degenerate triangles with $A = 0°$. Figure 4 looks very similar when plotted with the segment-area formulae: the line $A + 2B = 180°$ is a tangent to the curve and the point corresponding to $P$ satisfies $A = \frac{1}{4}\pi + \sin A \cos^3 A$ which may be solved by function iteration to give $A = B \approx 54.27°, C \approx 71.47°$.

2. If, instead of circles, we use the (isosceles) triangles of greatest area in each segment, then the triangle opposite angle $A$ has area $\alpha$ given by $\alpha = R^2(1 - \cos A)\sin A$ or $\alpha = (1 - \cos A)\sin A$ with appropriate units of area. Once again, Figure 4 looks very similar when plotted with the triangle-area formulae: the line $A + 2B = 180°$ is a tangent to the curve and the point corresponding to $P$ satisfies $2\cos^3 A + \cos A - 1 = 0$ which solves to give $A = B \approx 53.86°$, $C \approx 72.28°$. But there is a surprise in store when we look at the sum of areas $\alpha + \beta$ if angle $C$ is fixed. Here, when $C > 60°$, $\alpha + \beta$ is maximised for degenerate triangles with $A = 0°$. However, when $2\sin^{-1}\frac{1}{4} \approx 28.96° < C < 60°$, there is a maximum value for $\alpha + \beta$ with $A < B$. And when $C < 2\sin^{-1}\frac{1}{4} \approx 28.96°$, the maximum value of $\alpha + \beta$ occurs when $A = B$.

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NICK LORD

*Tonbridge School, Kent TN9 1JP*

e-mail: *njl@tonbridge-school.org*

## Three cameos involving *e*

The first two cameos below were inspired by The Mathematical Association's A-level poster called '*e*'. This occupies a prominent position on my classroom wall and features aspects of $e$ such as its decimal expansion to 12 decimal places and the graphs of $y = e^x$ and $y = \ln x$ shown in Figure 1.

*Cameo* 1

I had just completed my usual introduction to $e$, ending up with a heuristic derivation of the formula $e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$ to give us a way of working out $e$. Normally, I just pick a large value of $n$ such as a million for this calculation, but this time I decided to be more systematic and produced Table 1 below. The initial reaction of the class was that the first $n$ initial digits of $e$ were produced from using $10^n$ in the formula for $e$, but this stumbled at $n = 13$, so we quickly modified it to 'correct significant figures' meaning that, for example, when $n = 5$, both $e$ and the approximation $\left(1 + 10^{-5}\right)^{10^5}$ agree as 2.7183 when both are rounded to 5 significant figures.

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| $n$ | $(1 + 10^{-n})^{10^n}$ | Correct initial digits | Correct SFs |
|---|---|---|---|
| 1 | 2.5937424601 | 1 | 1 |
| 2 | 2.70481382942152609 | 2 | 2 |
| 3 | 2.71692393223589245 | 3 | 3 |
| 4 | 2.71814592682522486 | 4 | 4 |
| 5 | 2.71826823717448966 | 5 | 5 |
| 6 | 2.71828046931937688 | 6 | 6 |
| 7 | 2.71828169254496627 | 7 | 7 |
| 8 | 2.71828181486763621 | 8 | 8 |
| 9 | 2.71828182709990432 | 9 | 9 |
| 10 | 2.71828182832313114 | 10 | 10 |
| 11 | 2.71828182844545382 | 11 | 11 |
| 12 | 2.71828182845768609 | 12 | 12 |
| 13 | 2.71828182845890932 | 12 | 13 |
| 14 | 2.71828182845903164 | 14 | 14 |
| 15 | 2.71828182845904387 | 15 | 15 |

TABLE 1

Proving the properties noted in Table 1 needs a relatively weak bound for the difference between $e$ and $(1 + \frac{1}{n})^n$ which we state and prove below.

*Claim*: For $n \geqslant 3$, we have $\dfrac{1}{n} < e - \left(1 + \dfrac{1}{n}\right)^n < \dfrac{e}{n}$.

For the right-hand side, we use the fact that $\left(1 + \dfrac{1}{n}\right)^n < e < \left(1 + \dfrac{1}{n}\right)^{n+1}$ which rearranges to give $0 < e - \left(1 + \dfrac{1}{n}\right)^n < \dfrac{1}{n}\left(1 + \dfrac{1}{n}\right)^n < \dfrac{e}{n}$: this is valid for all $n \geqslant 1$.

To justify the left-hand side, we have to work a bit harder.

For $n \geqslant 3$, using the series for $e$ and the binomial expansion for $\left(1 + \dfrac{1}{n}\right)^n$, we have

$$e - \left(1 + \frac{1}{n}\right)^n - \frac{1}{n} > \sum_{k=0}^{n}\left[\frac{1}{k!} - \frac{n(n-1)\ldots(n-k+1)}{n^k} \cdot \frac{1}{k!}\right] - \frac{1}{n}$$

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$$> \sum_{k=0}^{4} \left[ \frac{1}{k!} - \frac{n(n-1)\ldots(n-k+1)}{n^k} \cdot \frac{1}{k!} \right] - \frac{1}{n}$$

$$= \frac{6(n-3)^2 + 17(n-3) + 3}{24n^3}$$

which is certainly positive for $n \geqslant 3$.

These bounds establish the bounds noticed in the table. For the right-hand side guarantees that $e$ and $(1 + 10^{-n})^{10^n}$ agree to $n$ significant figures and the left-hand side shows that they differ in the $(n + 1)$-th significant figure. The initial $n$ digits are correct until a 0 appears in the decimal expansion of $e$ : this first occurs in the 13th decimal place.

[It is worth noting that there are tighter bounds than those of our claim in the literature, such as $\frac{e}{2n+2} < e - \left(1 + \frac{1}{n}\right)^n < \frac{e}{2n+1}$ from [1].]

*Cameo* 2

Figure 1 provoked the following question from a Sixth Former, Will Miller: "Is there a choice of base $a$ for which the mutually inverse graphs $y = a^x$ and $y = \log_a x$ touch?". We investigated this using the graphical software package *Autograph*, using its constant controller facility to search for the value of $a$ that produces Figure 2. This turned out to be $a \approx 1.4447$. This number did not ring a bell, but what leapt out from Figure 2 is that, when the curves touch, they do so at the point $(e, e)$. Justifying this provided a useful reinforcement of the standard classwork we had just done on $e^x$. For if the graphs in Figure 2 touch, they also touch the line $y = x$, so we have the two conditions

$$a^x = x \tag{1}$$

and, differentiating,

$$(\ln a)\, a^x = 1. \tag{2}$$

Substituting $a = x^{1/x}$ from (1) into (2) shows that $\ln x = 1$ and $x = e$. Moreover, $a = x^{1/x} = e^{1/e} \approx 1.4447$.

For values of $a$ with $1 < a < e^{1/e}$, the graphs of $y = a^x$ and $y = \log_a x$ intersect twice but, because $a^e < e < \frac{1}{\ln a} = \log_a e$, the point $(e, e)$ always lies in the region enclosed by these graphs.

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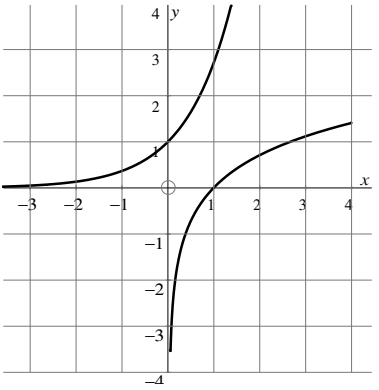



FIGURE 1: Graphs of $y = e^x$ and $y = \ln x$

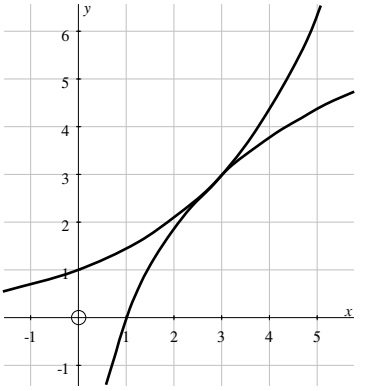



FIGURE 2: Touching graphs of $y = a^x$ and $y = \log_a x$

*Cameo* 3

Our third cameo arose from the observation that sketching the graph given implicitly by the equation $x^y y^x = k$ (where $k > 0$ is a constant) proves to be a challenge for some graph packages. Typical results are shown in Figure 3: notice that $(1, k)$ is on the graph, so it is easy to read off the value of $k$ used to plot the graph. A quick way to see why this range of behaviour occurs is to consider the intersections of $x^y y^x = k$ with the family of straight lines $x + y = c$.

For we have

$$k^{\frac{1}{x+y}} = x^{\frac{y}{x+y}} y^{\frac{x}{x+y}} \leqslant \frac{2xy}{x+y} \leqslant \frac{x+y}{2}$$

where the first inequality comes from the generalised AM-GM inequality, $x^\lambda y^{1-\lambda} \leqslant \lambda x + (1-\lambda)y$ for $x, y > 0$, $0 \leqslant \lambda \leqslant 1$, and the second inequality is a rearrangement of $(x-y)^2 \geqslant 0$. It follows that $k \leqslant \left(\frac{1}{2}c\right)^c$ and, from the graph of $z = \left(\frac{1}{2}c\right)^c$ in Figure 4, which has a minimum of $z = \left(\frac{1}{e}\right)^{2/e}$ when $x = \frac{2}{e}$, we read off:

- $k \geqslant 1$:

  Only sufficiently large values of $c \geqslant 2$ are valid here, corresponding to the single unbounded component shown in Figure 3(a).

- $\left(\frac{1}{e}\right)^{2/e} < k < 1$:

  Here there is an interval of valid values of $c$ near the origin together with some other sufficiently large values: this corresponds to the two separate components shown in Figure 3(b).

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- $k = \left(\frac{1}{e}\right)^{2/e} \approx 0.4791$:

  Here the two components of the graph merge. (Notice that the critical value of $k$ is $\frac{1}{a^2}$ where $a \approx 1.4447$ has the value from Cameo 2.)

- $0 < k < \left(\frac{1}{e}\right)^{2/e}$:

  Here there are no restrictions on $c$, giving the single unbounded component "sniffing the origin" as in Figure 3(c).

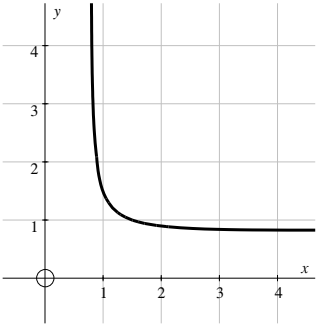



FIGURE 3(a): $k = 1.5$

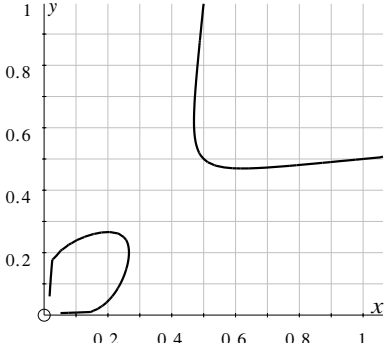



FIGURE 3(b): $k = 0.5$

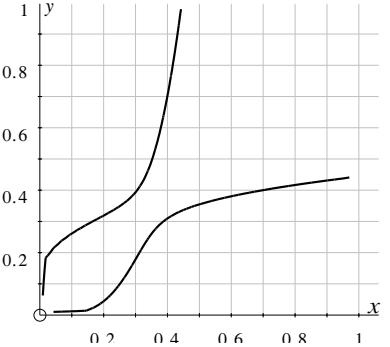



FIGURE 3(c): $k = 0.47$

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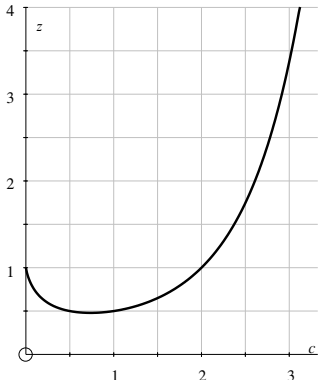



FIGURE 4: Graph of $z = \left(\frac{1}{2}c\right)^c$

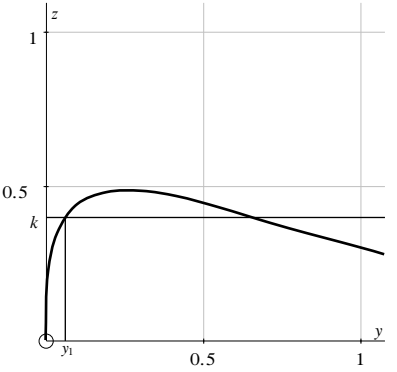



FIGURE 5: Graph showing solutions to $0.3^y y^{0.3} = 0.4$

The reason that graph plotters struggle near O is that the graph approximates a right-angled bend there. This is because, for fixed (small) values of $x$, $z = x^y y^x$ has the shape of graph typified by Figure 5 with a maximum when $\dfrac{dz}{dy} = x^y y^{x-1}(x + y \ln x) = 0$ or $y = -\dfrac{x}{\ln x}$. It follows that $y_1$, the smaller of the two $y$-values with $x^y y^x = k$, satisfies $0 < y_1 < -\dfrac{x}{\ln x}$. Thus $0 < \dfrac{y_1}{x} < -\dfrac{1}{\ln x}$ which tends to 0 as $x \downarrow 0$.

Finally, we note that one of the drawbacks of implicit representations of curves is that natural questions (such as, in Figure 3(b), what is the area of the bounded component?) usually do not have neat analytical solutions.

*Reference*

1. G. Pólya and G. Szegö, *Problems and theorems in analysis I*, Springer (1972) p. 38, Problem 170.

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NICK LORD
*Tonbridge School, Kent TN9 1JP*
e-mail: *njl@tonbridge-school.org*

## Two trajectories: a dynamic monkey tale

The "monkey drop" experiment is an engaging and instructive demonstration of basic kinematic principles. A toy stuffed monkey is suspended at height $b$ above the horizontal floor and at horizontal distance $a$ from a toy cannon on the floor which is loaded with a golf ball projectile. The cannon is aimed directly at the monkey, that is at the angle $\alpha = \arctan(b/a)$ relative to the floor. A trigger mechanism simultaneously fires the cannon and releases the monkey. To the amusement, not to say astonishment, of onlookers the projectile arches gracefully through the air and strikes the monkey before it reaches the floor [1]. A trial in which the launch speed of the projectile is increased, but the firing angle is unchanged,

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