Ergod. Th. & Dynam. Sys. (2005), **25**, 731–757
doi:10.1017/S0143385704000859
© *2005 Cambridge University Press*
Printed in the United Kingdom

Multifractal analysis on the flexed Sierpiński gasket

KRZYSZTOF BARAŃSKI†
Institute of Mathematics, Warsaw University, ul. Banacha 2, 02-097 Warszawa, Poland
(e-mail: baranski@mimuw.edu.pl)

(*Received* 20 *May* 2004 *and accepted in final form* 23 *September* 2004)

Abstract. We determine the Hausdorff and box dimensions of the flexed Sierpiński gasket Λ, which is the limit set for a non-Moran-like non-affine geometric construction in the plane. We also provide a multifractal formalism for the Bernoulli measures on Λ, computing explicitly the pointwise dimension spectra.

1. *Introduction*

The flexed Sierpiński gasket Λ was defined in 1988 by Rudnik [**15**]. It is an interesting example of the limit set of a geometric construction in the plane, which is a non-affine modification of the standard Sierpiński gasket. By an (inductive) geometric construction in $\mathbb{R}^k$ we mean a family of compact sets $\Delta_{\varepsilon_1,\dots,\varepsilon_n} \subset \mathbb{R}^k$, where $n = 1, 2, \dots$ and $\varepsilon_m \in \{1, \dots, d\}$ for $m = 1, \dots, n$ and some $d > 0$, such that:

- $\Delta_{\varepsilon_1,\dots,\varepsilon_n,\varepsilon_{n+1}} \subset \Delta_{\varepsilon_1,\dots,\varepsilon_n}$;
- $\operatorname{diam} \Delta_{\varepsilon_1,\dots,\varepsilon_n} \to 0$ as $n \to \infty$.

Usually, some additional conditions are required for the sets $\Delta_{\varepsilon_1,\dots,\varepsilon_n}$ (see [**13**] for various examples). The limit set of the construction is defined as

$$\Lambda = \bigcap_{n=1}^{\infty} \bigcup_{\varepsilon_1,\dots,\varepsilon_n \in \{1,\dots,d\}} \Delta_{\varepsilon_1,\dots,\varepsilon_n}.$$

An important class of examples consists of geometric constructions generated by a finite number of contractions $S_1, \dots, S_d$. In this case $\Delta_{\varepsilon_1,\dots,\varepsilon_n} = S_{\varepsilon_1} \circ \cdots \circ S_{\varepsilon_n}(\Delta)$ for some set $\Delta \subset \mathbb{R}^k$ and the limit set Λ is the unique non-empty compact set such that $\Lambda = \bigcup_{l=1}^{d} S_l(\Lambda)$. When S_l are similarities in $\mathbb{R}^k$, we say that Λ is self-similar and when they are affine maps, we call Λ a self-affine set. In the self-similar case, if some separation conditions are satisfied, e.g. the open set condition saying that $U \supset \bigcup_{l=1}^{d} S_l(U)$ for a non-empty bounded open set U such that the union is disjoint, then the Hausdorff and box

† Current address: Institute of Mathematics, Polish Academy of Sciences, ul. Śniadeckich 8, PO Box 21, 00-956 Warszawa 10, Poland.

https://doi.org/10.1017/S0143385704000859 Published online by Cambridge University Press

dimensions of Λ are equal and coincide with the similarity dimension, which is the unique number D such that $\sum_{l=1}^{d} \lambda_l^D = 1$, where $\lambda_l \in (0, 1)$ are the ratios of the similarities S_l. (A well-known example of a self-similar set is the Sierpiński gasket.) There are a number of generalizations of the result to the non-self-similar Moran-like constructions. Here, by a Moran-like construction, we mean a construction for which there exists a constant $c > 0$ such that for every n and every $\varepsilon_1, \ldots, \varepsilon_n \in \{1, \ldots, d\}$ we have

$$R_{\varepsilon_1,\ldots,\varepsilon_n} / r_{\varepsilon_1,\ldots,\varepsilon_n} < c,$$

where $R_{\varepsilon_1,\ldots,\varepsilon_n}$ (respectively $r_{\varepsilon_1,\ldots,\varepsilon_n}$) is the infimum (respectively supremum) of the set of $r > 0$ for which there exists an open ball of radius r containing (respectively contained in) $\Delta_{\varepsilon_1,\ldots,\varepsilon_n}$. In these generalizations, the Hausdorff and box dimension of Λ are usually equal and coincide with the root of Bowen's equation.

The self-affine case is more difficult to handle and there are quite few examples, where the Hausdorff dimension of the limit set can be computed rigorously. In 1984, Bedford [**2**] and McMullen [**12**] determined the Hausdorff and box dimensions of some self-affine limit sets in the plane known as the general Sierpiński carpets. In this case, the affine contractions S_l map the unit square onto d different rectangles chosen from rs ones obtained by a partition of the unit square into r vertical strips of equal width and s horizontal strips of equal width. The result was generalized by Kenyon and Peres [**9**] to the higher-dimensional case and by Lalley and Gatzouras [**11**] to a larger class of planar self-affine sets. In these examples, the Hausdorff dimension is typically strictly smaller than the box dimension. Some other examples of self-affine limit sets for which the Hausdorff and box dimensions can be determined are given [**13**]. In [**7**], Falconer and Lammering computed the box dimension for a family of so-called generalized Sierpiński triangles, which are some self-affine modifications of the standard Sierpiński gasket. Moreover, there are a number of papers studying self-affine graphs of real functions on the interval (see, e.g., [**3, 14, 17**]).

In [**6**], Falconer proved that if the affine maps S_l have the form $T_l + a_l$, where T_l are linear maps in $\mathbb{R}^k$ such that $\|T_l\| < 1/3$ and $a_l \in \mathbb{R}^k$ are the translation vectors, then for almost all $a = (a_1, \ldots, a_d) \in \mathbb{R}^{kd}$ the Hausdorff dimension of the limit set is determined in terms of singular value functions for the map T_l. In [**16**], Solomyak replaced the constant $1/3$ by $1/2$ (the maximal one) and [**5**], Falconer determined the box dimension for a class of self-affine limit sets.

The flexed Sierpiński gasket is a planar limit set homeomorphic to the standard Sierpiński gasket by an isometry in the internal metric, which is not an isometry in the Euclidean one (see [**15**] for the proof). In fact, there exists a continuous path of Sierpiński gaskets Λ^φ, $\varphi \in (0, \pi/3]$, such that $\Lambda^{\pi/3}$ is the standard one and Λ^φ for $\varphi \in (0, \pi/3)$ are the flexed ones. In the inductive construction of the flexed Sierpiński gasket, instead of considering three equilateral triangles contained in a larger one, we take three isosceles triangles contained in a larger one, starting from an isosceles triangle of angle φ. In this way at each step we obtain a family of isosceles triangles with various angles, which generally change in a nonlinear way. Moreover, most triangles have an angle tending to zero. It follows that the construction is not self-affine and not Moran-like. A detailed description of the flexed Sierpiński gasket is given in §3.

https://doi.org/10.1017/S0143385704000859 Published online by Cambridge University Press

In 1995, [**8**] it was conjectured that the Hausdorff dimension of the flexed Sierpiński gasket Λ^φ does not depend on $\varphi \in (0, \pi/3)$ and is smaller than $\log 3/\log 2$, which is the Hausdorff dimension of the standard Sierpiński gasket. In this paper we prove this conjecture, determining the Hausdorff and box dimensions of Λ. It appears that they are indeed smaller than $\log 3/\log 2$ and do not depend on $\varphi \in (0, \pi/3)$. Moreover, the Hausdorff dimension is strictly smaller than the box dimension. The reason for the constancy of the dimension with respect to φ is explained by the fact that Λ^{φ_1} and Λ^{φ_2} are bi-Lipschitz homeomorphic for every $\varphi_1, \varphi_2 \in (0, \pi/3)$. This follows easily from Lemma 3.2 and the proof of Lemma 5.2. Moreover, it is easy to notice that for any $\varphi \in (0, \pi/3)$ the set Λ^φ contains similar copies of the sets Λ^{φ_n} for a sequence φ_n tending to 0.

The computation of the box dimension of the flexed Sierpiński gasket can be essentially done by the use of Falconer's result from [**5**]. Here we give another, very explicit, proof. The method we use to compute the Hausdorff dimension is similar to the one used [**11, 12**]. In fact, our proof extends to a quite general class of planar geometric constructions, including the affine examples from [**11, 12**] together with some non-affine ones, in particular certain generalizations of the flexed Sierpiński gasket (see Remark 6.7). This is described in [**1**].

Apart from determining the Hausdorff dimension of Λ, we provide a multifractal formalism for a family of the Bernoulli measures on Λ, computing explicitly the pointwise dimension spectra $f_\mu(\alpha)$. Recall that

$$f_\mu(\alpha) = \mathrm{HD}\left(\left\{x \in A : \lim_{r\to 0} \log \mu(B_r(x))/\log r = \alpha\right\}\right).$$

(For a general exposition on the multifractal formalism refer, e.g., to [**4, 13**].)

Again, there are not many examples of measures for which the dimension spectrum can be determined rigorously. The classical examples contain the equilibrium measures for smooth conformal expanding maps and Axiom A diffeomorphisms on surfaces together with some generalizations and the Bernoulli measures on some self-affine sets. In [**10**], the dimension spectrum is computed for the Bernoulli measures on the general Sierpiński carpets. In most of these examples, the function $f_\mu(\alpha)$ is smooth and concave. In [**7**], another spectrum $\beta(q)$ is determined for the Bernoulli measures on the generalized Sierpiński triangles.

Here we determine the dimension spectrum $f_\mu(\alpha)$ for the Bernoulli measures on the flexed Sierpiński gasket. In the study of their multifractal properties, there appear some interesting phenomena. Although for some measures the dimension spectrum is analytic and concave, for others it has discontinuities or points of non-differentiability. Moreover, the measure of the full dimension (i.e. of the Hausdorff dimension equal to the Hausdorff dimension of Λ) has a non-trivial spectrum.

The plan of the paper is as follows. Section 2 contains basic definitions and results concerning the Hausdorff and box dimensions and the multifractal formalism. In §3 we give a detailed description of the construction of the flexed Sierpiński gasket Λ and its basic properties. In §4 we compute the box dimension of Λ (Theorem A). Section 5 contains some lemmas used in the proof of the main results. The most important one is Lemma 5.2, describing the geometry and combinatorics of Λ. Finally, in §6 we determine

https://doi.org/10.1017/S0143385704000859 Published online by Cambridge University Press

the Hausdorff dimension of a one-parameter family of the Bernoulli measures on Λ (Theorem B), compute their pointwise dimension spectra (Theorem C) and the Hausdorff dimension of Λ (Theorem D).

2. *Preliminaries*

Notation. The open disc in the plane of radius r centred at x will be denoted by $B_r(x)$. For a point x and sets A, B in the plane we define

$$\begin{aligned}\operatorname{dist}(x, A) &= \inf\{\varrho(x, y) : y \in A\},\\ \operatorname{dist}(A, B) &= \inf\{\varrho(y, z) : y \in A, z \in B\},\\ \operatorname{diam} A &= \sup\{\varrho(y, z) : y, z \in A\},\end{aligned}$$

where ϱ is the Euclidean distance. For $t \in \mathbb{R}$ we denote by $\lfloor t \rfloor$ the largest integer not greater than t and by $\lceil t \rceil$ the smallest integer not smaller than t. The symbol card denotes cardinality.

Recall some basic definitions and facts concerning the Hausdorff and box dimensions and the multifractal formalism.

Definition 2.1. For $A \subset \mathbb{R}^k$ and $\delta > 0$ the (outer) δ-Hausdorff measure of A is defined as

$$\mathcal{H}_\delta(A) = \lim_{\delta \to 0}\left(\inf \sum_{U \in \mathcal{U}} (\operatorname{diam} U)^\delta\right),$$

where the infimum is taken over all countable coverings $\mathcal{U}$ of A by open sets of diameter smaller than δ.

The Hausdorff dimension of A is defined as

$$\operatorname{HD}(A) = \sup\{\delta > 0 : \mathcal{H}_\delta(A) = +\infty\} = \inf\{\delta > 0 : \mathcal{H}_\delta(A) = 0\}.$$

Let $N_\delta(A)$ be the minimal number of balls of diameter δ needed to cover A (equivalently, we can use k-dimensional cubes of side δ). Define the lower and upper box dimensions as

$$\underline{\operatorname{BD}}(A) = \liminf_{\delta \to 0} \frac{\log N_\delta(A)}{-\log \delta}, \quad \overline{\operatorname{BD}}(A) = \limsup_{\delta \to 0} \frac{\log N_\delta(A)}{-\log \delta}.$$

If $\underline{\operatorname{BD}}(A) = \overline{\operatorname{BD}}(A)$, then we denote their common value by $\operatorname{BD}(A)$ and call it the box dimension of A. The box dimension is also called the box-counting or Minkowski dimension.

It is easy to check that

$$\operatorname{HD}\left(\bigcup_{m=1}^{\infty} A_m\right) = \sup_m \operatorname{HD}(A_m).$$

Moreover,

$$\operatorname{HD}(A) \le \underline{\operatorname{BD}}(A).$$

https://doi.org/10.1017/S0143385704000859 Published online by Cambridge University Press

Let μ be a finite Borel measure in $\mathbb{R}^n$. Define the Hausdorff dimension of μ as

$$\mathrm{HD}(\mu) = \inf\{\mathrm{HD}(A) : A \text{ has full measure } \mu\}.$$

The lower and upper pointwise dimensions of μ at a point x are defined as

$$\underline{d}_\mu(x) = \liminf_{r\to 0} \frac{\log \mu(B_r(x))}{\log r}, \quad \overline{d}_\mu(x) = \limsup_{r\to 0} \frac{\log \mu(B_r(x))}{\log r}.$$

If the lower and upper pointwise dimensions coincide, then we denote their common value by $d_\mu(x)$ and call it the pointwise dimension of μ at x.

Let

$$\mathcal{J}_\mu(\alpha) = \{x \in A : d_\mu(x) = \alpha\}$$

and define the (pointwise) dimension spectrum of μ by

$$f_\mu(\alpha) = \mathrm{HD}(\mathcal{J}_\mu(\alpha)).$$

We will estimate the Hausdorff dimension using the following well-known fact, which we call Frostman's lemma.

FROSTMAN'S LEMMA. *Let μ be a finite Borel measure in $\mathbb{R}^n$ and let $A \subset \mathbb{R}^n$. Then:*

- *if $\underline{d}_\mu(x) \geq D$ for μ-almost every x, then $\mathrm{HD}(\mu) \geq D$;*
- *if $\underline{d}_\mu(x) \leq D$ for μ-almost every x, then $\mathrm{HD}(\mu) \leq D$;*
- *if $\underline{d}_\mu(x) \leq D$ for every $x \in A$, then $\mathrm{HD}(A) \leq D$.*

3. *Construction of the flexed Sierpiński gasket*

In this section we recall the definition of the flexed Sierpiński gasket introduced [**15**]. In the description we will use the following convention: we say that a vertex of an isosceles non-equilateral triangle is *principal* if it is contained in the axis of symmetry of the triangle. Similarly, *the principal angle* of an isosceles triangle is the angle whose vertex is the principal vertex of the triangle and *the principal height* is the height containing the principal vertex.

Fix $\varphi \in (0, \pi/3)$ and consider an isosceles triangle Δ^φ of the principal angle φ and two longer sides of length 1. Then define three isosceles triangles $\Delta_0, \Delta_1, \Delta_2 \subset \Delta^\varphi$ in the following way. The triangles Δ^φ and Δ_0 have common principal vertex and Δ_0 has two longer sides of length $1/2$ contained respectively in two longer sides of Δ^φ. The principal vertices of the triangles Δ_1 and Δ_2 coincide respectively with two non-principal vertices of Δ^φ, moreover Δ_1 and Δ_2 have one common non-principal vertex and the remaining non-principal vertices of Δ_1 and Δ_2 coincide respectively with two non-principal vertices of Δ_0 (see Figure 1). We choose which of the two triangles is labelled Δ_1 or Δ_2 in such a way that the principal vertices of $\Delta_0, \Delta_1, \Delta_2$ are ordered counter-clockwise around the boundary of Δ^φ.

Then the triangle Δ_0 is geometrically similar to Δ^φ with ratio $1/2$, while Δ_1 and Δ_2 (which are congruent) have two longer sides of length $1/2$ and principal angle $\tilde{\varphi}$ for some $\tilde{\varphi} < \varphi$. By elementary trigonometry we get

$$2\sin(\varphi/2) = \sin(\varphi/2 + \tilde{\varphi}),$$

https://doi.org/10.1017/S0143385704000859 Published online by Cambridge University Press

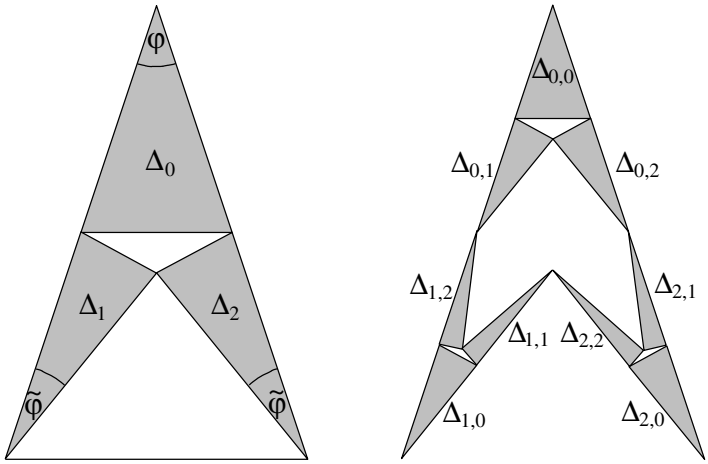


FIGURE 1. The first and second step of the construction of the flexed Sierpiński gasket.

which implies

$$\tilde{\varphi} = \arcsin(2\sin(\varphi/2)) - \varphi/2. \tag{1}$$

Then we repeat the construction inductively. Having defined the triangles $\Delta_{\varepsilon_1,\dots,\varepsilon_n}$ for $\varepsilon_1, \dots, \varepsilon_n \in \{0, 1, 2\}$ at the nth step, we take $\Delta_{\varepsilon_1,\dots,\varepsilon_n}$ as Δ^{φ} at the first step of the construction and define $\Delta_{\varepsilon_1,\dots,\varepsilon_n,0}$, $\Delta_{\varepsilon_1,\dots,\varepsilon_n,1}$, $\Delta_{\varepsilon_1,\dots,\varepsilon_n,2}$ in the same way as Δ_0, Δ_1, Δ_2 respectively (see Figure 1).

Now define the flexed Sierpiński gasket Λ as the limit set of the above construction, i.e.

$$\Lambda = \Lambda^{\varphi} = \bigcap_{n=1}^{\infty} \bigcup_{\varepsilon_1,\dots,\varepsilon_n \in \{0,1,2\}} \Delta_{\varepsilon_1,\dots,\varepsilon_n}.$$

We introduce the following notation. Let

$$\mathcal{D}_n = \{\Delta_{\varepsilon_1,\dots,\varepsilon_n} : \varepsilon_1, \dots, \varepsilon_n \in \{0, 1, 2\}\}.$$

For $n > 0$ and $\varepsilon_1, \dots, \varepsilon_n \in \{0, 1, 2\}$ let

$$k_{\varepsilon_1,\dots,\varepsilon_n} = \operatorname{card}\{m \le n : \varepsilon_m > 0\}$$

and for $k = 0, \dots, n$ let

$$\mathcal{D}_n^{(k)} = \{\Delta_{\varepsilon_1,\dots,\varepsilon_n} \in \mathcal{D}_n : k_{\varepsilon_1,\dots,\varepsilon_n} = k\}.$$

Definition 3.1. Denote by $S_{\varepsilon_1,\dots,\varepsilon_n}$ the length of the shortest side of $\Delta = \Delta_{\varepsilon_1,\dots,\varepsilon_n}$ (it is the side perpendicular to the principal height of Δ) and by $L_{\varepsilon_1,\dots,\varepsilon_n}$ the length of each of two longer sides of Δ. Also let $\varphi_{\varepsilon_1,\dots,\varepsilon_n}$ be the measure of the principal angle of Δ.

By construction,

$$L_{\varepsilon_1,\dots,\varepsilon_n} = 2^{-n} \quad \text{for every } \Delta_{\varepsilon_1,\dots,\varepsilon_n} \in \mathcal{D}_n.$$

https://doi.org/10.1017/S0143385704000859 Published online by Cambridge University Press

Moreover, analogously to (1), we see that

$$\varphi_{\varepsilon_1,\dots,\varepsilon_n} = h^{k_{\varepsilon_1,\dots,\varepsilon_n}}(\varphi), \tag{2}$$

where

$$h(\theta) = \arcsin(2\sin(\theta/2)) - \theta/2 \quad \text{for } \theta \in [0, \pi/3]$$

and h^k denotes the kth iterate of h.

LEMMA 3.2. *There exists a constant $c > 0$ such that for every n, k and every $\Delta_{\varepsilon_1,\dots,\varepsilon_n} \in \mathcal{D}_n^{(k)}$,*

$$2^{-k}/c < \varphi_{\varepsilon_1,\dots,\varepsilon_n} < c2^{-k},$$
$$2^{-n-k}/c < S_{\varepsilon_1,\dots,\varepsilon_n} < c2^{-n-k}.$$

Proof. By (2), it is sufficient to look at the function h. It is easy to check that $h(\theta) < \theta$ for $\theta \in (0, \pi/3)$. Moreover, h extends to an analytic function in a neighbourhood of 0 with $h(0) = 0$ and $h'(0) = 1/2$. Hence, there exists an analytic homeomorphism H in a neighbourhood of 0, such that $h(\theta) = H(H^{-1}(\theta)/2)$, so $h^k(\theta) = H(H^{-1}(\theta)/2^k)$. The formula extends for $\theta \in [0, \pi/3)$. Therefore, there exists a constant $c_0 > 0$ such that

$$2^{-k}/c_0 < h^k(\varphi) < c_0 2^{-k},$$

which easily implies the lemma. □

4. *The box dimension of* Λ

In this section we compute the box dimension of Λ.

THEOREM A.

$$\mathrm{BD}(\Lambda) = \frac{\log(\sqrt{17}+1)}{\log 2} - 1.$$

In the proof we use the two following facts. The first one is straightforward.

LEMMA 4.1. $\operatorname{card} \mathcal{D}_n^{(k)} = \binom{n}{k} 2^k$.

Proof. We have

$$\operatorname{card} \mathcal{D}_n^{(k)} = \operatorname{card}\{(\varepsilon_1, \dots, \varepsilon_n) : \varepsilon_1, \dots, \varepsilon_n \in \{0,1,2\} \text{ and } \operatorname{card}\{m \le n : \varepsilon_m \in \{1,2\}\} = k\}.$$

There are $\binom{n}{k}$ choices for m such that $\varepsilon_m \in \{1,2\}$. Given such a choice, there are 2^k possibilities for ε_m to be equal to 1 or 2. □

The second fact describes a division of Λ into some convenient subsets. Let

$$\mathcal{U}_m = \bigcup_{n=\lceil m/2 \rceil}^{m} \mathcal{D}_n^{(m-n)}.$$

LEMMA 4.2. *For every $m > 1$, the triangles from $\mathcal{U}_m$ have disjoint interiors. Moreover, $\mathcal{U}_m \cup \mathcal{U}_{m+1}$ is a cover of Λ.*

https://doi.org/10.1017/S0143385704000859 Published online by Cambridge University Press

Proof. It is clear that

$$\mathcal{U}_m = \bigcup_{\{(n,k):0\le k\le n \text{ and } n+k=m\}} \mathcal{D}_n^{(k)}.$$

To show the first part of the lemma, note that the interiors of any two triangles Δ, Δ' from $\bigcup_n \mathcal{D}_n$ either are disjoint or one of them contains the other. In the second case, we have $\Delta = \Delta_{\varepsilon_1,\dots,\varepsilon_n}$ and $\Delta' = \Delta_{\varepsilon_1,\dots,\varepsilon_n,\dots,\varepsilon_{n+l}}$ for some $l \ge 0$. Suppose Δ, $\Delta' \in \mathcal{U}_m$. Then

$$m = n + k_{\varepsilon_1,\dots,\varepsilon_n} = n + l + k_{\varepsilon_1,\dots,\varepsilon_n,\dots,\varepsilon_{n+l}},$$

which implies $l = 0$ and $\Delta = \Delta'$.

To deal with the second part, take $x \in \Lambda$. Then $x = \bigcap_n \Delta_{\varepsilon_1,\dots,\varepsilon_n}$ for some $\varepsilon_1, \varepsilon_2, \dots$. Consider the sequence $a_n = n + k_{\varepsilon_1,\dots,\varepsilon_n}$ for $n > 0$. Then $a_1 \le 2 \le m$ and $a_{m+1} \ge m+1$. Moreover, $a_{n+1} - a_n \in \{1, 2\}$ for all n. Hence, there exists n_0 such that $a_{n_0} \in \{m, m+1\}$, which means that $\Delta_{\varepsilon_1,\dots,\varepsilon_{n_0}} \in \mathcal{U}_m \cup \mathcal{U}_{m+1}$. □

Proof of Theorem A. Fix a large constant $C > 0$ and take a small $\delta > 0$. Let

$$m = m(\delta) = \left\lfloor -\frac{\log(C\delta)}{\log 2} \right\rfloor.$$

Then

$$\frac{C\delta}{2} < 2^{-m} \le C\delta. \tag{3}$$

Consider $\Delta_{\varepsilon_1,\dots,\varepsilon_n} \in \mathcal{U}_m$. By Lemma 3.2,

$$S_{\varepsilon_1,\dots,\varepsilon_n} < c2^{-m}.$$

Moreover, the principal height of $\Delta_{\varepsilon_1,\dots,\varepsilon_n}$ has length smaller than $L_{\varepsilon_1,\dots,\varepsilon_n} = 2^{-n}$. Hence, we can cover $\Delta_{\varepsilon_1,\dots,\varepsilon_n}$ by $\lceil C\rceil\lceil cC\rceil 2^{m-n}$ squares of side δ, so by Lemmas 4.1 and 4.2,

$$\begin{aligned} N_\delta(\Lambda) &< \lceil C\rceil\lceil cC\rceil\left(\sum_{n=\lceil m/2\rceil}^{m} 2^{m-n} \operatorname{card} \mathcal{D}_n^{(m-n)} + \sum_{n=\lceil (m+1)/2\rceil}^{m+1} 2^{m+1-n} \operatorname{card} \mathcal{D}_n^{(m+1-n)} \right) \\ &= \lceil C\rceil\lceil cC\rceil\left(\sum_{n=\lceil m/2\rceil}^{m} \binom{n}{m-n} 4^{m-n} + \sum_{n=\lceil (m+1)/2\rceil}^{m+1} \binom{n}{m+1-n} 4^{m+1-n} \right) \\ &= \lceil C\rceil\lceil cC\rceil\left(\sum_{k=0}^{\lfloor m/2\rfloor} \binom{m-k}{k} 4^k + \sum_{k=0}^{\lfloor (m+1)/2\rfloor} \binom{m+1-k}{k} 4^k \right). \end{aligned} \tag{4}$$

On the other hand, for each $\Delta_{\varepsilon_1,\dots,\varepsilon_n} \in \mathcal{U}_m$, the longer sides of the triangle $\Delta_{\varepsilon_1,\dots,\varepsilon_n,1,1}$ have length 2^{-n-2} and are contained in Λ, so by (3) we need more than $\lfloor C/8\rfloor 2^{m-n}$ squares of side δ to cover $\Lambda \cap \Delta_{\varepsilon_1,\dots,\varepsilon_n,1,1}$. Moreover, by (2) and trigonometry,

$$\begin{aligned} &\operatorname{dist}(\Delta_{\varepsilon_1,\dots,\varepsilon_n,1,1}, \partial\Delta_{\varepsilon_1,\dots,\varepsilon_n}) \\ &\quad = 2^{-n-2} \min\left(\sin\left(\frac{\pi - \varphi_{\varepsilon_1,\dots,\varepsilon_n}}{2} - h(\varphi_{\varepsilon_1,\dots,\varepsilon_n}) \right), \sin(h^2(\varphi_{\varepsilon_1,\dots,\varepsilon_n})) \right) \end{aligned}$$

(cf. Figure 1). By Lemma 3.2, it is easy to see that this expression is smaller than $c'2^{-m}$ for a constant $c' > 0$ independent of $\varepsilon_1, \varepsilon_2, \dots$. Choosing the constant C in (3) sufficiently large, we can assume

$$\operatorname{dist}(\Delta_{\varepsilon_1,\dots,\varepsilon_n,1,1}, \partial\Delta_{\varepsilon_1,\dots,\varepsilon_n}) > \sqrt{2}\,\delta.$$

https://doi.org/10.1017/S0143385704000859 Published online by Cambridge University Press

Since, by Lemma 4.2, the interiors of triangles from $\mathcal{U}_m$ are disjoint, this implies that any squares of side δ covering points from different triangles $\Delta_{\varepsilon_1,\dots,\varepsilon_n,1,1}$ are disjoint. Hence,

$$N_\delta(\Lambda) > \lfloor C/8 \rfloor \sum_{n=\lceil m/2 \rceil}^{m} 2^{m-n} \operatorname{card} \mathcal{D}_n^{(m-n)} = \lfloor C/8 \rfloor \sum_{k=0}^{\lfloor m/2 \rfloor} \binom{m-k}{k} 4^k. \tag{5}$$

By (4) and (5) we easily get

$$\mathrm{BD}(\Lambda) = \lim_{m\to\infty} \left[\log \max_{k=0,\dots,\lfloor m/2 \rfloor} \binom{m-k}{k} 4^k \Big/ m \log 2 \right] \tag{6}$$

(provided this limit exists). To compute this limit, note that

$$\binom{m-(k+1)}{k+1} 4^{k+1} \geq \binom{m-k}{k} 4^k \iff 4(m-2k)(m-2k-1) \geq (m-k)(k+1),$$

which for sufficiently large m is equivalent to

$$k \leq \frac{17m - 9 - \sqrt{17m^2 + 34m + 81}}{34}.$$

Hence,

$$\max_{k=0,\dots,\lfloor m/2 \rfloor} \binom{m-k}{k} 4^k = \binom{m-k_{\max}}{k_{\max}} 4^{k_{\max}}$$

for

$$k_{\max} = \left\lceil \frac{17m - 9 - \sqrt{17m^2 + 34m + 81}}{34} \right\rceil.$$

Note that

$$\lim_{m\to\infty} \frac{k_{\max}}{m} = \frac{1}{2} - \frac{\sqrt{17}}{34}.$$

It is easy to check that if $\lim_{m\to\infty} k/m = t$ for some $t \in (0, 1)$, then

$$\lim_{m\to\infty} \frac{\log \binom{m-k}{k}}{m} = (1-t)\log(1-t) - t\log t - (1-2t)\log(1-2t).$$

Therefore, by (6),

$$\mathrm{BD}(\Lambda) = 2t + \frac{(1-t)\log(1-t) - t\log t - (1-2t)\log(1-2t)}{\log 2}$$

for $t = 1/2 - \sqrt{17}/34$. After easy calculations, we obtain

$$\mathrm{BD}(\Lambda) = \frac{\log(\sqrt{17}+1)}{\log 2} - 1. \qquad \square$$

5. *Supporting lemmas*

In this section we prove the supporting lemmas used in the proof of the main results in §6. We begin with some definitions.

For $n > 0$ and $\varepsilon_1, \dots, \varepsilon_n \in \{0, 1, 2\}$ let

$$\varrho_{\varepsilon_1,\dots,\varepsilon_n} = \frac{k_{\varepsilon_1,\dots,\varepsilon_n}}{n}$$

https://doi.org/10.1017/S0143385704000859 Published online by Cambridge University Press

and

$$M = M_{\varepsilon_1,\dots,\varepsilon_n} = \min\{m \le n : m + k_{\varepsilon_1,\dots,\varepsilon_m} \ge n\}.$$

Note that

$$M + k_{\varepsilon_1,\dots,\varepsilon_M} - 1 \le n \le M + k_{\varepsilon_1,\dots,\varepsilon_M} \tag{7}$$

and

$$n/2 \le M \le n. \tag{8}$$

Moreover, Lemma 3.2 implies that

$$2^{-n}/(2c) < S_{\varepsilon_1,\dots,\varepsilon_M} < c2^{-n}. \tag{9}$$

Definition 5.1. We say that $\Delta_{\varepsilon_1,\dots,\varepsilon_n}, \Delta_{\varepsilon'_1,\dots,\varepsilon'_n} \in \mathcal{D}_n$ are of the same type if, for every $m = 1, \dots, n$, we have

$$\varepsilon_m > 0 \text{ if and only if } \varepsilon'_m > 0.$$

Otherwise, we say that $\Delta_{\varepsilon_1,\dots,\varepsilon_n}, \Delta_{\varepsilon'_1,\dots,\varepsilon'_n}$ are of different types.

Define

$$Q_{\varepsilon_1,\dots,\varepsilon_n} = \bigcup\{\Delta_{\varepsilon'_1,\dots,\varepsilon'_n} : \varepsilon'_m = \varepsilon_m \text{ for } m = 1, \dots, M_{\varepsilon_1,\dots,\varepsilon_n} \text{ and } \Delta_{\varepsilon_1,\dots,\varepsilon_n}, \Delta_{\varepsilon'_1,\dots,\varepsilon'_n} \text{ are of the same type}\}$$

and

$$\mathcal{Q}_n = \{Q_{\varepsilon_1,\dots,\varepsilon_n} : \Delta_{\varepsilon_1,\dots,\varepsilon_n} \in \mathcal{D}_n\}.$$

Note that different sequences $\varepsilon_1, \dots, \varepsilon_n$ can give the same element of $\mathcal{Q}_n$, in fact $Q = Q_{\varepsilon_1,\dots,\varepsilon_n} = Q_{\varepsilon'_1,\dots,\varepsilon'_n}$ if and only if $\Delta_{\varepsilon_1,\dots,\varepsilon_n}, \Delta_{\varepsilon'_1,\dots,\varepsilon'_n} \subset Q$. It is easy to see that the sets $Q \in \mathcal{Q}_n$ are pairwise disjoint.

It is clear that if $Q = Q_{\varepsilon_1,\dots,\varepsilon_n} \in \mathcal{Q}_n$, then $M_{\varepsilon_1,\dots,\varepsilon_n}$ and $\varrho_{\varepsilon_1,\dots,\varepsilon_n}$ do not depend on the choice of $\varepsilon_1, \dots, \varepsilon_n$. Hence, we can define

$$M(Q) = M_{\varepsilon_1,\dots,\varepsilon_n}, \quad \varrho_m(Q) = \varrho_{\varepsilon_1,\dots,\varepsilon_m} \quad \text{for } m \le n.$$

The following lemma, which describes the geometry and combinatorics of the sets $Q \in \mathcal{Q}_n$, will be the main technical tool in studying multifractal properties of the Bernoulli measures on Λ.

LEMMA 5.2. *There exist constants $c_1, c_2, K > 0$ such that for every n:*

(a) $\operatorname{diam} Q < c_1 2^{-n}$ *for every* $Q \in \mathcal{Q}_n$;

(b) *every disc of radius 2^{-n} intersects at most K sets from $\mathcal{Q}_n$ and for every two such sets Q, Q' we have $|\varrho_n(Q) - \varrho_n(Q')|, |\varrho_{M(Q)}(Q) - \varrho_{M(Q')}(Q')| < c_2/n$.*

Proof. Take $\Delta = \Delta_{\varepsilon_1,\dots,\varepsilon_n}, \Delta' = \Delta_{\varepsilon'_1,\dots,\varepsilon'_n} \in \mathcal{D}_n$. Let $Q = Q_{\varepsilon_1,\dots,\varepsilon_n}, Q' = Q_{\varepsilon'_1,\dots,\varepsilon'_n} \in \mathcal{Q}_n$. We will show the following claim.

CLAIM. *There exist constants $c_1, c_3, K_0 > 0$ and a set $\mathcal{K} \subset \{1, \dots, n\}$ of cardinality K_0 depending only on $\varepsilon_1, \dots, \varepsilon_n$, such that if $Q = Q'$, then $\operatorname{diam}(\Delta \cup \Delta') < c_1 2^{-n}$ and if $Q \ne Q'$, then either $\operatorname{dist}(\Delta, \Delta') > c_3 2^{-n}$ or there exists $\Delta_{\tilde\varepsilon_1,\dots,\tilde\varepsilon_n} \subset Q'$ such that $\tilde\varepsilon_m \ne \varepsilon_m$ only for $m \in \mathcal{K}$.*

https://doi.org/10.1017/S0143385704000859 Published online by Cambridge University Press

The point (a) of the lemma follows directly from the claim. Moreover, if B is a disc of radius 2^{-n}, then $\operatorname{diam} B \le c_3 2^{-(n+\lfloor \log_2 c_3 \rfloor - 1)}$ and the claim asserts that B intersects at most 3^{K_0} sets from $\mathcal{Q}_{n+\lfloor \log_2 c_3 \rfloor - 1}$, so B intersects at most 3^{K_1} sets from $\mathcal{Q}_n$, where $K_1 = K_0 - \lfloor \log_2 c_3 \rfloor + 1$. For every two such sets Q, Q' the claim implies that $Q = Q_{\varepsilon_1,\dots,\varepsilon_n}$, $Q' = Q_{\tilde\varepsilon_1,\dots,\tilde\varepsilon_n}$ for $\varepsilon_m, \tilde\varepsilon_m$ such that $\varepsilon_m \ne \tilde\varepsilon_m$ for at most K_1 indices m. Hence,

$$|k_{\varepsilon_1,\dots,\varepsilon_m} - k_{\tilde\varepsilon_1,\dots,\tilde\varepsilon_m}| \le K_1, \quad |\varrho_{\varepsilon_1,\dots,\varepsilon_m} - \varrho_{\tilde\varepsilon_1,\dots,\tilde\varepsilon_m}| \le K_1/m \tag{10}$$

for every $m \le n$. In particular, $|\varrho(Q) - \varrho(Q')| \le K_1/n$. Moreover, we have

$$|M(Q) - M(Q')| \le K_1. \tag{11}$$

To show (11), suppose that $M(Q) - M(Q') > K_1$ (the other case is analogous). Then $M(Q') + K_1 \le n$ and by (10),

$$M(Q') + K_1 + k_{\varepsilon_1,\dots,\varepsilon_{M(Q')+K_1}} \ge M(Q') + k_{\tilde\varepsilon_1,\dots,\tilde\varepsilon_{M(Q')+K_1}} \ge M(Q') + k_{\tilde\varepsilon_1,\dots,\tilde\varepsilon_{M(Q')}} \ge n,$$

so $M(Q') + K_1 \ge M(Q)$, which is a contradiction. Hence, (11) holds.

By (7) and (8),

$$\left| \varrho_{M(Q)}(Q) - \frac{n}{M(Q)} + 1 \right| \le \frac{2}{n},$$

so by (11) and (8),

$$|\varrho_{M(Q)}(Q) - \varrho_{M(Q')}(Q')| \le n \left| \frac{1}{M(Q)} - \frac{1}{M(Q')} \right| + \frac{4}{n} \le \frac{4(K_1+1)}{n},$$

which gives the point (b) of the lemma.

Now we prove the claim.

Proof of the claim. The case $\Delta = \Delta'$ is trivial, so suppose otherwise. Let

$$n_0 = \min\{m \le n : \varepsilon_m \ne \varepsilon'_m\}.$$

Note that if $n - n_0 \le K_0$ for some constant K_0, then the claim is straightforward, so we can assume $n - n_0 > K_0$ for a large K_0. By symmetry, we can also assume $\varepsilon_{n_0} < \varepsilon'_{n_0}$. Let $M = M_{\varepsilon_1,\dots,\varepsilon_n}$. We split the proof into three cases:

(i) $\varepsilon_{n_0} = 0, \varepsilon'_{n_0} > 0$;

(ii) $\varepsilon_{n_0} = 1, \varepsilon'_{n_0} = 2, M < n_0$;

(iii) $\varepsilon_{n_0} = 1, \varepsilon'_{n_0} = 2, M \ge n_0$.

Case (i). In this case Δ, Δ' are of different types, so $Q \ne Q'$. Let l be the straight line containing two non-principal vertices of the triangle $\Delta_{\varepsilon_1,\dots,\varepsilon_{n_0}}$. Note that Δ and Δ' are situated on different sides of l. Define

$$S = \{x \in \mathbb{R}^2 : \operatorname{dist}(x, l) \le c_3 2^{-n}\}$$

for a small constant c_3. If $\Delta \cap S = \emptyset$ or $\Delta' \cap S = \emptyset$, then $\operatorname{dist}(\Delta, \Delta') > c_3 2^{-n}$, so we can suppose that Δ, Δ' intersect S. Note that $\operatorname{dist}(\Delta_{\varepsilon_1,\dots,\varepsilon_{n_0},0}, l) = 2^{-n_0-1} \cos(\varphi_{\varepsilon_1,\dots,\varepsilon_{n_0}}/2) > (\sqrt{3}/4)2^{-n_0} > c_3 2^{-n}$, so

$$S \cap \Delta_{\varepsilon_1,\dots,\varepsilon_{n_0},0} = \emptyset.$$

https://doi.org/10.1017/S0143385704000859 Published online by Cambridge University Press

Moreover, the angles between both longer sides of $\Delta_{\varepsilon_1,\dots,\varepsilon_{n_0},1}$, $\Delta_{\varepsilon_1,\dots,\varepsilon_{n_0},2}$ and l are not smaller than $(\pi-\varphi_{\varepsilon_1,\dots,\varepsilon_{n_0}})/2-h(\varphi_{\varepsilon_1,\dots,\varepsilon_{n_0}}) > \pi/2-\arcsin(2\sin(\varphi/2)) > 0$ (cf. Figure 1), which easily implies that

$$S\cap\Delta_{\varepsilon_1,\dots,\varepsilon_{n_0},i}\subset\Delta_{\varepsilon_1,\dots,\varepsilon_{n_0},\underbrace{i,0,\dots,0}_{n-n_0}}$$

for $i=1,2$ (provided c_3 is chosen small enough). In the same way we show that

$$S\cap\Delta_{\varepsilon'_1,\dots,\varepsilon'_{n_0},0}=\emptyset,\quad S\cap\Delta_{\varepsilon'_1,\dots,\varepsilon'_{n_0},i}\subset\Delta_{\varepsilon'_1,\dots,\varepsilon'_{n_0},\underbrace{i,0,\dots,0}_{n-n_0}}$$

for $i=1,2$. We conclude that

$$\Delta=\Delta_{\varepsilon_1,\dots,\varepsilon_{n_0},\underbrace{i,0,\dots,0}_{n-n_0}},\quad \Delta'=\Delta_{\varepsilon'_1,\dots,\varepsilon'_{n_0},\underbrace{j,0,\dots,0}_{n-n_0}}$$

for some $i,j\in\{1,2\}$. Then $n_0=\max\{m\le n:\varepsilon_m\ne 0\}-1$, so $\mathcal{K}=\{n_0,n_0+1\}$ does not depend on $\varepsilon'_1,\dots,\varepsilon'_n$ and $\varepsilon'_m=\varepsilon_m$ for $m\notin\mathcal{K}$, so the claim holds for $\tilde{\varepsilon}_m=\varepsilon'_m$.

Case (ii). In this case $\varepsilon_m=\varepsilon'_m$ for every $m\le M$, so $M_{\varepsilon'_1,\dots,\varepsilon'_n}=M$ and $Q,Q'\subset\Delta_{\varepsilon_1,\dots,\varepsilon_M}$. Moreover,

$$\varphi_{\varepsilon_1,\dots,\varepsilon_M}<c2^{-k_{\varepsilon_1,\dots,\varepsilon_M}}\le c2^{-(n-M)}<c2^{-(n-n_0)}<c2^{-K_0},\tag{12}$$

which is a small constant. Choose one of the longer sides of $\Delta_{\varepsilon_1,\dots,\varepsilon_M}$ and divide it into 2^{M-n} equal intervals I_i, $i=1,\dots,2^{M-n}$. Denote by L the straight line containing the intervals I_i. The construction of Λ easily implies that there exist $\hat{\Delta}=\Delta_{\hat{\varepsilon}_1,\dots,\hat{\varepsilon}_n}$, $\hat{\Delta}'=\Delta_{\hat{\varepsilon}'_1,\dots,\hat{\varepsilon}'_n}\in\mathcal{D}_n$ such that $\hat{\Delta}\subset Q$, $\hat{\Delta}'\subset Q'$ and the longer sides of $\hat{\Delta}$, $\hat{\Delta}'$ are equal to I_i and $I_{i'}$, respectively, for some i,i', where $i=i'\iff Q=Q'$.

Note that for every $m>0$ the angle between the axis of symmetry of any triangle from $\mathcal{D}_m$ of principal angle ψ and the axis of symmetry of a triangle from $\mathcal{D}_{m+1}$ contained in it is not greater than $(\psi+h(\psi))/2$. Hence, for every $m\in\{M+1,\dots,n\}$ the angles between straight lines containing the longer sides of $\Delta_{\varepsilon_1,\dots,\varepsilon_m}$, $\Delta_{\varepsilon'_1,\dots,\varepsilon'_m}$, $\Delta_{\hat{\varepsilon}_1,\dots,\hat{\varepsilon}_m}$, $\Delta_{\hat{\varepsilon}'_1,\dots,\hat{\varepsilon}'_m}$ and L are not greater than

$$\varphi_{\varepsilon_1,\dots,\varepsilon_M}+\sum_{j=1}^{m-M}h^j(\varphi_{\varepsilon_1,\dots,\varepsilon_M})=\varphi_{\varepsilon_1,\dots,\varepsilon_M}+\sum_{j=1}^{m-M}\varphi_{\varepsilon_1,\dots,\varepsilon_M,\underbrace{1,\dots,1}_{j}},$$

which is less than

$$\theta_0=(1+c^2)\varphi_{\varepsilon_1,\dots,\varepsilon_M}$$

by (2) and Lemma 3.2. Using (12), we see that θ_0 is arbitrarily small, if K_0 is chosen large enough.

For $m\in\{M,\dots,n\}$ denote the orthogonal projections of the triangles $\Delta_{\varepsilon_1,\dots,\varepsilon_m}$, $\Delta_{\varepsilon'_1,\dots,\varepsilon'_m}$, $\Delta_{\hat{\varepsilon}_1,\dots,\hat{\varepsilon}_m}$, $\Delta_{\hat{\varepsilon}'_1,\dots,\hat{\varepsilon}'_m}$ to L by L_m, L'_m, $\hat{L}_m$, $\hat{L}'_m$, respectively. Then for $m>M$ we have $L_m\subset L_{m-1}$, $L'_m\subset L'_{m-1}$, $\hat{L}_m\subset\hat{L}_{m-1}$, $\hat{L}'_m\subset\hat{L}'_{m-1}$, the length of L_m, L'_m, $\hat{L}_m$, $\hat{L}'_m$ is in $[2^{-m}\cos\theta_0,2^{-m}]$ and one of its endpoints (denote it by y_m, y'_m, $\hat{y}_m$, $\hat{y}'_m$, respectively) is closer than $S_{\varepsilon_1,\dots,\varepsilon_{m-1}}\sin\theta_0<2^{-m+1}\theta_0^2$ to one of the endpoints of L_{m-1},

https://doi.org/10.1017/S0143385704000859 Published online by Cambridge University Press

L'_{m-1}, $\hat{L}_{m-1}$, $\hat{L}'_{m-1}$, respectively. Since Δ and $\hat{\Delta}$ (respectively Δ' and $\hat{\Delta}'$) are of the same type, y_m (respectively y'_m) is the right (left) endpoint of L_m (respectively L'_m) if and only if $\hat{y}_m$ (respectively $\hat{y}'_m$) is the right (left) endpoint of $\hat{L}_m$ (respectively $\hat{L}'_m$) (according to a chosen orientation of L). Moreover, $L_M = \hat{L}_M = L'_M = \hat{L}'_M$ and $\hat{L}_n = I_i$, $\hat{L}'_n = I_{i'}$. Using these facts, by a simple induction on m we conclude that the distance between the right (left) endpoint of the orthogonal projection of Δ (respectively Δ') to L and the right (left) endpoint of I_i (respectively $I_{i'}$) is smaller than

$$\sum_{m=M+1}^{n} (2^{-m+1}\theta_0^2 + 2^{-m}(1-\cos\theta_0)) < 3\theta_0^2 2^{-M} = 3(1+c^2)\theta_0\varphi_{\varepsilon_1,\dots,\varepsilon_M}2^{-M}$$
$$< 3c(1+c^2)\theta_0 2^{-M-k_{\varepsilon_1,\dots,\varepsilon_M}} \le 3c(1+c^2)\theta_0 2^{-n} < 2^{-n}/3 \tag{13}$$

(the estimations in (13) are by Lemma 3.2, the definition of M and the fact that θ_0 can be arbitrarily small). Since the length of I_i, $I_{i'}$ is 2^{-n}, (13) implies that either I_i and $I_{i'}$ have a common endpoint or the distance between the orthogonal projections of Δ and Δ' to L (and hence $\operatorname{dist}(\Delta, \Delta')$) is greater than $2^{-n}/3$. In the second case the claim is true, so assume that I_i and $I_{i'}$ have a common endpoint. If $I_i = I_{i'}$, which means $\hat{\Delta} = \hat{\Delta}'$ and $Q = Q'$, then $\operatorname{diam}(\Delta \cup \Delta')$ is smaller then the sum of the lengths of the projections of $\Delta \cup \Delta'$ to L and to the shorter side of $\Delta_{\varepsilon_1,\dots,\varepsilon_M}$. By (13), this sum does not exceed $(5/3)2^{-n} + S_{\varepsilon_1,\dots,\varepsilon_M} < (5/3+c)2^{-n}$ by (9). If $I_i \neq I_{i'}$, which means $\hat{\Delta} \neq \hat{\Delta}'$ and $Q \neq Q'$, then we can define

$$n_1 = \min\{M < m \le n : \hat{\varepsilon}_m \neq \hat{\varepsilon}'_m\}$$

and it is easy to check that one of the four following possibilities holds:

$$\hat{\Delta} = \Delta_{\hat{\varepsilon}_1,\dots,\hat{\varepsilon}_{n_1-1},0,\underbrace{1,0,\dots,0}_{n-n_1}}, \quad \hat{\Delta}' = \Delta_{\hat{\varepsilon}_1,\dots,\hat{\varepsilon}_{n_1-1},1,\underbrace{2,0,\dots,0}_{n-n_1}},$$
$$\hat{\Delta} = \Delta_{\hat{\varepsilon}_1,\dots,\hat{\varepsilon}_{n_1-1},0,\underbrace{2,0,\dots,0}_{n-n_1}}, \quad \hat{\Delta}' = \Delta_{\hat{\varepsilon}_1,\dots,\hat{\varepsilon}_{n_1-1},2,\underbrace{1,0,\dots,0}_{n-n_1}},$$
$$\hat{\Delta} = \Delta_{\hat{\varepsilon}_1,\dots,\hat{\varepsilon}_{n_1-1},1,\underbrace{2,0,\dots,0}_{n-n_1}}, \quad \hat{\Delta}' = \Delta_{\hat{\varepsilon}_1,\dots,\hat{\varepsilon}_{n_1-1},0,\underbrace{1,0,\dots,0}_{n-n_1}},$$
$$\hat{\Delta} = \Delta_{\hat{\varepsilon}_1,\dots,\hat{\varepsilon}_{n_1-1},2,\underbrace{1,0,\dots,0}_{n-n_1}}, \quad \hat{\Delta}' = \Delta_{\hat{\varepsilon}_1,\dots,\hat{\varepsilon}_{n_1-1},0,\underbrace{2,0,\dots,0}_{n-n_1}}$$

(we allow the case $n = n_1$). By the definition of $\hat{\Delta}$, $\hat{\Delta}'$, we have $\varepsilon_m = \varepsilon'_m = \hat{\varepsilon}_m = \hat{\varepsilon}'_m$ for every $m \le M$ and $\varepsilon_m = 0 \iff \hat{\varepsilon}_m = 0$, $\varepsilon'_m = 0 \iff \hat{\varepsilon}'_m = 0$ for every $M < m \le n$. Hence, $\varepsilon_m = 0 \iff \varepsilon'_m = 0$ for every $m \neq n_1, n_1+1$, so if we define

$$\tilde{\varepsilon}_m = \begin{cases} \varepsilon_m & \text{for } m \in \{1,\dots,n\} \setminus \{n_1, n_1+1\}, \\ \hat{\varepsilon}'_m & \text{for } m \in \{n_1, n_1+1\}, \end{cases}$$

then $\Delta_{\tilde{\varepsilon}_1,\dots,\tilde{\varepsilon}_n} \in Q'$. Moreover, if $n_2 = \max\{m \le n : \varepsilon_m \neq 0\}$, then the set $\mathcal{K} = \{n_2 - 1, n_2, n\} \cap \{1,\dots,n\}$ depends only on $\varepsilon_1,\dots,\varepsilon_n$ and contains the set $\{n_1, n_1+1\} \cap \{1,\dots,n\}$, which proves the claim in this case.

https://doi.org/10.1017/S0143385704000859 Published online by Cambridge University Press

Case (iii). Let $M' = M_{\varepsilon'_1,\dots,\varepsilon'_n}$. Then $\Delta_{\varepsilon_1,\dots,\varepsilon_M} \subset \Delta_{\varepsilon_1,\dots,\varepsilon_{n_0}}$ and $\Delta_{\varepsilon'_1,\dots,\varepsilon'_{M'}} \subset \Delta_{\varepsilon'_1,\dots,\varepsilon'_{n_0}}$, in particular $Q \neq Q'$.

Let l be the axis of symmetry of $\Delta_{\varepsilon_1,\dots,\varepsilon_{n_0-1}}$ (or Δ^φ, if $n_0 = 1$). Then $\Delta_{\varepsilon_1,\dots,\varepsilon_{n_0}}$ and $\Delta_{\varepsilon'_1,\dots,\varepsilon'_{n_0}}$ are symmetric with respect to l. Define S by the same formula as in Case (i). Again, if $\Delta \cap S = \emptyset$ or $\Delta' \cap S = \emptyset$, then $\operatorname{dist}(\Delta, \Delta') > c_3 2^{-n}$, so we can suppose that Δ, Δ' intersect S. Note that $\operatorname{dist}(\Delta_{\varepsilon_1,\dots,\varepsilon_{n_0},0} \cup \Delta_{\varepsilon_1,\dots,\varepsilon_{n_0},2}, l)$ and $\operatorname{dist}(\Delta_{\varepsilon'_1,\dots,\varepsilon'_{n_0},0} \cup \Delta_{\varepsilon'_1,\dots,\varepsilon'_{n_0},1}, l)$ are equal to

$$\min\left(\frac{S_{\varepsilon_1,\dots,\varepsilon_{n_0}}}{2}, 2^{-n_0-2}\sin\left(\frac{\varphi_{\varepsilon_1,\dots,\varepsilon_{n_0}}}{2} + h(\varphi_{\varepsilon_1,\dots,\varepsilon_{n_0}})\right)\right)$$
$$> \frac{1}{4}S_{\varepsilon_1,\dots,\varepsilon_{n_0}} > \frac{1}{4c}2^{-n_0-k_{\varepsilon_1,\dots,\varepsilon_{n_0}}} > \frac{1}{16c}2^{-n}$$

by Lemma 3.2 and the fact that $M \geq n_0$. Hence,

$$S \cap (\Delta_{\varepsilon_1,\dots,\varepsilon_{n_0},0} \cup \Delta_{\varepsilon_1,\dots,\varepsilon_{n_0},2} \cup \Delta_{\varepsilon'_1,\dots,\varepsilon'_{n_0},0} \cup \Delta_{\varepsilon'_1,\dots,\varepsilon'_{n_0},1}) = \emptyset$$

(provided c_3 is sufficiently small), so

$$S \cap \Delta_{\varepsilon_1,\dots,\varepsilon_{n_0}} \subset \Delta_{\varepsilon_1,\dots,\varepsilon_{n_0},1}, \quad S \cap \Delta_{\varepsilon'_1,\dots,\varepsilon'_{n_0}} \subset \Delta_{\varepsilon'_1,\dots,\varepsilon'_{n_0},2}.$$

Moreover, the angles between longer sides of $\Delta_{\varepsilon_1,\dots,\varepsilon_{n_0},1}$, $\Delta_{\varepsilon'_1,\dots,\varepsilon'_{n_0},2}$ and l are greater then $h(\varphi_{\varepsilon_1,\dots,\varepsilon_{n_0}})$, which together with (9) easily implies

$$S \cap \Delta_{\varepsilon_1,\dots,\varepsilon_{n_0},1} \subset \Delta_{\varepsilon_1,\dots,\varepsilon_{n_0},\underbrace{1,0,\dots,0}_{M-n_0}}, \quad S \cap \Delta_{\varepsilon'_1,\dots,\varepsilon'_{n_0},2} \subset \Delta_{\varepsilon'_1,\dots,\varepsilon'_{n_0},\underbrace{2,0,\dots,0}_{M-n_0}}$$

(we allow the case $M = n_0$). Since Δ, Δ' intersect S, we have

$$\Delta_{\varepsilon_1,\dots,\varepsilon_M} = \Delta_{\varepsilon_1,\dots,\varepsilon_{n_0},\underbrace{1,0,\dots,0}_{M-n_0}}, \quad \Delta_{\varepsilon'_1,\dots,\varepsilon'_{M'}} = \Delta_{\varepsilon'_1,\dots,\varepsilon'_{n_0},\underbrace{2,0,\dots,0}_{M-n_0}}.$$

In particular, $M = M'$ and $\Delta_{\varepsilon_1,\dots,\varepsilon_M}$, $\Delta_{\varepsilon'_1,\dots,\varepsilon'_M}$ have a common principal vertex and are symmetric with respect to l. Moreover, $k_{\varepsilon_1,\dots,\varepsilon_M} = k_{\varepsilon'_1,\dots,\varepsilon'_M} \geq k_{\varepsilon_1,\dots,\varepsilon_{n_0}} + 1$ and (as in Case (i)), n_0 does not depend on $\varepsilon'_1, \dots, \varepsilon'_n$.

Fix a small constant $\theta_0 > 0$. Suppose first that $\varphi_{\varepsilon_1,\dots,\varepsilon_{n_0}} \geq \theta_0$. Then $\varphi_{\varepsilon_1,\dots,\varepsilon_M}$ and $\varphi_{\varepsilon'_1,\dots,\varepsilon'_M}$ are not smaller than $h(\theta_0)$, so $n - M \leq k_{\varepsilon_1,\dots,\varepsilon_M} < -\log_2(h(\theta_0)/c)$ by Lemma 3.2. This implies that $\varepsilon_m \neq \varepsilon'_m$ only if $m \in \{n_0, n_0+1\}$ or $m > n + \log_2(h(\theta_0)/c)$, which proves the claim.

Suppose now that $\varphi_{\varepsilon_1,\dots,\varepsilon_{n_0}} < \theta_0$. Repeating the proof in the case $M < n_0$, we find intervals I_i, $I'_{i'}$ of length 2^{-n}, such that $I_i \subset L$, $I'_{i'} \subset L'$, where L, L' are longer sides of the triangles $\Delta_{\varepsilon_1,\dots,\varepsilon_M}$, $\Delta_{\varepsilon'_1,\dots,\varepsilon'_M}$, respectively, L, L' are symmetric with respect to l and the endpoints of the orthogonal projection of Δ (respectively Δ') to L (respectively L') are closer than $2^{-n}/4$ to the appropriate endpoints of I_i (respectively $I'_{i'}$). Let $\tilde{I}_i$, $\tilde{I}_{i'}$ be the orthogonal projections of I_i, $I'_{i'}$ to l. Then the endpoints of the orthogonal projection of Δ (respectively Δ') to l are closer than $2^{-n}/4$ to the appropriate endpoints of $\tilde{I}_i$ (respectively $\tilde{I}_{i'}$). Since the angles between L, L' and l are arbitrarily small if θ_0 is small enough, we can assume that the length of $\tilde{I}_i$, $\tilde{I}_{i'}$ is greater than $(3/4)2^{-n}$. Then either $\tilde{I}_i$

https://doi.org/10.1017/S0143385704000859 Published online by Cambridge University Press

and $\tilde{I}_{i'}$ have a common endpoint or the distance between the orthogonal projections of Δ and Δ' to l (and hence $\operatorname{dist}(\Delta, \Delta')$) is greater than $2^{-n}/4$. Now we end the proof in the similar way as in Case (ii), replacing L by l and I_i, $I_{i'}$ respectively by $\tilde{I}_i$, $\tilde{I}_{i'}$. □

□

We will also use the following observation.

LEMMA 5.3. *For every $\varepsilon_1, \varepsilon_2, \ldots$ there exist $\tilde{q} \in [0, 1]$ and a subsequence n_l such that*

$$\lim_{l\to\infty} \varrho_{\varepsilon_1,\ldots,\varepsilon_{n_l}} = \lim_{l\to\infty} \varrho_{\varepsilon_1,\ldots,\varepsilon_{M_{\varepsilon_1,\ldots,\varepsilon_{n_l}}}} = \tilde{q}.$$

Proof. To simplify notation, write $\varrho_m = \varrho_{\varepsilon_1,\ldots,\varepsilon_m}$ and $M(m) = M_{\varepsilon_1,\ldots,\varepsilon_m}$ for $m > 0$. Let

$$a_n = \varrho_n - \varrho_{M(n)}$$

and suppose that 0 is not an accumulation point of the sequence a_n. Then there exist $a > 0$ and n_0 such that $|a_n| > a$ for every $n \geq n_0$. Note that, by the definition of M, we have $M(n+1) \in \{M(n), M(n)+1\}$, which together with (8) easily implies

$$|a_{n+1} - a_n| \leq |\varrho_{n+1} - \varrho_n| + |\varrho_{M(n+1)} - \varrho_{M(n)}| \leq \frac{1}{n} + \frac{2}{M(n)} < \frac{5}{n} \to 0,$$

so there exists n_1 such that $a_n > a$ for every $n \geq n_1$ or $a_n < -a$ for every $n \geq n_1$. Suppose the first case holds (the other case is analogous). Let

$$i_j = \underbrace{M(M(\cdots(M}_{j \text{ times}}(n_2)\cdots),$$

where we fix n_2 so large that $i_{\lceil 1/a\rceil} > n_1$. Then $i_0, \ldots, i_{\lceil 1/a\rceil} > n_1$, so

$$\begin{aligned}\varrho_{i_0} &\geq \varrho_{i_0} - \varrho_{i_{\lceil 1/a\rceil}+1} = \varrho_{i_0} - \varrho_{i_1} + \varrho_{i_1} - \varrho_{i_2} + \cdots + \varrho_{i_{\lceil 1/a\rceil}} - \varrho_{i_{\lceil 1/a\rceil}+1} \\ &= a_{i_0} + \cdots + a_{i_{\lceil 1/a\rceil}} > (\lceil 1/a\rceil + 1)a > 1,\end{aligned}$$

which is a contradiction, since $\varrho_n \in [0, 1]$ for every n. Therefore, there exists a subsequence n_l such that $a_{n_l} \to 0$. Taking a subsequence of n_l we can assume that ϱ_{n_l} and $\varrho_{M(n_l)}$ converge for $l \to \infty$, so they must have a common limit $\tilde{q} \in [0, 1]$. □

Now we define and describe some functions which appear in studying multifractal properties of the Bernoulli measures on Λ.

For $q \in [0, 1]$ and $p \in (0, 1)$ let

$$\begin{aligned}G_1(q, p) &= -\frac{q \log p + (1-q)\log(1-p)}{\log 2}, \\ G_2(q) &= \frac{q}{q+1}, \\ G(q, p) &= G_1(q, p) + G_2(q).\end{aligned}$$

Also define

$$g(q) = G(q, q) = \frac{q}{q+1} - \frac{q \log q + (1-q)\log(1-q)}{\log 2}$$

for $q \in (0, 1)$ and extend g to a continuous function on $[0, 1]$ putting $g(0) = 0$, $g(1) = 1/2$ (see Figure 2).

https://doi.org/10.1017/S0143385704000859 Published online by Cambridge University Press

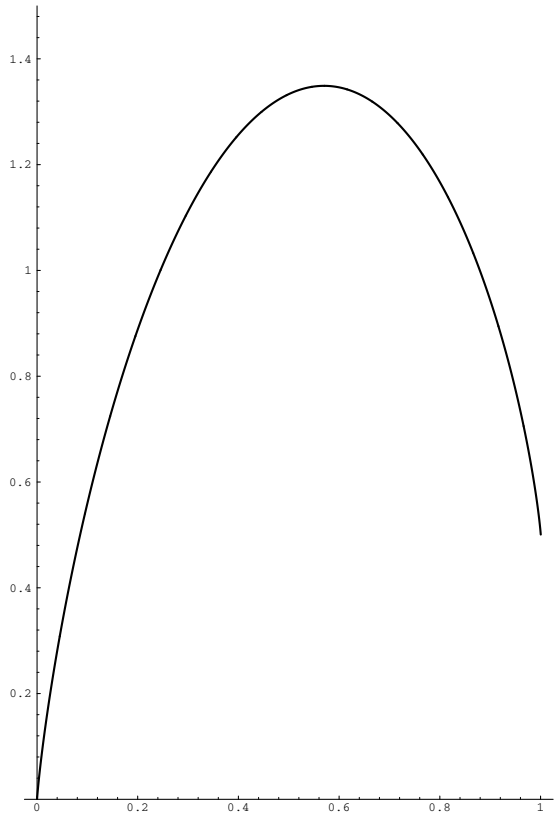


FIGURE 2. The graph of g.

LEMMA 5.4. *Let $G_p(q) = G(q, p)$ for $q \in [0, 1]$, $p \in (0, 1)$. Then:*

- *if $p \le (2)^{1/4}/[1 + (2)^{1/4}]$, then G_p is increasing and*

$$G_p([0, 1]) = \left[-\frac{\log(1-p)}{\log 2}, \frac{1}{2} - \frac{\log p}{\log 2}\right],$$

- *if $(2)^{1/4}/[1 + (2)^{1/4}] < p \le 2 - \sqrt{2}$, then G_p is increasing on*

$$\left[0, \sqrt{\frac{\log 2}{\log p - \log(1-p)}} - 1\right],$$

and decreasing on

$$\left[\sqrt{\frac{\log 2}{\log p - \log(1-p)}} - 1, 1\right];$$

moreover, $G_p(0) \le G_p(1)$ and

$$G_p([0, 1]) = \left[-\frac{\log(1-p)}{\log 2}, 1 + \frac{\log p - 2\log(1-p)}{\log 2} - 2\sqrt{\frac{\log p - \log(1-p)}{\log 2}}\right];$$

https://doi.org/10.1017/S0143385704000859 Published online by Cambridge University Press

- *if* $2-\sqrt{2} < p \le \frac{2}{3}$, *then* G_p *is increasing on*

$$\left[0, \sqrt{\frac{\log 2}{\log p - \log(1-p)}} - 1\right],$$

and decreasing on

$$\left[\sqrt{\frac{\log 2}{\log p - \log(1-p)}} - 1, 1\right];$$

moreover, $G_p(0) > G_p(1)$ *and*

$$G_p([0,1]) = \left[\frac{1}{2} - \frac{\log p}{\log 2}, 1 + \frac{\log p - 2\log(1-p)}{\log 2} - 2\sqrt{\frac{\log p - \log(1-p)}{\log 2}}\right];$$

- *if* $p > \frac{2}{3}$, *then* G_p *is decreasing and*

$$G_p([0,1]) = \left[\frac{1}{2} - \frac{\log p}{\log 2}, -\frac{\log(1-p)}{\log 2}\right].$$

Moreover, g *is concave and has exactly one critical point* $q_0 = 0.569\,85\ldots$, *so* $\max g = g(q_0) = 1.348\,87\ldots$.

Proof. The proof is a straightforward calculation, using the following:

$$G_p'(q) = \frac{1}{(q+1)^2} - \frac{\log p - \log(1-p)}{\log 2},$$

$$G_p(0) = -\frac{\log(1-p)}{\log 2}, \quad G_p(1) = \frac{1}{2} - \frac{\log p}{\log 2},$$

$$g'(q) = \frac{1}{(q+1)^2} - \frac{\log q - \log(1-q)}{\log 2}, \quad g''(q) = -\frac{2}{(q+1)^3} - \frac{1}{q(1-q)\log 2}. \qquad \square$$

6. *Multifractal formalism*

Let $\Sigma_3^+ = \{0,1,2\}^{\mathbb{N}}$ and let

$$\mathcal{C}_{\varepsilon_1,\ldots,\varepsilon_n} = \{(\varepsilon_1', \varepsilon_2', \ldots) \in \Sigma_3^+ : \varepsilon_1' = \varepsilon_1, \ldots, \varepsilon_n' = \varepsilon_n\}$$

be the cylinders. Define $\pi : \Sigma_3^+ \to \Lambda$ by

$$\pi((\varepsilon_1, \varepsilon_2, \ldots)) = \bigcap_{n=1}^{\infty} \Delta_{\varepsilon_1,\ldots,\varepsilon_n}.$$

For $p \in [0,1]$ let ν_p be the Bernoulli measure on Σ_3^+, such that $\nu_p(\mathcal{C}_0) = 1-p$ and $\nu_p(\mathcal{C}_1) = \nu_p(\mathcal{C}_2) = p/2$. Let

$$\mu_p = \pi(\nu_p).$$

Observe that for $p \in (0,1)$ the measure μ_p is supported on the entire set Λ, while μ_1 is supported on $\bigcap_n \bigcup_{\varepsilon_1,\ldots,\varepsilon_n \in \{1,2\}} \Delta_{\varepsilon_1,\ldots,\varepsilon_n}$ and μ_0 has a one-point distribution at the principal vertex of Δ^{φ}. It is easy to check that

$$\mu_p(\Delta_{\varepsilon_1,\ldots,\varepsilon_n}) = \nu_p(\mathcal{C}_{\varepsilon_1,\ldots,\varepsilon_n}) \tag{14}$$

for every $\varepsilon_1, \ldots, \varepsilon_n$. Note that triangles of the same type have equal measure μ_p.

https://doi.org/10.1017/S0143385704000859 Published online by Cambridge University Press

LEMMA 6.1. *For every $p \in (0,1)$ and every $\varepsilon_1, \varepsilon_2, \ldots$ we have*

$$\frac{\log \mu_p(Q_{\varepsilon_1,\ldots,\varepsilon_n})}{\log 2^{-n}} - (G_1(\varrho_{\varepsilon_1,\ldots,\varepsilon_n}, p) + G_2(\varrho_{\varepsilon_1,\ldots,\varepsilon_{M_{\varepsilon_1,\ldots,\varepsilon_n}}})) \to 0$$

as $n \to \infty$ uniformly with respect to $\varepsilon_1, \varepsilon_2, \ldots$. Moreover,

$$\frac{\log \mu_1(Q_{\varepsilon_1,\ldots,\varepsilon_n})}{\log 2^{-n}} \geq \frac{1}{2}$$

and, if $\varrho_{\varepsilon_1,\ldots,\varepsilon_n} = 1$ for every n, then

$$\frac{\log \mu_1(Q_{\varepsilon_1,\ldots,\varepsilon_n})}{\log 2^{-n}} \to \frac{1}{2}.$$

Proof. For simplicity, write $k_m = k_{\varepsilon_1,\ldots,\varepsilon_m}$, $\varrho_m = \varrho_{\varepsilon_1,\ldots,\varepsilon_m}$ and $M(m) = M_{\varepsilon_1,\ldots,\varepsilon_m}$ for $m > 0$. Suppose first $p < 1$. By definition, there are $2^{k_n - k_{M(n)}}$ triangles from $\mathcal{D}_n$ in $Q_{\varepsilon_1,\ldots,\varepsilon_n}$, so by (14),

$$\mu_p(Q_{\varepsilon_1,\ldots,\varepsilon_n}) = 2^{k_n - k_{M(n)}} \nu_p(\mathcal{C}_{\varepsilon_1,\ldots,\varepsilon_n}) = 2^{k_n - k_{M(n)}} (p/2)^{k_n} (1-p)^{n-k_n}.$$

This gives

$$\frac{\log \mu_p(Q_{\varepsilon_1,\ldots,\varepsilon_n})}{\log 2^{-n}} = -\frac{\varrho_n \log p + (1-\varrho_n)\log(1-p)}{\log 2} + \frac{k_{M(n)}}{n} = G_1(\varrho_n, p) + \frac{k_{M(n)}}{n}.$$

By (7),

$$\begin{aligned} G_2(\varrho_{M(n)}) = \frac{\varrho_{M(n)}}{\varrho_{M(n)}+1} &\leq \frac{k_{M(n)}}{n} \leq \frac{\varrho_{M(n)}}{\varrho_{M(n)}+1-(1/M(n))} \\ &\leq \frac{\varrho_{M(n)}}{\varrho_{M(n)}+1} + \frac{1}{M(n)} = G_2(\varrho_{M(n)}) + \frac{1}{M(n)}. \end{aligned}$$

This and (8) imply

$$0 \leq \frac{\log \mu_p(Q_{\varepsilon_1,\ldots,\varepsilon_n})}{\log 2^{-n}} - (G_1(\varrho_n, p) + G_2(\varrho_{M(n)})) \leq \frac{2}{n},$$

which ends the proof in this case.

Suppose now $p = 1$. Then

$$\mu_1(Q_{\varepsilon_1,\ldots,\varepsilon_n}) = \begin{cases} 0 & \text{if } k_n < n, \\ 2^{-M(n)} & \text{if } k_n = n, \end{cases}$$

so by (8),

$$\frac{\log \mu_1(Q_{\varepsilon_1,\ldots,\varepsilon_n})}{\log 2^{-n}} \geq \frac{1}{2}.$$

If $\varrho_n = 1$ (i.e. $k_n = n$) for every n, then by (7) we have $n/2 \leq M(n) \leq (n+1)/2$, so

$$\frac{1}{2} \leq \frac{\log \mu_1(Q_{\varepsilon_1,\ldots,\varepsilon_n})}{\log 2^{-n}} \leq \frac{n+1}{2n},$$

which ends the proof. □

https://doi.org/10.1017/S0143385704000859 Published online by Cambridge University Press

Definition 6.2. Take $x \in \Lambda$. Then there exists a sequence $\varepsilon_1, \varepsilon_2, \ldots$ (not necessarily unique) such that $x = \bigcap_n \Delta_{\varepsilon_1,\ldots,\varepsilon_n}$. Let

$$\varrho_n(x) = \varrho_{\varepsilon_1,\ldots,\varepsilon_n}$$

and

$$\varrho_{M(n)}(x) = \varrho_{M_{\varepsilon_1,\ldots,\varepsilon_n}}(x).$$

The construction of Λ implies that if the sequence $\varepsilon_1, \varepsilon_2, \ldots$ is not unique, then $\varepsilon_n = 0$ for sufficiently large n. Hence, the asymptotic properties of $\varrho_n(x)$ and $\varrho_{M(n)}(x)$ do not depend on the choice of $\varepsilon_1, \varepsilon_2, \ldots$.

The following proposition is a direct consequence of Lemmas 5.2 and 6.1.

PROPOSITION 6.3. *For every $p \in (0, 1)$ and every $x \in \Lambda$,*

$$\frac{\log \mu_p(B_{2^{-n}}(x))}{\log 2^{-n}} - (G_1(\varrho_n(x), p) + G_2(\varrho_{M(n)}(x))) \to 0$$

as $n \to \infty$. Moreover, if $\varrho_n(x) = 1$ for every n, then

$$\lim_{n\to\infty} \frac{\log \mu_1(B_{2^{-n}}(x))}{\log 2^{-n}} = \frac{1}{2}.$$

Proof. Take $p \in (0, 1]$, $x \in \Lambda$ and a sequence $\varepsilon_1, \varepsilon_2, \ldots$ such that $x = \bigcap_n \Delta_{\varepsilon_1,\ldots,\varepsilon_n}$. By Lemma 5.2, we have

$$B_{2^{-n}}(x) \supset B_{c_1 2^{-(n+\lceil \log_2 c_1 \rceil)}}(x) \supset Q_{\varepsilon_1,\ldots,\varepsilon_{n+\lceil \log_2 c_1 \rceil}},$$

so $\log \mu_p(B_{2^{-n}}(x))/\log 2^{-n} \le \log \mu_p(Q_{\varepsilon_1,\ldots,\varepsilon_{n+\lceil \log_2 c_1 \rceil}})/\log 2^{-n}$. For $p < 1$, Lemma 6.1 implies

$$\limsup_{n\to\infty} \left(\frac{\log \mu_p(B_{2^{-n}}(x))}{\log 2^{-n}} - (G_1(\varrho_n(x), p) + G_2(\varrho_{M(n)}(x))) \right) \le 0.$$

If $p = 1$ and $\varrho_n(x) = 1$ for every n, then by Lemma 6.1,

$$\limsup_{n\to\infty} \frac{\log \mu_1(B_{2^{-n}}(x))}{\log 2^{-n}} \le \frac{1}{2}.$$

On the other hand, Lemma 5.2 implies that $B_{2^{-n}}(x)$ intersects at most K sets $Q \in \mathcal{Q}_n$. Suppose first $p < 1$. Choose among those sets the one which has the largest measure μ_p and denote it by Q_n. Then by Lemma 6.1,

$$\begin{aligned}
&\liminf_{n\to\infty} \left(\frac{\log \mu_p(B_{2^{-n}}(x))}{\log 2^{-n}} - (G_1(\varrho_n(x), p) + G_2(\varrho_{M(n)}(x))) \right) \\
&\quad \ge \liminf_{n\to\infty} \left(\frac{\log(K \mu_p(Q_n))}{\log 2^{-n}} - (G_1(\varrho_n(x), p) + G_2(\varrho_{M(n)}(x))) \right) \\
&\quad = \liminf_{n\to\infty} (G_1(\varrho_n(Q_n), p) + G_2(\varrho_{M(Q_n)}(Q_n)) - G_1(\varrho_n(x), p) - G_2(\varrho_{M(n)}(x))),
\end{aligned}$$

where

$$|\varrho_n(Q_n) - \varrho_n(x)|, |\varrho_{M(Q_n)}(Q_n) - \varrho_{M(n)}(x)| \to 0$$

https://doi.org/10.1017/S0143385704000859 Published online by Cambridge University Press

as $n \to \infty$. Since $G_1(\cdot, p)$ and G_2 are uniformly continuous on $[0, 1]$, this implies

$$G_1(\varrho_n(Q_n), p) + G_2(\varrho_{M(Q_n)}(Q_n)) - G_1(\varrho_n(x), p) + G_2(\varrho_{M(n)}(x)) \to 0,$$

which ends the proof in the case $p < 1$.

For $p = 1$, Lemmas 5.2 and 6.1 imply

$$\frac{\log \mu_1(B_{2^{-n}}(x))}{\log 2^{-n}} \geq \frac{1}{2} - \frac{\log K}{n \log 2},$$

so

$$\liminf_{n\to\infty} \frac{\log \mu_1(B_{2^{-n}}(x))}{\log 2^{-n}} \geq \frac{1}{2}. \qquad \square$$

Let

$$\underline{d}_{\mu_p}(x) = \liminf_{r\to 0} \frac{\log \mu_p(B_r(x))}{\log r}, \quad \overline{d}_{\mu_p}(x) = \limsup_{r\to 0} \frac{\log \mu_p(B_r(x))}{\log r}$$

be the lower and upper pointwise dimensions of μ_p at $x \in \Lambda$. If $\underline{d}_{\mu_p}(x) = \overline{d}_{\mu_p}(x)$ then we denote their common value by $d_{\mu_p}(x)$ and call it the pointwise dimension of μ_p at x.

THEOREM B. *For every* $p \in [0, 1]$,

$$\mathrm{HD}(\mu_p) = g(p).$$

Proof. The case $p = 0$ is trivial. Suppose $p \in (0, 1)$. Let

$$\Lambda_p = \{x \in \Lambda : \lim_{n\to\infty} \varrho_n(x) = p\}.$$

By the strong large numbers law, Λ_p has full measure μ_p. Moreover, by Proposition 6.3, for every $x \in \Lambda_p$ we have

$$\frac{\log \mu_p(B_{2^{-n}}(x))}{\log 2^{-n}} \to G_1(p, p) + G_2(p) = g(p)$$

as $n \to \infty$, which easily implies $d_{\mu_p}(x) = g(p)$. Similarly, if

$$\Lambda_1 = \{x \in \Lambda : \varrho_n(x) = 1 \text{ for every } n\},$$

then Λ_1 has full measure μ_1 and, for every $x \in \Lambda_1$, Proposition 6.3 gives

$$\frac{\log \mu_1(B_{2^{-n}}(x))}{\log 2^{-n}} \to \frac{1}{2} = g(1),$$

so $d_{\mu_1}(x) = g(1)$. By Frostman's lemma, we conclude that

$$\mathrm{HD}(\Lambda_p) = \mathrm{HD}(\mu_p) = g(p) \tag{15}$$

for every $p \in (0, 1]$. $\square$

Let

$$\mathcal{J}_{\mu_p}(\alpha) = \{x \in \Lambda : d_{\mu_p}(x) = \alpha\}$$

and define the (pointwise) dimension spectrum of μ_p by

$$f_{\mu_p}(\alpha) = \mathrm{HD}(\mathcal{J}_{\mu_p}(\alpha)).$$

In the following two theorems we compute the dimension spectra of μ_p for $p \in (0, 1)$ and the Hausdorff dimension of Λ. Note that, in contrast to the case of the standard Sierpiński gasket, the measure μ_{q_0}, where q_0 is such that $g(q_0) = \max g$, is the measure of full dimension (i.e. $\mathrm{HD}(\mu_{q_0}) = \mathrm{HD}(\Lambda)$) and has non-trivial spectrum.

https://doi.org/10.1017/S0143385704000859 Published online by Cambridge University Press

THEOREM C. *For every* $p \in (0,1)$,

$$f_{\mu_p}(\alpha) = \max_{\{q:G(q,p)=\alpha\}} g(q) \quad \textit{for every } \alpha \in I_p,$$

and $\mathcal{J}_{\mu_p}(\alpha) = \emptyset$ *for every* $\alpha \notin I_p$, *where*

$$I_p = \begin{cases} \left[-\dfrac{\log(1-p)}{\log 2}, \dfrac{1}{2} - \dfrac{\log p}{\log 2}\right] \\ \quad \textit{for } p \le \dfrac{(2)^{1/4}}{1+(2)^{1/4}}, \\ \left[-\dfrac{\log(1-p)}{\log 2}, 1 + \dfrac{\log p - 2\log(1-p)}{\log 2} - 2\sqrt{\dfrac{\log p - \log(1-p)}{\log 2}}\right] \\ \quad \textit{for } \dfrac{(2)^{1/4}}{1+(2)^{1/4}} < p \le 2-\sqrt{2}, \\ \left[\dfrac{1}{2} - \dfrac{\log p}{\log 2}, 1 + \dfrac{\log p - 2\log(1-p)}{\log 2} - 2\sqrt{\dfrac{\log p - \log(1-p)}{\log 2}}\right] \\ \quad \textit{for } 2-\sqrt{2} < p < \dfrac{2}{3}, \\ \left[\dfrac{1}{2} - \dfrac{\log p}{\log 2}, -\dfrac{\log(1-p)}{\log 2}\right] \\ \quad \textit{for } p \ge \dfrac{2}{3}. \end{cases}$$

Moreover, for every $\alpha \in I_p$, *if* $q_p(\alpha) \in [0,1]$ *is such that* $g(q_p(\alpha)) = \max_{\{q:G(q,p)=\alpha\}} g(q)$, *then the set* $\mathcal{J}_{\mu_p}(\alpha)$ *has full measure* $\mu_{q_p(\alpha)}$ *and* $f_{\mu_p}(\alpha) = \mathrm{HD}(\mu_{q_p(\alpha)})$.

Theorem C implies the following immediately.

COROLLARY 6.4. *For every* $p \in (0,1)$ *and every* $\alpha \in I_p$:

- $f_{\mu_p}(\alpha) \le \alpha$;
- $f_{\mu_p}(\alpha) = \alpha \iff \alpha = g(p) = \mathrm{HD}(\mu_p)$;
- $\max f_{\mu_p} = \max g = f_{\mu_p}(G(q_0, p))$, *where* q_0 *is such that* $g(q_0) = \max g$.

Proof. It is easy to check that $G(q,p) > g(q)$ for $p \neq q$. This implies that $f_{\mu_p}(\alpha) \le \alpha$ with equality only if $\alpha = g(p)$ (which equals $\mathrm{HD}(\mu_p)$ by Theorem B). The third point of the corollary is straightforward. □

Remark 6.5. For some values of p the function f_{μ_p} is analytic and concave. For instance, Lemma 5.4 implies that for $p \le (2)^{1/4}/[1+(2)^{1/4}]$ and $p > \frac{2}{3}$ we have $f_{\mu_p} = g \circ G_p^{-1}$, so f_{μ_p} is analytic. In particular,

$$f_{\mu_{\frac{1}{2}}}(\alpha) = \alpha - 1 + \frac{(2-\alpha)\log(2-\alpha) - (3-2\alpha)\log(3-2\alpha) - (\alpha-1)\log(\alpha-1)}{(2-\alpha)\log 2}$$

for $\alpha \in [1, 3/2]$. Moreover, it is easy to check that $f_{\mu_{1/2}}$ is concave. However, for other values of p the function f_{μ_p} can have discontinuities. Indeed, take p slightly smaller

https://doi.org/10.1017/S0143385704000859 Published online by Cambridge University Press

than $2-\sqrt{2}$. Then, by Lemma 5.4, $G_p(0)$ is slightly smaller than $G_p(1)$, so there exists $q_1 > 0$ close to 0, such that $G_p(q_1) = G_p(1)$, $G_p(q) < G_p(1)$ for every $q < q_1$ and $G_p(q) \geq G_p(1)$ for every $q > q_1$. By Theorem C, for $\alpha \in [G_p(0), G_p(1))$ the values $f_{\mu_p}(\alpha)$ are close to $g(0) = 0$, while $f_{\mu_p}(G_p(1)) = g(1) = 1/2$. Such a situation holds, e.g., for $p = 0.583$. For some other values (e.g. $p = 0.579$), the function f_{μ_p} is continuous but not concave and has a point of non-differentiability. See Figure 3.

Proof of Theorem C. Take $x \in \mathcal{J}_{\mu_p}(\alpha)$ for $\alpha \in \mathbb{R}$. To simplify notation, we will write $M(m) = M_{\varepsilon_1,\dots,\varepsilon_m}$ for $m > 0$, where $\varepsilon_1, \varepsilon_2, \dots$ are such that $x = \bigcap_n \Delta_{\varepsilon_1,\dots,\varepsilon_n}$. By Proposition 6.3, $G_1(\varrho_n(x), p) + G_2(\varrho_{M(n)}(x)) \to \alpha$ as $n \to \infty$. Taking $\tilde{q} = \tilde{q}(x)$ and the subsequence $n_l = n_l(x)$ from Lemma 5.3 we obtain $\varrho_{n_l}(x), \varrho_{M(n_l)}(x) \to \tilde{q}$, so

$$G(\tilde{q}, p) = G_1(\tilde{q}, p) + G_2(\tilde{q}) = \alpha.$$

Note that by Lemma 5.4, $I_p = \{G(q, p) : q \in [0, 1]\}$. Hence, the set $\mathcal{J}_{\mu_p}(\alpha)$ is not empty if and only if $\alpha \in I_p$.

Take $\alpha \in I_p$. Then the set $\{q : G(q, p) = \alpha\}$ is not empty and $q_p(\alpha)$ is well defined. Let $D_\alpha = g(q_p(\alpha)) = \max_{\{q:G(q,p)=\alpha\}} g(q)$. First we show $f_{\mu_p}(\alpha) \leq D_\alpha$.

By Lemma 5.4, the set $\{q : G(q, p) = \alpha\}$ has at most two elements, say q_1, q_2, so $\tilde{q}(x) = q_1$ or $\tilde{q}(x) = q_2$. Hence, to prove $f_{\mu_p}(\alpha) \leq D_\alpha$, it is sufficient to show

$$\mathrm{HD}(\{x \in \mathcal{J}_{\mu_p}(\alpha) : \tilde{q}(x) = q_i\}) \leq D_\alpha \tag{16}$$

for $i = 1, 2$.

First, suppose $q_i \in (0, 1)$. Then for every $x \in \mathcal{J}_{\mu_p}(\alpha)$ such that $\tilde{q}(x) = q_i$ we have

$$G_1(\varrho_{n_l}(x), q_i) + G_2(\varrho_{M(n_l)}(x)) \to G_1(q_i, q_i) + G_2(q_i) = g(q_i),$$

so by Proposition 6.3, $\underline{d}_{\mu_{q_i}}(x) \leq g(q_i) \leq D_\alpha$ and by Frostman's lemma, (16) holds.

Consider now the case $q_i \in \{0, 1\}$. Since G_2 and g are continuous on $[0, 1]$, we can take $q' \in (0, 1)$ near q_i such that for every q in the closed interval of endpoints q_i and q' we have $|G_2(q) - G_2(q')| < \varepsilon$ and $|g(q) - g(q')| < \varepsilon$ for a small ε. For every $x \in \mathcal{J}_{\mu_p}(\alpha)$ such that $\tilde{q}(x) = q_i$ and for sufficiently large l, the points $\varrho_{n_l}(x)$ and $\varrho_{M(n_l)}(x)$ are contained in this interval. Hence,

$$G_2(\varrho_{M(n_l)}(x)) < G_2(q') + \varepsilon.$$

Moreover, it is easy to check that the function $q \mapsto G_1(q, p)$ is increasing (respectively decreasing) for $p < 1/2$ (respectively $p > 1/2$), so

$$G_1(\varrho_{n_l}(x), q') < G_1(q', q').$$

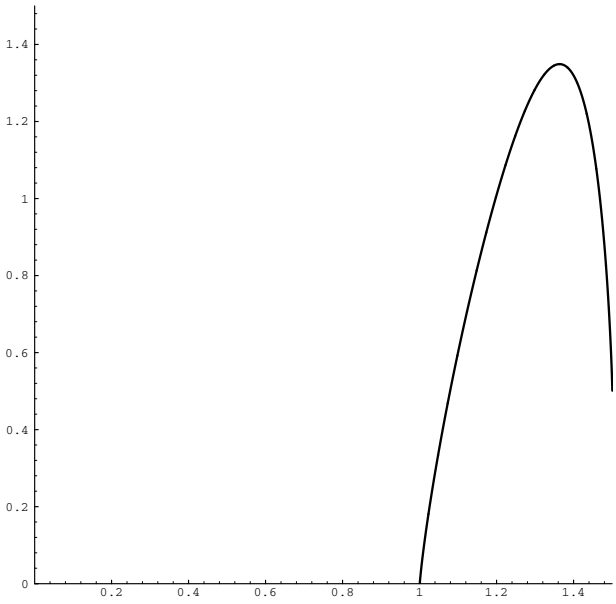
We conclude that

$$G_1(\varrho_{n_l}(x), q') + G_2(\varrho_{M(n_l)}(x)) < G_1(q', q') + G_2(q') + \varepsilon = g(q') + \varepsilon < g(q_i) + 2\varepsilon.$$

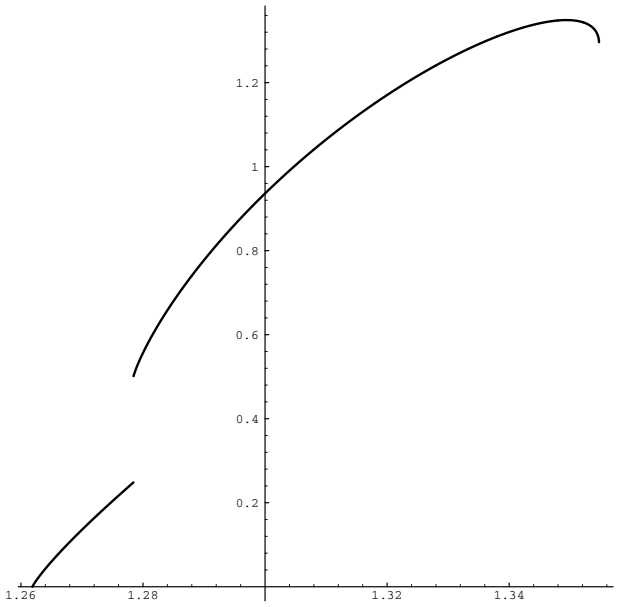
Again by Proposition 6.3, $\underline{d}_{\mu_{q'}}(x) \leq g(q_i) + 2\varepsilon \leq D_\alpha + 2\varepsilon$. Since ε was arbitrarily small, using Frostman's lemma we obtain (16). This shows $f_{\mu_p}(\alpha) \leq D_\alpha$.

Now we prove $f_{\mu_p}(\alpha) \geq D_\alpha$. First, suppose $q_p(\alpha) = 0$. Then $D_\alpha = g(0) = 0$, so the estimate is trivial and $f_{\mu_p}(\alpha) = 0 = \mathrm{HD}(\mu_0)$. Moreover, by Lemma 5.4,

https://doi.org/10.1017/S0143385704000859 Published online by Cambridge University Press



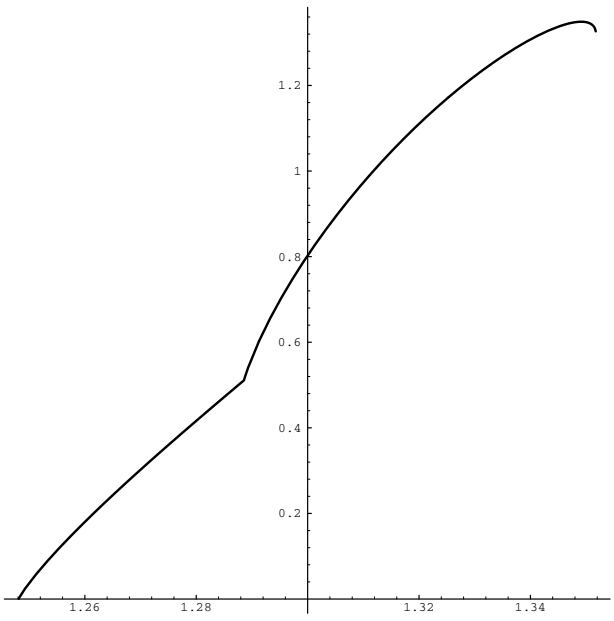
(a)



(b)

FIGURE 3. The graphs of f_{μ_p} for $p = 1/2$ (a), $p = 0.583$ (b) and $p = 0.579$ (c). The second and third graph are stretched horizontally for better exposition.

https://doi.org/10.1017/S0143385704000859 Published online by Cambridge University Press



(c)

FIGURE 3. Continued.

$\{q : G(q, p) = \alpha\} = \{0\}$, so $\alpha = G(0, p)$. If x_0 is the principal vertex of Δ^{φ}, then $\varrho_n(x_0) = 0$ for every n and by Proposition 6.3,

$$\frac{\log \mu_p(B_{2^{-n}}(x_0))}{\log 2^{-n}} \to G_1(0, p) + G_2(0) = G(0, p) = \alpha,$$

which easily implies $x_0 \in \mathcal{J}_{\mu_p}(\alpha)$. Since the measure μ_0 is concentrated at x_0, the set $\mathcal{J}_{\mu_p}(\alpha)$ has full measure μ_0.

Suppose now $q_p(\alpha) \in (0, 1]$ and consider the set $\Lambda_{q_p(\alpha)}$ defined in the proof of Theorem B. Then for every $x \in \Lambda_{q_p(\alpha)}$,

$$G_1(\varrho_n(x), p) + G_2(\varrho_{M(n)}(x)) \to G_1(q_p(\alpha), p) + G_2(q_p(\alpha)) = G(q_p(\alpha), p) = \alpha,$$

so Proposition 6.3 implies

$$\frac{\log \mu_p(B_{2^{-n}}(x))}{\log 2^{-n}} \to \alpha,$$

which easily gives

$$\Lambda_{q_p(\alpha)} \subset \mathcal{J}_{\mu_p}(\alpha).$$

Hence, by (15), we have

$$f_{\mu_p}(\alpha) \geq \mathrm{HD}(\Lambda_{q_p(\alpha)}) = \mathrm{HD}(\mu_{q_p(\alpha)}) = g(q_p(\alpha)) = D_\alpha.$$

Moreover, $\Lambda_{q_p(\alpha)}$ has full measure $\mu_{q_p(\alpha)}$, so the same holds for $\mathcal{J}_{\mu_p}(\alpha)$. □

THEOREM D. $\mathrm{HD}(\Lambda) = \max g = g(q_0)$, *where* $q_0 \in (0, 1)$ *is the unique number such that* $(q_0 + 1)^2 \log[q_0/1 - q_0] = \log 2$.

https://doi.org/10.1017/S0143385704000859 Published online by Cambridge University Press

Proof. The fact $\max g = g(q_0)$ follows from Lemma 5.4. The estimate $\mathrm{HD}(\Lambda) \geq \max g$ is a consequence of Theorem B. To show $\mathrm{HD}(\Lambda) \leq \max g$, let $x \in \Lambda$ and take $\tilde{q} = \tilde{q}(x)$ and $n_l = n_l(x)$ from Lemma 5.3. Fix a small $\varepsilon > 0$ and let $q_1, q_2, \ldots \in (0, 1)$ be a dense set in $(0, 1)$. By the continuity of G_1 and G_2, for every $q \in (0, 1)$ there exists $i > 0$ such that

$$|G_1(q, q_i) - G_1(q_i, q_i)|, |G_2(q) - G_2(q_i)| < \varepsilon.$$

Then

$$\Lambda = \Lambda'_0 \cup \Lambda'_1 \cup \bigcup_{i=1}^{\infty} \tilde{\Lambda}_i,$$

where

$$\tilde{\Lambda}_i = \{x \in \Lambda : |G_1(\tilde{q}(x), q_i) - G_1(q_i, q_i)|, |G_2(\tilde{q}(x)) - G_2(q_i)| < \varepsilon\}$$

and

$$\Lambda'_0 = \{x \in \Lambda : \tilde{q}(x) = 0\}, \quad \Lambda'_1 = \{x \in \Lambda : \tilde{q}(x) = 1\}.$$

Set $M(m) = M_{\varepsilon_1,\ldots,\varepsilon_m}$ for $m > 0$, where $\varepsilon_1, \varepsilon_2, \ldots$ are such that $x = \bigcap_n \Delta_{\varepsilon_1,\ldots,\varepsilon_n}$. Then for every $x \in \tilde{\Lambda}_i$,

$$\begin{aligned} G_1(\varrho_{n_l}(x), q_i) + G_2(\varrho_{M(n_l)}(x)) &\to G_1(\tilde{q}(x), q_i) + G_2(\tilde{q}(x)) \\ &< G_1(q_i, q_i) + G_2(q_i) + 2\varepsilon = g(q_i) + 2\varepsilon \leq \max g + 2\varepsilon, \end{aligned}$$

so by Proposition 6.3, $\underline{d}_{\mu_{q_i}}(x) \leq \max g + 2\varepsilon$ and by Frostman's lemma, $\mathrm{HD}(\tilde{\Lambda}_i) \leq \max g + 2\varepsilon$.

Recall that the function $q \mapsto G_1(q, p)$ is increasing (respectively decreasing) for $p < 1/2$ (respectively $p > 1/2$). Hence, if we take i_0, i_1 such that $q_{i_0} < 1/2$, $q_{i_1} > 1/2$ and $|G_2(q_{i_0}) - G_2(0)|, |G_2(q_{i_1}) - G_2(1)| < \varepsilon$, then for $j = 0, 1$ and every $x \in \Lambda'_j$,

$$\begin{aligned} G_1(\varrho_{n_l}(x), q_{i_j}) + G_2(\varrho_{M(n_l)}(x)) &\to G_1(\tilde{q}(x), q_{i_j}) + G_2(\tilde{q}(x)) \\ &< G_1(q_{i_j}, q_{i_j}) + G_2(q_{i_j}) + \varepsilon = g(q_{i_j}) + \varepsilon \leq \max g + \varepsilon, \end{aligned}$$

so by Proposition 6.3, $\underline{d}_{\mu_{q_{i_j}}}(x) \leq \max g + \varepsilon$ and by Frostman's lemma, $\mathrm{HD}(\Lambda'_j) \leq \max g + \varepsilon$. We conclude that $\mathrm{HD}(\Lambda) \leq \max g + 2\varepsilon$. Since ε can be arbitrarily small, this ends the proof. □

Remark 6.6. Note that the Hausdorff and box dimensions of Λ and the dimension spectra of the measures μ_p do not depend on the angle φ in the definition of Λ. Moreover, $\mathrm{HD}(\Lambda) = 1.348\,87\ldots < \mathrm{BD}(\Lambda) = 1.357\,01\ldots < \log 3/\log 2 = 1.584\,96\ldots$. The latter number is the Hausdorff dimension of the standard Sierpiński gasket.

Remark 6.7. We can modify the construction of the flexed Sierpiński gasket in the following way. In the first step of construction, instead of taking the longer sides of the triangles Δ_0, Δ_1, Δ_2 of length $1/2$, fix a constant $t \in (0, 1)$ and take Δ_0 with longer sides of length $1 - t$ and Δ_1, Δ_2 with longer sides of length t. See Figure 4. Then repeat the procedure inductively, dividing at each step the longer sides of the triangles $\Delta_{\varepsilon_1,\ldots,\varepsilon_n}$ in the ratio $1 - t$ to t, such that $L_{\varepsilon_1,\ldots,\varepsilon_n,0} = (1 - t)L_{\varepsilon_1,\ldots,\varepsilon_n}$. In this case we can determine

https://doi.org/10.1017/S0143385704000859 Published online by Cambridge University Press

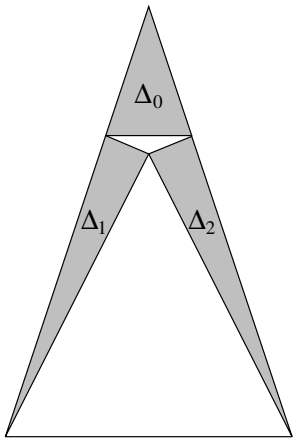


FIGURE 4. A generalization of the flexed Sierpiński gasket.

the Hausdorff and box dimensions of the limit set and the dimension spectra in the same way as previously, by an easy modification of the proofs (of course the formulas will be different and dependent on t). In fact, the proofs show the equivalence of these non-affine constructions to some affine ones generalizing the general Sierpiński carpets. For details see [**1**].

Acknowledgements. The author thanks Feliks Przytycki for drawing his attention to the problem. This research was supported by Polish KBN Grant No 2 P03A 034 25.

REFERENCES

[1] K. Barański. Hausdorff dimension of some geometric constructions. *Preprint*, 2003.
[2] T. Bedford. Crinkly curves, Markov partitions and box dimension in self-similar sets. *PhD Thesis*, University of Warwick, 1984.
[3] T. Bedford and M. Urbański. The box and Hausdorff dimension of self-similar sets. *Ergod. Th. & Dynam. Sys.* **10** (1990), 627–644.
[4] K. J. Falconer. *Fractal Geometry: Mathematical Foundations and Applications*. Wiley, 1990.
[5] K. J. Falconer. The dimension of self-affine fractals II. *Math. Proc. Camb. Phil. Soc.* **111** (1992), 169–179.
[6] K. J. Falconer. The Hausdorff dimension of self-affine fractals. *Math. Proc. Camb. Phil. Soc.* **103** (1988), 339–350.
[7] K. J. Falconer and B. Lammering. Fractal properties of generalized Sierpiński triangles. *Fractals* **6** (1998), 31–41.
[8] I. Herburt, M. Moszyńska and K. Rudnik. On intrinsic invariants of fractal dimensions. *Geom. Dedicata* **57** (1995), 1–10.
[9] R. Kenyon and Y. Peres. Measures of full dimension on affine-invariant sets. *Ergod. Th. & Dynam. Sys.* **16** (1996), 307–323.
[10] J. F. King. The singularity spectrum for general Sierpinski carpets. *Adv. Math.* **116** (1995), 1–11.
[11] S. P. Lalley and D. Gatzouras. Hausdorff and box dimensions of certain self-affine fractals. *Indiana Univ. Math J.* **41** (1992), 533–568.
[12] C. McMullen. The Hausdorff dimension of general Sierpiński carpets. *Nagoya Math. J.* **96** (1984), 1–9.

https://doi.org/10.1017/S0143385704000859 Published online by Cambridge University Press

[13] Ya. B. Pesin. *Dimension Theory in Dynamical Systems: Contemporary Views and Applications (Chicago Lectures in Mathematics Series)*. University of Chicago Press, Chicago, 1997.

[14] F. Przytycki and M. Urbański. On the Hausdorff dimension of some fractal sets. *Studia Math.* **93** (1989), 155–186.

[15] K. Rudnik. Concerning the rigidity problem for subsets of E^2. *Bull. Acad. Pol. Sci.* **37** (1989), 251–254.

[16] B. Solomyak. Measure and dimension for some fractals families. *Math. Proc. Camb. Phil. Soc.* **124** (1998), 531–546.

[17] M. Urbański. The Hausdorff dimension of the graphs of continuous self-affine functions. *Proc. Amer. Math. Soc.* **108** (1990), 921–930.

https://doi.org/10.1017/S0143385704000859 Published online by Cambridge University Press