Ergod. Th. & Dynam. Sys. (2013), **33**, 144–157 © *Cambridge University Press, 2012*
doi:10.1017/etds.2012.131

Decompositions of factor codes and embeddings between shift spaces with unequal entropies

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(*Received* 6 *January* 2012 *and accepted in revised form* 27 *March* 2012)

Abstract. Given a factor code between sofic shifts X and Y, there is a family of decompositions of the original code into factor codes such that the entropies of the intermediate subshifts arising from the decompositions are dense in the interval from the entropy of Y to that of X. Furthermore, if X is of finite type, we can choose those intermediate subshifts as shifts of finite type. In the second part of the paper, given an embedding from a shift space to an irreducible sofic shift, we characterize the set of the entropies of the intermediate subshifts arising from the decompositions of the given embedding into embeddings.

1. *Introduction*

Problems concerning decompositions of one code into several codes were considered in symbolic dynamics for coding purposes. The decomposition theorem of Williams states that every conjugacy between two shifts of finite type can be decomposed into a composition of splitting codes and amalgamation codes [**21**]. Since closing codes form an important class of finite-to-one codes, Adler and Marcus [**1**] asked whether every finite-to-one factor code between irreducible shifts of finite type can be represented as a composition of closing codes. This is the case in the eventual category, but not true in general [**12**]. Previous research on decompositions has concentrated on finite-to-one factor codes [**4, 19, 20**]. In particular, Boyle proved that up to conjugacy there are only finitely

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https://doi.org/10.1017/etds.2012.131 Published online by Cambridge University Press

many decompositions of a finite-to-one factor code between irreducible shifts of finite type into factor codes between irreducible shifts of finite type [**4**]. But so far, not much is known about decompositions of codes between shift spaces with unequal entropies, even in the category of irreducible shifts of finite type.

In this paper, we consider decompositions of codes between shift spaces with unequal entropies. Let $\phi : X \to Y$ be a factor code between shift spaces. We are interested in the set $\mathcal{S}(\phi)$ of all the entropies of the intermediate subshifts arising from the decompositions of ϕ, defined by

$$\mathcal{S}(\phi) = \{h(\phi_1(X)) : \phi = \phi_2 \circ \phi_1 \text{ with } \phi_1 \text{ and } \phi_2 \text{ factor codes}\},$$

where h denotes topological entropy. Since the class of shifts of finite type is a fundamental and tractable class of shift spaces, if X is of finite type we are also interested in the set $\mathcal{S}_0(\phi)$ of all the entropies of the intermediate shifts of finite type arising from the decompositions, defined by

$$\mathcal{S}_0(\phi) = \{h(\phi_1(X)) : \phi = \phi_2 \circ \phi_1 \text{ with } \phi_1 \text{ and } \phi_2 \text{ factor codes and } \phi_1(X) \text{ of finite type}\}.$$

In [**7**], Boyle and Tuncel showed that if $\phi : X \to Y$ is a factor code between irreducible shifts of finite type with unequal entropies then $h(X)$, and hence each element in $\mathcal{S}_0(\phi)$ other than $h(Y)$, are limit points of $\mathcal{S}_0(\phi)$. Thus up to conjugacy there are infinitely many decompositions of a factor code between irreducible shifts of finite type with unequal entropies into factor codes. In §3, we extend this result as follows (see Theorems 3.2 and 3.5).

THEOREM 1.1. *Let $\phi : X \to Y$ be a factor code between sofic shifts. Then $\mathcal{S}(\phi)$ is dense in the interval $[h(Y), h(X)]$. If X is of finite type, then $\mathcal{S}_0(\phi)$ is also dense in $[h(Y), h(X)]$.*

We remark that by the result of Lindenstrauss [**16**], if a factor map $\phi : X \to Y$ between shift spaces is given, then for any $h \in [h(Y), h(X)]$ we can find a topological dynamical system (Z, T) such that there are surjective homomorphisms $\phi_1 : (X, \sigma) \to (Z, T)$ and $\phi_2 : (Z, T) \to (Y, \sigma)$ with $\phi = \phi_2 \circ \phi_1$ and $h(T) = h$. However, even when X and Y are shifts of finite type, the constructed system Z is far from a shift space. Indeed, if X is a sofic shift then $\mathcal{S}(\phi)$ must be a countable set since factors of X are sofic and there are countably many real numbers that can appear as the entropies of sofic shifts, namely, rational multiples of logarithms of Perron numbers.

The existence of factor codes and that of embeddings (especially between irreducible shifts of finite type with different entropies) are closely related in symbolic dynamics [**2, 14**]. From this viewpoint, we turn to decompositions of embeddings in §4. Let $\phi : X \to Y$ be an embedding into an irreducible shift space. We are interested in the set $\mathcal{T}(\phi)$ of all the entropies of the intermediate shift spaces arising from decomposing ϕ into embeddings, defined by

$$\mathcal{T}(\phi) = \{h(\mathrm{dom}(\phi_2)) : \phi = \phi_2 \circ \phi_1 \text{ with } \phi_1 \text{ and } \phi_2 \text{ embeddings and } \mathrm{dom}(\phi_2) \text{ irreducible}\},$$

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where $\mathrm{dom}(\phi_2)$ is the domain of ϕ_2. We impose an irreducibility condition on the domain of ϕ_2 since otherwise the problem of finding a decomposition $\phi = \phi_2 \circ \phi_1$ into embeddings with $h(\mathrm{dom}(\phi_2)) = h$ is reduced to that of finding a subshift of Y with entropy h, and this problem is completely understood in symbolic dynamics. Define $\mathcal{T}_0(\phi)$ (respectively $\mathcal{T}_1(\phi)$) as for $\mathcal{T}(\phi)$ with the additional condition that $\mathrm{dom}(\phi_2)$ is of finite type (respectively sofic). To exclude a subtle problem at the point $h(X)$, we define $\mathcal{T}'(\phi) = \mathcal{T}(\phi)\backslash\{h(X)\}$ and similarly for $\mathcal{T}_0'(\phi)$ and $\mathcal{T}_1'(\phi)$. It is well-known that if Y is a mixing shift of finite type then $\mathcal{T}_0(\phi)$ is dense in $[h(X), h(Y)]$ (see [**8**]). We refine this result and characterize these sets as follows (see Corollaries 4.9, 4.10 and 4.15). In what follows, $\mathcal{P}$ (respectively $\mathcal{P}^w$) denotes the set of all real numbers $h \geq 0$ such that e^h is a Perron number (respectively weak Perron number).

THEOREM 1.2. *Let $\phi : X \to Y$ be an embedding from a shift space X into an irreducible sofic shift Y. Then we have*

$$\mathcal{T}'(\phi) = (h(X), h(Y)] \quad \textit{and} \quad \mathcal{T}_1'(\phi) = (h(X), h(Y)] \cap \mathcal{P}^w.$$

If X and Y are irreducible shifts of finite type with periods p and q, respectively, then

$$\mathcal{T}_0(\phi) = [h(X), h(Y)] \cap \{h \in \mathbb{R} : r \cdot h \in \mathcal{P} \textit{ for some } r \in \mathbb{N} \textit{ with } q|r \textit{ and } r|p\}.$$

We also present characterizations of the sets $\mathcal{T}$, $\mathcal{T}_0$ and $\mathcal{T}_1$ for other cases in §4.

2. *Background*

We introduce some terminology and known results. For further details, see [**15**]. A *shift space* (or *subshift*) is a closed σ-invariant subset of a full shift over some finite set of symbols. For a subshift X, denote by $\mathcal{B}_n(X)$ the set of all words of length n appearing in the points of X and $\mathcal{B}(X) = \bigcup_{n\geq 0} \mathcal{B}_n(X)$. Also let $\mathcal{A}_X = \mathcal{B}_1(X)$. For a finite set $\mathcal{A}$ and $l \in \mathbb{N}$, denote by $\mathcal{A}^l$ the set of all words of length l over $\mathcal{A}$ and let $\mathcal{A}^* = \bigcup_{l\geq 0} \mathcal{A}^l$.

A subshift X is called *non-wandering* if for all $u \in \mathcal{B}(X)$, there is a word w with $uwu \in \mathcal{B}(X)$. It is called *irreducible* if for all $u, v \in \mathcal{B}(X)$, there is a word w with $uwv \in \mathcal{B}(X)$. It is called *mixing* if for all $u, v \in \mathcal{B}(X)$, there is an integer $N \in \mathbb{N}$ such that whenever $n \geq N$, we can find $w \in \mathcal{B}_n(X)$ with $uwv \in \mathcal{B}(X)$. The *period* of X, denoted by $\mathrm{per}(X)$, is the greatest common divisor of periods of all periodic points of X. The *entropy* of a shift space is defined by $h(X) = \lim_{n\to\infty}(1/n) \log |\mathcal{B}_n(X)|$, which equals the topological entropy of (X, σ) as a dynamical system. By the variational principle, topological entropy is concentrated on the non-wandering set in the sense that if Z is the maximal non-wandering set of X then $h(Z) = h(X)$.

A *code* $\phi : X \to Y$ is a continuous σ-commuting map between shift spaces. It is called a *factor code* (respectively an *embedding*) if it is surjective (respectively injective). Every code can be recoded to be a 1-block code, i.e., a code for which x_0 determines $\phi(x)_0$.

Let X be a subshift over a finite set $\mathcal{A}$. Then one can find a set $\mathcal{F}_X$ of *forbidden words* so that X is the set of all sequences in $\mathcal{A}^{\mathbb{Z}}$ which do not contain any words in $\mathcal{F}_X$. If there is such a set $\mathcal{F}_X$ of words all of which have length $k + 1$ with $k \geq 0$, then X is called a (k-*step*) *shift of finite type*. Equivalently, X is k-step if every word $v \in \mathcal{B}_k(X)$ is a *synchronizing word* for X, i.e., if uv and vw are in $\mathcal{B}(X)$ then we have $uvw \in \mathcal{B}(X)$.

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A *sofic shift* is a factor of a shift of finite type. Given a sofic shift X, there is a factor code $\pi : \widetilde{X} \to X$ where $\widetilde{X}$ is of finite type with $h(\widetilde{X}) = h(X)$. Furthermore, if X is irreducible (respectively mixing), then $\widetilde{X}$ can be chosen to be irreducible (respectively mixing). Every sofic shift X contains a family of shifts of finite type whose entropies are dense in $[0, h(X)]$ (see [**17**]).

A real number ≥ 1 is called a *Perron number* if it is an algebraic integer which strictly dominates all the other algebraic conjugates. A real number λ is a Perron number if and only if there is a mixing shift of finite type X with $h(X) = \log \lambda$. Similarly, there is an irreducible shift of finite type X of period $p \in \mathbb{N}$ with $h(X) = \log \lambda$ if and only if λ^p is a Perron number. A real number $\lambda \geq 1$ is called a *weak Perron number* if λ^p is a Perron number for some $p \in \mathbb{N}$.

3. *Decompositions of factor codes*

In [**7**, Proposition 7.3], Boyle and Tuncel considered the notion of a magic diamond to investigate properties of Markovian codes, and obtained the following result.

PROPOSITION 3.1. [**7**] *Let $\phi : X \to Y$ be a factor code between irreducible shifts of finite type. Then every element in $\mathcal{S}_0(\phi)\backslash\{h(Y)\}$ is a limit point of $\mathcal{S}_0(\phi)$ from the left.*

In this section, we extend this result and show that under the condition of Proposition 3.1, $\mathcal{S}_0(\phi)$ is indeed dense in the interval $[h(Y), h(X)]$. We first consider the sofic case in which we do not require intermediate subshifts to be of finite type. This result can also be obtained from the result of Lindenstrauss (see Remark 3.3 (2)).

THEOREM 3.2. *Let $\phi : X \to Y$ be a factor code between sofic shifts. Then $\mathcal{S}(\phi)$ is dense in $[h(Y), h(X)]$.*

Proof. Since the case where $h(X) = h(Y)$ is clear, assume that $h(X) > h(Y)$. Let $h \in (h(Y), h(X))$ and $\epsilon > 0$. We claim that there is a decomposition $\phi = \phi_2 \circ \phi_1$ of ϕ into factor codes such that $h(\phi_1(X))$ is ϵ-close to h. First, find a shift of finite type $Z \subsetneq X$ with $h(Z) > h(Y)$ and $|h(Z) - h| < \epsilon/2$. By passing to higher block shifts and renaming symbols, we may assume that:

(a) ϕ is a 1-block code;
(b) Z is 1-step;
(c) $\mathcal{A}_Z$ and $\mathcal{A}_Y$ are disjoint;
(d) if $a, b \in \mathcal{A}_Z$ and $ab \in \mathcal{B}(X)$, then $ab \in \mathcal{B}(Z)$.

Let $\mathcal{A} = \mathcal{A}_Z \cup \mathcal{A}_Y$. Define a 1-block code $\theta : X \to \mathcal{A}^{\mathbb{Z}}$ by

$$\theta(x)_i = \begin{cases} \phi(x_i) & \text{if } x_i \notin \mathcal{A}_Z, \\ x_i & \text{if } x_i \in \mathcal{A}_Z, \end{cases}$$

and let $\tilde{Z}_0 = \theta(X)$. By the condition (d), each point in $\tilde{Z}_0$ is uniquely factored as (possibly infinite) Z-words and Y-words (hence if $u \in \mathcal{A}_Z^*$ occurs in $\tilde{Z}_0$, then $u \in \mathcal{B}(\tilde{Z}_0)$).

Also define a (3-block) code $\alpha : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ by

$$\alpha(x)_i = \begin{cases} \phi(x_i) & \text{if } x_i \in \mathcal{A}_Z \text{ and if } x_{i-1} \in \mathcal{A}_Y \text{ or } x_{i+1} \in \mathcal{A}_Y, \\ x_i & \text{otherwise,} \end{cases}$$

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and define $\tilde{Z}_n = \alpha(\tilde{Z}_{n-1})$ inductively for $n \in \mathbb{N}$. Note that for each $n \geq 1$, $\tilde{Z}_n$ is a sofic shift whose language contains no word of the form awb where $a, b \in \mathcal{A}_Z$ and $w \in \mathcal{B}(Y)$ with $1 \leq |w| \leq 2n$. Also $Z \subset \tilde{Z}_n$ for each $n \in \mathbb{N}$. Finally, for each $n \in \mathbb{N}$ define a shift space $\hat{Z}_n$ over $\mathcal{A}$ obtained by forbidding the set of words

$$\mathcal{F}_n = (\mathcal{F}_Y \cup \mathcal{F}_Z) \cup \bigcup_{j=1}^{2n} \mathcal{A}_Z \mathcal{A}_Y^j \mathcal{A}_Z,$$

where $\mathcal{F}_Y$ (respectively $\mathcal{F}_Z$) is a set of forbidden words of Y (respectively Z) over $\mathcal{A}_Y$ (respectively $\mathcal{A}_Z$). By (d), one can see that $\tilde{Z}_n \subset \hat{Z}_n$ for each $n \in \mathbb{N}$. By letting $\hat{Z} = \bigcap_{n\in\mathbb{N}} \hat{Z}_n$, we have $h(\hat{Z}) = \lim_n h(\hat{Z}_n)$ since $\{\hat{Z}_n\}_{n\in\mathbb{N}}$ is a decreasing sequence of shift spaces. Since $\hat{Z}$ contains no word of the form awb with $a, b \in \mathcal{A}_Z$ and $w \in \mathcal{B}(Y)$, it follows that

$$\mathcal{B}(\hat{Z}) = \{uvw : u, w \in \mathcal{B}(Y) \text{ and } v \in \mathcal{B}(Z)\}.$$

So the non-wandering set of $\hat{Z}$ is contained in $Y \cup Z$. Since the entropy is concentrated on the non-wandering set and $h(Y) < h(Z)$, we have

$$h(Z) \leq \lim_n h(\hat{Z}_n) = h(\hat{Z}) \leq h(Z).$$

Now take $N \in \mathbb{N}$ such that $|h(\tilde{Z}_N) - h(Z)| < \epsilon/2$ and let $\phi_1 = \alpha^N \circ \theta : X \to \tilde{Z}_N$. Also define a 1-block code $\phi_2 : \tilde{Z}_N \to Y$ by letting $\phi_2(z)_i = z_i$ if $z_i \in \mathcal{A}_Y$ and $\phi_2(z)_i = \phi(z_i)$ if $z_i \in \mathcal{A}_Z$. Then $\phi = \phi_2 \circ \phi_1$ and ϕ_2 is indeed a factor code. The following inequality completes the proof:

$$|h(\phi_1(X)) - h| = |h(\tilde{Z}_N) - h| \leq |h(\tilde{Z}_N) - h(Z)| + |h(Z) - h| < \epsilon. \qquad \square$$

Remark 3.3. (1) The proof of Theorem 3.2 also applies to the case where Y is a shift space and X is a shift space such that there are shifts of finite type $X_n \subset X$ with $\lim_{n\to\infty} h(X_n) = h(X)$ (for example, X may be an *almost specified shift* [**10**]).

(2) By corollary of Lindenstrauss' result stated in §1, the conclusion of Theorem 3.2 holds for arbitrary factor code $\phi : X \to Y$ between shift spaces. To see this, suppose that we have a factorization $\phi = \phi_2 \circ \phi_1$ where $\phi_1 : X \to Z$ and $\phi_2 : Z \to Y$ are surjective homomorphisms as in [**16**, §4]. Then Z is zero-dimensional and can be presented as an inverse limit of shift spaces Z_n. By uniform continuity of ϕ_2, there exists $N \in \mathbb{N}$ such that for every $n \geq N$, if π_n is a projection from Z onto Z_n, then there is γ_n such that $\phi_2 = \gamma_n \circ \pi_n$. So ϕ can be factored through the shift space Z_n. Since $h(Z) = \lim h(Z_n)$, it follows that $\mathcal{S}(\phi)$ is dense in $[h(Y), h(X)]$.

We now prove the case of shifts of finite type. The heart of Theorem 3.5 lies in the following proposition. For a shift space X and $n \in \mathbb{N}$, denote by $X^{[n]}$ the *nth higher block shift* of X (see [**15**, §1.4]).

PROPOSITION 3.4. *Let $\phi : X \to Y$ be a factor code from a shift of finite type X with $h(X) > h(Y)$. Then for any $\epsilon > 0$, there exists a decomposition $\phi = \phi_2 \circ \phi_1$ of ϕ into factor codes such that $|h(\phi_1(X)) - h(Y)| < \epsilon$ and $\phi_1(X)$ is of finite type.*

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Proof. We may assume that X is 1-step, ϕ is 1-block, and that $\mathcal{A}_X$ and $\mathcal{A}_Y$ are disjoint. Let $\mathcal{F}_Y$ be a set of forbidden words of Y over $\mathcal{A}_Y$.

For each $n \in \mathbb{N}$ and a symbol a not in $\mathcal{A}_Y$, define the set $\mathcal{F}_n(a)$ of words in $(\mathcal{A}_Y \cup \{a\})^*$ by

$$\mathcal{F}_n(a) = \mathcal{F}_Y \cup \left(\bigcup_{i=0}^{[n/4]-1} a\mathcal{A}_Y^i a \right),$$

(by convention we let $\mathcal{F}_n(a) = \mathcal{F}_Y$ for $n < 4$) and the shift space $\mathsf{X}_{\mathcal{F}_n(a)}$ over the alphabet $\mathcal{A}_Y \cup \{a\}$ by forbidding the set $\mathcal{F}_n(a)$. As $\mathcal{F}_n(a)$ increases with n, the shift space $\mathsf{X}_{\mathcal{F}_n(a)}$ decreases with n. Since $Y \subset \mathsf{X}_{\mathcal{F}_n(a)}$ for each $n \in \mathbb{N}$, and the non-wandering set of $\bigcap_{n\in\mathbb{N}} \mathsf{X}_{\mathcal{F}_n(a)}$ is contained in Y, we have

$$h(Y) \le \lim_n h(\mathsf{X}_{\mathcal{F}_n(a)}) = h\left(\bigcap_{n\in\mathbb{N}} \mathsf{X}_{\mathcal{F}_n(a)} \right) \le h(Y).$$

Choose $n \in \mathbb{N}$ such that $h(\mathsf{X}_{\mathcal{F}_n(a)}) < h(Y) + \epsilon$. Then indeed we have $h(\mathsf{X}_{\mathcal{F}_n(\bar{a})}) < h(Y) + \epsilon$ for any symbol $\bar{a}$ not in $\mathcal{A}_Y$. We may assume that n is a multiple of 4. Since X is 1-step, so is $X^{[n]}$ and each symbol of $X^{[n]}$ is a synchronizing word for $X^{[n]}$. Let $\mathcal{A} = \mathcal{A}_{X^{[n]}} \cup \mathcal{A}_Y$. In the remainder of the proof, we will regard an element in $\mathcal{B}_1(X^{[n]})$ as a symbol of $X^{[n]}$ and also as a word in $\mathcal{B}_n(X)$, depending on the context.

Now we partition $\mathcal{A}_{X^{[n]}} = \mathcal{B}_1(X^{[n]})$ into disjoint sets $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_N$ so that $\mathcal{E}_N$ contains all X-words of length n each of which has a 'large' self-overlap. The precise definitions of the $\mathcal{E}_j$ are as follows.

(1) For $j = 1, \ldots, N-1$, each set $\mathcal{E}_j$ contains exactly one symbol $a^{(j)}$ which, as an X-word, does not overlap itself, or overlaps itself only after a shift of more than $n/4$ symbols to the right. We require that $\bigcup_{j=1}^{N-1} \mathcal{E}_j$ exhausts all such symbols.

(2) The set $\mathcal{E}_N = \mathcal{B}_1(X^{[n]}) \setminus \bigcup_{j=1}^{N-1} \mathcal{E}_j$. Note that, each $b \in \mathcal{E}_N$, as an X-word, can overlap itself after a shift of at most $n/4$ symbols to the right (hence each b overlaps itself in at least $3n/4$ symbols).

Let $m \in \mathbb{N}$. Define a $(2m+1)$-block code $\phi^{(m)}$ from $X^{[n]}$ to $\mathcal{A}^{\mathbb{Z}}$ by

$$\phi^{(m)}(x)_0 = \begin{cases} x_0 & \text{if } x_i \in \bigcup_{k\ge j} \mathcal{E}_k \text{ for all } i = -m, \ldots, m, \\ \phi((x_0)_1) & \text{otherwise,} \end{cases}$$

where j is the unique number with $x_0 \in \mathcal{E}_j$ and $(x_0)_1$ is the first symbol of x_0 as an X-word of length n. (Intuitively, consider the case $n = 1$. Then the $\mathcal{E}_i$ give an order on the set $\mathcal{A}_X$ defined by $a < b$ if $a \in \mathcal{E}_i$ and $b \in \mathcal{E}_j$ with $i < j$. If we do not see a symbol 'less than' x_0 in the $[-m, m]$ neighborhood of x_0, we leave x_0 unchanged. If there is a symbol 'less than' x_0 in the neighborhood, we map x_0 using ϕ. The idea of $\phi^{(m)}$ is to 'leave' X symbols unchanged syndetically for each point in $\phi^{(m)}(X)$.)

CLAIM. *The image $Z_m = \phi^{(m)}(X^{[n]})$ is a $(2mN+1)$-step shift of finite type.*

Proof. First, we prove that any Z_m-word of length $(2mN+1)$ contains a symbol of $X^{[n]}$. Let $w \in \mathcal{B}_{2mN+1}(Z_m)$. Take $z \in Z_m$ with $z_{[-mN,mN]} = w$ and $x \in X^{[n]}$ with $\phi^{(m)}(x) = z$. Also for each $i = -mN, \ldots, mN$, let j_i be the unique index such that $x_i \in \mathcal{E}_{j_i}$. If $z_0 \in \mathcal{A}_{X^{[n]}}$, we are done. If not, then there exists $i_1 \in [-m, m]$ such that $j_{i_1} < j_0$. If $z_{i_1} \in \mathcal{A}_{X^{[n]}}$, we are done. If not, then there exists $i_2 \in [i_1 - m, i_1 + m]$ such that $j_{i_2} < j_{i_1}$ and

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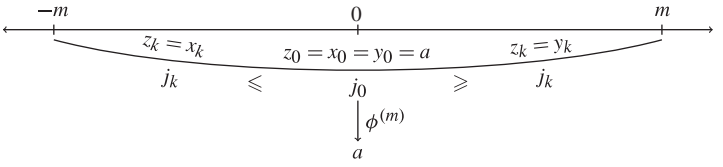


FIGURE 1. The 0th coordinate of $\phi^{(m)}(z)$.

so on. This process eventually terminates after at most $(N-1)$ steps and there is $i \in [-(N-1)m, (N-1)m]$ such that $j_i \leq j_k$ for $k = i-m, \ldots, i+m$. Then $z_i = x_i \in \mathcal{A}_{X^{[n]}}$, as desired.

Second, we show that each symbol in $\mathcal{A}_{X^{[n]}}$ is also a synchronizing symbol for Z_m. To show this, let $ua, av \in \mathcal{B}(Z_m)$ with $a \in \mathcal{A}_{X^{[n]}}$. Take $x, y \in X^{[n]}$ with $\phi^{(m)}(x)_{[-|u|,0]} = ua$ and $\phi^{(m)}(y)_{[0,|v|]} = av$. Since $a \in \mathcal{A}_{X^{[n]}}$ we have $x_0 = y_0 = a$. Then $z = x_{(-\infty,0)}y_{[0,\infty)} \in X^{[n]}$ since a is a synchronizing word for $X^{[n]}$. We claim that $\phi^{(m)}(z)_{[-|u|,|v|]} = uav$. Since $\phi^{(m)}$ has memory and anticipation m, we have $\phi^{(m)}(z)_i = \phi^{(m)}(y)_i$ for $i \geq m$. Let j_0 be the unique index with $a \in \mathcal{E}_{j_0}$. Since $ua, av \in \mathcal{B}(Z_m)$, we have $x_k, y_k \in \bigcup_{j \geq j_0} \mathcal{E}_j$ for each $k = -m, \ldots, m$ and hence $\phi^{(m)}(z)_0 = a$ (see Figure 1).

Now given $i = 1, \ldots, m-1$, since $z_i = y_i \in \bigcup_{j \geq j_0} \mathcal{E}_j$ we have the following cases (see Figure 2).

(1) If $z_i \in \bigcup_{j > j_0} \mathcal{E}_j$, then $\phi^{(m)}(z)_i = \phi((z_i)_1) = \phi((y_i)_1) = \phi^{(m)}(y)_i$.

(2) If $z_i \in \mathcal{E}_{j_0}$ and $z_k \in \bigcup_{j \geq j_0} \mathcal{E}_j$ for all $k \in [m+1, m+i]$, then

$$\phi^{(m)}(z)_i = z_i = y_i = \phi^{(m)}(y)_i.$$

(3) If $z_i \in \mathcal{E}_{j_0}$ and $z_k \in \bigcup_{j < j_0} \mathcal{E}_j$ for some $k \in [m+1, m+i]$, then

$$\phi^{(m)}(z)_i = \phi((z_i)_1) = \phi((y_i)_1) = \phi^{(m)}(y)_i.$$

Hence $\phi^{(m)}(z)_i = \phi^{(m)}(y)_i$ for $i \geq 0$. Similarly we have $\phi^{(m)}(z)_i = \phi^{(m)}(x)_i$ for $i \leq 0$, so $\phi^{(m)}(z)_{[-|u|,|v|]} = uav$, as desired.

Since every word in Z_m of length $(2mN+1)$ contains a synchronizing word for Z_m, it follows that Z_m is a $(2mN+1)$-step shift of finite type. □

Now we give an upper bound of the entropy of $Z_m = \phi^{(m)}(X^{[n]})$. We claim that $Z_m \subset \mathsf{X}_{\tilde{\mathcal{F}}_m}$, where $\mathsf{X}_{\tilde{\mathcal{F}}_m}$ is the subshift over $\mathcal{A}$ obtained by forbidding the set of words

$$\tilde{\mathcal{F}}_m = \mathcal{F}_Y \cup \left(\bigcup_{\substack{i,j=1 \\ i \neq j}}^{N} \bigcup_{k=0}^{m-1} \mathcal{E}_i \mathcal{A}_Y^k \mathcal{E}_j \right) \cup \left(\bigcup_{i=1}^{N-1} \bigcup_{k=0}^{(n/4)-1} \mathcal{E}_i \mathcal{A}_Y^k \mathcal{E}_i \right) \cup \left(\bigcup_{k=1}^{m-1} \mathcal{E}_N \mathcal{A}_Y^k \mathcal{E}_N \right).$$

To prove the claim, suppose that $z \in Z_m$.

(a) Since any word in $\mathcal{A}_Y^*$ appearing in z comes from applying ϕ on an X-word of the same length, it follows that a word in $\mathcal{F}_Y$ cannot occur in z.

(b) Suppose that a word of the form $a^{(i)}wa^{(j)}$, with $a^{(i)} \in \mathcal{E}_i$, $a^{(j)} \in \mathcal{E}_j$, $i \neq j$ and $|w| < m$ occurs in z. If $i > j$, then the definition of $\phi^{(m)}$ forces $a^{(i)}$ to be mapped by $\phi^{(m)}$ to $\phi((a^{(i)})_1)$ (note that $\phi^{(m)}$ has memory and anticipation m), a contradiction. Similarly we have a contradiction if $i < j$.

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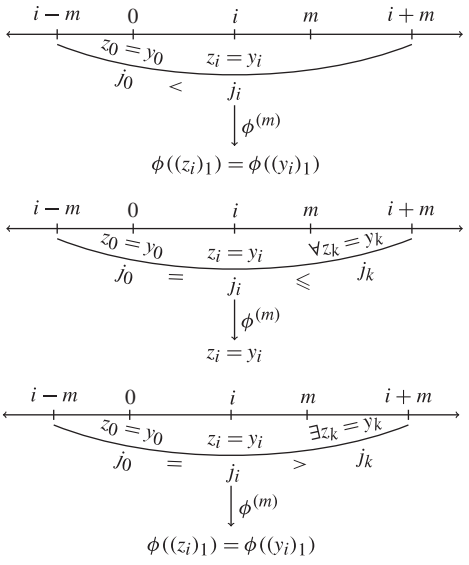


FIGURE 2. The ith coordinate of $\phi^{(m)}(z)$.

(c) If a word of the form $a^{(i)}wa^{(i)}$ with $|w| < n/4$ and $i < N$ occurs in z, then the last $(n - |w| - 1)$ symbols of $a^{(i)}$ (as an X-word) must equal the first $(n - |w| - 1)$ symbols of $a^{(i)}$ because of the overlapping property of $X^{[n]}$. But this is impossible, since each $a^{(i)}$, $i = 1, \ldots, N-1$, does not overlap itself in $\geq 3n/4$ symbols (as an X-word).

(d) Suppose that a word awb with $a, b \in \mathcal{E}_N$ and $w \in \bigcup_{k=1}^{m-1} \mathcal{A}_Y^k$ occurs in z. Let γ be a preimage of awb under $\phi^{(m)}$. Then γ must contain a subword of the form $aub \in \mathcal{B}(X^{[n]})$ in the center. However if u contains a symbol in $\bigcup_{k=1}^{N-1} \mathcal{E}_k$, then a and b must be mapped by $\phi^{(m)}$ to the symbols $\phi(a_1)$ and $\phi(b_1)$, respectively. Also, if u consists of symbols in $\mathcal{E}_N$, then since u is mapped to w by $\phi^{(m)}$, some symbol in $\bigcup_{k=1}^{N-1} \mathcal{E}_k$ appears to the left of a or to the right of b within distance m in γ. Thus at least one of a and b also has to be mapped by $\phi^{(m)}$ to $\phi(a_1)$ or $\phi(b_1)$, which is a contradiction.

Thus $z \in \mathsf{X}_{\tilde{\mathcal{F}}_m}$ and the claim holds.

Now $\bigcap_m \mathsf{X}_{\tilde{\mathcal{F}}_m}$ equals the shift space $\mathsf{X}_{\tilde{\mathcal{F}}}$ defined by forbidding the set of words $\tilde{\mathcal{F}} = \bigcup_{m\in\mathbb{N}} \tilde{\mathcal{F}}_m$. Note that in $\mathsf{X}_{\tilde{\mathcal{F}}}$ there is no transition of the form $\mathcal{E}_i \to \mathcal{E}_j$ or $\mathcal{E}_i \to Y \to \mathcal{E}_j$ with $i \neq j$, and also no transition of the form $\mathcal{E}_N \to Y \to \mathcal{E}_N$. Thus the non-wandering set of $\mathsf{X}_{\tilde{\mathcal{F}}}$ is contained in the set

$$Y \cup \left(\bigcup_{i=1}^{N-1} \mathsf{X}_{\mathcal{F}_n(a^{(i)})} \right) \cup \left(X^{[n]} \cap (\mathcal{E}_N)^{\mathbb{Z}} \right).$$

By the choice of n, we have $h(\mathsf{X}_{\mathcal{F}_n(a^{(i)})}) < h(Y) + \epsilon$ for each $i = 1, \ldots, N-1$. Also, the shift space $X^{[n]} \cap (\mathcal{E}_N)^{\mathbb{Z}}$ consists only of periodic points. To see this, let $x \in X^{[n]} \cap (\mathcal{E}_N)^{\mathbb{Z}}$. Then each x_i has a self-overlap of more than $3n/4$ symbols and for

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each $i \in \mathbb{Z}$, x_i and x_{i+1} overlap progressively. It follows that x_0 determines the whole point x and x must be periodic.

Hence it follows that

$$\lim_m h(\mathsf{X}_{\tilde{\mathcal{F}}_m}) = h(\mathsf{X}_{\tilde{\mathcal{F}}}) \leq \max_{1 \leq i < N} \{h(\mathsf{X}_{\mathcal{F}_n(a^{(i)})}), h(Y), h(X^{[n]} \cap (\mathcal{E}_N)^{\mathbb{Z}})\} < h(Y) + \epsilon$$

as desired. Take $M \in \mathbb{N}$ so that $h(\mathsf{X}_{\tilde{\mathcal{F}}_M}) < h(Y) + \epsilon$ and let $\phi_1 = \phi^{(M)} \circ \beta$, where $\beta : X \to X^{[n]}$ is the nth higher block code. Finally define a 1-block code $\phi_2 : Z_M \to Y$ by sending all symbols $a \in \mathcal{A}_{X^{[n]}}$ to $\phi((a)_1)$ and $a \in \mathcal{A}_Y$ to a. Then ϕ_1 and ϕ_2 satisfy all the conditions. □

Combining Theorem 3.2 and Proposition 3.4, we obtain the main theorem in this section.

THEOREM 3.5. *Let $\phi : X \to Y$ be a factor code with X of finite type. Then $\mathcal{S}_0(\phi)$ is dense in $[h(Y), h(X)]$.*

Proof. The case $h(X) = h(Y)$ is clear. Let $h \in (h(Y), h(X))$ and $\epsilon > 0$. Then by Theorem 3.2, there is a decomposition $\phi = \xi \circ \psi$ such that $|h(\psi(X)) - h| < \epsilon/2$. Now by Proposition 3.4, there is a decomposition $\psi = \eta \circ \phi_1$ such that $\phi_1(X)$ is of finite type and $|h(\phi_1(X)) - h(\psi(X))| < \epsilon/2$. By letting $\phi_2 = \xi \circ \eta$, we are done. □

There are some cases in which we can characterize the sets $\mathcal{S}(\phi)$ and $\mathcal{S}_0(\phi)$. For example, if X is a mixing shift of finite type and $Y = \{0^\infty\}$, then $\mathcal{S}(\phi) = \mathcal{S}_0(\phi) = [0, h(X)] \cap \mathcal{P}$, where $\mathcal{P}$ is the set of all numbers $h \geq 0$ such that e^h is a Perron number as in §1.

As relaxed restrictions are given in the construction of an infinite-to-one factor code between irreducible shifts of finite type with unequal entropies [**2, 7, 10**], we present the following conjecture. Note that in the following $\mathcal{S}(\phi)$ is always contained in $[h(Y), h(X)] \cap \mathcal{P}$.

CONJECTURE 3.6. *Let $\phi : X \to Y$ be a factor code between mixing sofic shifts. Then $\mathcal{S}(\phi) = [h(Y), h(X)] \cap \mathcal{P}$. If X is of finite type, then $\mathcal{S}_0(\phi) = [h(Y), h(X)] \cap \mathcal{P}$.*

4. *Decompositions of embeddings*

Let $\phi : X \to Y$ be an embedding. If we construct a subshift Z with $\phi(X) \subset Z \subset Y$, then this gives us a decomposition $\phi = \phi_2 \circ \phi_1$ of ϕ into embeddings where $\phi_2 : Z \to Y$ is the canonical embedding given by the inclusion and $\phi_1 : X \to Z$ is equal to ϕ except that the codomain of ϕ_1 is Z. Thus finding a decomposition of an embedding into embeddings is closely related to the construction of subshifts lying between two shift spaces. As there are many results on constructing new shift spaces in mixing shifts of finite type, we already know more about decomposition of embeddings than that of factors.

PROPOSITION 4.1. [**8**, §26] *Let $\phi : X \to Y$ be an embedding with Y of finite type. Then $\mathcal{T}_0(\phi)$ is dense in the interval $[h(X), h(Y)]$.*

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In this section, we refine this result and give characterizations of the sets $\mathcal{T}$, $\mathcal{T}_0$ and $\mathcal{T}_1$ for several cases. In contrast to decompositions of factor codes, it is easier to characterize these sets in finite-type case than in sofic case. We recall some definitions and results.

For a nonnegative integral square matrix A, we denote by X_A the *edge shift* defined by A, i.e., the shift space which consists of all bi-infinite trips on the directed graph with the adjacency matrix A. A shift of finite type is conjugate to an edge shift. An edge shift X is mixing if and only if $X = \mathsf{X}_A$ for some A which is *primitive*, i.e., $A^n > 0$ for some $n \in \mathbb{N}$. For a shift space X, denote by $q_k(X)$ the number of periodic points of X with least period k. If X is a mixing shift of finite type, then $h(X) = \lim_{k\to\infty} k^{-1} \log q_k(X)$.

The following proposition can be found in [**1**, Corollary 10.2] or [**13**].

PROPOSITION 4.2. *Let X be a mixing shift of finite type. Then there is a mixing shift of finite type $\widetilde{X}$ such that $\widetilde{X}$ has a fixed point and $h(\widetilde{X}) = h(X)$.*

The following lemma is referred to as the Blowing up Lemma.

LEMMA 4.3. [**2**] *Let X be an irreducible shift of finite type with $h(X) > 0$ and $q_n(X) > 0$. Let $M_1, \ldots, M_k \geq 1$. Then there is an irreducible shift of finite type $\widetilde{X}$ such that:*
(1) $q_n(\widetilde{X}) = q_n(X) - n + n \cdot |\{i : M_i = 1\}|$;
(2) $q_m(\widetilde{X}) = q_m(X) + n \cdot |\{i : nM_i = m\}|$ *if* $m = nM_i$ *for some* $M_i > 1$; *and*
(3) $q_i(\widetilde{X}) = q_i(X)$ *for all other* i. *Furthermore, if X is mixing then so is $\widetilde{X}$.*

LEMMA 4.4. [**11**] *Let $X \subset \widetilde{X}$ be shift spaces and $\phi : X \to Y$ a conjugacy. Then there exist a shift space $\widetilde{Y} \supset Y$ and a conjugacy $\tilde{\phi} : \widetilde{X} \to \widetilde{Y}$ such that $\tilde{\phi}|_X = \phi$.*

We also need the following lemma equivalent to the extension theorem of Boyle, which, in turn, is a slight extension of Krieger's Embedding Theorem [**14**].

LEMMA 4.5. [**3**, Lemma 2] *Let ϕ be an embedding from a shift space X into a mixing shift of finite type Y. If Z is a shift space with $X \subset Z$, $h(Z) < h(Y)$ and $q_k(Z) \leq q_k(Y)$ for all $k \in \mathbb{N}$ then there is an embedding $\tilde{\phi} : Z \to Y$ with $\tilde{\phi}|_X = \phi$.*

The following theorem is a key ingredient in characterizing the sets $\mathcal{T}$ and $\mathcal{T}_0$ for several cases.

THEOREM 4.6. *Let $X \subsetneq Y$ be shift spaces with Y mixing and sofic. Then for any $h \in (h(X), h(Y))$, there is a mixing shift space Z such that $X \subset Z \subset Y$ and $h(Z) = h$. Furthermore, the following hold.*
(1) If $h \in \mathcal{P}$, then Z can be chosen to be sofic.
(2) If $h \in \mathcal{P}$ and Y is of finite type, then Z can be chosen to be of finite type.

Proof. First, assume that Y is a shift of finite type and $h \in \mathcal{P}$, i.e., e^h is a Perron number. Then there is a mixing shift of finite type X_1 such that $h(X_1) = \log e^h = h$. By Proposition 4.2, one can find a mixing shift of finite type X_2 such that X_2 has a fixed point and $h(X_2) = h(X_1) = h$. Since $h(X) < h(X_2)$, it follows that $q_n(X) < q_n(X_2)$ for all sufficiently large $n \in \mathbb{N}$. Thus by applying the Blowing up Lemma to a fixed point in X_2, we can find a mixing shift of finite type W_1 such that $h(W_1) = h(X_2) = h$ and $q_n(X) \leq q_n(W_1)$ for all $n \in \mathbb{N}$.

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Since $h(W_1) < h(Y)$, we have $q_n(W_1) < q_n(Y)$ for all large $n \in \mathbb{N}$. By applying the Blowing up Lemma repeatedly to points in W_1 having low periods, it is possible to obtain a mixing shift of finite type W_2 such that $h(W_2) = h$ and

$$q_n(X) \leq q_n(W_2) \leq q_n(Y) \quad \text{for all } n \in \mathbb{N}.$$

(We remark that this is an argument that appeared in [**2**, §2].)

By Krieger's Embedding Theorem [**14**], there is an embedding from X into W_2, and by Lemma 4.4 there is a mixing shift of finite type W such that W is conjugate to W_2 and $X \subset W$. Since Y is of finite type, we can extend the inclusion map $i : X \to Y$ to an embedding $\bar{i} : W \to Y$ using Lemma 4.5. Then $Z = \bar{i}(W)$ is the desired subshift, which proves (2).

Next, assume that $h \in \mathcal{P}$ and Y is merely sofic. Let $\pi : \widetilde{Y} \to Y$ be a factor code such that $\widetilde{Y}$ is a mixing shift of finite type and $h(\widetilde{Y}) = h(Y)$ (e.g., a code given by its minimal right resolving presentation [**15**]). Let $\widetilde{X} = \pi^{-1}(X)$. Since $h(\widetilde{X}) = h(X)$, by (2) we can find a mixing shift of finite type $\widetilde{Z}$ such that $\widetilde{X} \subset \widetilde{Z} \subset \widetilde{Y}$ and $h(\widetilde{Z}) = h$. Then $Z = \pi(\widetilde{Z})$ satisfies all the conditions, which proves (1).

Finally, we prove the general case. Since $\mathcal{P}$ is dense in $[0, \infty)$, there is a strictly increasing sequence $\{\lambda_i\}_{i \in \mathbb{N}}$ in $\mathcal{P}$ and a strictly decreasing sequence $\{\eta_i\}_{i \in \mathbb{N}}$ in $\mathcal{P}$ such that $\lambda_i \to h$ and $\eta_i \to h$. Without loss of generality we can assume that $h(X) < \lambda_1 < \eta_1 < h(Y)$. Let $X_0 = X$ and $Y_0 = Y$. For each $n \in \mathbb{N}$, define X_n and Y_n inductively as follows: let X_n be a mixing sofic shift such that $X_{n-1} \subset X_n \subset Y_{n-1}$ and $h(X_n) = \lambda_n$. Also let Y_n be a mixing sofic shift such that $X_n \subset Y_n \subset Y_{n-1}$ and $h(Y_n) = \eta_n$. Note that (1) guarantees the existence of X_n and Y_n for each $n \in \mathbb{N}$.

Now let Z be the closure of the union of X_n, $n \in \mathbb{N}$. Since $\{X_n\}_{n \in \mathbb{N}}$ is an increasing sequence of mixing shift spaces, Z is also a mixing shift space (which follows from the fact that $\mathcal{B}(Z) = \bigcup_n \mathcal{B}(X_n)$). Also we have $h(Z) \geq \lim_{n \to \infty} h(X_n) = h$. Since $Z \subset Y_n$ for all $n \in \mathbb{N}$, it follows that $h(Z) = h$, which completes the proof. □

Remark 4.7. A shift space X is called a *coded system* if it contains an increasing sequence of irreducible shifts of finite type whose union is dense in X. In fact, one can take Z to be a coded system in Theorem 4.6.

The following is just a restatement of Theorem 4.6 (apply the argument in the beginning of this section). Note that $h(Y)$ is clearly contained in the sets $\mathcal{T}(\phi)$, $\mathcal{T}_0(\phi)$ and $\mathcal{T}_1(\phi)$ in the following corollary.

COROLLARY 4.8. *Let $\phi : X \to Y$ be an embedding into a mixing sofic shift Y. Then $\mathcal{T}(\phi) \supset (h(X), h(Y)]$ and $\mathcal{T}_1(\phi) \supset (h(X), h(Y)] \cap \mathcal{P}$. If Y is of finite type, then $\mathcal{T}_0(\phi) \supset (h(X), h(Y)] \cap \mathcal{P}$.*

We now give characterizations of $\mathcal{T}$ where Y is sofic, and $\mathcal{T}_0$ where X and Y are of finite type. By reducing from the irreducible sofic case to the mixing finite type case, the following corollaries are immediate.

COROLLARY 4.9. *Let $\phi : X \to Y$ be an embedding into an irreducible sofic shift. Then $\mathcal{T}'(\phi) = (h(X), h(Y))$. If X is irreducible, then $\mathcal{T}(\phi) = [h(X), h(Y)]$.*

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COROLLARY 4.10. *Let $\phi : X \to Y$ be an embedding between irreducible shifts of finite type X and Y with periods p and q, respectively. Then*

$$\mathcal{T}_0(\phi) = [h(X), h(Y)] \cap \{h \in \mathbb{R} : r \cdot h \in \mathcal{P} \text{ for some } r \in \mathbb{N} \text{ with } q|r \text{ and } r|p\}.$$

Remark 4.11. Let $\phi : X \to Y$ be an embedding from a non-wandering shift of finite type X into an irreducible shift of finite type Y. Then for each $\epsilon > 0$ one can find an irreducible shift of finite type $\widetilde{X}$ such that the period of $\widetilde{X}$ is equal to that of X, $\phi(X) \subset \widetilde{X} \subset Y$, and $h(\widetilde{X})$ is ϵ-close to $h(X)$. Thus as in Corollary 4.10 we have

$$\mathcal{T}_0'(\phi) = (h(X), h(Y)] \cap \{h \in \mathbb{R} : r \cdot h \in \mathcal{P} \text{ for some } r \in \mathbb{N} \text{ with } q|r \text{ and } r|p\}.$$

However this equality does not hold if X is merely of finite type. For a simple example, let $Y = \{0, 1, 2, 3, 4, 5, 6\}^{\mathbb{Z}}$ be the full 7-shift and X the shift of finite type defined by $X = \{\sigma^k(x) : x \in A \text{ and } k \in \mathbb{Z}\}$, where

$$A = \{(10)^\infty, (23)^\infty, (10)^\infty.4(23)^\infty, (10)^\infty.56(23)^\infty\}.$$

To prove the case of sofic shifts, we need the following lemmata. In what follows, a *bi-closing* code is a code which does not collapse two distinct right (or left) asymptotic points. A bi-closing code is topologically conjugate to a *bi-resolving* code, which is well-studied in symbolic dynamics. Also, a code is bi-closing if and only if it uniformly separates the fibers, i.e., there is an $\epsilon > 0$ such that each preimage of a point in the range is an ϵ-separated set (e.g., see [**9**, Lemma 2.6]). For more on closing codes and bi-closing codes, see [**9, 12, 18**].

LEMMA 4.12. [**6**, Lemma 2.4] *Let $X \subset \widetilde{X}$ be shifts of finite type and $\pi : X \to Y$ a bi-closing factor code. Then there exist a shift space $\widetilde{Y} \supset Y$ and a bi-closing factor code $\tilde{\pi} : \widetilde{X} \to \widetilde{Y}$ extending π such that for any $x \in \widetilde{X} \setminus X$, $\tilde{\pi}(x)$ has a unique preimage.*

LEMMA 4.13. [**8**, Theorem 26.17] *Let X be a mixing shift of finite type and Z a proper subshift of X. Then for given $h < h(X)$, there is a mixing shift of finite type $V \subset X$ such that $h(V) > h$ and $V \cap Z = \emptyset$.*

THEOREM 4.14. *Let X be a shift space and Y a mixing shift of finite type with $X \subsetneq Y$. Then for $h \in (h(X), h(Y)) \cap \mathcal{P}^w$, there is an irreducible sofic shift Z with*

$$X \subset Z \subset Y \quad \text{and} \quad h(Z) = h.$$

Proof. By Theorem 4.6 (2), there is a mixing shift of finite type $\widetilde{X}$ such that $X \subset \widetilde{X} \subset Y$ and $h(\widetilde{X}) < h$. By passing to a higher block shift, we may assume that $\widetilde{X} = \mathsf{X}_B$ for a primitive matrix B. Also, by Lemma 4.13, there is a mixing shift of finite type $V \subset Y$ disjoint from $\widetilde{X}$ with $h(V) > h$.

Let $m \in \mathbb{N}$ be the size of B. For each $n \in \mathbb{N}$ define an $nm \times nm$ matrix B_n by

$$B_n = \begin{pmatrix} 0 & B & 0 & \cdots & 0 \\ 0 & 0 & B & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & B \\ B & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

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Note that B_n is irreducible and $\mathrm{per}(B_n) = n$ for each $n \in \mathbb{N}$. Since

$$h(\mathsf{X}_{B_n}) = h(\mathsf{X}_B) < h < h(V),$$

by the proof of [**10**, Lemma 4.1] the shift space X_{B_n} embeds into V for all large n. Take $n \in \mathbb{N}$ large so that $n \cdot h \in \mathcal{P}$ and X_{B_n} embeds into V. For brevity, we also denote by X_{B_n} the embedded image in V.

Then by Corollary 4.10, there is an irreducible shift of finite type W such that $\mathsf{X}_{B_n} \subset W \subset V$ and $h(W) = h$. Note that the natural projection code π from X_{B_n} onto X_B is bi-resolving (e.g., see [**18**]). Thus by Lemma 4.12, we can find a (sofic) shift space $Z_1 \supset \mathsf{X}_B$ and a bi-closing factor code $\tilde{\pi} : W \to Z_1$ extending π. Since bi-closing codes preserve entropy, we have $h(Z_1) = h(W) < h(Y)$. Note that $\tilde{\pi}$ is one-to-one from $W \backslash \mathsf{X}_{B_n}$ onto $Z_1 \backslash \mathsf{X}_B$. Also, $W \cap \mathsf{X}_B \subset V \cap \mathsf{X}_B = \emptyset$. It follows that

$$q_k(Z_1) \le q_k(W) + q_k(\mathsf{X}_B) \le q_k(Y)$$

for each $k \in \mathbb{N}$. Thus we can extend an embedding $i : \mathsf{X}_B \to Y$ (given by the inclusion in the beginning of the proof) to an embedding $\bar{i} : Z_1 \to Y$ by Lemma 4.5. The sofic shift $Z = \bar{i}(Z_1)$ is a desired one (indeed, this is an AFT (almost finite type) shift [**5**] since $\tilde{\pi}$ is bi-closing). □

The following result follows from the reduction to the finite type and mixing case.

COROLLARY 4.15. *Let $\phi : X \to Y$ be an embedding into an irreducible sofic shift Y. Then $\mathcal{T}_1'(\phi) = (h(X), h(Y)] \cap \mathcal{P}^w$. If X is irreducible and sofic, then $\mathcal{T}_1(\phi) = [h(X), h(Y)] \cap \mathcal{P}^w$.*

Remark 4.16. In general, $\mathcal{T}_0(\phi)$ is not even dense under the assumption of Corollary 4.15. For a simple example, let Y be the even shift [**15**, §1.2] and $X = \{0^\infty\}$. If Z is a shift of finite type which contains X and is contained in Y, then $Z = X$. Thus $\mathcal{T}_0(\phi)$, where $\phi : X \to Y$ is the inclusion map, consists only of one point (compare with Proposition 4.1).

Proof of Theorem 1.2. The result follows from Corollaries 4.9, 4.10, and 4.15. □

Acknowledgements. We thank Sujin Shin for her good advice. We also thank the referee for useful suggestions and comments, especially for pointing out the result stated in Remark 3.3 (2). The first named author was supported by Basal Grant-CMM and Fondap 15090007. The second named author was supported by TJ Park Postdoctoral Fellowship.

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