Student Problems

Students up to the age of 19 are invited to send solutions to either of the following problems to Stan Dolan, 126A Harpenden Road, St. Albans, Herts AL3 6BZ.

Two prizes will be awarded – a first prize of £25, and a second prize of £20 – to the senders of the most impressive solutions for either problem. It is, therefore, not necessary to submit solutions to both. Solutions should arrive by September 20th 2015. Please give your School year, the name and address of your School or College, and the name of a teacher through whom the award may be made. Please print your own name clearly! The names of all successful solvers will be published in the November 2015 edition.

Problem 2015.3

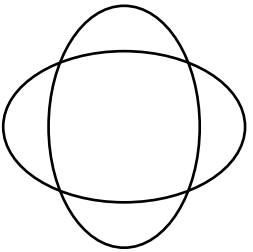
For any quadruplet (a, b, c, d) of numbers, define a second quadruplet to be

$$\left(\frac{a+b}{2}, \frac{b+c}{2}, \frac{c+d}{2}, \frac{d+a}{2}\right).$$

Repeat this process indefinitely and describe what happens.

Problem 2015.4

The overlapping ellipses $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ (with $a > b > 0$) form five bounded regions as shown in the diagram.



What value of $\frac{b}{a}$ makes all five regions have the same area?

Solutions to 2015.1 and 2015.2

No correct solutions were received for Problem 2015.2. Problem 2015.1 was solved by Shikha Chandrashekhar (Lakshmipat Singhania Academy) and Nathan Day (The Sixth Form College Farnborough).

Problem 2015.1

Prove that $\tan 20° - \tan 40° + \tan 80° = 3\sqrt{3}$.

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Solution

This problem was proposed by Chris Starr and had a very neat solution. Both Shikha and Nathan first established the result that, for any angle θ,

$$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}.$$

Shikha then noted that $\theta = 20°, -40°$ and $80°$ give three different values of $t = \tan\theta$ such that $\tan 3\theta = \sqrt{3}$. In each case,

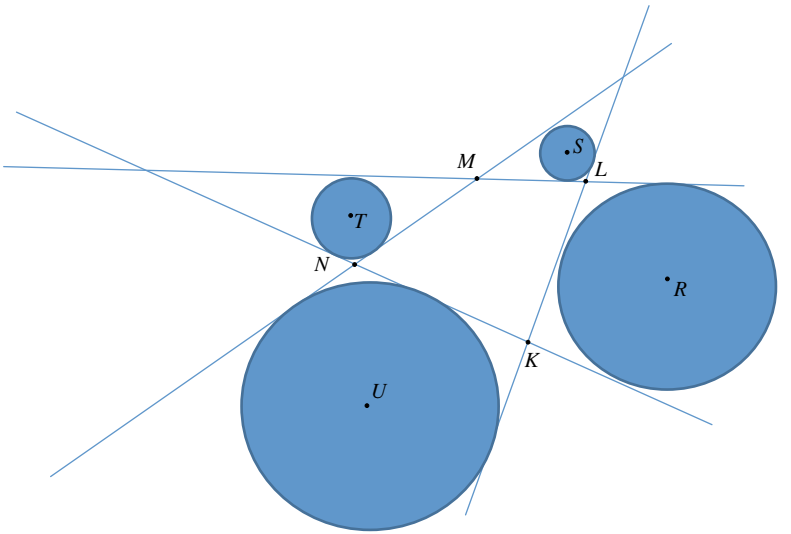
$$\sqrt{3} = \frac{3t - t^3}{1 - 3t^2} \Rightarrow t^3 - 3\sqrt{3}t^2 - 3t + \sqrt{3} = 0.$$

Therefore the sum of the roots of this cubic is

$$\tan 20° - \tan 40° + \tan 80° = 3\sqrt{3}.$$

Problem 2015.2

In the diagram, $KLMN$ is a convex quadrilateral, the sides are extended and circles inscribed as shown.



(a) Prove that points R, S, T and U, the centres of the circles, form a cyclic quadrilateral.

(b) Apply the same construction to quadrilateral $RSTU$ and repeat this process indefinitely. Describe what happens.

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Solution

This attractive problem was proposed by Paul Stephenson. Denote the angles of quadrilateral $KLMN$ by K, L, M and N, in the natural way. Similarly denote the angles of quadrilateral $RSTU$ by R, S, T and U.

The sides of $RSTU$ bisect angles at K, L, M and N. Then the angles of triangle RLK are R, $\frac{1}{2}(\pi - L)$ and $\frac{1}{2}(\pi - K)$ and so $R = \frac{1}{2}(K + L)$. Similar equations hold for S, T and U. In particular, $T = \frac{1}{2}(M + N)$.

Part (a) now follows easily, since $R + T = \frac{1}{2}\big(K + L + M + N\big)$ The quadrilateral $RSTU$ is therefore cyclic since opposite angles sum to π.

The solution to part (b), that the quadrilaterals approach a square, also follows from the results for the angles R, S, T and U, together with another result which has been separated out as Problem 2015.3.

The first prize of £25 is awarded to Shikha Chandrashekhar. The second prize of £20 is awarded to Nathan Day.

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