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A general Bieberbach inequality

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A proof is given of a general Bieberbach inequality which has the Bieberbach, Urysohn and harmonic Urysohn inequalities as special cases. This establishes a stronger as well as more general inequality than the previously mentioned inequalities. Generalizations (as well as strengthened versions) of other inequalities previously proven by the author are also obtained.

The setting for this paper will be Euclidean n-dimensional space, R^n. Compact convex sets with non-empty interiors are called convex bodies and the space of all convex bodies, endowed with the Hausdorff topology, will be denoted by $\mathscr{K}^n$. The surface and volume of the unit n-ball will be denoted by Ω and ω_n, respectively. For a convex body A, we let $k_A(u)$ denote the width of A in the direction $u \in \Omega$. The diameter $D(A)$ and breadth $d(A)$ of A are defined to be, respectively, the maximum and minimum (taken over all Ω) of the width of A. The volume of A will be written as $V(A)$. If $*$ is a point in the interior of A, we shall use A^* to denote the polar body of A with respect to the unit sphere centred at $*$. For reference see Hadwiger (2).

For a convex body A and a real number $p \neq 0$, we define the p-width of A by

$$\bar{b}_p(A) = \left[\frac{1}{n\omega_n}\int_\Omega k_A^p(u)\,dS(u)\right]^{1/p},$$

where dS denotes the area element on Ω. For p equal to $-\infty$, 0 or ∞ we define the p-width of A by

$$\bar{b}_p(A) = \lim_{s\to p} \bar{b}_s(A).$$

It thus follows that $\bar{b}_\infty(A) = D(A)$, $\bar{b}_{-\infty}(A) = d(A)$, while

$$\bar{b}_0(A) = \exp\left[\frac{1}{n\omega_n}\int_\Omega \log k_A(u)\,dS(u)\right].$$

For a fixed p, the p-width is a map

$$\bar{b}_p : \mathscr{K}^n \to R.$$

It is positive, continuous, bounded, invariant under motions, homogeneous of degree one and monotone under set inclusion.

The p-widths of a convex body are continuous in p. The proposition that follows shows that the p-widths of a convex body are monotone in p. It follows directly from Jensen's inequality ((3), p. 144).

Proposition. $\bar{b}_p(A) \leqslant \bar{b}_q(A)$ $[-\infty \leqslant p < q \leqslant \infty, A \in \mathscr{K}^n]$ with equality if and only if A is of constant width.

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Consider the general Bieberbach inequality:

$$V(A) \leqslant 2^{-n}\omega_n \bar{b}_p(A)^n \quad [A \in \mathscr{K}^n]$$

with equality if and only if A is an n-ball.

Bieberbach (1) established this general inequality for $p = \infty$, thus proving the famous Bieberbach inequality. This was later improved by Urysohn (7) when he proved the Urysohn inequality thereby establishing the general Bieberbach inequality for $p = 1$. We obtained a further improvement by proving the harmonic Urysohn inequality (5) which established the general Bieberbach inequality for $p = -1$. In this paper we show that the general Bieberbach inequality holds for p if and only if $-n < p \leqslant \infty$.

Associated with the general Bieberbach inequality is the general dual Bieberbach inequality:

$$2^n\omega_n \bar{b}_p(A)^{-n} \leqslant V(A^*) \quad [A \in \mathscr{K}^n]$$

with equality if and only if A is an n-ball and $*$ is its centre.

The general dual Bieberbach inequality for $p = \infty$ (dual Bieberbach inequality) and for $p = 1$ (dual Urysohn inequality) was established by us in (4). Later in (5), we improved this by proving the general dual Bieberbach inequality for $p = -1$ (harmonic dual Urysohn inequality). We shall prove that the general dual Bieberbach inequality holds for p, if and only if $-n < p \leqslant \infty$.

LEMMA 1. $$2^n\omega_n \bar{b}_{-n}(A)^{-n} \leqslant V(A^*) \quad [A \in \mathscr{K}^n]$$

with equality if and only if A is symmetric with respect to $*$.

Proof. Let H_A be the support function of A for the origin $*$. We apply Minkowski's inequality ((3), p. 146) to

$$\bar{b}_{-n}(A) = \left[\frac{1}{n\omega_n}\int_\Omega [H_A(u) + H_A(-u)]^{-n}\, dS(u)\right]^{-1/n}$$

and obtain

$$\bar{b}_{-n}(A) \geqslant 2\left[\frac{1}{n\omega_n}\int_\Omega H_A^{-n}(u)\, dS(u)\right]^{-1/n}$$

with equality if and only if $H_A(u) = H_A(-u)$ for all $u \in \Omega$. This is the desired result.

If we combine Lemma 1 with the proposition we obtain:

THEOREM 1. $$2^n\omega_n \bar{b}_p(A)^{-n} \leqslant V(A^*) \quad [-n < p \leqslant \infty, A \in \mathscr{K}^n]$$

with equality if and only if A is an n-ball and $*$ is its centre.

By combining Theorem 1 with Lemma 1, we see that the general dual Bieberbach inequality holds for p if and only if $-n < p \leqslant \infty$. If $p < -n$, a counter-example is given by any non-spherical A symmetric with respect to $*$; this follows from the proposition and Lemma 1. If the conditions for equality were omitted from the general dual Bieberbach inequality then Lemma 1 in conjunction with the proposition shows that it would hold for p if and only if $-n \leqslant p \leqslant \infty$.

Santaló (6) has shown that for a convex body A

$$\operatorname{Inf}\{V(A)\, V(A^*) \mid * \in \operatorname{int} A\} \leqslant \omega_n^2$$

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with equality if and only if A is an n-dimensional ellipsoid. This result can be combined with Lemma 1 to yield:

LEMMA 2. $V(A) \leqslant 2^{-n}\omega_n \bar{b}_{-n}(A)^n \quad [A \in \mathscr{K}^n]$

with equality if and only if A is an n-dimensional ellipsoid.

By combining Lemma 2 with the proposition we obtain:

THEOREM 2. $V(A) \leqslant 2^{-n}\omega_n \bar{b}_p(A)^n \quad [-n < p \leqslant \infty, A \in \mathscr{K}^n]$

with equality if and only if A is an n-ball.

By combining Theorem 2 and Lemma 2, we see that the general Bieberbach inequality holds for p if and only if $-n < p \leqslant \infty$. If the conditions for equality were omitted from the general Bieberbach inequality then Lemma 2 in conjunction with the proposition shows that it would hold for p if and only if $-n \leqslant p \leqslant \infty$.

Combining Theorems 1 and 2 we obtain:

THEOREM 3. $1 \leqslant [\bar{b}_p(A)/2][\bar{b}_p(A^*)/2] \quad [-n < p \leqslant \infty, A \in \mathscr{K}^n]$

with equality if and only if A is an n-ball and $*$ is its centre.

Theorem 3 generalizes inequalities obtained by us in (4) and (5). We note that for $p = -n$, the inequality of Theorem 3 would still hold, however, the conditions for equality change to A being an n-dimensional ellipsoid with $*$ as its centre.

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