$$

Then by taking intersections we have measure one sets $A^m_{n+1}, A^m_{n+2}, \ldots, A^m_m$, where each $A^m_i \in E_{i,\alpha_i}$, such that for all $\vec{\nu} \in [\prod_{n<i\leq m} A^m_i]^{<\omega}$, we have that the above equality holds.

For $i > n$, let

$$A^p_i = \bigcap_{i\leq m<\omega} A^m_i.$$

For $i \leq n$, let $f^p_i = f^r_i \restriction \mathrm{dom}(f^p_i)$.

For $\vec{\nu}$ in $[\prod_{i>n} A^p_i]^{<\omega}$, let

$$f^p_n(x)(\nu_{n+1}, \ldots, \nu_m) = f^m_n(x)(\nu_{n+1}), \ldots, (\nu_m).$$

Then p is as desired. ⊣

Corollary 2.11. *$(\mathbb{P}_0, \leq^*)$ preserves all cardinals in the interval $[\kappa^{++}_n, \kappa_{n+1}]$ for each n, and so preserves κ_ω and μ.*

Proof. (Sketch) Otherwise, suppose that for some n, we have some regular V-cardinal $\tau \in [\kappa^{++}_n, \kappa_{n+1}]$ that is collapsed. Let $p \in \mathbb{P}_0$, and $\lambda < \tau$ be such that $p \Vdash_{\mathbb{P}_0} \dot{h} : \lambda \to \tau$ is onto. Using that $\mathbb{P}_{0n+1}$ has the κ^{++}_n c.c. and $(\mathbb{P}_0 \restriction [n+1, \omega), \leq^*)$ is κ_{n+1}-closed, we build conditions $\langle q_\alpha \mid \alpha < \lambda \rangle$, such that:

(1) each $q_\alpha \restriction n+1 = p \restriction n+1$,
(2) $\langle q_\alpha \restriction [n+1, \omega) \mid \alpha < \kappa \rangle$ is $\leq^*$-decreasing,
(3) each $q_\alpha \Vdash \dot{h}(\alpha) \in X_\alpha$ for some $X_\alpha \subset \tau$ in V with $|X_\alpha| < \kappa^{++}_n$.

Let $X = \bigcup_{\alpha<\lambda} X_\alpha$, and let $q \leq^* p$ be such that for all $\alpha < \lambda$, $q \leq^* q_\alpha$. Then $q \Vdash_{\mathbb{P}_0} \mathrm{ran}(\dot{h}) \subset X$, but $|X| < \tau$. Contradiction. ⊣

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THEOREM 2.12 (Assaf Sharon, [8]). *In $V^{\mathbb{P}}$, $2^{\omega} < \kappa_{\omega}$, $\kappa_{\omega}^{\omega} = \kappa_{\omega}^{++}$, and there are no very good scales at κ_{ω}.*

In the generic extension, κ_{ω} is no longer strong limit, because $(\mathbb{P}, \leq^*)$ is only κ_0-closed. So there are no bounded subsets added to κ_0 and 2^{κ_0} becomes μ^+.

§3. The bad scale. In this section, we prove the following theorem.

THEOREM 3.1. *If κ_0 is indestructibly supercompact in V, then in $V^{\mathbb{P}}$ there is a bad scale of length μ in $\prod_{n<\omega} \kappa_n^{++}$ and so weak square at κ_{ω} fails.*

We assume that V is prepared so that the supercompactness of κ_0 is preserved by forcing with $\mathbb{P}_0$. Fix a scale $\langle g_{\alpha} \mid \gamma < \mu \rangle \in V$ in $\prod_n \kappa_n^{++}$ of length μ. Set $S := \{\gamma < \mu \mid \omega < \mathrm{cf}(\gamma) < \kappa_0, \gamma \text{ is a bad point for } \langle g_{\gamma} \mid \gamma < \mu \rangle\}$. By standard reflection arguments, S is stationary in V. Also, since $\mathbb{P}_0$ preserves μ and is κ_0-directed closed, $\langle g_{\gamma} \mid \gamma < \mu \rangle$ remains a bad scale after forcing with $\mathbb{P}_0$. More precisely, if G_0 is $\mathbb{P}_0$-generic, a point of cofinality less than κ_0 is bad in V iff it is bad in $V[G_0]$, and the set S is stationary in $V[G_0]$ (since κ_0 remains supercompact in $V[G_0]$).

DEFINITION 3.2. For $p \in \mathbb{P}$, set $t(p) = p \restriction [\mathrm{lh}(p), \omega)$. Let $\mathbb{D} := \{t(p) \mid p \in \mathbb{P}\}$, and $t(p) \leq_{\mathbb{D}} t(q)$ if for some large enough n, $1^\frown p \restriction [n, \omega) \leq_{\mathbb{P}} 1^\frown q \restriction [n, \omega)$.

Note that both $\mathbb{P}$ and $\mathbb{P}_0$ project to $\mathbb{D}$.

LEMMA 3.3. *$\mathbb{P}/\mathbb{D}$ has the μ-chain condition.*

PROOF. Let H^* be $\mathbb{P}_0$-generic and let H be the induced $\mathbb{D}$-generic object. It is enough to show that $\mathbb{P}/H$ has the μ-cc in $V[H^*]$. First note that for any $p \in \mathbb{P}$ with $p \restriction [n, \omega) \in H$ for some n, there are an n^* and $p' \in H^*$ such that $p' \leq_{\mathbb{P}_0} 1^\frown p \restriction [n^*, \omega)$.

Let $\langle p_{\alpha} \mid \alpha < \mu \rangle$ be a sequence of elements of $\mathbb{P}/H$ in $V[H^*]$. By the above note, for all α, there is an n_{α}, such that $1^\frown p_{\alpha} \restriction [n_{\alpha}, \omega) \in H^*$. By refining our list of conditions, we can assume that there is n^* such that $n_{\alpha} = n^*$ for all α. By extending each condition, we can assume it has length n^*. Now if we take the first $\kappa_{n^*-1}^{++}$ conditions, then by Proposition 2.10 there are two which are compatible. ⊣

So if H is $\mathbb{D}$-generic, since $\mathbb{P}_0$ projects to $\mathbb{D}$, we have that S is stationary in $V[H]$. Then by the μ-chain condition of $\mathbb{P}/H$, S is stationary after forcing with $\mathbb{P}$.

Let G be $\mathbb{P}$-generic.

LEMMA 3.4. *Let $\tau < \kappa_0$ be a regular uncountable cardinal in V (and so in $V[G]$), and suppose $V[G] \models A \subset ON, o.t.(A) = \tau$. Then there is a $B \in V$ such that B is an unbounded subset of A.*

PROOF. Let $p \in G$, $p \Vdash \dot{h} : \tau \to \dot{A}$ enumerate $\dot{A}$. By the Prikry lemma, define a $\leq^*$-decreasing sequence $\langle p_{\alpha} \mid \alpha < \tau \rangle$, such for every $\alpha < \tau$, $p_{\alpha} \leq^* p$ and there is $n_{\alpha} < \omega$, such that every $q \leq p_{\alpha}$ with length n_{α} decides $\dot{h}(\alpha)$. Then there is an unbounded $I \subset \tau$ and $n < \omega$ such that for all $\alpha \in I$, $n = n_{\alpha}$. Let p' be stronger than all p_{α} for $\alpha < \tau$. By appealing to density, we may assume that $p' \in G$. Let $q \leq p'$ be a condition in G with length n, and set $B = \{\gamma \mid (\exists \alpha \in I) q \Vdash \dot{h}(\alpha) = \gamma\}$. Then B is as desired. ⊣

LEMMA 3.5. *Suppose that in $V[G]$, $h \in \prod_n \kappa_n^{++}$. Then there is a function $\bar{h} \in \prod_n \kappa_n^{++}$ in V, such that for all large n, $h(n) \leq \bar{h}(n)$.*

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PROOF. Let p force that $\dot{h}$ is as in the statement of the lemma. For simplicity assume that the length of p is 0. For all n by the Prikry property there is some $p_n \leq^* p$ and k_n, such that every $r \leq p_n$ of length k_n decides the value of $\dot{h}(n)$. Since there are only κ_n^{++} such possibilities, by shrinking the measure one sets of $p_n \restriction [n+1, \omega)$, we may assume that $k_n = n+1$. In other words, every $n+1$-step extension of p_n decides $\dot{h}(n)$.

Doing this inductively, we arrange that $\langle p_n \mid n < \omega \rangle$ are $\leq^*$- decreasing. Let q be a lower bound, of length 0. For every $n+1$-step extension $q^\frown \vec{\nu}$, let $\gamma_{\vec{\nu}}$ be the value of $\dot{h}(n)$ decided by $q^\frown \vec{\nu}$. Define $\bar{h}(n) = \sup_{\vec{\nu}:|\vec{\nu}|=n+1} \gamma_{\vec{\nu}} < \kappa_n^{++}$.

Then q forces $\dot{h}(n) \leq \bar{h}(n)$. ⊣

COROLLARY 3.6. *$\langle g_\beta \mid \beta < \mu \rangle$ is a bad scale in $V[G]$.*

§4. Two metatheorems. In this section, we prove two metatheorems which roughly characterize when a diagonal Prikry forcing forces weak square to hold. We start with an abstract definition of what it means for a poset to be a *diagonal Prikry forcing*.

DEFINITION 4.1. A poset $(\mathbb{P}, \leq, \leq^*)$ is a *diagonal Prikry forcing* if there is an increasing sequence of regular cardinals $\langle \kappa_n \mid n < \omega \rangle$ where we define $\mu = (\sup_{n<\omega} \kappa_n)^+$ such that:

(1) Every $p \in \mathbb{P}$ is of the form $p = \langle p_n \mid n < \omega \rangle$.
(2) There is a function $\ell : \mathbb{P} \to \omega$ such that $p \leq^* q$ if and only if $\ell(p) = \ell(q)$ and $p \leq q$.
(3) If we define $\mathbb{P}^n = \{p \restriction [n, \omega) \mid p \in \mathbb{P}, \ell(p) = 0\}$ with the induced ordering from $\leq$ and $\mathbb{D} = \bigcup_{n<\omega} \mathbb{P}^n$ where we let $p \restriction [n, \omega) \leq q \restriction [m, \omega)$ if there is $k \geq \max(m, n)$ such that $p \restriction [k, \omega) \leq q \restriction [k, \omega)$ in $\mathbb{P}^k$, then we have the following:
   (a) $\mathbb{P}^n$ is countably closed and $< \kappa_n$-distributive.
   (b) For $n < m$, the map taking $p \restriction [n, \omega)$ to $p \restriction [m, \omega)$ is a good projection from $\mathbb{P}^n$ to $\mathbb{P}^m$.
   (c) $\mathbb{P}$ induces a generic for $\mathbb{D}$.
(4) Working in $V[\mathbb{P}^0]$, for all cardinals $\delta < \kappa_0$ which are successors of singular cardinals of cofinality ω, there is a condition $p \in \mathbb{P}/\mathbb{D}$ such that p forces that "δ is preserved and for all ordinals γ with $\mathrm{cf}(\gamma) = \delta$ and all $A \subseteq \gamma$ unbounded, there is $B \subseteq A$ unbounded with B in V."
(5) It is forced by $\mathbb{P}$ that "μ is preserved and its predecessor μ^- has the property that for all $\alpha < \mu^-$, $\alpha^\omega < \mu^-$.

We note that most posets which we think of as Diagonal Prikry forcing fit this definition. Examples include Diagonal Prikry forcing (see [4]), Gitik and Sharon's forcing [6], Gitik and Magidor's extender based forcing [5], the main forcing from this paper and any variation of these posets with collapses added.

We are now ready to prove two metatheorems about diagonal Prikry forcing and weak square. For the first one we will use the following fact, which appears in the last section of [9]. For the benefit of the reader we sketch the main points of the proof, see [9] for the complete argument.

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FACT 4.2. *Suppose that κ is a singular strong limit, and $\langle \tau_n \mid n < \omega \rangle$ is an increasing sequence of regular cardinals with limit κ. Then $\prod_n Col(\tau_n, \kappa^+)/fin$ adds weak square at κ.*

PROOF. Denote $\mathbb{C}_n := Col(\tau_n, \kappa^+)$ and let $\dot{C}_n$ be the $\mathbb{C}_n$-name for the generic club of κ^+ of order type τ_n. Let H be $\prod_n \mathbb{C}_n/fin$-generic. We say that a point γ is *good* if there is $[c_n]/fin \in H$ such that for all large n, $c_n \Vdash_{\mathbb{C}_n} \gamma \in \lim \dot{C}_n$. Then in $V[H]$, the set $X := \{\gamma < \kappa^+ \mid \gamma \text{ is good}\}$ is $> \omega$ -club. Now for every $\gamma \in X$, let $\vec{c}_\gamma$ and n_γ witness that $\gamma \in X$. That is for all $n \geq n_\gamma$, $\vec{c}_\gamma(n) \Vdash_{\mathbb{C}_n} \gamma \in \lim \dot{C}_n$. By density we may also assume that each $\vec{c}_\gamma(n)$ decides $\dot{C}_n \cap \gamma$, and let $Z_{\gamma,n}$ be that value. For $\gamma \in X$, define

$$\mathcal{C}_\gamma := \{ Y \subset \gamma \mid Y \text{ is a club in } \gamma, (\exists k \geq n_\gamma)(Y \subset \bigcap_{n \geq k} Z_{\gamma,n})\}.$$

And for $\gamma \in \text{cof}(\omega) \setminus X$, set $\mathcal{C}_\gamma = \{Y\}$, where Y is any cofinal ω sequence in γ. Then one can show that $\langle \mathcal{C}_\gamma \mid \gamma \in X \cup \text{cof}(\omega)\rangle$ is a $\square^*_\kappa$ sequence. ⊣

THEOREM 4.3. *Suppose that $\mathbb{P}$ is a diagonal Prikry forcing and that there is a model V_0 with the same cardinals as V where μ^- is strong limit, $\mathbb{P}^0 \in V_0$, $V = V_0[A]$ for some generic A and $\mathbb{P}^0$ is mutually generic with the forcing to add A. If for all $n < \omega$ forcing with $\mathbb{P}^n$ over V_0 collapses μ, then forcing with $\mathbb{P}$ over V adds a $\square^*_{\mu^-}$-sequence.*

PROOF. By clause (3)(c) of Definition 4.1 and the conditions of the theorem, it is enough to show that forcing with $\mathbb{D}$ over V_0 adds a weak square sequence. We can assume that for all large n, $\langle \mathbb{P}^n, \leq^* \rangle$ collapses μ to have cofinality κ_n. Then by the above mentioned fact, it follows that weak square is added by $\mathbb{D}$. We remark that the proof is essentially a reconstruction of the proof of the following fact. ⊣

An application of the above theorem is that the standard way of replacing long extenders with short extenders in the modified EBF will add weak square. In this application, we take A to be trivial. We remark that forcing with short extenders is the usual method of interleaving collapses in extender based forcing.

The natural way of combining short extenders with modified EBF is to replace the $a \subseteq \kappa_\omega^{++}$ in the definition of $\mathbb{Q}_{n0}$ with an order preserving function $a : \kappa_\omega^{++} \to \kappa_n^{+n+2}$ with extra properties. Without further modifications, this poset will not have the κ_ω^{++}-cc. To recover the κ_ω^{++} chain condition, we have to define an equivalence relation and take a quotient of the poset. We refer the reader to [3] or [7] for details of this method. We let $\mathbb{P}_s$ be this *modified short extender based forcing* and we will show that it adds weak square by applying Theorem 4.3.

We isolate a relevant part of $\mathbb{P}_s$ given by the following definition.

DEFINITION 4.4. Let $\mathbb{Q}^*_{n0}$ be the poset whose conditions are of the form $\langle a, A, f \rangle$ where a is a function as in the definition of the short extender forcing (see [2] Definition 2.1 and the modification Definition 2.11), $A \in E_{n \max(\text{ran}(a))}$ and f is as in clause (2) of Definition 2.4 and the ordering is the natural one combining $\leq^*$ from the short extender forcing and the modified EBF.

It is easy to see that $\mathbb{Q}^*_{n0}$ adds an order preserving map from κ_ω^{++} to κ_n^{+n+2}. Unfortunately, it is not true in this case that $\mathbb{D}$ is $\prod_{n<\omega} \mathbb{Q}^*_{n0}/fin$. The issue is that the final order we use on $\mathbb{P}_s$ involves the equivalence relation required to regain the κ_ω^{++}-cc. To this end, we will show that even if we quotient $\mathbb{Q}^*_{n0}$ by the restriction

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of the equivalence relation on $\mathbb{P}_s$, it still collapses μ. This will be enough to apply Theorem 4.3, since we can take $\mathbb{D}$ to be the mod finite product of the restricted $\mathbb{Q}^*_{n0}$. For definiteness, we denote this restricted poset $(\mathbb{Q}^*_{n0}, \to)$ and for those familiar with this type of forcing we note that we are using the equivalence relation $\leftrightarrow_{n,3}$ (See for example Definition 22 of [7]).

Before we prove that this poset collapses μ, we prove an easy claim that seems to be required in the proof.

CLAIM 4.5. *Let f be a part of some condition in $\mathbb{Q}^*_{n0}$. For all $m > n$, the set $\{\alpha < \kappa_\omega^{++} \mid \exists A_{n+1} \cdots \times A_m \forall \vec{\nu} \in \vec{A}, \alpha \in \mathrm{dom}(f(\vec{\nu}))\}$ has size at most κ_n.*

PROOF. If the set had size greater than κ_n, then using the completeness of the associated measures we would have a single sequence of measure one sets witnessing membership for more than κ_n different αs. This is impossible, since then there will be a $\vec{\nu}$ such that $|\mathrm{dom}(f(\vec{\nu}))| > \kappa_n$. ⊣

CLAIM 4.6. $(\mathbb{Q}^*_{n0}, \to)$ *collapses μ.*

PROOF. Let $R \subseteq \mu^+ \cap \mathrm{cf}(\omega)$ be such that $|R| = \kappa_n^+$ and for every $\gamma \in R$, there is an unbounded set $R_\gamma \subseteq \gamma$ of order type κ_ω. For every $c \in \gamma^\omega$ let the equivalence class of c be the set of $\{d \in \gamma^\omega \mid c, d \text{ coincide on a tail end}\}$. The number of these equivalence classes is μ, so we enumerate them as $\langle c^\gamma_\eta \mid \eta < \mu\rangle$.

Let H be generic for $(\mathbb{Q}^*_{n0}, \to)$. In $V[H]$ define $\phi : R \to \mu$ by setting $\phi(\gamma) = \eta$ if there are $p \in H$ and $c \in \gamma^\omega$ such that $c \in c^\gamma_\eta$, $c \subseteq \mathrm{dom}(a^p)$ and $a^p(c)$ coincides on a tail end with the least ω sequence of 2-good ordinals in its supremum. Here we take the least in some canonical well-ordering fixed in advance.

First, we show that ϕ is well defined. Suppose that for some γ we have conditions $p, q \in H$ and c, c' in γ^ω such that

- $c \subseteq \mathrm{dom}(a^p)$, $c' \subseteq \mathrm{dom}(a^q)$,
- $a^p(c)$ and $a^q(c')$ both coincide on a tail end with the least ω sequence in their respective supremum.

Let $r \in H$ be a common extension of p and q. Since a^r is order preserving, we define $\bar{\gamma} = \sup(a^r(c)) = \sup(a^r(c'))$. Using the definition of $\leftrightarrow_{n,3}$, we have that $\mathrm{ran}(a^r)$ contains a tail end of the least ω sequence of 2-good ordinals in $\bar{\gamma}$ and each of c and c' must map to this ω-sequence. So c and c' coincide on a tail.

Next we show that the range of ϕ is unbounded in μ. For every $\delta < \mu$, we claim that the set $D = \{p \mid (\exists \gamma \in R)(\exists \eta > \delta)(\exists c \in c^\gamma_\eta)$ such that p and c witness $\phi(\gamma) = \eta\}$ is dense. Given a condition $q = \langle a, A, f\rangle$, let $d = \mathrm{dom}(a) \cup \{\alpha < \kappa_\omega^{++} \mid \exists A_{n+1} \cdots \times A_m \forall \vec{\nu} \in \vec{A}, \alpha \in \mathrm{dom}(f(\vec{\nu}))\}$. By Claim 4.5 this has size at most κ_n. Now let $\gamma \in R$ be such that $|R_\gamma \setminus \beta| = \kappa_\omega$, where $\beta := \sup(d \cap R)$. Denote $R'_\gamma := R_\gamma \setminus \beta$. $(R'_\gamma)^\omega$ has size μ, and so there is some $\eta, \delta < \eta < \mu$, such that $c^\gamma_\eta \subset R'_\gamma$. Construct an extension $\langle a', A', f'\rangle$, such that $c^\gamma_\eta \subset \mathrm{dom}(a')$, and $a'(c^\gamma_\eta) = b$, where $\min(b) > \sup(\mathrm{ran}(a))$, and b coincides on a tail end with the least ω-sequence of 2-good ordinals cofinal in $\sup(b)$. We also shrink the measure one sets A^q_i to ensure that for all relevant $\vec{\nu}$, $\mathrm{dom}(f'(\vec{\nu})) \cap \mathrm{dom}(a') = \emptyset$.

It follows that ϕ collapses μ. ⊣

So by Theorem 4.3 we get:

COROLLARY 4.7. $\mathbb{P}_s$ *adds a weak square sequence.*

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It follows that the natural attempt to bring the construction form the previous two sections to $\aleph_\omega$ will add weak square, since such a forcing will induce a generic for $\mathbb{P}_s$.

The inclusion of A in Theorem 4.3 allows us to apply the theorem to situations like the one in the authors' previous paper [9] where μ^- is first forced not to be strong limit and then diagonal Prikry forcing is defined afterward.

We now prove the second of our two metatheorems.

THEOREM 4.8. *Let $\mathbb{P}$ be a diagonal Prikry forcing. If for some n, $\Vdash_{\mathbb{P}^n}$ "μ is preserved, $\mathbb{P}/\mathbb{D}$ is μ-cc and there is a μ-supercompactness embedding with critical point κ_0.", then it is forced by $\mathbb{P}$ that $\square^*_{\mu^-}$ fails.*

PROOF. Let H^* be $\mathbb{P}^n$-generic over V and let H be the induced $\mathbb{D}$-generic. By hypothesis $\mathbb{P}/H$ is μ-cc and there is $j : V[H^*] \to M$ witnessing that κ_0 is μ-supercompact.

We assume that it is forced by $\mathbb{P}/H$ over $V[H]$ and that there is a weak square sequence $\langle \mathcal{C}_\alpha \mid \alpha < \mu \rangle$. We may assume that for all $\alpha < \mu$, there is $\dot{D}_\alpha \in \mathcal{C}_\alpha$ such that $\mathrm{ot}(\dot{D}_\alpha) = \mathrm{cf}(\alpha)$ and for all $C \in \mathcal{C}_\alpha$, $\mathrm{ot}(C) < \mu^-$.

Work in M with the forcing $j(\mathbb{P}/H)$. Let $\gamma = \sup j``\mu$. By clause (4) of Definition 4.1 there is a $p^* \in j(\mathbb{P}/H)$ such that p^* forces μ is preserved and for all unbounded subsets $A \subseteq \gamma$ there is a $B \subseteq A$ unbounded with $B \in M$. So p^* forces that there is $\dot{E}_\gamma \in j(\vec{\mathcal{C}})_\gamma$ of ordertype μ and extending p^* if necessary, we can assume that there is a club $E \subseteq \gamma$ in M which p^* forces to be a subset of E_γ.

Let $D = \{\alpha < \mu \mid j(\alpha) \in E\}$. By standard arguments D is $< \kappa_0$-club. Let ζ be such that $D \cap \zeta$ has ordertype μ^-.

CLAIM 4.9. *For all $x \subseteq D \cap \zeta$ with $\mathrm{ot}(x) = \omega$, there is $p_x \in \mathbb{P}/H$ such that p_x forces $x \subseteq C$ for some $C \in \mathcal{C}_\zeta$.*

The claim is easy by noticing that $j(x) = j``x \subseteq E \cap j(\zeta)$ which p^* forces is a subset of $E_\gamma \cap j(\zeta) \in j(\vec{\mathcal{C}})_{j(\zeta)}$. Using the fact that $\mathbb{P}/H$ is μ-cc, we can find a generic G for $\mathbb{P}/H$ which contains p_x for μ many x. This is a contradiction as there are only μ^- many possibilities for the C of the claim and each has ordertype less than μ^- so by clause (5) of Definition 4.1 there can be at most μ^- such subsets. ⊣

We note that the assumption $\Vdash_{\mathbb{P}^n}$ $\mathbb{P}/\mathbb{D}$ is μ-cc is typical for diagonal Prikry forcings. The modified extender based forcing from this paper will satisfy it by the proof of Lemma 3.3.

Theorems 4.3 and 4.8 provide a dichotomy for approaching solutions to Woodin's question, "Is it consistent that both SCH and weak square fail at $\aleph_\omega$?" Suppose that one constructs a diagonal Prikry forcing as above, which forces the failure of SCH at $\aleph_\omega$. The above two theorems taken together say roughly that weak square holds in the extension if and only if each $\mathbb{P}^n$ collapses μ.

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