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ON TRUNCATION PROPERTIES OF FINITE-BUFFER QUEUES AND QUEUEING NETWORKS

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We show that several truncation properties of queueing systems are consequences of a simple property of censored stochastic processes. We first consider a discrete-time stochastic process and show that its censored process has a truncated stationary distribution. When the stochastic process has continuous time, we present a similar result under the additional condition that the process is locally balanced. We apply these results to single-server batch arrival batch service queues with finite buffers and queueing networks with finite buffers and batch movements, and extend the well-known results on truncation properties of the $M^X/G/1/k$ queues and queueing networks with jump-over blocking.

1. INTRODUCTION

In some queueing models, restricting the state transitions to a subset of the state space results in a truncated stationary distribution of the Markov process of the original model. Such results are well known when the Markov process is reversible (see Kelly [8]), but it also holds in some nonreversible systems. For instance, consider an $M/G/1/k$ queue with $0 < k \leq \infty$, and let $\{\pi^{(k)}(x); x = 0,1,\ldots,k\}$ be the

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stationary queue length probabilities. Keilson [7] showed that if $0 < k' < k$ and $0 \leq x_1, x_2 < k'$, then

$$\frac{\pi^{(k')}(x_1)}{\pi^{(k')}(x_2)} = \frac{\pi^{(k)}(x_1)}{\pi^{(k)}(x_2)}.$$

Some related results and extensions can be found in Glasserman and Gong [5] and Miyazawa [10] and the references therein. A similar truncation property holds for some queueing networks as well. In queueing networks, however, the restriction of state space is obtained by restricting the routing probabilities, rather than restricting the transition rates. For instance, consider an open Jackson network and assume that there is a finite waiting capacity B_j at node j (including those in service). Modify the routing probabilities at this node such that an arriving customer at the node who observes this node full jumps over the node and immediately becomes a departure from the node. This routing scheme is referred to as jump-over blocking in the literature (see, e.g., van Dijk [14]). Clearly, jump-over blocking restrains the number of customers in node j to less than or equal to B_j. Pittel [12] and van Dijk [13] studied closed and open Jackson networks, respectively, and showed that a network with jump-over blocking has a stationary distribution that is the truncation of the product form stationary distribution of the corresponding Jackson network with an infinite buffer at each station. This property has been extended by Boucherie [2] to the class of product form networks with batch movements.

The aim of this paper is to give a unified view for these truncation properties of queueing systems and give some extensions to these results. To that end, we define a censored process for state-space truncation of an arbitrary stochastic process (not necessarily Markovian), which is obtained by cutting off its sample paths whenever the original process is out of a given region. This is equivalent to watching the process only when it is in the restricted region. It can be easily shown that the censored process has the truncated stationary distribution of the original process. To use the censored processes to construct systems with restricted state spaces that are of interest, we focus on two models. The first is to modify a discrete-time Markov chain, which directly uses the censored process (Lemma 2.2), and the second is to consider a continuous-time Markov chain that satisfies certain local balance equations (Lemma 2.3). In both cases, we show that the system with restricted state space has truncated stationary distributions.

We apply these results to two queueing systems: $M^X/G^Y/1/k$ queues with batch Poisson arrivals and geometric batch services, and queueing networks with finite buffers and batch movements. We first consider $M^X/G^Y/1/k'$ and show that the stationary queue length distribution just after a departure is the truncation of the corresponding distribution of an $M^X/G^Y/1/k$ queue for $k' < k$. Using a duality property, this result is extended to $G^Y/M^X/1/k$ queues. We then consider a queueing network with finite buffer and batch movements. Under a jump-over blocking mechanism, we show that the stationary distribution of the network with a finite buffer is the truncation of the stationary distribution of the corresponding network with an infinite buffer, as long as a local balance condition is satisfied. Furthermore, the

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stationary distribution of the network with an infinite buffer does not have to have a product form. Indeed, the truncation property is a result of the censored continuous-time Markov chain that satisfies local balance equations, and not the form of its stationary distributions. These results extend those of Keilson [7], Glasserman and Gong [5], Miyazawa [10], van Dijk [13,14], and Boucherie [2].

This paper is composed of four sections. In Section 2, we introduce a censored process and present two lemmas on truncation properties. These results are applied in Sections 3 and 4 to $M^X/G^Y/1/k$ queues and queueing networks with finite buffers and blocking, respectively.

2. A CENSORED PROCESS AND ITS APPLICATIONS

Let X_n be a discrete-time stochastic process with countable state space S and assume that it has a unique limiting distribution π. Let S' be a subset of S such that the nth exiting time from S' goes to infinity as $n \to \infty$, where $X_0 \in S'$.

Let τ_n be the first time epoch that the accumulated time that X_ℓ is in S' reaches n; that is,

$$\tau_n = \inf\left\{\ell \geq n; \sum_{j=0}^{\ell} 1[X_j \in S'] = n\right\},$$

where $1[A]$ is the indicator function of the statement A. By definition, we have

$$\sum_{j=0}^{\tau_n} 1[X_j \in S'] = n,$$

thus

$$\lim_{n\to\infty} \frac{n}{\tau_n} = \lim_{n\to\infty} \frac{1}{\tau_n} \sum_{j=0}^{\tau_n} 1[X_j \in S'] = \sum_{x\in S'} \pi(x).$$

Define the stochastic process Z_n with state space S' as

$$Z_n = X_{\tau_n}, \qquad n \geq 0;$$

that is, Z_n is the truncated process of X_n whose state space is limited to S'. We refer to Z_n as the *censored process* of X_n on the truncated state space S'.

Obviously, we have, for $x \in S'$,

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n} 1[Z_j = x] &= \lim_{n\to\infty} \frac{\tau_n}{n} \frac{1}{\tau_n} \sum_{j=0}^{n} 1[X_{\tau_j} = x] \\ &= \lim_{n\to\infty} \frac{\tau_n}{n} \frac{1}{\tau_n} \sum_{j=0}^{\tau_n} 1[X_j = x] \\ &= \frac{\pi(x)}{\sum_{x'\in S'} \pi(x')}.\end{aligned}$$

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Thus, we have the following result.

LEMMA 2.1: *The stationary distribution of the censored stochastic process Z_n of X_n on S' is the truncated distribution of π on S'.*

Remark 2.1: When the stochastic process is a Markov chain, this result can be found in Kemeny and Snell [9] (see also Zhao and Liu [16]). In this paper, we also need this generalized version of the result. See Remark 3.3.

Now, assume that X_n is a discrete-time Markov chain with transition probabilities $p(x, y)$, $x, y \in S$. We censor this Markov chain on a subset $S' \subset S$ and denote the censored process by Z_n. Then, Z_n is again a Markov chain. To calculate the transition probabilities of Z_n, for $x \in S\backslash S'$ and $y \in S'$, let $h(x, y)$ be the probability that, starting from x, the Markov chain first enters set S' at state y. Then, the transition probability of Z_n, denoted by $\tilde{p}_{S'}(x, y)$, is

$$\tilde{p}_{S'}(x, y) = p(x, y) + \sum_{z \in S\backslash S'} p(x, z)h(z, y), \qquad x, y \in S', \tag{2.1}$$

where the probabilities $h(x, y)$ are uniquely determined by

$$h(x, y) = p(x, y) + \sum_{z \in S\backslash S'} p(x, z)h(z, y), \qquad x \in S\backslash S', y \in S'. \tag{2.2}$$

LEMMA 2.2: *Suppose the discrete-time Markov chain with transition probabilities p has stationary distribution $\pi(x), x \in S$. For $x \in S\backslash S'$, $y \in S'$, if $p(x, y)/\sum_{z \in S'} p(x, z)$ is independent of x, then its censored process on state space S' is a Markov chain with transition probabilities*

$$\tilde{p}_{S'}(x, y) = p(x, y) + \frac{p(x_0, y)}{\sum_{z \in S'} p(x_0, z)} \sum_{z \in S\backslash S'} p(x, z), \qquad x, y \in S', \tag{2.3}$$

and this Markov chain has the truncated stationary distribution of π restricted to S', where x_0 is an arbitrary state in $S\backslash S'$.

PROOF: It suffices to show that

$$h(x, y) \equiv \frac{p(x, y)}{\sum_{z \in S'} p(x, z)} \tag{2.4}$$

satisfies (2.2). From the assumption, $h(x, y)$ is independent of $x \in S\backslash S'$ for each $y \in S'$. Hence, it can be written as $h(y)$, and it satisfies the equations

$$h(y) = p(x, y) + \sum_{z \in S\backslash S'} p(x, z)h(y), \qquad x \in S\backslash S', y \in S'.$$

Thus, it follows from Lemma 2.1 that the stationary distribution of the Markov chain with transition probabilities (2.3) is the truncated distribution of π on S'. ■

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The following special case of Lemma 2.2 has been proved in Miyazawa and Shanthikumar [11].

COROLLARY 2.1: *If there exists a state $a \in S'$ such that $p(x,y) = 0$ for $x \in S\backslash S'$ and $y \in S'$ unless $y = a$, then the condition of Lemma* 2.2 *is satisfied and the truncation property holds.*

PROOF: When the Markov chain can only enter S' from $S\backslash S'$ via a port a, we have, by (2.4), $h(x,y) = 0$ for $x \in S\backslash S'$, $y \in S'\backslash\{a\}$ and $h(x,a) = 1$ for all $x \in S\backslash S'$; that is, $h(x,y)$ is independent of $x \in S\backslash S'$. ■

We next consider continuous-time Markov chains. To present a similar truncation result, we need to introduce an additional condition on the stationary distribution π, called local balance.

Consider a continuous-time Markov chain $X(t)$. Denote the transition rate function of $X(t)$ by $q(x,y)$ and assume that it admits stationary distribution $\pi(x)$, $x \in S$. Let $\mathcal{J}$ be a countable index set and $q_j(x,y)$, $j \in \mathcal{J}$ be a decomposition of $q(x,y)$; that is,

$$q(x,y) = \sum_{j\in\mathcal{J}} q_j(x,y).$$

The Markov chain is said to be *locally balanced* with respect to q_j if for each $x \in S$,

$$\pi(x)\sum_{y\in S} q_j(x,y) = \sum_{y\in S} \pi(y)q_j(y,x), \qquad x \in S. \tag{2.5}$$

See Chao, Miyazawa, and Pinedo [3] and Whittle [15] for more discussions on local balances.

The local balance (2.5) defines a subset of states, denoted by S_j, that communicates with x via transition rates $q_j(x,y)$. For $j \in \mathcal{J}$, define a transition probability $p_j(x,y)$; $x,y \in S_j$ by

$$p_j(x,y) = \frac{q_j(x,y)}{q_j(x)}, \qquad x,y \in S_j,$$

where

$$q_j(x) = \sum_{y\in S} q_j(x,y), \qquad x \in S_j.$$

Clearly, (2.5) is the global balance equation for the discrete-time Markov chain with transition probabilities $p_j(x,y)$, $x,y \in S_j$, with $\{\pi(x)q_j(x);\ x \in S_j\}$ being its stationary measure. We refer to this Markov chain as a *local process* of $X(t)$.

Now, for a subset $S' \subset S$, we consider the censored process of the local Markov chain $\{p_j(x,y);\ x,y \in S_j\}$ on subset $S' \cap S_j$. Let $\tilde{p}_j(x,y)$, $x,y \in S' \cap S_j$, be the transition probabilities of the censored process; then, it follows from Lemma 2.1 that $\{\pi(x)q_j(x);\ x \in S' \cap S_j\}$ is the stationary measure for the censored Markov chain $\{\tilde{p}_j(x,y);\ x,y \in S' \cap S_j\}$; that is,

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$$\pi(x)q_j(x) = \sum_{y \in S' \cap S_j} \pi(y)q_j(y)\tilde{p}_j(y,x), \qquad x \in S' \cap S_j. \tag{2.6}$$

Let $\tilde{q}_{S'}$ be the transition rate of the censored continuous-time Markov chain on S' defined by

$$\tilde{q}_{S'}(x,y) = \sum_{j \in \mathcal{J}} q_j(x)\tilde{p}_j(x,y), \qquad x, y \in S'.$$

Note that $\tilde{q}_{S'}$ denotes a continuous-time Markov chain such that at each transition epoch, there may be multiple instantaneous state changes. It can be obtained from the original Markov process $q(x,y)$; $x, y \in S$ as follows. When the process is in state $x \in S'$ and a transition from x to $y \in S$ occurs due to rate $q_j(x,y)$ for some $j \in \mathcal{J}$, if $y \in S'$, then it stays there, otherwise the state jumps immediately to state $z \in S_j$ with probability $p_j(y,z)$, and if $z \in S'$, it stays there, otherwise it jumps immediately to another state $z' \in S_j$ with probability $p_j(z,z')$, and so on. This instantaneous process continues until the first time it hits a state in S'. Thus, in a way, the process with transition rates $\tilde{q}_{S'}$ is a censored process on state space S'.

LEMMA 2.3: *If the continuous-time Markov chain with transition rates q and state space S has stationary distribution π and it satisfies local balance* (2.5), *then the censored continuous-time Markov chain with state space S' and transition rates $\tilde{q}_{S'}$ has a stationary distribution that is the truncated distribution of π on S'.*

PROOF: Summing (2.6) over $j \in \mathcal{J}$ gives the global balance equation for the censored Markov process $\tilde{q}_{S'}$. Thus, $\pi(x)$; $x \in S'$ is the stationary measure of the censored process, and the truncated distribution of π on S' is the stationary distribution for $\tilde{q}_{S'}$. ■

An extreme case of Lemma 2.3 is when the continuous-time Markov chain is reversible. In this case, the detailed balance equation is satisfied; the truncation property always holds.

In the subsequent two sections, we illustrate the usefulness of Lemmas 2.2 and 2.3 by applying them to $M^X/G^Y/1/k$ queues and queueing networks with finite buffer and blocking.

3. $M^X/G^Y/1/k$ QUEUES

Consider an $M^X/G^Y/1/k$ queue where batches of customers arrive according to a Poisson process with rate λ. The capacity k includes the server. When $k = \infty$, we assume that $\lambda E(X)/E(Y) < 1$, assuring that the stationary distribution exists. Partial acceptance policy is considered; that is, when a batch of customers does not find enough space in the system for the entire batch, the system accepts as many as it can. Furthermore, for the services, when the size of a service batch is Y and there are at least Y customers in the system upon the service completion, the number of customers in the system reduces by Y; otherwise the system is emptied. Let $X_1, X_2, \ldots$ be the sequence of i.i.d. arrival batch sizes, and $Y_1, Y_2, \ldots$ be the sequence of i.i.d. potential

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service batch sizes. We assume that the service batch sizes have geometric distribution; that is, $P(Y_1 = i) = (1-p)p^{i-1}$, $i = 1,2,\ldots$.

We use the superscript k to indicate that the system under consideration is $M^X/G^Y/1/k$. Let $Q^{(k)}(t)$ be the number of customers in system at time t, $T_n^{(k)}$ be the epoch of the nth service completion, and

$$L_n^{(k)} = (Q^{(k)}(T_n^{(k)}-) - Y_n^{(k)})^+$$

be the number of customers left behind by the nth service completion, where $Y_n^{(k)}$ is the nth potential service batch size which is geometrically distributed, and $x^+ = \max(x,0)$. Let ξ_n be the number of arrivals during the nth service. Clearly, we have

$$L_n^{(k)} = (\max(L_{n-1}^{(k)} + \xi_n, k) - Y_n^{(k)})^+. \tag{3.1}$$

For convenience, we assume that the system is initially empty with $L_0^{(k)} = 0$. As is well known, the process $\{L_n^{(k)}; n = 0,1,\ldots\}$ is a discrete-time Markov chain, referred to as the embedded Markov chain. This Markov chain has state space $S = \{0,1,\ldots,k-1\}$.

Let $0 < k' < k$. We compare queueing systems with different buffer sizes k' and k. The following result generalizes that of Keilson [7], Cohen [4], Glasserman and Gong [5], and Miyazawa [10].

THEOREM 3.1: *Let $\pi^{(k)}$ be the stationary distribution of $\{L_n^{(k)}; n \geq 0\}$ in the $M^X/G^Y/1/k$ queue; then, for $k' < k$ and $0 \leq x_1, x_2 < k'$, we have*

$$\frac{\pi^{(k')}(x_1)}{\pi^{(k')}(x_2)} = \frac{\pi^{(k)}(x_1)}{\pi^{(k)}(x_2)}. \tag{3.2}$$

Thus,

$$\pi^{(k')}(x) = \frac{\pi^{(k)}(x)}{\sum_{y=0}^{k'-1} \pi^{(k)}(y)}, \qquad x = 0,1,\ldots,k'-1.$$

PROOF: We apply Lemma 2.2 to the process $L_n^{(k)}$ with state space $S = \{0,1,\ldots,k-1\}$ and $S' = \{0,1,\ldots,k'-1\}$. To prove Theorem 3.1, we need to show the following two facts:

(a) The conditional probability from $x \in S\backslash S'$ to $y \in S'$, given it enters S', does not depend on x.
(b) The transition probabilities for the censored Markov chain of $L_n^{(k)}$ in $M^X/G^Y/1/k$ [i.e., (2.3)] are the same as that of $\{L_n^{(k')}; n \geq 0\}$ for the $M^X/G^Y/1/k'$ queue.

Once (a) and (b) are established, it follows from Lemma 2.2 that the stationary distributions of the two queues satisfy (3.2).

We first prove (a). Let $k' \leq x < k$ and assume that $L_n^{(k)} = x \in S\backslash S'$. Let $a(i) = P(\xi_1 = i)$ be the distribution of the number of arrivals during a service time. Let

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$x' \in S'$. We only consider the case $x' > 0$. The case $x' = 0$ can be proved similarly. We have

$$P(L_{n+1}^{(k)} = x' \,|\, L_n^{(k)} = x)$$

$$= \frac{\displaystyle\sum_{\ell=\max\{x,x'\}+1}^{k-1} a(\ell - x)(1-p)p^{\ell-x'-1} + \bar{a}(k-x)(1-p)p^{k-x'-1}}{\displaystyle\sum_{x'\in S'}\sum_{\ell=\max\{x,x'\}+1}^{k-1} a(\ell - x)(1-p)p^{\ell-x'-1} + \bar{a}(k-x)(1-p)p^{k-x'-1}}$$

$$= \frac{p^{-x'}}{\displaystyle\sum_{x'\in S'} p^{-x'}},$$

where $\bar{a}(n) = \sum_{\ell=n}^{\infty} a(\ell)$. Clearly this is independent of x.

We then prove (b). We calculate the transition probabilities for the censored Markov chain of $\{L_n^{(k)}; n \geq 0\}$ on $S' = [0,1,\ldots,k'-1\}$. Let L_1^- be the number of customers in the $M^X/G^Y/1/k$ system just before the first departure and $L_0^{(k)} = x \in S'$. Note that in the censored Markov chain, a transition from $x \in S'$ to $x' \in S'$ has to take one of the following three paths (see Fig. 1):

i. $L_1^- < k'$; that is, the process does not jump out of S', and $L_1^{(k)} = x'$.
ii. $L_1^- \geq k'$ and $L_1^{(k)} = x'$; that is, just before the first departure, there is at least k' customers in the system, and the departure brings the number of customers to x'.
iii. $L_1^- \geq k'$ and $L_1^{(k)} \geq k'$; in this case, the number of customers for the Markov chain $\{L_n^{(k)}; n \geq 0\}$ first jumps from x to $S\backslash S'$, and we need to calculate the probability of the censored Markov chain that it first jumps back to S' at x'.

We now consider the transition probabilities for $\{L_n^{(k')}; n \geq 0\}$ in the $M^X/G^Y/1/k'$ queue. Let $L_0^{(k')} = x < k'$ and $x' < k'$. In contrast to the transition probabilities for the $M^X/G^Y/1/k$ queue, the transition probability from $L_0^{(k')} = x \in S'$ to $L_1^{(k')} = x' \in S'$ has the following two possibilities (see Fig. 2):

i′. The number of customers in the system just before the next departure is less than k', and $L_1^{(k')} = x'$.
ii′. The number of customers in the system just before the next departure is k', and $L_1^{(k')} = x'$.

Again, we only consider the case $x' > 0$. It is easily seen that the probabilities for (i) and (i′) in the two queues are the same, both given by

$$\sum_{\ell=\max\{x,x'\}+1}^{k'-1} a(\ell - x)(1-p)p^{\ell-x'-1}.$$

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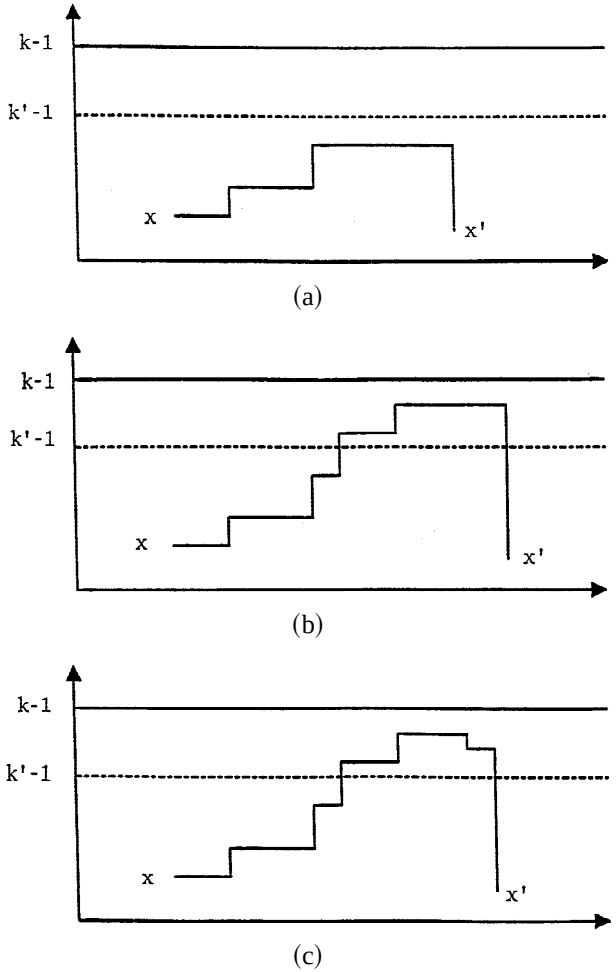


FIGURE 1.

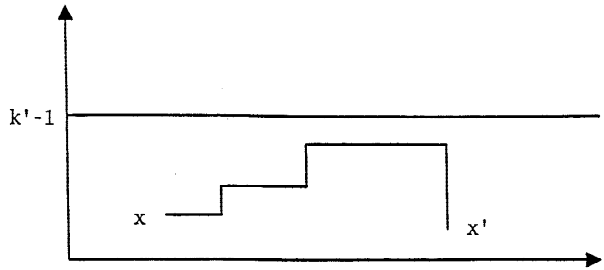
Thus, it suffices to show that sum of probabilities for (ii) and (iii) of the $M^X/G^Y/1/k$ queue is the same as the probability for (ii′) of the $M^X/G^Y/1/k'$ queue.

Note that (ii) and (iii) give the conditional probability that the censored Markov chain of $M^X/G^Y/1/k$ queue, given $L_1^- \geq k'$, first returns to S' at x'. To calculate this probability, define

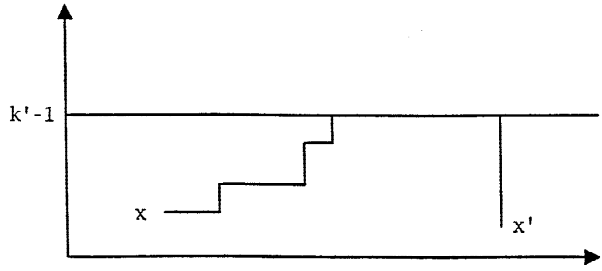
$$A_j = \{L_1^{(k)} \geq k', L_2^{(k)} \geq k', \ldots, L_{j-1}^{(k)} \geq k', L_j^{(k)} < k'\};$$

that is, A_j is the event that after $j-1$ steps, $L_n^{(k)}$ returns to S'. Then, using the Markov property, we obtain that, given $L_1^- \geq k'$, the censored Markov chain first returns S' for the first time at x' with probability

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(a)



(b)

FIGURE 2.

$$\sum_{j=1}^{\infty} P(L_j^{(k)} = x', A_j | L_1^- \geq k')$$

$$= \sum_{j=1}^{\infty} P(L_j^{(k)} = x' | A_j, L_1^- \geq k') P(A_j | L_1^- \geq k')$$

$$= P(L_j^{(k)} = x' | L_1^- \geq k', L_1^{(k)} < k') P(A_1 | L_1^- \geq k')$$

$$+ \sum_{j=2}^{\infty} P(L_j^{(k)} = x' | L_{j-1}^{(k)} \geq k', L_j^{(k)} < k') P(A_j | L_1^- \geq k')$$

$$= (1-p)p^{k'-x'-1},$$

where the last equality follows from the geometric service size distribution that

$$P(L_1^{(k)} = x' | L_1^- \geq k', L_1^{(k)} < k') = (1-p)p^{k'-x'-1},$$

$$P(L_j^{(k)} = x' | L_{j-1}^{(k)} \geq k', L_j^{(k)} < k') = (1-p)p^{k'-x'-1}.$$

Thus, the transition probability for the $M^X/G^Y/1/k$ queue due to (ii) and (iii) is

$$(1-p)p^{k'-x'-1} P(L_1^- \geq k' | L_0^{(k)} = x) = (1-p)p^{k'-x'-1} \bar{a}(k'-x).$$

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On the other hand, it is obvious that the transition probability for $M^X/G^Y/1/k'$ from x to x' due to (ii) is also $(1-p)p^{k'-x'-1}\bar{a}(k'-x)$. This proves (b) and Theorem 3.1. ■

Remark 3.1: It can be easily seen from the proof of Theorem 3.1 that the result holds when the server is subject to vacations for a variety of vacation mechanisms.

Remark 3.2: It follows from Theorem 3.1 that if $L^{(k)}$ is the stationary number of customers left behind a departure then, for $k' < k$,

$$P(L^{(k)} = x | L^{(k)} < k') = P(L^{(k')} = x), \quad x = 0,1,\ldots,k'-1.$$

Remark 3.3: Equality (3.2) of Theorem 3.1 also holds when $\pi^{(k)}$ is replaced by the distribution of the time stationary number of customers in the system [i.e., the stationary distribution of the stochastic process $\{Q^{(k)}(t); t \geq 0\}$, where $Q^{(k)}(t)$ is the number of customers in the system at time t in the $M^X/G^Y/1/k$ system]. Note that $Q^{(k)}(t)$ is not a Markov process because of the arbitrarily distributed service times. Furthermore, the state space for $Q^{(k)}(t)$ is $\{0,1,\ldots,k-1,k\}$, which is different from the state space of $L^{(k)}$. To prove this result, consider an $M^X/G^Y/1/k$ queue and its censored process on $\{0,1,\ldots,k'-1\}$. Because the service batch sizes are geometrically distributed, it can be shown that the censored process stochastically represents the corresponding $M^X/G^Y/1/k'$. Hence, it follows from a continuous-time version of Lemma 2.1 that the stationary distribution satisfies the proportionality property.

Remark 3.4: By considering the number of customers before each departure occurs, we can establish a result (3.2) for the stationary number of customers in the system before departures.

Remark 3.5: Using a duality relationship between $M^X/G^Y/1/k$ and $G^Y/M^X/1/k+1$ (see Miyazawa [10]), it can be easily shown that if $\tilde{L}^{(k)}$ is the stationary number of customers immediately after the arrival of a batch in the $G^Y/M^X/1/k$ queue, then $\tilde{L}^{(k)}$ and $k+1-L^{(k)}$ are equal in law, where $L^{(k)}$ is the stationary number of customers before a departure in the $M^X/G^Y/1/k+1$ queue. Hence, the proportionality property in Theorem 3.1 can be extended to $G^Y/M^X/1/k+1$ queues if $\pi^{(k)}(x)$ is replaced by $\pi^{(k)}(k+1-x)$; that is, for $0 \leq x_1, x_2 < k'$,

$$\frac{\pi^{(k')}(k'+1-x_1)}{\pi^{(k')}(k'+1-x_2)} = \frac{\pi^{(k)}(k+1-x_1)}{\pi^{(k)}(k+1-x_2)}.$$

4. QUEUEING NETWORKS WITH FINITE BUFFERS

Various blocking protocols have been studied for queueing networks (e.g., see van Dijk [14]). It is known that if a Jackson network has a finite buffer at each station, then under the jump-over blocking mechanism, the stationary distribution of the finite-buffer network is the truncation of the stationary distribution of the corresponding Jackson network with infinite buffers. This result was first obtained by Pittel [12] for closed queueing networks and by van Dijk [13] for the Jackson type of

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open networks. It is further extended by Boucherie [2] to networks with batch movements.

We show that these results are consequences of Lemma 2.3. Furthermore, we show that the truncation property is not a property for networks with a product form solution (i.e., the stationary distribution of the network is the product of the marginal distributions—possibly with a normalizing constant), it is a property of any queueing network satisfying local balances.

To illustrate, consider a queueing network with N nodes. Let x_j be the number of customers at node j, $j = 1,\ldots,N$, and let

$$\boldsymbol{x} = (x_1,\ldots,x_N)$$

be the state of the network with state space S. For convenience, let node 0 represent the outside. Let U be a set of $(N+1)$-dimensional vectors, which denotes the set of batches that may move together. For instance, if only one customer can depart from any node, including node 0 (the outside), then $U = \{\bar{\boldsymbol{e}}_0, \bar{\boldsymbol{e}}_1,\ldots,\bar{\boldsymbol{e}}_N\}$, where $\bar{\boldsymbol{e}}_j$ is the $(N+1)$-dimensional vector with a 1 in position j and a 0 everywhere else, including node 0. Here, we use $\bar{\boldsymbol{e}}_j$ instead of $\boldsymbol{e}_j$, as $\boldsymbol{e}_j$ is used as the N-dimensional unit vector without node 0, whereas $\bar{\boldsymbol{e}}_j$ is the $(N+1)$-dimensional unit vector including node 0. In general, for a batch $\bar{\boldsymbol{a}} = (a_0, a_1,\ldots,a_N) \in U$, we use $\boldsymbol{a}$ to denote the vector $\bar{\boldsymbol{a}}$ with the first element removed; that is,

$$\boldsymbol{a} = (a_1,\ldots,a_N).$$

The following is the system dynamics when the state of the network is $\boldsymbol{x}$:

i. The departure rate of a vector $\bar{\boldsymbol{a}} \in U$ is $\mu(\boldsymbol{x},\bar{\boldsymbol{a}})$, where

$$\bar{\boldsymbol{a}} = (a_0, a_1,\ldots,a_N)$$

and a_j is the number of departures from node j, $j = 0,1,\ldots,N$; thus $a_j \le x_j$, $j = 1,\ldots,N$. The first component, a_0, is the number of arrivals to the network. The state of the network immediately after the departure is $\boldsymbol{x}' = \boldsymbol{x} - \boldsymbol{a}$.

ii. When a departure vector $\bar{\boldsymbol{a}}$ leaves the network in state $\boldsymbol{x}'$, it is transformed into another $(N+1)$-dimensional arrival vector $\bar{\boldsymbol{a}}' = (a_0', a_1',\ldots,a_N')$ with probability $r_{\boldsymbol{x}'}(\bar{\boldsymbol{a}},\bar{\boldsymbol{a}}')$, where a_j' is the number of arrivals at node j, $j = 0,1,\ldots,N$, and a_0', which is the number of arrivals at node 0, is the number of customers departing from the network. The state of the network upon the arrival of $\bar{\boldsymbol{a}}'$ is $\boldsymbol{x}' + \boldsymbol{a}'$. We assume

$$\sum_{\bar{\boldsymbol{a}}' \in U} r_{\boldsymbol{x}'}(\bar{\boldsymbol{a}},\bar{\boldsymbol{a}}') = 1, \qquad \bar{\boldsymbol{a}} \in U,$$

$$\sum_{j=0}^{N} a_j = \sum_{j=0}^{N} a_j'.$$

Assume that the network process has a stationary distribution $\pi(\boldsymbol{x})$, $\boldsymbol{x} \in S$.

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Now, consider the same queueing network but with restricted state space $S' \subset S$. For instance, if the restricted network is a network with a finite buffer B_j at each node j, then

$$S' = \{(x_1, \ldots, x_N);\ 0 \le x_j \le B_j,\ j = 1, \ldots, N\}.$$

However, S' can be more general.

The dynamics of the network with state space S' are the following when it is in state $\boldsymbol{x} \in S'$.

- i′. The departure rate of a vector $\bar{\boldsymbol{a}} \in U$ is $\mu(\boldsymbol{x}, \bar{\boldsymbol{a}})$. If $\boldsymbol{x}' = \boldsymbol{x} - \boldsymbol{a} \notin S'$, then the network remains in state $\boldsymbol{x}$ (i.e., the departure transition is prohibited).
- ii′. When a departure vector $\bar{\boldsymbol{a}}$ leaves the network in state $\boldsymbol{x}' \in S'$, it is transformed into an $(N+1)$-dimensional arrival vector $\bar{\boldsymbol{a}}' = (a_0', a_1', \ldots, a_N')$ with probability $r_{\boldsymbol{x}'}(\bar{\boldsymbol{a}}, \bar{\boldsymbol{a}}')$, where a_j' is the number of arrivals at node j, $j = 0, 1, \ldots, N$, and a_0', which is the number of arrivals at node 0, is the number of customers departing from the network. The network state becomes $\boldsymbol{x}' + \boldsymbol{a}'$ if $\boldsymbol{x}' + \boldsymbol{a}' \in S'$.
- iii′. If $\boldsymbol{x}' + \boldsymbol{a}' \notin S'$, then the vector $\bar{\boldsymbol{a}}'$ jumps over state $\boldsymbol{x}'$ and transforms into an arrival vector $\boldsymbol{a}''$ with probability $r_{\boldsymbol{x}'}(\bar{\boldsymbol{a}}', \bar{\boldsymbol{a}}'')$, and the network state changes to $\boldsymbol{x}' + \boldsymbol{a}''$, provided $\boldsymbol{x}' + \boldsymbol{a}'' \in S'$; otherwise, this process continues until the first moment the state after the arrival transition is in S'.

This is an extension of the jump-over blocking protocol defined by van Dijk [13,14] and Boucherie [2] (see also Henderson and Taylor [6]). The following theorem extends the results of van Dijk and Boucherie.

THEOREM 4.1: *If the batch movement network with state space S satisfies the local balance equations*

$$\pi(\boldsymbol{x} + \boldsymbol{a})\mu(\boldsymbol{x} + \boldsymbol{a}, \bar{\boldsymbol{a}}) = \sum_{\bar{\boldsymbol{a}}' \in U} \pi(\boldsymbol{x} + \boldsymbol{a}')\mu(\boldsymbol{x} + \boldsymbol{a}', \bar{\boldsymbol{a}}')r_{\boldsymbol{x}}(\bar{\boldsymbol{a}}', \bar{\boldsymbol{a}}),$$
$$\boldsymbol{x} \in S,\ \bar{\boldsymbol{a}} \in U, \tag{4.1}$$

then the network with state space S′ and jump-over blocking [*i.e., the dynamics as defined by* (i′)–(iii′)] *has the stationary distribution* $\tilde{\pi}(\boldsymbol{x})$, $\boldsymbol{x} \in S'$, *which is the truncation of* $\pi(\boldsymbol{x})$; *that is,*

$$\tilde{\pi}(\boldsymbol{x}) = \frac{\pi(\boldsymbol{x})}{\sum_{\boldsymbol{x}' \in S'} \pi(\boldsymbol{x}')}, \qquad \boldsymbol{x} \in S'.$$

PROOF: Let $\mathcal{J} = \{(\boldsymbol{x}, \ell);\ \boldsymbol{x} \in S,\ \ell = 1, 2, \ldots\}$. For each $(\boldsymbol{x}, \ell) \in \mathcal{J}$, define the transition rate $q_{(\boldsymbol{x}, \ell)}$ as

$$q_{(\boldsymbol{x}, \ell)}(\boldsymbol{x} + \boldsymbol{a}, \boldsymbol{x} + \boldsymbol{a}') = 1[|\bar{\boldsymbol{a}}| = |\bar{\boldsymbol{a}}'| = \ell]\mu(\boldsymbol{x} + \boldsymbol{a}, \bar{\boldsymbol{a}})r_{\boldsymbol{x}}(\bar{\boldsymbol{a}}, \bar{\boldsymbol{a}}'), \qquad \bar{\boldsymbol{a}}, \bar{\boldsymbol{a}}' \in U.$$

Clearly, $q_{(\boldsymbol{x}, \ell)}$ determines a communication class $S_{(\boldsymbol{x}, \ell)}$ given by

$$S_{(\boldsymbol{x}, \ell)} = \{\boldsymbol{x} + \boldsymbol{a}; \bar{\boldsymbol{a}} \in U, |\bar{\boldsymbol{a}}| = \ell\}.$$

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Since $q_{(\boldsymbol{x},\ell)}(\boldsymbol{x}+\boldsymbol{a},\boldsymbol{y})$ and $q_{(\boldsymbol{x},\ell)}(\boldsymbol{y},\boldsymbol{x}+\boldsymbol{a})$ vanish unless $\boldsymbol{y}$ takes the form $\boldsymbol{x}+\boldsymbol{a}$ with $|\bar{\boldsymbol{a}}'| = |\bar{\boldsymbol{a}}'|$ and $\bar{\boldsymbol{a}}' \in U$, (4.1) can be rewritten as

$$\pi(\boldsymbol{x}+\boldsymbol{a}) \sum_{\boldsymbol{y}\in S} q_{(\boldsymbol{x},\ell)}(\boldsymbol{x}+\boldsymbol{a},\boldsymbol{y}) = \sum_{\boldsymbol{y}\in S} \pi(\boldsymbol{y}) q_{(\boldsymbol{x},\ell)}(\boldsymbol{y},\boldsymbol{x}+\boldsymbol{a}), \qquad \boldsymbol{x}\in S, \bar{\boldsymbol{a}}\in U. \tag{4.2}$$

Applying Lemma 2.3 proves the theorem. ■

Note that the relevant results on queueing networks reported in the literature on jump-over blocking are only for networks with state-independent routing and with specific forms of stationary distributions (e.g., product form) or departure functions (e.g., see Sect. 3 of Boucherie [2]). In our above result, the network has state-dependent routing probabilities, and no assumptions are made on the form of the stationary distribution of the original network.

Another special case of our result in Section 3 is networks with arbitrary service times. Consider a network with N nodes and I classes of customers with arbitrarily distributed class-dependent service requirements. The routing probabilities of a class u customer at node j is $r_{ju,kv}$ (e.g., see, Baskett et al. [1], Kelly [8], and Chao et al. [3]). Under the stability condition and symmetric service disciplines, the stationary distribution is known to have a product form. Now consider the same network but with a finite buffer B_j at node j. An arrival of a class u customer at node j when the buffer size is full jumps over the node and joins node k as a class v customer with probability $r_{ju,kv}$, as if its service has been completed, and if the buffer at node k is full, it jumps over node k again and so on.

Let x_{ju} be the number of class u customers at node j, and

$$\boldsymbol{x}_j = (x_{ju}; u = 1,..,I), \qquad \boldsymbol{x} = (\boldsymbol{x}_j; j = 1,\ldots,N).$$

The truncation property of this network, presented in the following corollary, has not been previously reported in the literature. For the local balance equations of this network with arbitrarily distributed service requirements and symmetric service disciplines, the reader is referred to Chapter 6 of Chao et al. [3].

COROLLARY 4.1: *Queueing networks with arbitrary processing requirements, symmetric service discipline, finite buffer and jump-over blocking have a truncated stationary distribution* $\tilde{\pi}(\boldsymbol{x})$:

$$\tilde{\pi}(\boldsymbol{x}) = K \prod_{j=1}^{N} \pi_j(\boldsymbol{x}_j) \quad \text{for} \sum_{u=1}^{I} x_{ju} \le B_j, j = 1,\ldots,N,$$

where π_j *is the marginal distribution of node j when it has an infinite buffer and K is the normalization constant.*

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References

1. Baskett, F., Chandy, K.M., Muntz, R.R., & Palacios, F.G. (1975). Open, closed and mixed networks of queues with different classes of customers. *Journal of ACM* 22: 248–260.
2. Boucherie, R. (1996). Batch routing queueing networks with jump-over blocking. *Probability in the Engineering and Informational Sciences* 10: 287–297.
3. Chao, X., Miyazawa, M., & Pinedo, M. (1999). *Queueing networks: Customers, signals, and product form solutions.* Chichester: Wiley.
4. Cohen, J.W. (1970). *On regenerative processes in queueing theory.* Lecture Notes in Economics and Mathematical Systems. Berlin: Springer-Verlag.
5. Glasserman, P. & Gong, W. (1991). Time-varying and truncating *K*-capacity queues from one *k* to another. *Journal of Applied Probability* 28: 647–655.
6. Henderson, W. & Taylor, P.G. (1999). State dependent coupling of quasi-reversible nodes. Preprint.
7. Keilson, J. (1966). The ergodic queue length distribution for queueing systems with finite capacity. *Journal of the Royal Statistical Society* B28: 190–201.
8. Kelly, F. (1979). *Reversibility and stochastic networks.* New York: Wiley.
9. Kemeny, J.G. & Snell, J.L. (1960). *Finite Markov chains*. Princeton, NJ: Van Nostrand Reinhold.
10. Miyazawa, M. (1990). Complementary generating functions for the $M^X/GI/1/k$ and $GI/M^Y/1/k$ queues and their applications to the comparison of loss probabilities. *Journal of Applied Probability* 27: 684–692.
11. Miyazawa, M. & Shanthikumar, J.G. (1991). Monotonicity of the loss probability of single server finite queue with respect to convex order arrival or service processes. *Probability in the Engineering and Informational Sciences* 5: 43–52.
12. Pittel, B. (1979). Closed exponential networks with saturation: The Jackson type stationary distributions and its asymptotic analysis. *Mathematics of Operations Research* 5: 357–378.
13. van Dijk, N.M. (1988). On Jackson's product form with "jump-over" blocking. *Operations Research Letters* 7: 233–235.
14. van Dijk, N.M. (1993). *Queueing networks and product forms: A systems approach.* Chichester: Wiley.
15. Whittle, P. (1986). *Systems in stochastic equilibrium.* New York: Wiley.
16. Zhao, Y. & Liu, D. (1996). The censored Markov chains and the best augmentation. *Journal of Applied Probability* 33: 623–629.

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