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Affine cubic functions

III. The real plane

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The classification of affine cubic functions in the real case is a fairly easy corollary of that in the complex case (9). However as the results can be easily interpreted by diagrams, one can obtain a much richer understanding. For example, the question of which types of cubic curve occur as level curves of which types of function is now much less trivial. This will lead us first to re-examine the classification of cubic curves going back to Newton (4). Next the 'dynamic' approach of considering these curves as members of families leads to the diagrams associated with the umbilic catastrophes of Thom (8). However the consideration of functions rather than of curves gives a 1-dimensional foliation of these diagrams which we describe next. We conclude by placing the results back in a projective setting.

CLASSIFICATION

We shall use the results in the complex case, and determine the real forms of the list obtained in (9). For division I we put the binary cubic formed from the terms of degree 3 in the given affine cubic f in the standard form x^3+y^3. In the real case we have two types, corresponding respectively to the elliptic and hyperbolic umbilic, with standard forms

$$\text{(hyperbolic)} \quad x^3+y^3 \text{ or set } x=u+v,\ y=u-v \text{ to yield } 2(u^3+3uv^2)$$
$$\text{(elliptic)} \quad 2(u^3-3uv^2).$$

The corresponding curves have 1 resp. 3 real asymptotes and are called by Newton (4) defective resp. redundant hyperbolas. We will label the cases I^- resp. I^+.

Division I^- (*defective hyperbolas*)

For the function

$$x^3+y^3+2axy+bx+cy$$

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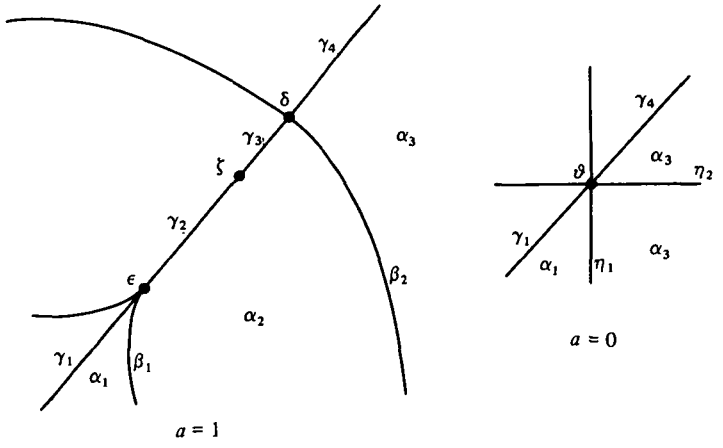


Fig. 1

we found that if there are two critical points at the same level, or more generally if any level curve is reducible, then $b^3 = c^3$. In the real case, this implies $b = c$. We have a critical point of multiplicity at least 2 if Δ vanishes, where

$$\Delta = a^3 - 6a^4bc - 4a^2(b^3 + c^3) - 3b^2c^2.$$

To indicate this on a diagram, note that Δ is weighted homogeneous, where $w(a) = 1$, $w(b) = w(c) = 2$. It thus suffices to illustrate the sections $a = 1$ and $a = 0$ (see Fig. 1).

On $a = 1$, the curve $\Delta = 0$ has two components, β_1 and β_2, dividing the plane into three regions; α_1, α_2 and α_3. The line $b = c$ is an axis of symmetry, meeting the curve in two points (δ and ϵ) and containing a third catastrophic point (ζ) at the origin. These points divide the line into four segments: γ_1, γ_2, γ_3 and γ_4. These labels are taken from ((9), Table 1) which we repeat here for convenience.

As a tends to 0 the parabolas asymptotic to $\Delta = 0$ converge to lines, which correspond to functions of type η.

Table 1

Species	Conditions	Singular levels	Species	Conditions	Singular levels
α	$\Delta \neq 0, b^3 \neq c^3$	$A_1/A_1/A_1/A_1$	ϵ	$b^3 = c^3 = -a^6 \neq 0$	A_3/A_1
β	$\Delta = 0, b^3 \neq c^3, a \neq 0$	$A_2/A_1/A_1$	ζ	$b = c = 0, a \neq 0$	$3A_1/A_1$
γ	$b^3 = c^3 \neq 0, -a^6, a^6/27$	$2A_1/A_1/A_1$	η	$a = bc = 0, b \neq c$	A_2/A_2
δ	$b^3 = c^3 = a^6/27 \neq 0$	$A_2/2A_1$	θ	$a = b = c = 0$	D_4

Division I^+ (redundant hyperbolas)

Substituting $x = u + v$, $y = u - v$ in the above yields

$$f = 2(u^3 + 3uv^2) + 2a(u^2 - v^2) + (b + c)u + (b - c)v;$$

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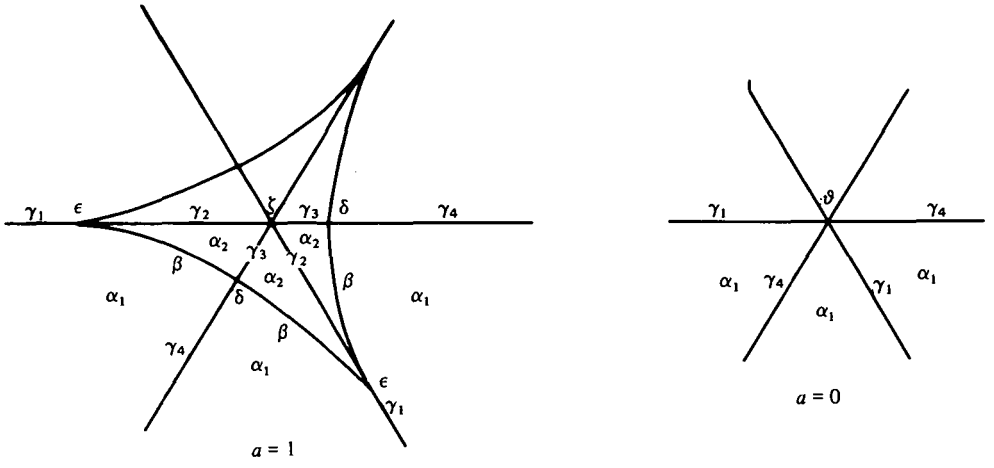


Fig. 2

we set $b+c = 2r$, $b-c = 2s$ and divide by 2 to obtain

$$\tfrac{1}{2}f = u^3 + 3uv^2 + a(u^2 - v^2) + ru + sv,$$
$$\Delta = a^8 - 6a^4(r^2 - s^2) - 8a^2(r^3 + 3rs^2) - 3(r^2 - s^2)^2.$$

To pass to case I^+ we replace v by iv, s by $-is$ so that

$$\tfrac{1}{2}f = u^3 - 3uv^2 + a(u^2 + v^2) + ru + sv,$$
$$\Delta = a^8 - 6a^4(r^2 + s^2) - 8a^2(r^3 - 3rs^2) - 3(r^2 + s^2)^2.$$

The curve $\Delta = 0$ on the level $a = 1$ is now a 3-cusped hypocycloid, as we could anticipate from papers of Porteous(6) and Zeeman(11), and the entire figure has the symmetry of a triangle (Fig. 2). On account of this symmetry the regions do not all correspond to distinct types. When $a = 0$, the hypocycloid collapses to a point. Species η does not occur.

To summarize, the list of species in division I is

$$I^- \ \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \delta, \epsilon, \zeta, \eta_1, \eta_2, \theta.$$
$$I^+ \ \alpha_1, \alpha_2, \beta, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \delta, \epsilon, \zeta, \theta.$$

Division II (*parabolic hyperbolas*)

Both points at infinity must be real: they have different multiplicities so cannot be complex conjugates. We may thus take the terms of degree 3 as x^2y and, as in the complex case, $f(x, y) = x^2y + y^2 + bx + cy$. Again, recall (9) the table for the complex case:

Table 2

Species	Conditions	Singular levels
α	$b \neq 0 \quad 4c^2 + 27b^2 \neq 0$	$A_1/A_1/A_1$
β	$b \neq 0 \quad 4c^3 + 27b^2 = 0$	A_2/A_1
γ	$b = 0 \quad c \neq 0$	$2A_1/A_1$
δ	$b = 0 \quad c = 0$	A_3

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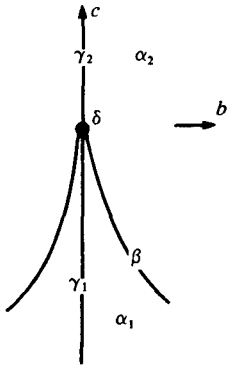


Fig. 3

In the figure (Fig. 3) there is a symmetry (change the signs of x and b). We thus have just six species: II $\alpha_1, \alpha_2, \beta, \gamma_1, \gamma_2, \delta$.

Division III (*diverging parabolas*)

Changing the sign of f if necessary, we arrive easily at the normal form $f = x^3+y^2+3ax$. We thus have three species: $\alpha_1\ (a < 0)$, $\beta(a = 0)$, $\alpha_2(a > 0)$.

Division IV (*hyperbolisms*)

Again we arrive at the same normal form as before $f(x, y) = x^2y+bx+cy$; there is a symmetry, changing the signs of b and x. As for division II, we have six species.

$\alpha_1(c < 0, b \neq 0)$, $\gamma_1(c < 0, b = 0)$ give curves with 3 real asymptotes: hyperbolisms of the hyperbola;

$\alpha_2(c > 0, b \neq 0)$, $\gamma_2(c > 0, b = 0)$ give curves with 1 real asymptote: hyperbolisms of the ellipse; and

$\beta(c = 0, b \neq 0)$, $\delta(c = 0, b = 0)$ give curves with 2 real asymptotes: hyperbolisms of the parabola.

Divisions V, VI

It is elementary to list these.

$$V\ \alpha\ x(y-x^2) \quad \beta\ (y-x^3) \quad \gamma_1\ (x^3-x) \quad \gamma_2\ (x^3+x) \quad \delta\ (x^3)$$
$$VI\ \alpha_1\ (x^2-y^2) \quad \alpha_2\ (x^2+y^2) \quad \beta\ (x^2+y) \quad \gamma\ (x^2) \quad \delta\ (x).$$

The irreducible levels of $V\alpha$ give trident curves; of $V\beta$ cubical parabolae.

MODULI AND AUTOMORPHISMS

The description of connected groups of automorphisms is essentially as in the complex case: a stratum of dimension $n \leqslant 7$ is a single orbit under the affine group, and (except for $V\gamma$) the level curves of f fall into finitely many orbits; if $n \geqslant 8$, the

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stabilizer of f is finite and it takes $(n-8)$ moduli to determine f or a special level curve; $(n-7)$ for a general level. The cases are as follows:

Species	I	α	β	γ	δ	ϵ	ζ	η	θ	II	α	β	γ	δ	III	α	β
n		10	9	9	8	8	8	8	7		9	8	8	7		8	7

Species	IV	α	β	γ	δ	V	α	β	γ	δ	VI	α	β	γ	δ
n		8	7	7	6		7	6	5	4		6	5	4	3

Next we ask which f are equivalent to $-f$ (possibly plus some constant) under $Aff_2(\mathbb{R})$: in our previous terminology, whether the image of α_f contains any negative numbers. The answer is no for all f of divisions II, III (because of the y^2 term), $VI\,\alpha$, β, γ; yes for $VI\,\delta$, all f of division V (change sign of x and/or y), IV (change signs of x, y) and (ditto) those f in division I with $a = 0$. For other f in division I the answer is no (as in the complex case).

This discussion of those elements in $Aff_2(\mathbb{R})$ which preserve f (i.e. G_{0f} in the notation of (9)) is virtually identical with the complex case when we have the same normal form (divisions II–VI): one merely has to exclude cases where non-real numbers appear explicitly (e.g. $x' = \omega x$ or ix). Division I is more interesting. For I^-, as we have the same normal form and must exclude complex cube roots of unity we have a single symmetry of order 2 for species of types γ, δ, ϵ, ζ, θ: none for α, β, η. This is given by interchanging x and y. For I^+, the 'same' symmetry is given in the same cases by changing the sign of v. In addition, when $r = s = 0$ (species $I^+\,\zeta$, θ) there is a threefold rotational symmetry.

Note that in the cases where f *is* equivalent to $-f$, (but not to $f+1$), the level curve $f = 0$ has an extra symmetry. If this (as affine curve) is singular this is less interesting, but for species $V\gamma$, $IV\alpha$, $I\eta$ and cases of $I\alpha, \gamma$ with $a = 0$ we have a non-singular curve.

Classification of cubic curves

The best way to see the geometry of the different types of functions is to draw a set of level curves for each of the 51 species. However we can make some of it explicit by describing the types of levels encountered. To do this we need some classification of cubic curves themselves. This is the classical part of the theory, but I shall give it in more descriptive notation. We will restrict to irreducible cubics, and to divisions I–III, as the other cases are more simply approached directly.

First consider the equisingularity classification of real projective cubics: we have five types, non-singular with one (E) or two (S, O) real branches; crunodal (L, X), acnodal (A) or cuspidal (C). The type of intersection of the projective curve Γ with the line at infinity z is given by the division: I^- (one transverse intersection), I^+ (three transverse intersections), II (tangent), III (inflexional tangent): we are assuming z not through a singular point. Moreover when the curve has a loop (either a separate component or with a crunode) we must distinguish cases O, L when z does not meet the loop from cases S, X when it does. Of course S and X cannot occur in divisions I^-, III. Otherwise all do occur, and are roughly illustrated in Fig. 4. Here the real projective

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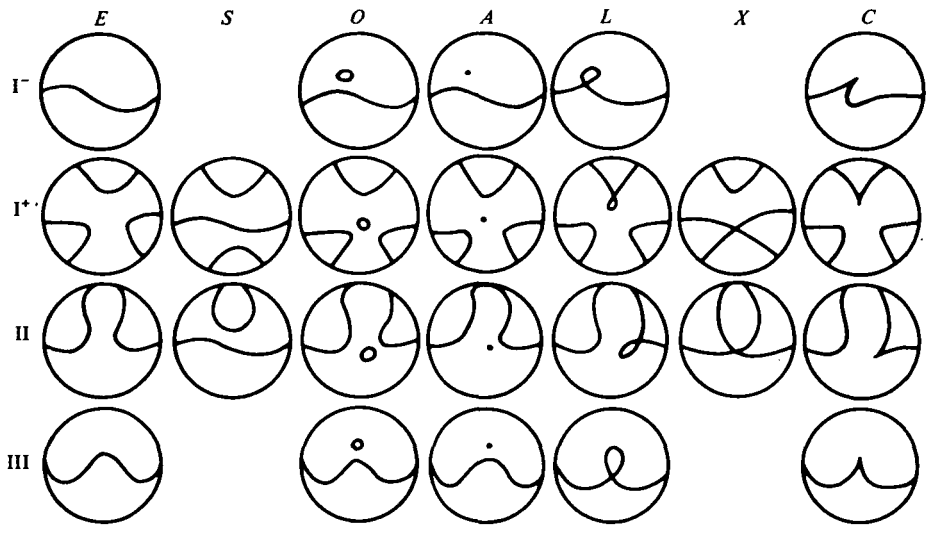


Fig. 4

plane is represented as a disc: the boundary corresponds to the line at infinity and opposite points on it are to be identified.

One can get a somewhat better geometric picture by taking into account curvature and inflexions. If we refine the divisions to take into account the number of *real* inflexion points at infinity we obtain eight: I_0^+, I_1^+, I_3^+, I_0^-, I_1^-, II_0, II_1 and III. For each transverse intersection point $P = \bar{\Gamma} \cap z$, the tangent l to $\bar{\Gamma}$ at P is an asymptote in the affine plane, In general, the two branches of Γ lie on opposite sides of l at opposite ends, but if l is an inflexional tangent, the two branches are on the same side and the asymptote is said to be osculating.

Of the 56 combinations we have already noted that I_0^-, I_1^-, III cannot have S or X curves: nor in fact can I_3^+; while I_3^+ and I_1 cannot have L or C curves. This reduces us to 44 cases. On the other hand, $I_1^+ E$, 0 and A each split into two subcases a and b according as the two finite inflexions are on different branches or the same branch; and I_1^- 0, A and $II_1 S$ split into subcases c and d according as the conchoidal part of the curve is convex or concave towards the other component. This raises the number of cases back to 50.

This list should be compared with that of Newton(4). Newton's classification took symmetries into account, so in division I^+ the condition $a = 0$ (concurrence of the asymptotes) becomes important. These leads to a trifurcation of cases $I_0^+ E$, $I_0^+ S$, $I_0^+ X$, $I_1^+ Ea$, $I_1^+ Eb$, $I_1^+ S$, $I_1^+ X$ and $I_3^+ E$ and a doubling of case $I_0^- E$ increasing the number of cases to 67 of which Newton omitted 6 in error ($I_1^+ Ob$, $I_1^+ Ab$, $I_3^+ O$, $I_3^+ A$, $II_1 O$ and $II_1 A$), later added by Stirling(7) and Murdoch(3); also Gua de Malves(2). To these one adds the 7 hyperbolisms, increased to 9 by counting the symmetry in $IV\alpha$ as a catastrophe, trident and cubical parabola to obtain Newton's list.

Newton's classification was reconsidered by Plücker(5) and Cayley(1). For division

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I, the 3 points in which the curve meets its asymptotes lie on a line L, and a finer classification was suggested to take account of the position of this line. For the curve $x^3+y^3+6axy+bx+cy+d=0$ the asymptotes are $x+\omega^2y-2\omega a=0$ (where $\omega^3=1$) with collective equation $x^3+y^3-8a^3+6axy$, so L is given by $bx+cy+(d+8a^3)=0$. Clearly this is not related to our species, where d is ignored. In fact our classification pertains to the pencil $f+\lambda z^3$ (where z is the line at infinity) and Plücker's to the pencil $f+\lambda z^2L$.

Observe that the refinement of divisions to take account of inflexions at infinity corresponds to our classification of functions:

Species $I\alpha, \beta, \eta$	contain curves of type I_0,
Species $I\gamma, \delta, \epsilon$; $I^-\zeta, \theta$	contain curves of type I_1,
Species $I^+\zeta, \theta$	contain curves of type I_3,
Species $II\alpha, \beta$	contain curves of type II_0,
Species $II\gamma, \delta$	contain curves of type II_1.

Thus we may drop the suffices 0, 1, 3 from now on. Recall also from the last section that these conditions (osculating asymptotes) are just those under which extra symmetries are present.

Reducible cubics also appear in the cases where there is an osculating asymptote (clearly a component of Γ is itself an osculating asymptote for all levels of the same function. We shall denote these by the letter R.

Description of the level curves of the functions

The level curves of the function $f(x,y)$ are the curves $f(x,y)=t$ ($t\in\mathbb{R}$). Each point of the plane belongs to just one such curve. As t increases from $-\infty$ to $+\infty$, the curves change smoothly, except when t passes through one of the 'special levels' of f, when there is a transition to a different type. Curves labelled E, O, S are non-singular; A, X, L, C and R are all singular, hence transitional types. For each type of function, we will now list the sequence of types of level curve as t increases from $-\infty$ to $-\infty$. These can mostly be obtained by inspection. The following diagram of reducible curves has also helped the author (Fig. 5).

Division I^-

α_1	$E\,A\,O\,L\,E\,L\,O\,A\,E$	γ_1	$E\,A_d\,O_d\,R\,O_d\,A_d\,E$
β_1	$E\,A\,O\,L\,E\,C\,E$	ϵ	$E\,A_d\,O_d\,R\,E$
α_2	$E\,A\,O\,L\,E$	γ_2	$E\,A_d\,O_d\,R\,O_c\,L\,E$
β_2	$E\,C\,E$	ζ	$E\,R\,O_c\,L\,E$
α_3	E	γ_3	$E\,R\,E\,A_c\,O_c\,L\,E$
η_1	$E\,C\,E\,C\,E$	δ	$E\,R\,E\,C\,E$
θ	$E\,R\,E$	γ_4	$E\,R\,E$
η_2	E		

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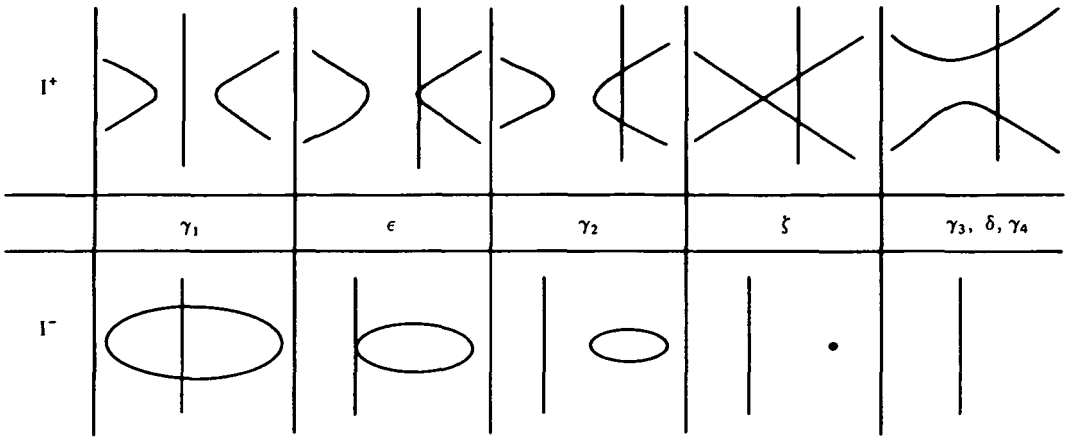


Fig. 5

Division I^+

α_1 $E\,X\,S\,X\,E$
β $E\,X\,S\,X\,E\,C\,E$
α_2 $E\,X\,S\,X\,E\,L\,O\,A\,E$
θ $E\,R\,E$

γ_1 $E_a\,X\,S\,R\,S\,X\,E_a$
ϵ $E_a\,R\,S\,X\,E_a$
γ_2 $E_a\,A_a\,O_a\,R\,S\,X\,E_a$
ζ $E\,A\,O\,R\,E$
γ_3 $E_b\,A_b\,O_b\,L\,E_b\,R\,E_b$
δ $E_b\,C\,E_b\,R\,E_b$
γ_4 $E_b\,R\,E_b$

Division II

α_1 $E\,A\,O\,L\,E\,X\,S$
β $E\,C\,E\,X\,S$
α_2 $E\,X\,S$

γ_1 $E\,A\,O\,R\,S_d$
δ $E\,R\,S_d$
γ_2 $E\,X\,S_c\,R\,S_d$

Division III

α_1 $E\,A\,O\,L\,E$
β $E\,C\,E$
α_2 E

Corresponding to the fact that our normal form for division III was the standard universal unfolding of an A_2 singularity, these curves fit a standard cusp diagram (Fig. 6), where the points on a vertical line represent curves corresponding to different levels of the same function. The cusp represents a curve of type C: points on the upper (resp. lower) branch of the curve correspond to curves of type L (resp. A) and points of the region inside (resp. outside) to curves of type O (resp. E).

Similarly in division II we have a universal unfolding of an A_3 singularity and a standard dovetail diagram (Fig. 7). The upper region here is labelled S, its frontier X and the lower region E. There is an 'interior' region O with upper boundary L and lower A: these meet along the cuspidal curve C. Two sheets cross along a curve R,

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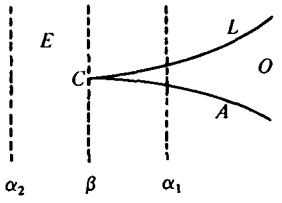


Fig. 6

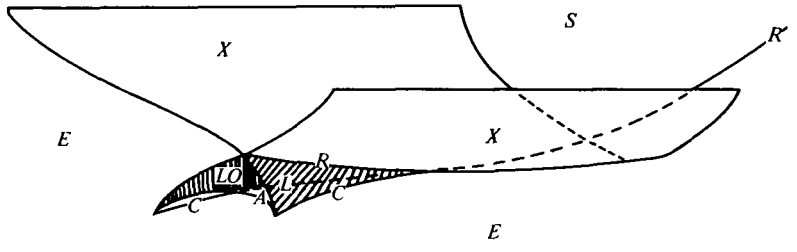
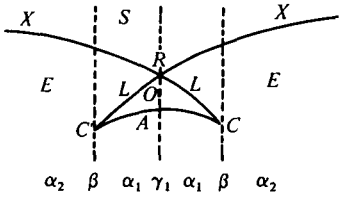
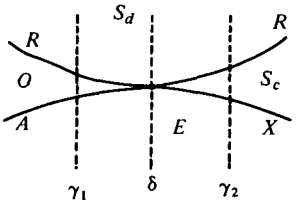


Fig. 7

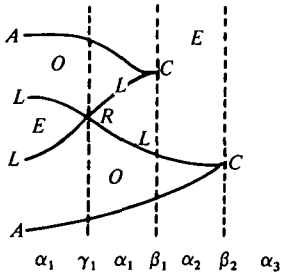


Section $c = -1$



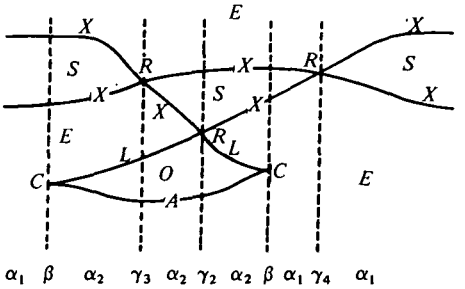
Section $b = 0$

Fig. 8



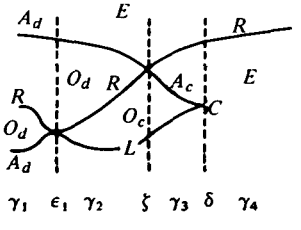
Section of I^- ($a = 1, c = -2$)

Fig. 9



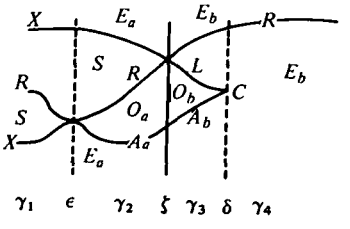
Section of I^+ ($a = 1, r = 10s + 1$)

Fig. 10



Section of I^- ($a = 1, b = c$)

Fig. 11



Section of I^+ ($a = 1, s = 0$)

Fig. 12

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which continues after the catastrophe as a spike in the S region. To see this more clearly, we draw the sections $c = -1$ and $b = 0$ (Fig. 8). The 'spike' corresponding to R in the γ_2 case is customarily omitted from catastrophe diagrams.

The perceptive reader may note a slight misfit here with the diagrams associated by Zeeman and some others with the corresponding catastrophes. This is explained as follows. The universal unfolding for a standard singularity of type A_n is

$$f_a(x) = x^{n+1} + \sum_1^n a_i x^{n-1}.$$

The critical levels ('catastrophe set') are given by $0 = \partial f_a(x)/\partial x$: an equation expressing a_{n-1} as function of $a_1, \ldots, a_{n-2}, x$. The projection on the parameter space is singular where $0 = \partial a_{n-1}/\partial x$ i.e. $0 = \partial^2 f_a/\partial x^2$: the critical locus is obtained by eliminating x from $0 = \partial f_a/\delta x = \delta^2 f_a/\partial x^2$. Now $g_a(x) = \partial f_a/\partial x$ is (up to unimportant constants) the standard universal unfolding for A_{n-1}; the critical points of the surface $g_a(x) = 0$ are given by $0 = g_a(x) = \partial g_a(x)/\partial x$, and the projection on the parameter space is as above.

In general, some curves of type R (in species $I^+\gamma_2$, γ_3, γ_4 and $I^-\gamma_1$) have crunodes and the corresponding lines separate regions of different types: the rest correspond to spikes as above. In a section transverse to $I\gamma$ the former give rise to crunodes and the latter to acnodes.

The diagrams for division I are also built from pieces like this. For I^- we indicate (in the plane $a = 1$) the sections by the line $c = -2$, (Fig. 9) and for I^+ by $s = \frac{1}{10}(r-1)$ (Fig. 10). This illustrates all the cases of type α, β and γ other than the acnodes. To get a better picture of the more special cases, we also look along a line of symmetry $b = c$ resp. $s = 0$ (of course this section is non-generic) (see Figs. 11 and 12). The reader with good geometrical feeling can compare these with figures for umbilics given in (10) (11). It is of interest to note that subspecies c and d are separated (Figs. 8, 11) by an R-curve; subspecies a and b in division I_1^+ by I_3^+, i.e. by species ζ.

These diagrams fit with the fact that there exists a geometric automorphism interchanging f and $-f$, hence reversing the levels, in no cases for divisions II, III and just for $a = 0$ with division I – i.e. for (some) functions of species $I^-\alpha_1$, α_3, γ_1, γ_4, η_1, η_2, θ, $I^+\alpha_1$, γ_1, γ_4, θ (but not for $I^-\beta_2$, $III\ \alpha_2$, β). Using this fact, it is easy to verify that each of the classes of curves we have defined is a connected subset of the space of all curves.

AFFINE AND PROJECTIVE CURVES

We conclude by describing, as in (9), for various curves in the projective plane, which choices of line to take 'at infinity' determine which types of function, as above. This helps illuminate those relationships between cases which are obscured by the normal forms for the separate divisions.

Threefold line. Only one real type: $V\delta$.

Repeated line x^2y. If $l = x$ $VI\delta$, if $l = y$ $VI\gamma$; if l through $x = y = 0$ $V\gamma_1$, otherwise $IV\delta$.

Concurrent lines. (a) *all real.* For l one of the lines $VI\gamma$, through the intersection $V\gamma_1$, otherwise $I^+\theta$.

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(*b*) *one real, two conjugate pairs.* l the real line $x\ VI\gamma$, l one of the two Hessian lines h_1, h_2 of the three lines $V\delta$, l in angle between h_1 and h_2 containing $x\ V\gamma_1$, in the other angle $V\gamma_2$, not through the intersection point $I^-\theta$.

Three general lines (*a*) *all real.* l one of the lines $VI\alpha_1$, through intersection of two $IV\gamma_1$, otherwise $I^+\zeta$.

(*b*) *one real.* l the real line $VI\alpha_2$, through intersection of other two $IV\gamma_2$, otherwise $I^-\zeta$.

Conic and tangent. l the tangent $VI\beta$, through the point of contact $IV\beta$, l touches conic $II\delta$, l cuts conic twice $I^+\epsilon$, l has no real intersection with conic $I^-\epsilon$.

This and subsequent cases can be illuminated by a diagram representating the dual plane, and indicating the family corresponding to each position of l on the diagram. As before, we represent the plane by a disc. The family of tangents to a conic is a conic in the dual plane. If we choose the line dual to the point of contact as the line at infinity, we have Fig. 13.

Conic and chord.

(*a*) *real conic, real intersection points.* The Hessian cubic is another conic, having double contact with the given one along the chord, together with the chord. The dual picture is similar: represent the Hessian as dotted. In Fig. 14 on account of the symmetry just half the dual plane is shown.

(*b*) *real conic, imaginary intersection points.* The conic component of the Hessian has no real points.

For l the given line $VI\alpha_2$; if l touches the conic $I\gamma_2$; and if l meets/doesn't meet the conic (i.e. the corresponding point is outside/inside the dual conic) we have $I^+\gamma_1/I^-\gamma_2$.

(*c*) *imaginary conic.* In this case the Hessian gives a real conic, not meeting the given line. For l the given line, $VI\alpha_2$; if l touches the Hessian conic $I^-\delta$; otherwise if l meets/doesn't meet it we have $I^-\gamma_4/I^-\gamma_3$.

Cuspidal cubic $y^2z = x^3$. Here it is convenient to draw (half) the dual plane (Fig. 15).

Crunodal cubic. The dual of this curve is essentially the same as the quartic illustrated by Fig. 1: here the line at infinity is dual to the node. The Hessian yields a similar curve, through the cusp of the first (see Fig. 16).

The main regions belong to families as indicated in Fig. 16. However this is not the whole story. A point in one of the shaded regions represents a curve of type L, otherwise (including the line of separation) of type X. A point in the lower left hand region (or its boundary curve) represents a curve which is an extreme critical level of the corresponding function; the other region of type $I^+\alpha_2$ (and curve of type $I^+\beta$) corresponds to one of the middle critical levels. There is a similar distinction in type $IV\alpha_1$.

The types occurring along the line of symmetry are shown in Fig. 18.

Acnodal cubic. Here, as we can now expect, we find two tricuspidal quartics. The Cayleyan (envelope of lines such that tangents at the 3 intersection points concur)

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is a circle inscribed to the dual of the Hessian and escribed to the other. This configuration first appears in Porteous(6).

For l through the acnode (on the edge of the diagram) we have $IV\gamma_2$ on the lines of symmetry, $IV\alpha_2$ otherwise. For the main regions see Fig. 17 (on account of the symmetry just a hexant is shown), and along a line of symmetry, see Fig. 18.

I shall not attempt to present the corresponding figures for non-singular cubics. We see from the complex case that harmonic and equianharmonic cubics will play special rôles: now as any real cubic can be uniquely put in the form

$$x^3+y^3+z^3+3\lambda xyz=0 \quad (\lambda\in\mathbb{R})$$

which is singular for $\lambda=-1$, harmonic for $\lambda=-1\pm\sqrt{3}$ and equianharmonic for $\lambda=0, 2$, we see that there will be ten cases to consider, which sounds too complicated to be worth the effort. However, this complication arises from the interplay of the curve with its Hessian. If we ignore the latter, the duals of the various types of irreducible curve divide the plane as in Fig. 19. In each case the regions which are heavily shaded, unshaded, the curve between them and the cusps on it correspond to affine curves of respective types I^+, I^-, II and III (with E, O, L, A, C for the diagrams in turn): the lightly shaded region and its boundary curve correspond to types I^+ and II with shapes S and X.

For species $I\beta$, the line at infinity touches the Hessian $H(f)$; the curve β in Figs. 1, 2 separates regions where $H(f)$ meets $z=0$ in 3 real points from those where there is just 1 real intersection. Since for $I\zeta$, $H(f)$ meets $z=0$ in the same three points as does f, we deduce that $H(f)$ meets $z=0$ in

3 real points for species $I^-\alpha_2$, γ_2, γ_3, ζ, $I^+\alpha_1$, γ_1, γ_4

1 real point for species $I^-\alpha_1$, α_3, γ_1, γ_4, $I^+\alpha_2$, γ_2, γ_3, ζ.

Of course for species $I\beta$, δ, ϵ, η, θ there is a repeated intersection, which must therefore be real.

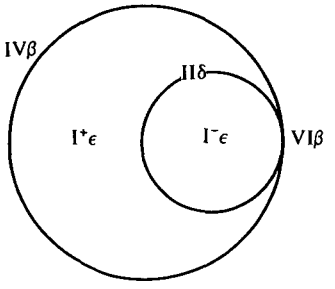


Fig. 13

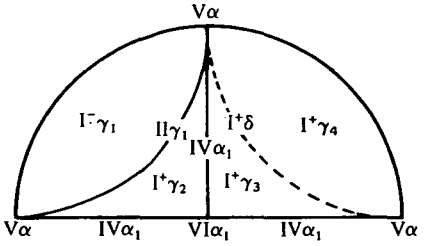


Fig. 14

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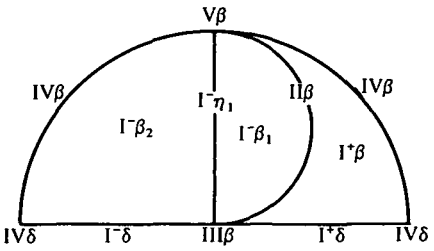


Fig. 15

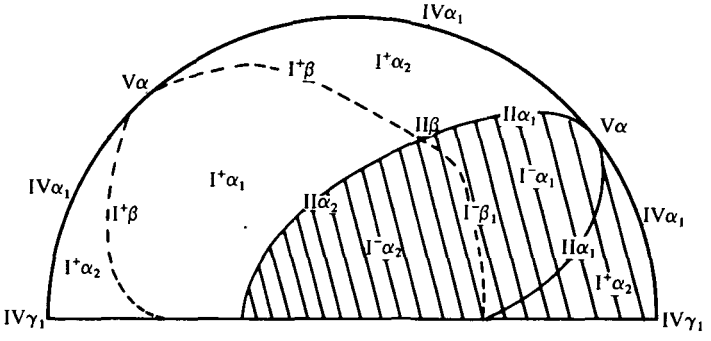


Fig. 16

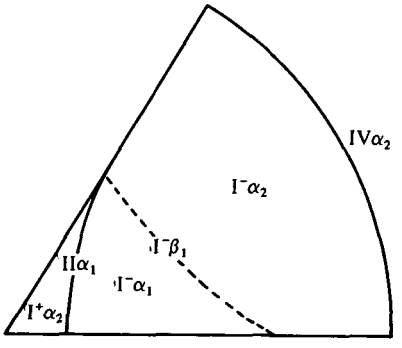


Fig. 17

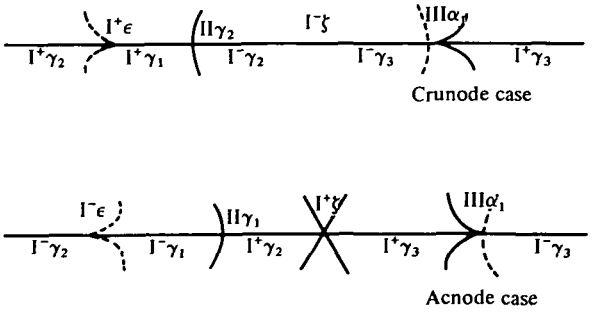


Fig. 18

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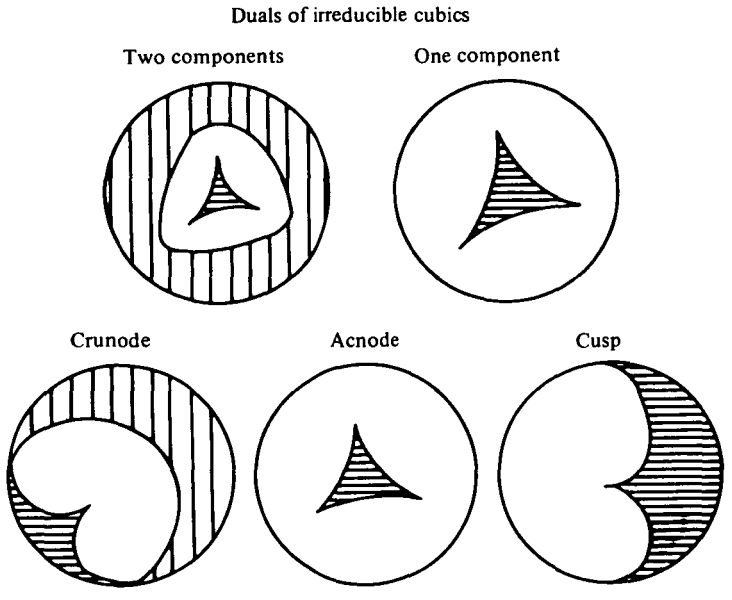


Fig. 19

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