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A dynamic elastic-visco-plastic unilateral contact problem with normal damped response and Coulomb friction

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We consider a dynamic frictional contact problem between an elastic-visco-plastic body and a foundation. The contact is modelled with a normal damped response condition of such a type that the normal velocity is restricted with unilateral constraint, associated with the Coulomb law in which the coefficient of friction may depend on the velocity. We derive a variational formulation of the problem which has the form of a system coupling an integro–differential equation for the stress field with an evolutionary variational inequality for the displacement field. This inequality is approximated by a variational equation using a smoothing of the friction and the penalty approximation of the unilateral condition. The existence of a weak solution to the variational equation is proved by the Galerkin method for an auxiliary problem with given viscoplastic part of the stress and a fixed point argument. The solvability of the original problem is proved by passing to the limit of the penalty parameter and the smoothing parameter. This convergence is based on a certain regularity of solutions which is verified with the use of a local rectification of the boundary and a translation method.

1 Introduction

The aim of this paper is the analysis of an elastic-visco-plastic contact problem with Coulomb friction and a contact model that combines a normal damped response model with a unilateral contact condition. The constitutive law for the elastic-visco-plastic material is given by an integro–differential relation,

$$\boldsymbol{\sigma}(t) = \mathscr{A}_1\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + \mathscr{A}_0\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathscr{G}(\boldsymbol{\sigma}(s) - \mathscr{A}_1\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(s)), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\,ds, \tag{1.1}$$

where $\boldsymbol{u}$ denotes the displacement field whereas $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}(\boldsymbol{u})$ represent the stress and the linearized strain tensor, respectively. In (1.1) and everywhere in this paper $\mathscr{A}_1$ and $\mathscr{A}_0$ are linear operators describing the purely viscous and the elastic properties of the material, respectively, and $\mathscr{G}$ is a non-linear constitutive function which describes the visco-plastic behaviour of the material; also, the dot above a variable represents the derivative with respect to the time variable t.

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Examples of constitutive laws of the form (1.1) can be constructed by using rheological arguments (see e.g. [10]). It follows from (1.1) that, at each time moment t, the stress tensor $\boldsymbol{\sigma}(t)$ is split into two parts: $\boldsymbol{\sigma}(t) = \boldsymbol{\sigma}^V(t) + \boldsymbol{\sigma}^R(t)$, where $\boldsymbol{\sigma}^V(t) = \mathscr{A}_1\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t))$ represents the purely viscous part of the stress whereas $\boldsymbol{\sigma}^R(t)$ satisfies a rate-type elastic-visco-plastic relation,

$$\boldsymbol{\sigma}^R(t) = \mathscr{A}_0\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathscr{G}(\boldsymbol{\sigma}^R(s), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\, ds. \tag{1.2}$$

When $\mathscr{G} = 0$ the constitutive law (1.1) reduces to the Kelvin–Voigt viscoelastic constitutive relation,

$$\boldsymbol{\sigma} = \mathscr{A}_1\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + \mathscr{A}_0\boldsymbol{\varepsilon}(\boldsymbol{u}). \tag{1.3}$$

Rate-type elastic-visco-plastic constitutive laws of the form (1.2) have been used in the literature to describe the behaviour of rubbers, metals, pastes and rocks, among others. Examples and mechanical interpretations can be found in [1, 6]. In particular, an application to a mining engineering problem was described in [6, Ch. 3]. It concerns the study of the creep of coal in a long wall, where self-advancing mechanized roof supports are used. Quasi-static contact problems for materials of the form (1.2) and (1.3) were investigated in a large number of papers, see [5, 12] for a survey. There, both the variational analysis and the numerical approach of the problems, including the study of semi-discrete and fully discrete schemes, were provided. Existence results in the study of dynamic problems with Kelvin–Voigt materials of the form (1.3) can be found in [2, 7–9] and, for more details, we refer to the monograph [4]. An existence result in the study of a dynamic frictional contact problem that combines (1.1) with a normal compliance model was obtained recently in [11].

This paper contains the analysis of a contact problem with friction for a viscoplastic material with the constitutive law (1.1) and the combination of a normal damped response condition with a unilateral contact model and friction modelled by the Coulomb law. The presence of friction and the unilateral contact model makes the analysis much more challenging than in [10, 11], where the normal compliance contact model was used, because it leads to an additional non-monotone and non-compact term in the variational formulation. In the analysis it will be important to achieve some further regularity of the solution to the problem, this is done by means of a special translation technique. The application of this technique to a non-linear viscoplastic constitutive material law is an important novelty of our approach.

The paper is organized as follows. In Section 2 we describe the considered contact problem, list the assumptions on the data and derive its variational formulation. Then, we turn to the weak solvability of the problem which is proved in several steps, as follows. First, in Section 3, we state and prove the weak solvability of two intermediate contact problems obtained by using a smoothing of the friction and the penalty approximation of the unilateral condition. Next, in Section 4, we prove some higher regularity of the solution with the help of the translation technique. Here it is important to carefully estimate the elasto-plastic part of the stress tensor. Finally, in Section 5, we perform the limit process which allows us to prove our main existence result, Theorem 6.

We end this introductory section by presenting the notation we shall use in the rest of the paper. We denote by r_+ and r_- the positive and negative part of r, i.e. $r_+ = \max\{0, r\}$,

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$r_- = \max\{0, -r\}$. We also denote by $\mathbb{S}^N$ the space of second-order symmetric tensors on $\mathbb{R}^N$ $(N = 2, 3)$, while '$\cdot$' and $|\cdot|$ will represent the inner product and the Euclidean norm on $\mathbb{S}^N$ and $\mathbb{R}^N$. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a Lipschitz boundary Γ and let $\boldsymbol{\nu}$ denote the unit outer normal on Γ. We assume that Γ is partitioned into three disjoint measurable parts Γ_1, Γ_2 and Γ_3. Everywhere in what follows the index i and j run from 1 to N, summation over repeated indices is implied and an index that follows a comma represents the partial derivative with respect to the corresponding component of the independent spatial variable. By $\mathbb{R}_+$ we denote the interval $[0, +\infty)$ and by c with possible index we denote positive constants whose values may change from line to line.

We use the standard notation L_p, $\boldsymbol{L}_p \equiv (L_p)^N$, $p \in [1, \infty]$ for Lebesgue spaces and W_p^k, $H^k \equiv W_2^k$, $\boldsymbol{H}^k \equiv (H^k)^N$, $k \geqslant 0$, $p \in [1, \infty]$ for Sobolev(-Slobodetskii) spaces. For the spaces with zero traces the notation $\mathring{H}^k$ and $\mathring{\boldsymbol{H}}^k = (\mathring{H}^k)^N$ is used when $k \notin \frac{1}{2} + \mathbb{N}$. For a domain $M \subset \mathbb{R}^d$ $(d \in \mathbb{N})$ and every real Banach space X we use the notation $C(M; X)$ for the space of bounded continuous functions from M to X and $B(M; X)$ for the space of bounded functions from M to X, both equipped with the appropriate sup-norm. Moreover, for $k \in \mathbb{N}$ and $p \in [1, \infty]$, we use the standard notation for the Lebesgue spaces $L_p(M; X)$ and for the Sobolev spaces $W_p^k(M; X)$. For $k \equiv (k_1, k_2) \in \mathbb{R}_+^2$ and some domains $M_i \in \mathbb{R}^{N_i}, i = 1, 2$ and $M \equiv M_1 \times M_2$, $H^k(M) \equiv L_2(M_1; H^{k_2}(M_2)) \cap H^{k_1}(M_1; L_{k_2}(M_2))$ is the corresponding anisotropic Sobolev(-Slobodetskii) space. A similar notation will be used for the spaces of functions with range in a Banach space X. For us M_1 will be a time interval J and M_2 the domain Ω, its boundary, or parts of its boundary. It is well known that functions in the space $H^{1/2,1}(J \times \Omega)$ take their (lateral) traces in $H^{1/4,1/2}(J \times \Gamma)$ (see Section 5.1 of [4] for details). In general, we denote by $\|\cdot\|_X$ the norm on a Banach space X and by X^* its dual. We use the notation $\langle u, v\rangle_M$ for the $L_2(M)$ scalar product of the functions u and v defined on a set M; we extend this notation for the duality pairing of u and v when u is now a functional, related to M. The notation $\langle \boldsymbol{u}, \boldsymbol{v}\rangle_{M,N}$ is used for vector valued functions $\boldsymbol{u}$ and $\boldsymbol{v}$ and $\langle \boldsymbol{\sigma}, \boldsymbol{\zeta}\rangle_{M,N\times N}$ for matrix-valued functions $\boldsymbol{\sigma}$ and $\boldsymbol{\zeta}$.

For every element $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ we also use the same symbol $\boldsymbol{v}$ to denote the trace of $\boldsymbol{v}$ on Γ and we denote by v_ν and $\boldsymbol{v}_\tau$ the normal and the tangential components of $\boldsymbol{v}$ on Γ given by

$$v_\nu = \boldsymbol{v} \cdot \boldsymbol{\nu}, \qquad \boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu \boldsymbol{\nu}.$$

We also denote by σ_ν and $\boldsymbol{\sigma}_\tau$ the normal and the tangential traces of a function $\boldsymbol{\sigma} \in L_2(\Omega; \mathbb{S}^N)$ which satisfies $\mathrm{Div}\,\boldsymbol{\sigma} \in \boldsymbol{L}_2(\Omega)$; we note that when $\boldsymbol{\sigma}$ is a regular function then

$$\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot \boldsymbol{\nu}, \qquad \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu},$$

and the following Green's formula holds:

$$\langle \boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v})\rangle_{\Omega,N\times N} + \langle \mathrm{Div}\,\boldsymbol{\sigma}, \boldsymbol{v}\rangle_{\Omega,N} = \langle \boldsymbol{\sigma}\boldsymbol{\nu}, \boldsymbol{v}\rangle_{\Gamma,N}, \qquad \forall \boldsymbol{v} \in \boldsymbol{H}^1(\Omega). \tag{1.4}$$

Finally, for given $\Gamma_1 \subset \Gamma$ we denote by $\boldsymbol{H}_0^1(\Omega)$ the closed subspace of $\boldsymbol{H}^1(\Omega)$ given by

$$\boldsymbol{H}_0^1(\Omega) = \{\, \boldsymbol{v} \in \boldsymbol{H}^1(\Omega)\,;\, \boldsymbol{v} = \boldsymbol{0} \ \text{ on } \ \Gamma_1 \,\}.$$

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2 Problem statement and variational formulation

In this section we present the classical and the variational formulation of the problem which describes the frictional contact process and present the assumptions on the data.

The physical setting is as follows. An elastic-visco-plastic body occupies a bounded domain $\Omega \subset \mathbb{R}^N$ $(N = 2, 3)$ with a regular boundary Γ that is partitioned into three disjoint measurable parts Γ_1, Γ_2 and Γ_3. Let $T > 0$ and let $J \equiv (0, T)$ denote the time interval of interest. A Dirichlet boundary condition $\boldsymbol{U}$ is given on $S_{1,T} = (0, T) \times \Gamma_1$. A volume force of density $\boldsymbol{f}_0$ acts in $Q_T = (0, T) \times \Omega$ and a surface traction of density $\boldsymbol{f}_2$ acts on $S_{2,T} = (0, T) \times \Gamma_2$. In the reference configuration the body is in a frictional contact on $S_{3,T} = (0, T) \times \Gamma_3$ with an obstacle, the so-called foundation. The contact is modelled with normal damped response in such a way that the normal velocity is limited. It is associated with a version of the Coulomb law of dry friction, in which the coefficient of friction may depend on the velocity. Under these conditions, the classical formulation of the problem is the following:

Problem $\mathscr{P}$ *Find a displacement field* $\boldsymbol{u} : [0, T] \times \Omega \to \mathbb{R}^N$ *and a stress field* $\boldsymbol{\sigma} : [0, T] \times \Omega \to \mathbb{S}^N$ *such that*

$$\boldsymbol{\sigma}(t) = \mathscr{A}_1 \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + \mathscr{A}_0 \boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathscr{G}(\boldsymbol{\sigma}(s) - \mathscr{A}_1 \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(s)), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\, ds \quad \text{in } Q_T, \tag{2.1}$$

$$\rho \ddot{\boldsymbol{u}} = \operatorname{Div} \boldsymbol{\sigma} + \boldsymbol{f}_0 \quad \text{in } Q_T, \tag{2.2}$$

$$\boldsymbol{u} = \boldsymbol{U} \quad \text{on } S_{1,T}, \tag{2.3}$$

$$\boldsymbol{\sigma} \boldsymbol{\nu} = \boldsymbol{f}_2 \quad \text{on } S_{2,T}, \tag{2.4}$$

$$\dot{u}_\nu \leqslant g, \quad \sigma_\nu + p(\dot{u}_\nu) \leqslant 0, \quad (\sigma_\nu + p(\dot{u}_\nu))(\dot{u}_\nu - g) = 0 \quad \text{on } S_{3,T}, \tag{2.5}$$

$$\left.\begin{aligned} \dot{\boldsymbol{u}}_\tau = \boldsymbol{0} &\Rightarrow |\boldsymbol{\sigma}_\tau| \leqslant \mathfrak{F}(\dot{\boldsymbol{u}})|\sigma_\nu| \\ \dot{\boldsymbol{u}}_\tau \neq \boldsymbol{0} &\Rightarrow \boldsymbol{\sigma}_\tau = -\mathfrak{F}(\dot{\boldsymbol{u}})|\sigma_\nu| \frac{\dot{\boldsymbol{u}}_\tau}{|\dot{\boldsymbol{u}}_\tau|} \end{aligned}\right\} \quad \text{on } S_{3,T}, \tag{2.6}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \dot{\boldsymbol{u}}(0) = \boldsymbol{u}_1 \quad \text{in } \Omega. \tag{2.7}$$

Equation (2.1) is the elastic-visco-plastic constitutive law already presented in Section 1, (2.2) represents the equation of motion in which ρ denotes the density of mass, whereas (2.3) and (2.4) are the displacement and traction boundary conditions, respectively. Condition (2.6) represents the Coulomb law of dry friction in which $\mathfrak{F}$ is the coefficient of friction, assumed to depend on the velocity, $\dot{\boldsymbol{u}}$. Finally, the functions $\boldsymbol{u}_0$ and $\boldsymbol{u}_1$ in (2.7) denote the initial displacement and the initial velocity, respectively.

Our main interest is on the contact condition (2.5). Here σ_ν denotes the normal stress, $\dot{u}_\nu$ is the normal displacement, $g \geqslant 0$ is given and p is a function which satisfies

$$\begin{cases} \text{(a) } p :]-\infty, g] \to \mathbb{R}. \\ \text{(b) There exists } L_p > 0 \text{ such that} \\ \qquad |p(r_1) - p(r_2)| \leqslant L_p |r_1 - r_2| \quad \forall r_1, r_2 \leqslant g. \\ \text{(c) } (p(r_1) - p(r_2))(r_1 - r_2) \geqslant 0 \quad \forall r_1, r_2 \leqslant g. \end{cases} \tag{2.8}$$

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Condition (2.5) shows that the normal velocity is restricted with a unilateral constraint, since $\dot{u}_\nu \leqslant g$. Moreover, when $\dot{u}_\nu < g$ then the reaction of the foundation is uniquely determined by the normal velocity (since $-\sigma_\nu = p(\dot{u}_\nu)$) and, when $\dot{u}_\nu = g$, the normal stress is not uniquely determined but is submitted to the restriction $-\sigma_\nu \geqslant p(g)$. Such a condition shows that the contact follows a normal damped response condition up to the limit g and then, when this limit is reached, the normal stress follows a unilateral condition. For these reasons we refer to the contact condition (2.5) as a normal damped response contact condition with unilateral constraint. Normal damped response contact conditions of the form $-\sigma_\nu = p(\dot{u}_\nu)$ were used in the literature in order to model the lubricated contact (see [5, 12] for details). An example of normal compliance function p which satisfies condition (2.8) is

$$p(r) = c_\nu r_+ + p_0,$$

where c_ν is a positive constant and p_0 represents the hydrostatic pressure of the lubricant. Also, note that when $g = 0$ and $p \equiv 0$, condition (2.5) becomes the Signorini-type contact condition in velocities,

$$\dot{u}_\nu \leqslant 0, \qquad \sigma_\nu \leqslant 0, \qquad \sigma_\nu \dot{u}_\nu = 0,$$

used in [4, 13]. This type of boundary condition was motivated in [13] by the modelling of the plowing process in which an obstacle, the plow, advances with a given velocity and is in contact with the ground.

We now describe the assumptions on the data we consider in the study of the mechanical problem (2.1)–(2.7). Thus, we assume that the operators $\mathscr{A}_\iota$, $\iota = 0, 1$ are linear, whereas the operator $\mathscr{G}$ may be non-linear and they satisfy the following conditions for $\iota = 0, 1$:

$$\begin{cases} \text{(a) } \mathscr{A}_\iota = (a_{ijk\ell}^{(\iota)}) : \Omega \times \mathbb{S}^N \to \mathbb{S}^N. \\ \text{(b) } a_{ijk\ell}^{(\iota)} \in L_\infty(\Omega),\ 1 \leqslant i, j, k, \ell \leqslant N. \\ \text{(c) } \mathscr{A}_\iota \boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \boldsymbol{\sigma} \cdot \mathscr{A}_\iota \boldsymbol{\tau},\ \forall\, \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{S}^N,\ \text{a.e. in } \Omega. \\ \text{(d) There exists } a_\iota, A_\iota > 0 \text{ such that} \\ \qquad \dfrac{\mathscr{A}_\iota \boldsymbol{\tau} \cdot \boldsymbol{\tau}}{|\boldsymbol{\tau}|^2} \in [a_\iota, A_\iota] \quad \forall\, \boldsymbol{\tau} \in \mathbb{S}^N \setminus \{\mathbf{0}\},\ \text{a.e. in } \Omega. \end{cases} \tag{2.9}$$

$$\begin{cases} \text{(a) } \mathscr{G} : \Omega \times \mathbb{S}^N \times \mathbb{S}^N \to \mathbb{S}^N. \\ \text{(b) There exists } L_{\mathscr{G}} > 0 \text{ such that} \\ \qquad |\mathscr{G}(\boldsymbol{x}, \boldsymbol{\sigma}_1, \boldsymbol{\varepsilon}_1) - \mathscr{G}(\boldsymbol{x}, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_2)| \leqslant L_{\mathscr{G}} (|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2| + |\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2|) \\ \qquad\qquad \forall\, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^N,\ \text{for a.e. } \boldsymbol{x} \in \Omega. \\ \text{(c) For any } \boldsymbol{\sigma}, \boldsymbol{\varepsilon} \in \mathbb{S}^N,\ \text{the mapping } \boldsymbol{x} \mapsto \mathscr{G}(\boldsymbol{x}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}) \\ \qquad \text{is measurable on } \Omega. \\ \text{(d) The mapping } \boldsymbol{x} \mapsto \mathscr{G}(\boldsymbol{x}, \mathbf{0}, \mathbf{0}) \text{ belongs to } L_2(\Omega; \mathbb{S}^N). \end{cases} \tag{2.10}$$

For simplicity of the presentation we assume that the mass density is equal to 1 and the prescribed displacement vanishes. We note that this assumption is not restrictive since there is no mathematical difficulty to extend our results below for a sufficiently smooth,

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space-dependent density and for $\boldsymbol{U}$ (cf. Assumptions (5.1.1) and (5.1.6) in [4]). The body forces and surface tractions have the regularity

$$\boldsymbol{f}_0 \in L_2\big(0,T;\boldsymbol{H}^{1/2}(\Omega)^*\big), \quad \boldsymbol{f}_2 \in L_2\big(0,T;\boldsymbol{H}^{1/2}(\Gamma_2)^*\big). \tag{2.11}$$

The coefficient of friction satisfies

$$\begin{cases} \text{(a) } \mathfrak{F} : \Gamma_3 \times \mathbb{R}^N \to \mathbb{R}_+. \\ \text{(b) There exists } C_{\mathfrak{F}} > 0 \text{ such that } \|\mathfrak{F}\|_{B(\Gamma_3\times\mathbb{R}^N)} < C_{\mathfrak{F}}. \\ \text{(c) The mapping } \mathfrak{F}(\boldsymbol{x},\cdot) \text{ is Lipschitz continuous for a.e. } \boldsymbol{x} \in \Gamma_3 \text{ with} \\ \quad \text{Lipschitz constant independent of } \boldsymbol{x}. \\ \text{(d) For any } \boldsymbol{v} \in \mathbb{R}^N, \text{ the mapping } \boldsymbol{x} \mapsto \mathfrak{F}(\boldsymbol{x},\boldsymbol{v}) \text{ is measurable on } \Gamma_3. \end{cases} \tag{2.12}$$

Finally, the initial data are assumed to satisfy

$$\boldsymbol{u}_0 \in \boldsymbol{H}_0^1(\Omega). \tag{2.13}$$

Assumptions (2.11) allow us, for a.e. $t \in (0,T)$, to define $\boldsymbol{f}(t) \in \boldsymbol{H}_0^1(\Omega)^*$ by

$$\langle \boldsymbol{f}(t), \boldsymbol{v}\rangle_{\Omega,N} = \langle \boldsymbol{f}_0(t), \boldsymbol{v}\rangle_{\Omega,N} + \langle \boldsymbol{f}_2(t), \boldsymbol{v}\rangle_{\Gamma_2,N} \quad \forall\, \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega), \tag{2.14}$$

and note that

$$\boldsymbol{f} \in L_2\big(0,T;\boldsymbol{H}_0^1(\Omega)^*\big). \tag{2.15}$$

Also, in the study of Problem $\mathscr{P}$ we need the set of admissible velocity fields given by

$$K = \{\ \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega)\ ;\ v_\nu \leqslant g \quad \text{a.e. on } \Gamma_3\ \} \tag{2.16}$$

and we reinforce assumption (2.13) with

$$\boldsymbol{u}_1 \in K. \tag{2.17}$$

We present now a brief description of the steps in the derivation of a variational formulation for the mechanical problem. To this end, assume that $(\boldsymbol{u},\boldsymbol{\sigma})$ are smooth functions satisfying (2.1)–(2.7). We introduce the set of test functions

$$\mathscr{K} = \{\ \boldsymbol{v} \in \boldsymbol{H}^1\big(0,T;\boldsymbol{H}_0^1(\Omega)\big)\ ;\ \boldsymbol{v}(t) \in K \quad \forall\, t \in [0,T]\ \}. \tag{2.18}$$

Let $t \in [0,T]$ and let $\boldsymbol{w} \in \mathscr{K}$. We take the dot product of equation (2.2) with $\boldsymbol{w}(t) - \dot{\boldsymbol{u}}(t)$, integrate the result over Ω and use Green's formula (1.4) to obtain

$$\begin{aligned} &\langle \ddot{\boldsymbol{u}}(t), \boldsymbol{w}(t) - \dot{\boldsymbol{u}}(t)\rangle_{\Omega,N} + \langle \boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{w}(t)) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t))\rangle_{\Omega,N\times N} \\ &\quad = \langle \boldsymbol{f}_0(t), \boldsymbol{w}(t) - \dot{\boldsymbol{u}}(t)\rangle_{\Omega,N} + \langle \boldsymbol{\sigma}(t)\boldsymbol{\nu}, \boldsymbol{w}(t) - \dot{\boldsymbol{u}}(t)\rangle_{\Gamma,N}. \end{aligned} \tag{2.19}$$

Next, we use the boundary conditions (2.4) and (2.6) and note that $\boldsymbol{w}(t) = \boldsymbol{0}$ on Γ_1; as a

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result we have

$$\int_\Gamma \boldsymbol{\sigma}(t)\boldsymbol{v}\cdot(\boldsymbol{w}(t)-\dot{\boldsymbol{u}}(t))\,d\boldsymbol{a} = \int_{\Gamma_2} \boldsymbol{f}_2(t)\cdot(\boldsymbol{w}(t)-\dot{\boldsymbol{u}}(t))\,d\boldsymbol{a} + \int_{\Gamma_3} \sigma_\nu(t)\,(w_\nu(t)-\dot{u}_\nu(t))\,d\boldsymbol{a} + \int_{\Gamma_3} \boldsymbol{\sigma}_\tau(t)\,(\boldsymbol{w}_\tau(t)-\dot{\boldsymbol{u}}_\tau(t))\,d\boldsymbol{a}. \tag{2.20}$$

Moreover, (2.5) yields

$$\int_{\Gamma_3} \sigma_\nu(t)\,(w_\nu(t)-\dot{u}_\nu(t))\,d\boldsymbol{a} \geqslant \int_{\Gamma_3} p(\dot{u}_\nu(t)(\dot{u}_\nu(t)-w_\nu(t))\,d\boldsymbol{a}, \tag{2.21}$$

and (2.6) implies that

$$\int_{\Gamma_3} \boldsymbol{\sigma}_\tau(t)\,(\boldsymbol{w}_\tau(t)-\dot{\boldsymbol{u}}_\tau(t))\,d\boldsymbol{a} \geqslant \int_{\Gamma_3} \mathfrak{F}(\dot{\boldsymbol{u}}(t))|\sigma_\nu(t)|\,\big(|\dot{\boldsymbol{u}}_\tau(t)|-|\boldsymbol{w}_\tau(t)|\big)\,d\boldsymbol{a}. \tag{2.22}$$

We combine now (2.19)–(2.22) and use (2.14) to find

$$\begin{aligned}&\langle\ddot{\boldsymbol{u}}(t),\boldsymbol{w}(t)-\dot{\boldsymbol{u}}(t)\rangle_{\Omega,N} + \langle\boldsymbol{\sigma}(t),\boldsymbol{\varepsilon}(\boldsymbol{w}(t))-\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t))\rangle_{\Omega,N\times N}\\ &\quad+\langle p(\dot{u}_\nu(t)),w_\nu(t)-\dot{u}_\nu(t)\rangle_{\Gamma_3} + \langle\mathfrak{F}(\dot{\boldsymbol{u}}(t))|\sigma_\nu(t)|,|\dot{\boldsymbol{w}}_\tau(t)|-|\dot{\boldsymbol{u}}_\tau(t)|\rangle_{\Gamma_3}\\ &\quad\geqslant\langle\boldsymbol{f}(t),\boldsymbol{w}(t)-\dot{\boldsymbol{u}}(t)\rangle_{\Omega,N}.\end{aligned} \tag{2.23}$$

Then, we integrate (2.23) over $[0,T]$ and combine the resulting inequality with the constitutive law (2.1), the unilateral constraint in (2.5) and the initial conditions (2.7). As a result, we obtain the following variational formulation of Problem $\mathscr{P}$.

Problem $\mathscr{P}^V$ *Find a displacement field* $\boldsymbol{u}:[0,T]\to\boldsymbol{H}_0^1(\Omega)$ *and a stress field* $\boldsymbol{\sigma}:[0,T]\to L_2(\Omega;\mathbb{S}^N)$ *such that* $\dot{\boldsymbol{u}}\in\mathscr{K}$,

$$\boldsymbol{\sigma}(t)=\mathscr{A}_1\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t))+\mathscr{A}_0\boldsymbol{\varepsilon}(\boldsymbol{u}(t))+\int_0^t\mathscr{G}(\boldsymbol{\sigma}(s)-\mathscr{A}_1\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(s)),\boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\,ds \tag{2.24}$$

for a.e. $t\in(0,T)$,

$$\begin{aligned}&\langle\ddot{\boldsymbol{u}},\boldsymbol{w}-\dot{\boldsymbol{u}}\rangle_{Q_T,N}+\langle\boldsymbol{\sigma},\boldsymbol{\varepsilon}(\boldsymbol{w}-\dot{\boldsymbol{u}})\rangle_{Q_T,N\times N}+\langle p(\dot{u}_\nu),w_\nu-\dot{u}_\nu\rangle_{S_{3,T}}\\ &\quad+\langle\mathfrak{F}(\dot{\boldsymbol{u}})|\sigma_\nu|,|\boldsymbol{w}_\tau|-|\dot{\boldsymbol{u}}_\tau|\rangle_{S_{3,T}}\geqslant\langle\boldsymbol{f},\boldsymbol{w}-\dot{\boldsymbol{u}}\rangle_{Q_T,N}\quad\forall\,\boldsymbol{w}\in\mathscr{K},\end{aligned} \tag{2.25}$$

$$\boldsymbol{u}(0)=\boldsymbol{u}_0,\qquad\dot{\boldsymbol{u}}(0)=\boldsymbol{u}_1. \tag{2.26}$$

We turn now to the study of Problem $\mathscr{P}^V$ which will be carried out in Sections 3–5 below; it is based on a limit procedure for the solutions of a sequence of intermediate frictional contact problems, similar to that used in [4, Ch. 4]. Since the modifications are straightforward, sometimes we omit the details. Everywhere below we assume that (2.8)–(2.13) and (2.17) hold.

3 Intermediate frictional contact problems

In this section we construct the intermediate frictional contact problems and provide their solvability. To this end, for every $\lambda>0$ and a fixed β with $0<\beta<2/N$, we consider the

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function $p_\lambda : \mathbb{R} \to \mathbb{R}$ defined by

$$p_\lambda(r) = \begin{cases} p(r) & \text{if } r \leqslant g, \\ \frac{1}{\lambda}(r-g)^\beta + p(g) & \text{if } r > g. \end{cases} \tag{3.1}$$

This special form of the penalty function simplifies the proof of the uniqueness of the solutions to the intermediate problems. Moreover, for every $\eta > 0$, let $\Phi_\eta : \mathbb{R}^N \to [0, \infty]$ be a convex differentiable function which has its minimum in $\boldsymbol{v} = \boldsymbol{0}$ and satisfies the approximation property

$$|\, \Phi_\eta(\boldsymbol{v}) - |\boldsymbol{v}|\,| < \eta \qquad \forall \boldsymbol{v} \in \mathbb{R}^N. \tag{3.2}$$

Such a function exists (see [4, p. 161]) and, moreover, we can assume that

$$|\nabla \Phi_\eta(\boldsymbol{v})| \leqslant 1 \qquad \forall \boldsymbol{v} \in \mathbb{R}^N, \tag{3.3}$$

where $\nabla \Phi_\eta$ denotes the gradient of Φ_η. Also, let $\boldsymbol{\theta} \in L_2\big(Q_T; \mathbb{S}^N\big)$ be given. We use the notation above to define the following intermediate frictional contact problem.

Problem $\mathscr{P}^V_{\lambda\eta\boldsymbol{\theta}}$. *Find a displacement field* $\boldsymbol{u} : [0, T] \to \boldsymbol{H}^1_0(\Omega)$ *and a stress field* $\boldsymbol{\sigma} : [0, T] \to L_2(\Omega; \mathbb{S}^N)$ *such that*

$$\boldsymbol{\sigma} \equiv \boldsymbol{\sigma}_0 + \boldsymbol{\theta} \text{ with } \boldsymbol{\sigma}_0 = \mathscr{A}_1 \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + \mathscr{A}_0 \boldsymbol{\varepsilon}(\boldsymbol{u}) \text{ a.e. in } Q_T, \tag{3.4}$$

$$\begin{aligned} &\langle \ddot{\boldsymbol{u}}, \boldsymbol{w} \rangle_{Q_T, N} + \langle \boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{w}) \rangle_{Q_T, N \times N} + \langle p_\lambda(\dot{u}_\nu), w_\nu \rangle_{S_{3,T}} \\ &\quad + \langle \mathfrak{F}(\dot{\boldsymbol{u}}) p_\lambda(\dot{u}_\nu) \nabla \Phi_\eta(\dot{\boldsymbol{u}}_\tau), \boldsymbol{w}_\tau \rangle_{S_{3,T}} = \langle \boldsymbol{f}, \boldsymbol{w} \rangle_{Q_T, N} \quad \forall\, \boldsymbol{w} \in \boldsymbol{H}^1_0(\Omega), \end{aligned} \tag{3.5}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \dot{\boldsymbol{u}}(0) = \boldsymbol{u}_1. \tag{3.6}$$

Clearly, problem $\mathscr{P}^V_{\lambda\eta\boldsymbol{\theta}}$ represents the variational formulation of a frictional contact problem in which the elastic-visco-plastic constitutive law (2.1) is replaced by the viscoelastic constitutive law (3.4), the contact condition (2.5) is replaced by the penalized condition $-\sigma_\nu = p_\lambda(\dot{u}_\nu)$ and the friction law (2.6) is replaced by a smoothed condition which involves $\Phi_\eta(\dot{\boldsymbol{u}}_\tau)$ instead of $|\dot{\boldsymbol{u}}_\tau|$. For this reason, we refer to Problem $\mathscr{P}^V_{\lambda\eta\boldsymbol{\theta}}$ as the penalized smoothed viscoelastic problem associated to the original frictional contact problem $\mathscr{P}^V$.

The existence of a solution to Problem $\mathscr{P}^V_{\lambda\eta\boldsymbol{\theta}}$ is given by the following result.

Proposition 1 *Assume that conditions* (2.8)–(2.13) *and* (2.17) *hold. Then, for all* $\lambda, \eta > 0$ *and* $\boldsymbol{\theta} \in L_2(Q_T; \mathbb{S}^N)$, *Problem* $\mathscr{P}^V_{\lambda\eta\boldsymbol{\theta}}$ *has a unique solution with regularity*

$$\boldsymbol{u} \in \boldsymbol{H}^1(Q_T) \cap C^1([0, T]; \boldsymbol{H}^1(\Omega)),\ \dot{\boldsymbol{u}} \in \boldsymbol{H}^{1/2,1}(Q_T),\ \ddot{\boldsymbol{u}} \in L_2\big(J; \boldsymbol{H}^1_0(\Omega)^*\big), \tag{3.7}$$

$$\boldsymbol{\sigma} \in L_2\big(Q_T; \mathbb{S}^N\big), \quad \mathrm{Div}\, \boldsymbol{\sigma} \in L_2\big(J; \boldsymbol{H}^1_0(\Omega)^*\big). \tag{3.8}$$

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Moreover, the solution satisfies

$$\|\dot{\boldsymbol{u}}\|_{L_\infty(0,T;\boldsymbol{L}_2(\Omega))} + \|\dot{\boldsymbol{u}}\|_{\boldsymbol{H}^{1/2,1}(Q_T)} + \|p_\lambda(\dot{u}_\nu)\dot{u}_\nu\|_{L_1(S_{3,T})} \leqslant c_1 + c_2\|\boldsymbol{\theta}\|_{L_2(Q_T;\mathbb{S}^N)} \tag{3.9}$$

with constants c_1, c_2 independent of λ, η and $\boldsymbol{\theta}$.

Proof In order to verify the existence of a solution we use the Galerkin method. Let $\mathscr{X} = \{\boldsymbol{v}_j\,;\, j \in \mathbb{N}\}$ be an orthonormal basis of the space $\boldsymbol{H}_0^1(\Omega)$ and denote by $\mathscr{X}_m = \operatorname{span}\{\boldsymbol{v}_1,\dots,\boldsymbol{v}_m\}$ the Galerkin space of order m. A function $\boldsymbol{u}_m \in C^1(J;\mathscr{X}_m)$ is a solution of the Galerkin approximation of the order m to Problem $\mathscr{P}^V_{\lambda\eta\boldsymbol{\theta}}$, if it solves (3.5) for every $\boldsymbol{w} \in \mathscr{X}_m$ and every $t \in (0,T)$, where $\boldsymbol{\sigma} = \boldsymbol{\sigma}_m$ is defined by (3.4) with $\boldsymbol{u}$ replaced by $\boldsymbol{u}_m$ and the initial conditions are satisfied in the approximate sense $\langle \boldsymbol{u}_m(0), \boldsymbol{w}_m\rangle_{\Omega,N} = \langle \boldsymbol{u}_0, \boldsymbol{w}_m\rangle_{\Omega,N}$ and $\langle \dot{\boldsymbol{u}}_m(0), \boldsymbol{w}_m\rangle_{\Omega,N} = \langle \boldsymbol{u}_1, \boldsymbol{w}_m\rangle_{\Omega,N}$ for every $\boldsymbol{w}_m \in \mathscr{X}_m$. The Galerkin solution $\boldsymbol{u}_m$ can be obtained by solving a system of m ordinary differential equations for the unknown coefficients of the functions $\boldsymbol{u}_m$ in the basis $\{\boldsymbol{v}_1,\dots,\boldsymbol{v}_m\}$ of the space $\mathscr{X}_m$. The existence of such a solution follows from the theory of ordinary differential equations.

Next, we proceed to *a priori* estimates. To this end, we use the test function $\boldsymbol{w}_m = \dot{\boldsymbol{u}}_m$ in the Galerkin discretization of (3.5), the ellipticity of the operators $\mathscr{A}_1$ and $\mathscr{A}_0$ combined with the monotonicity of the functions p_λ, the properties of the function Φ_η, $\mathfrak{F}$, and the Gronwall lemma; as a result we obtain the estimate

$$\|\dot{\boldsymbol{u}}_m\|^2_{L_\infty(0,T;\boldsymbol{L}_2(\Omega))} + \|\dot{\boldsymbol{u}}_m\|^2_{L_2(J;\boldsymbol{H}^1(\Omega))} + \|p_\lambda(\dot{u}_{m\nu})\dot{u}_{m\nu}\|_{L_1(S_{3,T})} \leqslant c_1 + c_2\|\boldsymbol{\theta}\|_{L_2(Q_T;\mathbb{S}^N)}$$

with c_1, c_2 independent of m, η and $\boldsymbol{\theta}$. A standard dual estimate then yields that $\|\ddot{\boldsymbol{u}}_m\|_{L_2(J;\boldsymbol{H}_0^1(\Omega)^*)}$ is bounded by $c_3 + c_4\|\boldsymbol{\theta}\|_{\boldsymbol{L}_2(Q_T)}$. Interpolation of $H^1\big(J;\boldsymbol{H}^1(\Omega)^*\big)$ and $L_2(J;\boldsymbol{H}^1(\Omega))$ gives

$$\|\dot{\boldsymbol{u}}_m\|^2_{\boldsymbol{H}^{1/2,1}(Q_T)} \leqslant c_5 + c_6\|\boldsymbol{\theta}\|_{L_2(Q_T;\mathbb{S}^N)},$$

see [4, pp. 273–276] for details. This estimate is sufficient in order to pass to the limit as $m \to +\infty$. We use the compact embedding theorems and trace theorems for Sobolev spaces to see that there exists a sequence $m_k \to +\infty$ such that the corresponding sequence of solutions $\boldsymbol{u}_k = \boldsymbol{u}_{m_k}$ converges to a function $\boldsymbol{u}$ which satisfies (3.6), (3.7) and, moreover,

$$\begin{aligned}
&\langle \ddot{\boldsymbol{u}}(t), \boldsymbol{w}\rangle_{\Omega,N} + \langle \mathscr{A}_1\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)), \boldsymbol{\varepsilon}(\boldsymbol{w})\rangle_{\Omega,N\times N} + \langle \mathscr{A}_0\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{w})\rangle_{\Omega,N\times N} \\
&\quad + \langle \boldsymbol{\theta}(t), \boldsymbol{\varepsilon}(\boldsymbol{w})\rangle_{\Omega,N\times N} + \langle p_\lambda(\dot{u}_\nu(t)), w_\nu\rangle_{\Gamma_3} \\
&\quad + \langle \mathfrak{F}(\dot{\boldsymbol{u}}(t))p_\lambda(\dot{u}_\nu(t)), \nabla\Phi_\eta(\dot{\boldsymbol{u}}_\tau(t)), \boldsymbol{w}_\tau\rangle_{\Gamma_3} = \langle \boldsymbol{f}(t), \boldsymbol{w}\rangle_{\Omega,N} \\
&\qquad\qquad \forall\, \boldsymbol{w} \in \boldsymbol{H}_0^1(\Omega), \quad \text{for a.e. } t \in (0,T).
\end{aligned} \tag{3.10}$$

Define now the function $\boldsymbol{\sigma}$ by (3.4). It follows that the pair $(\boldsymbol{u},\boldsymbol{\sigma})$ is a solution to Problem $\mathscr{P}^V_{\lambda\eta\boldsymbol{\theta}}$ and it satisfies (3.7)–(3.9), which concludes the existence part of the proof.

The uniqueness of the solution is obtained by taking $(\dot{\boldsymbol{u}}_1 - \dot{\boldsymbol{u}}_2)\chi_t$ with two solutions $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ and the indicator function χ_t of the time interval $I_t = (0,t)$ as test function in the difference of the equations for these two solutions and performing similar estimates as above. Let us denote $Q_t := (0,t)\times\Omega$ and $S_{3,t} := (0,t)\times\Gamma_3$. Due to the monotonicity of the function p_λ, the penalty term leads to a non-negative contribution on the left-hand

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side that can be neglected. The friction term can be represented by

$$\begin{aligned}
&\int_{S_{3,t}} \big(\mathfrak{F}(\dot{\boldsymbol{u}}_1)p_\lambda((\dot{\boldsymbol{u}}_1)_\nu)\nabla\Phi_\eta((\dot{\boldsymbol{u}}_1)_\tau) - \mathfrak{F}(\dot{\boldsymbol{u}}_2)p_\lambda((\dot{\boldsymbol{u}}_2)_\nu)\nabla\Phi_\eta((\dot{\boldsymbol{u}}_2)_\tau)\big)\cdot((\dot{\boldsymbol{u}}_1)_\tau - (\dot{\boldsymbol{u}}_2)_\tau)\,d\boldsymbol{a}\,ds \\
&\quad = \int_{S_{3,t}} \mathfrak{F}(\dot{\boldsymbol{u}}_1)p_\lambda((\dot{\boldsymbol{u}}_1)_\nu)\big(\nabla\Phi_\eta((\dot{\boldsymbol{u}}_1)_\tau) - \nabla\Phi_\eta((\dot{\boldsymbol{u}}_2)_\tau)\big)\cdot((\dot{\boldsymbol{u}}_1)_\tau - (\dot{\boldsymbol{u}}_2)_\tau)\,d\boldsymbol{a}\,ds \\
&\qquad + \int_{S_{3,t}} \mathfrak{F}(\dot{\boldsymbol{u}}_1)(p_\lambda((\dot{\boldsymbol{u}}_1)_\nu) - p_\lambda((\dot{\boldsymbol{u}}_2)_\nu))\nabla\Phi_\eta((\dot{\boldsymbol{u}}_2)_\tau)\cdot((\dot{\boldsymbol{u}}_1)_\tau - (\dot{\boldsymbol{u}}_2)_\tau)\,d\boldsymbol{a}\,ds \\
&\qquad + \int_{S_{3,t}} \big(\mathfrak{F}(\dot{\boldsymbol{u}}_1) - \mathfrak{F}(\dot{\boldsymbol{u}}_2)\big)p_\lambda((\dot{\boldsymbol{u}}_2)_\nu)\nabla\Phi_\eta((\dot{\boldsymbol{u}}_2)_\tau)\cdot((\dot{\boldsymbol{u}}_1)_\tau - (\dot{\boldsymbol{u}}_2)_\tau)\,d\boldsymbol{a}\,ds.
\end{aligned}$$

The first summand here is non-negative, due to the convexity of Φ_η. For the second and third terms we use the Lipschitz continuity of p_λ, the growth condition in (3.1) for p_λ and the property that $\nabla\Phi_\eta$ is bounded. This yields an estimate by

$$c_1\int_{S_{3,t}} \big(1 + |(\dot{\boldsymbol{u}}_2)_\nu|^\beta\big)|\dot{\boldsymbol{u}}_1 - \dot{\boldsymbol{u}}_2|^2\,d\boldsymbol{a}\,ds \leqslant c_2\big(1 + \|\dot{\boldsymbol{u}}_2\|^\beta_{L_q(S_{3,t})}\big)\|\dot{\boldsymbol{u}}_1 - \dot{\boldsymbol{u}}_2\|^2_{L_q(S_{3,t})}$$

with $q = 2 + \beta$. For the further estimate we use the trace theorem $W^\mu_q(\Omega) \hookrightarrow L_q(\Gamma_3)$ with $\mu > 1/q$, the interpolation estimates

$$\|v\|_{W^\mu_q(\Omega)} \leqslant c_1\|v\|_{H^\gamma(\Omega)} \leqslant c_2\|v\|^{1-\gamma}_{L_2(\Omega)}\|v\|^\gamma_{H^1(\Omega)}$$

with $\gamma - N/2 = \mu - N/q$ and

$$\|v\|_{L_q(I_t;W^\mu_q(\Omega))} \leqslant c\|v\|^{1-\gamma}_{L_r(I_t;L_2(\Omega))}\|v\|^\gamma_{L_2(I_t;H^1(\Omega))}$$

with r defined by $(1-\gamma)/r + \gamma/2 = 1/q$. By some algebra we see that $2 < r < \infty$, if $q < 2 + 2/N$ and μ is chosen sufficiently close to $1/q$. Using, moreover, some Hölder estimates, we obtain an estimate of the friction term by

$$c_1(\kappa)\|\dot{\boldsymbol{u}}_1 - \dot{\boldsymbol{u}}_2\|^2_{L_r(I_t;\boldsymbol{L}_2(\Omega))} + \kappa\|\dot{\boldsymbol{u}}_1 - \dot{\boldsymbol{u}}_2\|^2_{L_2(I_t;\boldsymbol{H}^1(\Omega))}.$$

Choosing κ sufficiently small and performing standard estimates for all the other terms, we obtain

$$\|(\dot{\boldsymbol{u}}_1 - \dot{\boldsymbol{u}}_2)(t)\|^2_{\boldsymbol{L}_2(\Omega)} + \|\dot{\boldsymbol{u}}_1 - \dot{\boldsymbol{u}}_2\|^2_{L_2(I_t;\boldsymbol{H}^1(\Omega))} \leqslant c\|\dot{\boldsymbol{u}}_1 - \dot{\boldsymbol{u}}_2\|^2_{L_r(I_t;\boldsymbol{L}_2(\Omega))} \tag{3.11}$$

with some constant c that is independent of t (if t is bounded by some T). Taking the $r/2$th power and applying the Gronwall lemma we obtain

$$\|\dot{\boldsymbol{u}}_1 - \dot{\boldsymbol{u}}_2\|^2_{L_\infty(0,T;\boldsymbol{L}_2(\Omega))} \leqslant 0,$$

which concludes the proof.

It is clear that the solutions $(\boldsymbol{u}, \boldsymbol{\sigma})$ obtained in Proposition 1 depend on the penalization parameter λ, on the smoothing parameter η and on the given function $\boldsymbol{\theta}$. In the next step

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of our analysis we replace the function $\boldsymbol{\theta}$ by the true viscoplastic stress. The viscoplastic stress is defined via the solution of the integral equation

$$\boldsymbol{\theta}(t) = \int_0^t \mathscr{G}(\boldsymbol{\theta}(s) + \mathscr{A}_0\boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \boldsymbol{\varepsilon}(\boldsymbol{u}(s))\, ds,\ t \in J \tag{3.12}$$

for given $\boldsymbol{u}$. The existence and uniqueness of a solution $\boldsymbol{\theta}$ is stated in Proposition 2.

Proposition 2 *Let* $\boldsymbol{u} \in H^1\big(J;\boldsymbol{H}^1(\Omega)\big)$ *be given. Then* (3.12) *has a unique solution* $\boldsymbol{\theta} \in L_2\big(Q_T;\mathbb{S}^N\big)$.

Proof We apply the Banach fixed point theorem to the operator

$$\boldsymbol{\theta} \mapsto \Theta(\boldsymbol{\theta}) \quad \text{with} \quad \Theta(\boldsymbol{\theta};t) := \int_0^t \mathscr{G}(\boldsymbol{\theta}(s) + \mathscr{A}_0\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{u}(s))\, ds$$

that maps $L_2\big(Q_T;\mathbb{S}^N\big)$ to $L_2\big(Q_T;\mathbb{S}^N\big)$. The contraction property will be proved for the norm

$$\|\boldsymbol{\theta}\|_\alpha := \left(\int_0^T e^{-\alpha t}\, \|\boldsymbol{\theta}(t)\|^2_{L_2(\Omega;\mathbb{S}^N)}\, dt\right)^{1/2}$$

with some $\alpha > 0$ to be specified later. This norm is equivalent to the $L_2\big(Q_T;\mathbb{S}^N\big)$–norm. For $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2 \in L_2\big(Q_T;\mathbb{S}^N\big)$ we obtain with the Lipschitz property of $\mathscr{G}$ the estimate

$$\|\Theta(\boldsymbol{\theta}_1;t) - \Theta(\boldsymbol{\theta}_2;t)\|^2_{L_2(\Omega;\mathbb{S}^N)} \leqslant c_1 \int_0^t \|\boldsymbol{\theta}_1(s) - \boldsymbol{\theta}_2(s)\|^2_{L_2(\Omega;\mathbb{S}^N)}\, ds$$

with some constant c_1. Multiplication by $e^{-\alpha t}$, integration over $t \in J$, and integration by parts leads to

$$\begin{aligned}\|\Theta(\boldsymbol{\theta}_1) - \Theta(\boldsymbol{\theta}_2)\|^2_\alpha &\leqslant c_1 \int_0^T e^{-\alpha t} \int_0^t \|\boldsymbol{\theta}_1(s) - \boldsymbol{\theta}_2(s)\|^2_{L_2(\Omega;\mathbb{S}^N)}\, ds\, dt \\ &= c_1 \int_0^T \frac{1}{\alpha}(e^{-\alpha t} - e^{-\alpha T})\|\boldsymbol{\theta}_1(t) - \boldsymbol{\theta}_2(t)\|^2_{L_2(\Omega;\mathbb{S}^N)}\, dt \leqslant \frac{c_1}{\alpha}\|\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2\|^2_\alpha.\end{aligned}$$

As a consequence, for $\alpha > c_1$ the mapping Θ is a contraction with respect to the norm $\|\cdot\|_\alpha$ and the Banach fixed point theorem proves the existence of a unique fixed point of Θ, which concludes the proof.

We now consider the following penalized smoothed viscoplastic contact problem.

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Problem $\mathscr{P}^V_{\lambda\eta}$ *Find a displacement field* $\boldsymbol{u} : [0,T] \to \boldsymbol{H}^1_0(\Omega)$ *and a stress field* $\boldsymbol{\sigma} : [0,T] \to L_2(\Omega;\mathbb{S}^N)$ *such that* $\dot{\boldsymbol{u}} \in L_2\big(J;\boldsymbol{H}^1_0(\Omega)\big)$,

$$\boldsymbol{\sigma}(t) = \mathscr{A}_1\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + \mathscr{A}_0\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathscr{G}(\boldsymbol{\sigma}(s) - \mathscr{A}_0\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(s)), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\,ds \tag{3.13}$$

$$\text{for a.e. } t \in (0,T),$$

$$\begin{aligned}\langle \ddot{\boldsymbol{u}}(t), \boldsymbol{w}\rangle_{\Omega,N} + \langle \boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{w})\rangle_{\Omega,N\times N} + \langle p_\lambda(\dot{u}_\nu(t)), w_\nu\rangle_{\Gamma_3} \\ + \langle \mathfrak{F}(\dot{\boldsymbol{u}}(t))p_\lambda(\dot{u}_\nu(t))\nabla\Phi_\eta(\dot{\boldsymbol{u}}_\tau(t)), \boldsymbol{w}_\tau\rangle_{\Gamma_3} = \langle \boldsymbol{f}(t), \boldsymbol{w}\rangle_{\Omega,N}\end{aligned} \tag{3.14}$$

$$\forall\, \boldsymbol{w} \in \boldsymbol{H}^1_0(\Omega), \quad \text{for a.e. } t \in (0,T),$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \qquad \dot{\boldsymbol{u}}(0) = \boldsymbol{u}_1. \tag{3.15}$$

Clearly, problem $\mathscr{P}^V_{\lambda\eta}$ represents the variational formulation of a frictional contact problem in which the contact condition (2.5) is replaced by the penalized condition $-\sigma_\nu = p_\lambda(\dot{u}_\nu)$ and the norm $|\dot{\boldsymbol{u}}_\tau|$ in the friction term is smoothed by the term $\Phi_\eta(\dot{\boldsymbol{u}}_\tau)$. For this reason, we refer to Problem $\mathscr{P}^V_{\lambda\eta}$ as the penalized smoothed viscoplastic problem associated to the original frictional contact problem $\mathscr{P}^V$.

The well posedness of this problem is given by the following existence and uniqueness result.

Proposition 3 *Let the assumptions of Proposition 1 be valid. Then problem* $\mathscr{P}^V_{\lambda\eta}$ *has a unique solution. This solution satisfies*

$$\|\dot{\boldsymbol{u}}\|_{L_\infty(0,T;\boldsymbol{L}_2(\Omega))} + \|\dot{\boldsymbol{u}}\|_{\boldsymbol{H}^{1/2,1}(Q_T)} + \|p_\lambda(\dot{u}_\nu)\dot{u}_\nu\|_{L_1(S_{3,T})} \leqslant c \tag{3.16}$$

with constant c *independent of* λ *and* η.

Proof For a given $\boldsymbol{\theta} \in L_2\big(Q_T;\mathbb{S}^N\big)$ we denote by $\boldsymbol{U}(\boldsymbol{\theta})$ the solution $\boldsymbol{u}$ of problem $\mathscr{P}^V_{\lambda\eta\boldsymbol{\theta}}$ guaranteed by Proposition 1 and, for a given $\boldsymbol{u} \in H^1(J;\boldsymbol{H}^1(\Omega))$ let $\Theta(\boldsymbol{u}) \in L_2\big(Q_T;\mathbb{S}^N\big)$ be the solution of (3.12), guaranteed by Proposition 2. We apply the Banach fixed point theorem to the operator

$$L_2\big(Q_T;\mathbb{S}^N\big) \ni \boldsymbol{\theta} \mapsto \Theta(\boldsymbol{U}(\boldsymbol{\theta})) \in L_2\big(Q_T;\mathbb{S}^N\big),$$

and to this end, again, we equip the space $L_2\big(Q_T;\mathbb{S}^N\big)$ with the norm $\|\cdot\|_\alpha$, for some appropriate $\alpha > 0$.

Let $\boldsymbol{u}_j = \boldsymbol{U}(\boldsymbol{\theta}_j)$ for $j = 1,2$ and let Θ_1, Θ_2 be the solutions of (3.12) to $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$. We take the differences of (3.5) for the two solutions $\boldsymbol{u}_1, \boldsymbol{u}_2$ with the two viscoplastic stresses $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2$ and use there the test function $\boldsymbol{w} = \dot{\boldsymbol{u}}_1 - \dot{\boldsymbol{u}}_2$. We integrate the result over the time interval $[0,t]$ for $t \in J$. The estimate of the volume terms is standard. For the penalty and friction terms we obtain the same results as in the proof of the uniqueness part to Proposition 1. This leads to the estimate

$$\begin{aligned}\|(\dot{\boldsymbol{u}}_1 - \dot{\boldsymbol{u}}_2)(t)\|^2_{L_2(\Omega)} + \|\dot{\boldsymbol{u}}_1 - \dot{\boldsymbol{u}}_2\|^2_{L_2(I_t;H^1(\Omega))} \\ \leqslant c_1\|\dot{\boldsymbol{u}}_1 - \dot{\boldsymbol{u}}_2\|^2_{L_r(I_t;\boldsymbol{L}_2(\Omega))} + c_2\|\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2\|^2_{L_2(I_t;L_2(\Omega;\mathbb{S}^N))}.\end{aligned}$$

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An application of the Gronwall lemma yields

$$\|\dot{\boldsymbol{u}}_1 - \dot{\boldsymbol{u}}_2\|_{L_2(I_t;\boldsymbol{H}^1(\Omega))} \leqslant c\|\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2\|_{L_2(I_t;L_2(\Omega;\mathbb{S}^N))}$$

and then, via $\boldsymbol{u}_j(t) = \boldsymbol{u}_0 + \int_0^t \dot{\boldsymbol{u}}_j(s)\,ds$, it follows that

$$\|(\boldsymbol{u}_1 - \boldsymbol{u}_2)(t)\|_{L_2(I_t;\boldsymbol{H}^1(\Omega))} \leqslant c\|\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2\|_{L_2(I_t;L_2(\Omega;\mathbb{S}^N))}. \tag{3.17}$$

From the Lipschitz property of $\mathscr{G}$ we conclude

$$\|(\Theta_1 - \Theta_2)(t)\|_{L_2(\Omega;\mathbb{S}^N)} \leqslant c\int_0^t \left(\|(\Theta_1 - \Theta_2)(s)\|_{L_2(\Omega;\mathbb{S}^N)} + \|(\boldsymbol{u}_1 - \boldsymbol{u}_2)(s)\|_{\boldsymbol{H}^1(\Omega)}\right) ds.$$

From (3.17) and the previous inequality we obtain by the Gronwall lemma

$$\|(\Theta_1 - \Theta_2)(t)\|^2_{L_2(\Omega;\mathbb{S}^N)} \leqslant c_1\|\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2\|^2_{L_2((0,t)\times\Omega;\mathbb{S}^N)}.$$

Therefore, proceeding as in the proof of Proposition 2 we obtain that

$$\|\Theta_1 - \Theta_2\|^2_\alpha \leqslant \frac{c_1}{\alpha}\|\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2\|^2_\alpha.$$

Then we apply the Banach fixed point theorem for $\alpha > c_2$ and conclude the proof.

4 A regularity result

In this section we state and prove a regularity result for the solution of Problem $\mathscr{P}^V_{\lambda\eta}$. This result is needed to perform the limit procedure as $\lambda, \eta \to 0$. It requires the following additional assumption.

Assumption 4 We assume that Γ_3 is a $C^{1,1}$ surface, that $\mathfrak{F}$ is zero on $(\Gamma_3 \setminus \Gamma_{\mathfrak{F}}) \times \mathbb{R}^N$, where $\Gamma_{\mathfrak{F}}$ is a subset of Γ_3 with $\operatorname{dist}(\Gamma_{\mathfrak{F}}, \Gamma_1 \cup \Gamma_2) > 0$. For some subdomain $\Omega_{\mathfrak{F}} \subset \Omega$ with $\operatorname{dist}(\Gamma_{\mathfrak{F}}, \Omega \setminus \Omega_{\mathfrak{F}}) > 0$ we also suppose $\mathscr{A}_1, \mathscr{A}_0 \in C^{0,1}(\Omega_{\mathfrak{F}})$ and $\boldsymbol{f}_{|J\times\Omega_{\mathfrak{F}}} \in \boldsymbol{H}^{-1/4,-1/2}(J \times \Omega_{\mathfrak{F}})$. Moreover, we assume that

$$\|\mathfrak{F}\|_{L_\infty(\Gamma_3\times\mathbb{R}^N)} < C_{\mathfrak{F}} \tag{4.1}$$

with the constant $C_{\mathfrak{F}}$ defined in (4.10) below.

Proposition 5 *Let the assumptions of Proposition 1 and Assumption 4 hold. Then the solution of Problem $\mathscr{P}^V_{\lambda\eta}$ satisfies $\boldsymbol{u} \in \boldsymbol{H}^{1/2,1}(J \times \Gamma_{\mathfrak{F}})$ and*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^{1/2,1}(S_{\mathfrak{F},T})} + \|p_\lambda(\dot{u}_\nu))\|_{L_2(S_{\mathfrak{F},T})} \leqslant C$$

where $S_{\mathfrak{F},T} = J \times \Gamma_{\mathfrak{F}}$ and C is a positive constant which is independent of λ and η.

Proof The proof is done with the method outlined in Section 5.1 of [4] and for this reasons, when the modifications are straightforward, we skip the details. The main difference is

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the presence of the additional viscoplastic stress $\boldsymbol{\theta}$ defined by (3.12). We use a localization technique and a local rectification of the boundary to transform the problem onto a half-space domain. Let χ_1 be an arbitrary member of a suitable smooth partition of unity of some neighbourhood of $\Gamma_{\mathfrak{F}}$ such that the support of χ_1 has a diameter of at most δ for a sufficiently small δ. Let $\chi_{0,L}$ be a cut-off function for the time variable of the form $\chi_{0,L}(t) = \chi_{0,0}(t-L)$ with a smooth function $\chi_{0,0}$ that satisfies $\chi_{0,0} = 0$ for $t > 0$ and $\chi_{0,0} = 1$ for $t < -1$. Finally, let $\chi(t, \boldsymbol{x}) = \chi_{0,L}(t)\chi_1(\boldsymbol{x})$. We use the notation $\mathfrak{G}(\boldsymbol{u}) = \mathfrak{F}(\boldsymbol{u}_\tau)\nabla\Phi_\eta(\boldsymbol{u}_\tau)$, $\check{\boldsymbol{u}} = \chi(\dot{\boldsymbol{u}} - \boldsymbol{u}_1)$ and $\mathring{\boldsymbol{u}} = \chi(\boldsymbol{u} - \boldsymbol{u}_0)$. Note that $\mathfrak{G}(\boldsymbol{u})$ is bounded in $L_\infty(J \times \Gamma_3)$ by the same bound as $\mathfrak{F}$ and $\check{\boldsymbol{u}}$, $\mathring{\boldsymbol{u}}$ satisfy the initial conditions 0 at $t = 0$. The functions $\check{\boldsymbol{u}}$ and $\mathring{\boldsymbol{u}}$ can be extended by $\boldsymbol{0}$ from the time interval J to $\mathbb{R}$. Using the test function $\chi\boldsymbol{v}$ in (3.5) and shifting the function χ from the 'right' side of each form to the 'left' side yields the following localized variational equation:

$$\begin{aligned}\langle \dot{\check{\boldsymbol{u}}}, \boldsymbol{v}\rangle_{Q_T,N} + \langle \mathscr{A}_1\boldsymbol{\varepsilon}(\check{\boldsymbol{u}}) + \mathscr{A}_0\boldsymbol{\varepsilon}(\mathring{\boldsymbol{u}}) + \mathring{\boldsymbol{\theta}}, \boldsymbol{\varepsilon}(\boldsymbol{v})\rangle_{Q_T,N\times N} + \langle \widetilde{p}_\lambda(\check{u}_v), v_v\rangle_{S_{3,T}} \\ + \langle \mathfrak{G}\,\widetilde{p}_\lambda(\check{u}_v), \boldsymbol{v}_\tau\rangle_{S_{3,T},N} = \langle \widetilde{\boldsymbol{f}}, \boldsymbol{v}\rangle_{Q_T,N}.\end{aligned} \tag{4.2}$$

Here $\mathring{\boldsymbol{\theta}} = \chi\boldsymbol{\theta}$ and $\widetilde{\boldsymbol{f}}$ denotes the modified linear functional given by

$$\begin{aligned}\langle \widetilde{\boldsymbol{f}}, \boldsymbol{v}\rangle_{Q_T,N} = \langle \chi\boldsymbol{f} + \dot{\chi}(\dot{\boldsymbol{u}} - \boldsymbol{u}_1), \boldsymbol{v}\rangle_{Q_T,N} - \langle \mathscr{A}_1\boldsymbol{\varepsilon}(\chi\boldsymbol{u}_1) + \mathscr{A}_0\boldsymbol{\varepsilon}(\chi\boldsymbol{u}_0), \boldsymbol{\varepsilon}(\boldsymbol{v})\rangle_{Q_T,N\times N} \\ + \langle \mathscr{A}_1\mathscr{B}\dot{\boldsymbol{u}} + \mathscr{A}_0\mathscr{B}\boldsymbol{u}, \boldsymbol{\varepsilon}(\boldsymbol{v})\rangle_{Q_T,N\times N} - \langle \mathscr{A}_1\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + \mathscr{A}_0\boldsymbol{\varepsilon}(\boldsymbol{u}), \mathscr{B}\boldsymbol{v}\rangle_{Q_T,N\times N} - \langle \boldsymbol{\theta}, \mathscr{B}\boldsymbol{v}\rangle_{Q_T,N\times N}.\end{aligned}$$

The matrix $\mathscr{B}$ above is defined by $\mathscr{B}\boldsymbol{v} = \boldsymbol{\varepsilon}(\chi\boldsymbol{v}) - \chi\,\boldsymbol{\varepsilon}(\boldsymbol{v})$ and $\widetilde{p}_\lambda(\check{u}_v) = \chi\, p_\lambda(\chi^{-1}\check{u}_v + u_{1v})$. It will be important that $\widetilde{p}_\lambda$ is monotone with respect to $\check{u}_v$. This follows from the representation

$$\widetilde{p}_\lambda(\check{u}_v) = \chi\, p_\lambda(\chi^{-1}\check{u}_v + u_{1v}) = \begin{cases} \chi\, p(\chi^{-1}\check{u}_v + u_{1v}) & \text{for } \check{u}_v \leqslant \chi(g - u_{1v}), \\ \frac{1}{\lambda}\chi^{1-\beta}(\check{u}_v + \chi(u_{1v} - g))^\beta + \chi\, p(g) & \text{for } \check{u}_v > \chi(g - u_{1v}). \end{cases}$$

In the second step we use a local rectification of the boundary. To a suitable 'centre point' $\boldsymbol{x}_0$ of $\operatorname{supp}\chi_1$ let R be the affine orthogonal transformation that maps $\boldsymbol{0} \in \mathbb{R}^N$ to $\boldsymbol{x}_0$, the plane $\{(\boldsymbol{y}', 0) \mid \boldsymbol{y}' \in \mathbb{R}^{N-1}\}$ to the tangent plane of Γ_3 at $\boldsymbol{x}_0$ and the unit vector $\boldsymbol{e}_N$ to $-\boldsymbol{\nu}(\boldsymbol{x}_0)$. Let $\psi : B_{N-1}(\delta) \to \mathbb{R}$ be the function that defines the surface Γ_3 as a 'graph over its tangent plane' and $\Psi(\boldsymbol{y}', y_N) = R(\boldsymbol{y}', y_N + \psi(\boldsymbol{y}'))$. We use the transform of coordinates $\boldsymbol{x} = \Psi(\bar{\boldsymbol{x}})$ for $\bar{\boldsymbol{x}} \in O_\delta := B_{N-1}(\delta) \times (0, \delta)$. By this procedure, (4.2) is transformed to the domain $J \times O_\delta$; then all the involved functions are extended from $J \times O_\delta$ to the half-space $Q := \mathbb{R} \times \mathbb{R}^{N-1} \times \mathbb{R}_+$, either by $\boldsymbol{0}$ (as for $\check{\boldsymbol{u}}$, $\mathring{\boldsymbol{u}}$ and $\mathring{\boldsymbol{\theta}}$), or in such a way that the extended functions have the same smoothness and the same upper and lower bounds as the original functions (as e.g. the coefficients of the tensors $\mathscr{A}_1$ and $\mathscr{A}_0$). For more details see Section 5.1 of [4]. For simplicity of the presentation we denote the transformed functions and coefficients by the same symbols as before and we omit the tilde at $\tilde{p}_\lambda$ and $\tilde{f}$. This leads to a variational formulation on a half-space,

$$\begin{aligned}\langle \dot{\check{\boldsymbol{u}}}, \boldsymbol{v}\rangle_{Q,N} + \langle \mathscr{A}_1\boldsymbol{\varepsilon}(\check{\boldsymbol{u}}) + \mathscr{A}_0\boldsymbol{\varepsilon}(\mathring{\boldsymbol{u}}) + \mathring{\boldsymbol{\theta}}, \boldsymbol{\varepsilon}(\boldsymbol{v})\rangle_{Q,N\times N} + \langle p_\lambda, v_v\rangle_S + \langle \mathfrak{G}p_\lambda, \boldsymbol{v}_\tau\rangle_{S,N} \\ = \langle \boldsymbol{f}, \boldsymbol{v}\rangle_{Q,N}.\end{aligned} \tag{4.3}$$

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Here $S = \mathbb{R} \times \mathbb{R}^{N-1} \times \{0\} = \partial Q$. The coefficients of $\mathscr{A}_1$ and $\mathscr{A}_0$ now also include some factors composed of derivatives of ψ; they come from the chain rule for ∇ applied to a function of the type $\boldsymbol{u}(\Psi(\boldsymbol{x}))$. Since $|\nabla\psi| \leqslant C\delta$ holds, this modification can be made arbitrarily small by choosing δ sufficiently small. Analogously, in the definition of $\widetilde{p}_\lambda$ we include the density of surface measure $\sqrt{1+|\nabla\psi|^2}$ that comes from the transformation formula for surface integrals. The volume measure does not change for the used transformation. For simplicity of the notation we omit the tilde at $\widetilde{p}$ in what follows. Equations (4.3) is the basis of all the further considerations.

For a shift parameter $\boldsymbol{h} \in \mathbb{R}^{N-1}$ and a function $w : S \to \mathbb{R}$ let $w_{\boldsymbol{h}}(t,\boldsymbol{x}) := w(t,\boldsymbol{x}+\boldsymbol{h})$ and $\Delta_{\boldsymbol{h}} w = w_{\boldsymbol{h}} - w$. Correspondingly, for a time shift q let $w_q(t,\boldsymbol{x}) = w(t+q,\boldsymbol{x})$ and $\Delta_q w(t,\boldsymbol{x}) = w_q - w$. Denote by

$$\widehat{w}(\vartheta,\boldsymbol{\xi}) = (2\pi)^{-N/2} \int_{\mathbb{R}} \int_{\mathbb{R}^{N-1}} w(t,\boldsymbol{x})\, e^{-i(t\vartheta+\boldsymbol{x}\cdot\boldsymbol{\xi})}\, d\boldsymbol{x}\, dt,$$

the Fourier transform of w; then the norm in the space $H^{\alpha,\beta}(S)$ with $0 \leqslant \alpha, \beta < 1$ has the two representations

$$\begin{aligned} \|w\|^2_{H^{\alpha,\beta}(S)} &= \|w\|^2_{L_2(S)} + \int_{\mathbb{R}} \frac{\|\Delta_q w\|^2_{L_2(S)}}{|q|^{1+2\alpha}}\, dq + \int_{\mathbb{R}^{N-1}} \frac{\|\Delta_{\boldsymbol{h}} w\|^2_{L_2(S)}}{|\boldsymbol{h}|^{N-1+2\beta}}\, d\boldsymbol{h} \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}^{N-1}} |\widehat{w}(\vartheta,\boldsymbol{\xi})|^2 \big(1 + c_1(\alpha)|\vartheta|^{2\alpha} + c_{N-1}(\beta)|\boldsymbol{\xi}|^{2\beta}\big)\, d\boldsymbol{\xi}\, d\vartheta \end{aligned} \tag{4.4}$$

with the renormation constants (see [4], Sections 1.7.1 and 5.1)

$$c_1(\alpha) = 2^{2-2\alpha} \int_{\mathbb{R}} \frac{\sin^2 z}{|z|^{1+2\alpha}}\, dz \quad \text{and} \quad c_{N-1}(\beta) = c_1(\beta) \int_{\mathbb{R}^{N-2}} (1+|\boldsymbol{z}|^2)^{-(N-1)/2-\alpha}\, d\boldsymbol{z}.$$

For a function $w : Q \to \mathbb{R}$ and $\boldsymbol{h} \in \mathbb{R}^{N-1}$ we define *tangential* space shifts $w_{\boldsymbol{h}}(t,\boldsymbol{x}',x_N) = w(t,\boldsymbol{x}'+\boldsymbol{h},x_N)$, $\boldsymbol{x}' \in \mathbb{R}^{N-1}$, $x_N > 0$, and corresponding definitions of $\Delta_{\boldsymbol{h}} w$, w_q and $\Delta_q w$. With the partial Fourier transform

$$\widetilde{w}(\vartheta,\boldsymbol{\xi},x_N) = (2\pi)^{-N/2} \int_{\mathbb{R}} \int_{\mathbb{R}^{N-1}} w(t,\boldsymbol{x}',x_N)\, e^{-i(t\vartheta+\boldsymbol{x}'\cdot\boldsymbol{\xi})}\, d\boldsymbol{x}'\, dt,$$

the norm of the anisotropic Sobolev space $H^{\alpha,\beta,0}(Q)$ has the corresponding representation

$$\begin{aligned} \|w\|^2_{H^{\alpha,\beta,0}(Q)} &= \|w\|^2_{L_2(Q)} + \int_{\mathbb{R}} \frac{\|\Delta_q w\|^2_{L_2(Q)}}{|q|^{1+2\alpha}}\, dq + \int_{\mathbb{R}^{N-1}} \frac{\|\Delta_{\boldsymbol{h}} w\|^2_{L_2(Q)}}{|\boldsymbol{h}|^{N-1+2\beta}}\, d\boldsymbol{h} \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}_+} |\widetilde{w}(\vartheta,\boldsymbol{\xi},x_N)|^2 \big(1 + c_1(\alpha)|\vartheta|^{2\alpha} + c_{N-1}(\beta)|\boldsymbol{\xi}|^{2\beta}\big)\, dx_N\, d\boldsymbol{\xi}\, d\vartheta. \end{aligned}$$

We insert the test function $\boldsymbol{v} = \Delta_{\boldsymbol{h}}\breve{\boldsymbol{u}}$ in (4.3), $\boldsymbol{v} = -\Delta_{\boldsymbol{h}}\breve{\boldsymbol{u}}$ in the 'shifted' equation $(4.3)_{\boldsymbol{h}}$, add the results, multiply by $|\boldsymbol{h}|^{-(N-1)-2\cdot 1/2}$ and integrate with respect to $\boldsymbol{h} \in \mathbb{R}^{N-1}$. Then we insert $\boldsymbol{v} = \Delta_q\breve{\boldsymbol{u}}$ in (4.3), $\boldsymbol{v} = -\Delta_q\breve{\boldsymbol{u}}$ in the shifted formulation $(4.3)_q$, add both inequalities, multiply by $|q|^{-3/2}$, integrate with respect to $q \in \mathbb{R}$ and add the result to the equation

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obtained from the space shifts. The aim is to derive an estimate for the expression

$$\mathfrak{A} := \mathfrak{A}_1 + \mathfrak{A}_2$$

with

$$\mathfrak{A}_1 = \int_{\mathbb{R}^{N-1}} \frac{\langle \mathscr{A}_1 \boldsymbol{\varepsilon}(\Delta_{\boldsymbol{h}} \check{\boldsymbol{u}}), \boldsymbol{\varepsilon}(\Delta_{\boldsymbol{h}} \check{\boldsymbol{u}}) \rangle_{Q,N\times N}}{|\boldsymbol{h}|^N} \, d\boldsymbol{h} \quad \text{and} \quad \mathfrak{A}_2 = \int_{\mathbb{R}} \frac{\langle \mathscr{A}_1 \boldsymbol{\varepsilon}(\Delta_q \check{\boldsymbol{u}}), \boldsymbol{\varepsilon}(\Delta_q \check{\boldsymbol{u}}) \rangle_{Q,N\times N}}{|q|^{3/2}} \, dq.$$

Here, the first term $\mathfrak{A}_1$ corresponds to the semi-norm of the anisotropic Sobolev space $L_2\big(\mathbb{R}; \boldsymbol{H}^{3/2,1}\big(\mathbb{R}^{N-1} \times \mathbb{R}_+\big)\big)$, where the first index (3/2) refers to the tangential space variable, and the second one (i.e. 1) to the normal space variable. The second term $\mathfrak{A}_2$ corresponds to the semi-norm of $H^{1/4}\big(\mathbb{R}; \boldsymbol{H}^1\big(\mathbb{R}^{N-1} \times \mathbb{R}_+\big)\big)$. The estimate of $\mathfrak{A}$ follows with the help of suitable Hölder inequalities and the Lipschitz continuity of the coefficients $\mathscr{A}_1$ and $\mathscr{A}_0$ and the normal vector $\boldsymbol{\nu}$ that arises implicitly in $\check{u}_\nu$ and $\check{\boldsymbol{u}}_\tau$. Note that $\boldsymbol{\nu}$ now corresponds to the transformed 'original' normal vector, and this transformed version is close to but not equal to the normal vector $-\boldsymbol{e}_N$ of the half-space domain. For simplicity of the notation we also use

$$\mathfrak{A}_3 := \int_{\mathbb{R}^{N-1}} \frac{\langle \mathscr{A}_0 \boldsymbol{\varepsilon}(\Delta_{\boldsymbol{h}} \mathring{\boldsymbol{u}}), \boldsymbol{\varepsilon}(\Delta_{\boldsymbol{h}} \mathring{\boldsymbol{u}}) \rangle_{Q,N\times N}}{|\boldsymbol{h}|^N} \, d\boldsymbol{h},$$

which arises from the space shifts for the form $\langle \mathscr{A}_0 \boldsymbol{\varepsilon}(\mathring{\boldsymbol{u}}), \boldsymbol{\varepsilon}(\boldsymbol{v}) \rangle_{Q,N\times N}$ (the time shifts for this form lead to some bounded term) and

$$\mathfrak{J} = \int_{\mathbb{R}^{N-1}} \frac{\langle \Delta_{\boldsymbol{h}}(\mathfrak{G} p_\lambda), \Delta_{\boldsymbol{h}} \check{u}_\nu \rangle_S}{|\boldsymbol{h}|^N} \, d\boldsymbol{h} + \int_{\mathbb{R}} \frac{\langle \Delta_q(\mathfrak{G} p_\lambda), \Delta_q \check{u}_\nu \rangle_S}{|q|^{3/2}} \, dq,$$

which arises form the friction term. The term $\langle \dot{\check{\boldsymbol{u}}}, \boldsymbol{v} \rangle_{Q,N}$ leads to no contribution, because the time integrals of $\langle \Delta_{\boldsymbol{h}} \dot{\check{\boldsymbol{u}}}, \Delta_{\boldsymbol{h}} \check{\boldsymbol{u}} \rangle_{\mathbb{R}^{N-1}\times\mathbb{R}^+,N}$ and $\langle \Delta_q \dot{\check{\boldsymbol{u}}}, \Delta_q \check{\boldsymbol{u}} \rangle_{\mathbb{R}^{N-1}\times\mathbb{R}^+,N}$ vanish. From the space shifts of the penalty functional $\langle p_\lambda(\check{u}_\nu), v_\nu \rangle_S$ we obtain

$$\langle p_\lambda(\check{u}_\nu), (\check{\boldsymbol{u}}_{\boldsymbol{h}} - \check{\boldsymbol{u}}) \cdot \boldsymbol{\nu} \rangle_S + \langle (p_\lambda(\check{u}_\nu))_{\boldsymbol{h}}, (\check{\boldsymbol{u}} - \check{\boldsymbol{u}}_{\boldsymbol{h}}) \cdot \boldsymbol{\nu}_{\boldsymbol{h}} \rangle_S. \tag{4.5}$$

Since $\Gamma_{\mathfrak{F}} \in C^{1,1}$ holds, the expression (4.5) can be represented as $\langle \Delta_{\boldsymbol{h}} p_\lambda(\check{u}_\nu), \Delta_{\boldsymbol{h}} \check{u}_\nu \rangle_S$ plus some terms of lower order that contribute to some additional expressions $c_3 \|p_\lambda\|_{L_2(S)}$ and $c_4(\kappa) \|\check{\boldsymbol{u}}\|_{\boldsymbol{H}^{1/2,1}(S)}$ in the estimate (4.7) below, for more details see Section 5.1 of [4]. A similar technique is used for the treatment of the friction term.

Using now (3.12), for the viscoplastic stress tensor $\mathring{\boldsymbol{\theta}}$ we obtain

$$\begin{aligned} &\left| \int_{\mathbb{R}^{N-1}} \frac{\langle \Delta_{\boldsymbol{h}} \mathring{\boldsymbol{\theta}}, \boldsymbol{\varepsilon}(\Delta_{\boldsymbol{h}} \check{\boldsymbol{u}}) \rangle_{Q,N\times N}}{|\boldsymbol{h}|^N} \, d\boldsymbol{h} + \int_{\mathbb{R}} \frac{\langle \Delta_q \mathring{\boldsymbol{\theta}}, \boldsymbol{\varepsilon}(\Delta_q \check{\boldsymbol{u}}) \rangle_{Q,N\times N}}{|q|^{3/2}} \, dq \right| \\ &\quad \leqslant c \mathfrak{X}_2^{1/2} (\mathfrak{A}_1 + \mathfrak{A}_2)^{1/2} \end{aligned}$$

with

$$\mathfrak{X}_2 = \int_{\mathbb{R}^{N-1}} \frac{\|\Delta_{\boldsymbol{h}} \mathring{\boldsymbol{\theta}}\|^2_{L_2(Q;\mathbb{S}^N)}}{|\boldsymbol{h}|^N} \, d\boldsymbol{h} + \int_{\mathbb{R}} \frac{\|\Delta_q \mathring{\boldsymbol{\theta}}\|^2_{L_2(Q;\mathbb{S}^N)}}{|q|^{3/2}} \, dq.$$

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The part of $\mathfrak{X}_2$ that arises from time shifts is estimated via the relations $\mathring{\boldsymbol{\theta}} = \chi\boldsymbol{\theta}$ and $\dot{\boldsymbol{\theta}} = \mathscr{G}(\boldsymbol{\theta} + \mathscr{A}_0\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{u}))$; as a result we have

$$\int_{\mathbb{R}} \frac{\|\Delta_q\mathring{\boldsymbol{\theta}}\|^2_{L_2(Q;\mathbb{S}^N)}}{|q|^{3/2}}\,dq \leqslant C,$$

where the constant C depends on $\|\boldsymbol{\theta}\|_{L_2(Q_T;\mathbb{S}^N)}$ and $\|\boldsymbol{u}\|_{L_2(J;\boldsymbol{H}^1(\Omega))}$. For the estimate of the part arising from the space shifts, we apply the shift technique to the time derivative of relation (3.12):

$$\dot{\boldsymbol{\theta}} = \mathscr{G}(\boldsymbol{\theta} + \mathscr{A}_0\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{u})).$$

Then we define $\breve{\boldsymbol{\theta}} = \chi\dot{\boldsymbol{\theta}}$ and obtain

$$\breve{\boldsymbol{\theta}} = \chi\,\mathscr{G}(\boldsymbol{\theta} + \mathscr{A}_0\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{u})). \tag{4.6}$$

Next, we take the difference of the space shifts of (4.6) and the original (4.6), multiply the result by $|\boldsymbol{h}|^{-N}\Delta_{\boldsymbol{h}}\mathring{\boldsymbol{\theta}}$, and integrate with respect to $(t, \boldsymbol{x}, \boldsymbol{h})$ over $Q\times\mathbb{R}^N$. Then the right-hand side of (4.6) is estimated via

$$\begin{aligned}|\Delta_{\boldsymbol{h}}(\chi\mathscr{G}(\boldsymbol{\theta} + \mathscr{A}_0\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{u})))| &\leqslant c_1\big(|\Delta_{\boldsymbol{h}}\mathring{\boldsymbol{\theta}}| + |\boldsymbol{\varepsilon}(\Delta_{\boldsymbol{h}}\mathring{\boldsymbol{u}})| + |\boldsymbol{\varepsilon}(\Delta_{\boldsymbol{h}}(\chi\boldsymbol{u}_0))| + |\Delta_h\boldsymbol{u}|\big)\\ &\quad + c_2|\boldsymbol{h}|\big(|\boldsymbol{\theta}| + |\boldsymbol{\theta}_h| + |\boldsymbol{\varepsilon}(\boldsymbol{u})| + |\boldsymbol{\varepsilon}(\boldsymbol{u}_h)| + |\boldsymbol{u}| + 1\big).\end{aligned}$$

As a consequence we obtain

$$\int_{\mathbb{R}^N} \frac{\|\Delta_{\boldsymbol{h}}\breve{\boldsymbol{\theta}}\|^2_{L_2(Q;\mathbb{S}^N)}}{|\boldsymbol{h}|^N}\,d\boldsymbol{h} \leqslant c_1\int_{\mathbb{R}^N} \frac{\|\Delta_{\boldsymbol{h}}\mathring{\boldsymbol{\theta}}\|^2_{L_2(Q;\mathbb{S}^N)}}{|\boldsymbol{h}|^N}\,d\boldsymbol{h} + c_2\mathfrak{A}_3 + c_3$$

and, therefore,

$$\mathfrak{X}_1 \leqslant c_1\mathfrak{X}_2 + c_2\mathfrak{A}_3 + c_3$$

with

$$\mathfrak{X}_1 := \int_{\mathbb{R}^N} \frac{\|\Delta_{\boldsymbol{h}}\breve{\boldsymbol{\theta}}\|^2_{L_2(Q;\mathbb{S}^N)}}{|\boldsymbol{h}|^N}\,d\boldsymbol{h}.$$

By these briefly outlined technicalities (that are partly explained in Section 5.1 in [4]), we derive the estimate

$$\begin{aligned}\mathfrak{A}_1 + \mathfrak{A}_2 + \mathfrak{X}_1 &\leqslant (1+\kappa)\mathfrak{J} + c_1(\kappa)\mathfrak{A}_3 + c_2(\kappa)\mathfrak{X}_2\\ &\quad + c_3(\kappa)\|p_\lambda\|_{L_2(S)} + c_4(\kappa)\|\breve{\boldsymbol{u}}\|_{\boldsymbol{H}^{1/2,1}(S)} + c_5(\kappa).\end{aligned} \tag{4.7}$$

Here κ is a constant that depends on δ and on the constants in some Hölder inequalities and it can be made arbitrarily small by choosing δ sufficiently small and choosing appropriate weights in Hölder inequalities.

The crucial term now is the friction term $\mathfrak{J}$. With the help of the Fourier transform and formula (4.4), it is estimated by

$$\begin{aligned}\mathfrak{J} &\leqslant \int_{\mathbb{R}\times\mathbb{R}^{N-1}} \Big(c_{N-1}\big(\tfrac{1}{2}\big)|\xi| + c_1\big(\tfrac{1}{4}\big)|\vartheta|^{1/2}\Big)\,\mathrm{Re}\,\Big(\widehat{\mathfrak{G}p_\lambda}\overline{\widehat{\breve{\boldsymbol{u}}}_\tau}(\vartheta,\xi)\Big)\,d\xi\,d\vartheta\\ &\leqslant \|\mathfrak{G}\|_{L_\infty(S)}\|p_\lambda\|_{L_2(S)}\mathfrak{H}^{1/2}\end{aligned}$$

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with

$$\mathfrak{H} = \int_{\mathbb{R}} \int_{\mathbb{R}^{N-1}} \left(c_{N-1}\left(\tfrac{1}{2}\right)|\boldsymbol{\xi}| + c_1\left(\tfrac{1}{4}\right)|\vartheta|^{1/2} \right)^2 |\widehat{\breve{\boldsymbol{u}}}_\tau(\vartheta, \boldsymbol{\xi})|^2 \, d\boldsymbol{\xi} \, d\vartheta.$$

Here $\|\mathfrak{G}\|_{L_\infty(S)}$ is bounded by $\|\mathfrak{F}\|_{L_\infty(\Gamma_3 \times \mathbb{R}^N)}$. The term $\mathfrak{H}$ is estimated by

$$\mathfrak{H} \leqslant (1+r) \int_{\mathbb{R}^{N-1}} \frac{\|\Delta_{\boldsymbol{h}} \breve{\boldsymbol{u}}\|^2_{\boldsymbol{H}^{0,1/2}(S)}}{|\boldsymbol{h}|^N} \, d\boldsymbol{h} + \left(1 + \tfrac{1}{r}\right) \int_{\mathbb{R}} \frac{\|\Delta_q \breve{\boldsymbol{u}}\|^2_{\boldsymbol{H}^{1/4,0}(S)}}{|q|^{3/2}} \, dq$$

with arbitrary parameter $r > 0$. Now we use two special trace type estimates that are valid for solutions of the equations

$$\dot{\breve{\boldsymbol{u}}} - \mathrm{Div}(\mathscr{A}_1 \nabla \breve{\boldsymbol{u}}) = \boldsymbol{f}$$

with given boundary data on S. These estimates are (see Sections 5.1–5.3 of [4])

$$\|\breve{\boldsymbol{u}}_\tau\|'_{\boldsymbol{H}^{0,1/2}(S)} \leqslant D_1 \|\breve{\boldsymbol{u}}\|_A + c_1 \|\boldsymbol{f}\|_{H^{-1/2,-1}(Q)} \quad \text{and}$$
$$\|\breve{\boldsymbol{u}}_\tau\|'_{\boldsymbol{H}^{1/4,0}(S)} \leqslant D_2 \|\breve{\boldsymbol{u}}\|_A + c_2 \|\boldsymbol{f}\|_{H^{-1/2,-1}(Q)}.$$

Here and in the sequel by prime we denote the semi-norm in $\boldsymbol{H}^k(Q)$ for which $(\|\cdot\|'_{\boldsymbol{H}^k(Q)})^2 = \|\cdot\|^2_{\boldsymbol{H}^k(Q)} - \|\cdot\|^2_{\boldsymbol{L}_2(Q)}$ holds. Note that only the tangential component of $\breve{\boldsymbol{u}}$ arises on the left-hand side here. We use these estimates with the right-hand side

$$\boldsymbol{v} \mapsto \langle \boldsymbol{f}, \boldsymbol{v} \rangle_{Q,N} - \langle \mathscr{A}_0 \boldsymbol{\varepsilon}(\mathring{\boldsymbol{u}}), \boldsymbol{\varepsilon}(\boldsymbol{v}) \rangle_{Q,N\times N} - \langle \mathring{\boldsymbol{\theta}}, \boldsymbol{\varepsilon}(\boldsymbol{v}) \rangle_{Q,N\times N}$$

instead of $\boldsymbol{f}$. Choosing the optimal value of the parameter r we obtain

$$\mathfrak{H} \leqslant (1+\kappa)\left(D_1 \mathfrak{A}_1^{1/2} + D_2 \mathfrak{A}_2^{1/2}\right)^2 + c_1(\kappa)\mathfrak{A}_3 + c_2(\kappa)\mathfrak{X}_2 + c_3(\kappa).$$

It remains to find an estimate for $\|p_\lambda\|_{L_2(S)}$. This is done from the representation

$$\|p_\lambda\|^2_{L_2(S)} = \|p_\lambda\|_{H^{-1/4,-1/2}(S)} + \int_{\mathbb{R}^{N-1}} \frac{\|\Delta_{\boldsymbol{h}} p_\lambda\|^2_{H^{-1/4,-1/2}(S)}}{|\boldsymbol{h}|^N} \, d\boldsymbol{h} + \int_{\mathbb{R}} \frac{\|\Delta_q p_\lambda\|^2_{H^{-1/4,-1/2}(S)}}{|q|^{3/2}} \, dq.$$

We apply the shift technique to Green's formula

$$-\langle p_\lambda, w_\nu \rangle_S = \langle \dot{\breve{\boldsymbol{u}}}, \boldsymbol{w} \rangle_{Q,N} + \langle \mathscr{A}_1 \boldsymbol{\varepsilon}(\breve{\boldsymbol{u}}) + \mathscr{A}_0 \boldsymbol{\varepsilon}(\mathring{\boldsymbol{u}}) + \mathring{\boldsymbol{\theta}}, \boldsymbol{\varepsilon}(\boldsymbol{w}) \rangle_{Q,N\times N} - \langle \boldsymbol{f}, \boldsymbol{w} \rangle_{Q,N} \tag{4.8}$$

that is valid for a function $\boldsymbol{w}$ with $\boldsymbol{w}_\tau = \boldsymbol{0}$ on S. We use a special $\boldsymbol{h}$-dependent test function $\boldsymbol{w}$ defined by the solution of the problem

$$\dot{\boldsymbol{w}} - \mathrm{Div}(\mathscr{A}_1 \boldsymbol{\varepsilon}(\boldsymbol{w})) = 0 \tag{4.9}$$

on Q with boundary data

$$\widehat{w_\nu}(\vartheta, \boldsymbol{\xi}) = \widehat{\Delta_{\boldsymbol{h}} p_\lambda}(\vartheta, \boldsymbol{\xi}) \left(1 + c_1\left(\tfrac{1}{4}\right)|\vartheta|^{1/2} + c_{N-1}\left(\tfrac{1}{2}\right)|\boldsymbol{\xi}|\right)^{-1}.$$

The boundary data satisfy

$$\|w_\nu\|^2_{H^{1/4,1/2}(S)} = \|\Delta_{\boldsymbol{h}} p_\lambda\|^2_{H^{-1/4,-1/2}(S)} = \langle \Delta_{\boldsymbol{h}} p_\lambda, w_\nu \rangle_S.$$

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Moreover, using the arguments in Sections 5.1–5.3 of [4] it follows that the solutions of (4.9) satisfy the estimate

$$\langle \dot{\boldsymbol{w}}, \boldsymbol{v}\rangle_{Q;N} + \langle \mathscr{A}_1 \boldsymbol{\varepsilon}(\boldsymbol{w}), \boldsymbol{\varepsilon}(\boldsymbol{v})\rangle_{Q,N\times N} \leqslant \big(D_0 \langle \mathscr{A}_1 \boldsymbol{\varepsilon}(\boldsymbol{w}), \boldsymbol{\varepsilon}(\boldsymbol{w})\rangle_{Q,N\times N}^{1/2},$$

for every test function $\boldsymbol{v}$ such that $\boldsymbol{v}_\tau = \boldsymbol{0}$ on S, with some positive constant D_0. Using the test function $\boldsymbol{v} = \boldsymbol{w}$ in the space difference (4.8)–(4.8)$_{\boldsymbol{h}}$ of the Green formula, multiplying the result by $|\boldsymbol{h}|^{-N}$ and integrating with respect to $\boldsymbol{h} \in \mathbb{R}^{N-1}$ we obtain

$$\int_{\mathbb{R}^{N-1}} \frac{\|\Delta_{\boldsymbol{h}} p_\lambda\|^2_{H^{-1/4,-1/2}(S)}}{|\boldsymbol{h}|^N}\, d\boldsymbol{h} \leqslant (1+\kappa) D_0^2 \mathfrak{A}_1 + c_1 \mathfrak{A}_3 + c_2 \mathfrak{X}_2 + c_3.$$

Analogously, using the q-dependent test function defined by

$$\widehat{w_\nu}(\vartheta, \boldsymbol{\xi}) = \widehat{\Delta_q p_\lambda}(\vartheta, \boldsymbol{\xi}) \big(1 + c_1\big(\tfrac{1}{4}\big)|\vartheta|^{1/2} + c_{N-1}\big(\tfrac{1}{2}\big)|\boldsymbol{\xi}|\big)^{-1}$$

in the time difference (4.8)–(4.8)$_q$, multiplying the result by $|q|^{-3/2}$ and integrating with respect to q yields

$$\int_{\mathbb{R}} \frac{\|\Delta_q p_\lambda\|^2_{H^{-1/4,-1/2}(S)}}{|q|^{3/2}}\, dq \leqslant (1+\kappa) D_0^2 \mathfrak{A}_2 + c_1.$$

The combination of the last two inequalities yields

$$\|p_\lambda\|^2_{L_2(S)} \leqslant (1+\kappa) D_0^2 (\mathfrak{A}_1 + \mathfrak{A}_2) + c_1 \mathfrak{A}_3 + c_2 \mathfrak{X}_2 + c_3.$$

With this estimate the relation (4.7) is modified to

$$\begin{aligned} \mathfrak{A}_1 + \mathfrak{A}_2 + \mathfrak{X}_1 \leqslant{}& (1+\kappa) \|\mathfrak{F}\|_{L_\infty(\Gamma_3\times\mathbb{R}^N)} D_0 \sqrt{\mathfrak{A}_1 + \mathfrak{A}_2} \big(D_1 \sqrt{\mathfrak{A}_1} + D_2 \sqrt{\mathfrak{A}_2}\big) \\ &+ c_1 \mathfrak{A}_3 + c_2 \mathfrak{X}_2 + c_3. \end{aligned}$$

This implies that

$$\mathfrak{A}_1 + \mathfrak{A}_2 + \mathfrak{X}_1 \leqslant (1+\kappa) \|\mathfrak{F}\|_{L_\infty(\Gamma_3\times\mathbb{R}^N)} C_{\mathfrak{F}}^{-1} (\mathfrak{A}_1 + \mathfrak{A}_2) + c_1 \mathfrak{A}_3 + c_2 \mathfrak{X}_2 + c_3,$$

with

$$C_{\mathfrak{F}}^{-1} = D_0 \sqrt{D_1^2 + D_2^2}. \tag{4.10}$$

If (4.1) holds and κ is chosen sufficiently small, then

$$\mathfrak{A}_1 + \mathfrak{A}_2 + \mathfrak{X}_1 \leqslant c_1 \mathfrak{A}_3 + c_2 \mathfrak{X}_2 + c_3. \tag{4.11}$$

The last task is the estimate of $\mathfrak{A}_3$ and $\mathfrak{X}_2$. Here the special form of the time dependent cut-off function $\chi(t, \boldsymbol{x}) = \chi_{0,0}(t - L)\chi_1(\boldsymbol{x})$ used for the localization is essential. Actually $\mathfrak{A}_1$, $\mathfrak{A}_2$, $\mathfrak{X}_1$, $\mathfrak{A}_3$, $\mathfrak{X}_2$ depend implicitly on the time parameter L. From the formula $\frac{\partial}{\partial L}\chi = -\frac{\partial}{\partial t}\chi$ and the definitions $\breve{\boldsymbol{u}} = \chi(\dot{\boldsymbol{u}} - \boldsymbol{u}_1)$ and $\mathring{\boldsymbol{u}} = \chi(\boldsymbol{u} - \boldsymbol{u}_0)$ we obtain, with the help of integration

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by parts and the notation $O = \mathbb{R}^{N-1} \times \mathbb{R}_+$ for the half-space

$$\begin{aligned}
&\frac{d}{dL}\mathfrak{A}_3(L) \\
&\quad = \int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}} \frac{\partial}{\partial L}(\chi_{0,0})^2 \frac{\langle \mathscr{A}_0 \boldsymbol{\varepsilon}(\Delta_{\boldsymbol{h}}(\chi_1(\boldsymbol{u}-\boldsymbol{u}_0))), \boldsymbol{\varepsilon}(\Delta_{\boldsymbol{h}}(\chi_1(\boldsymbol{u}-\boldsymbol{u}_0)))\rangle_{O,N\times N}}{|\boldsymbol{h}|^N}\, dt\, d\boldsymbol{h} \\
&\quad = \int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}} 2(\chi_{0,0})^2 \frac{\langle \mathscr{A}_0 \boldsymbol{\varepsilon}(\Delta_{\boldsymbol{h}}(\chi_1 \dot{\boldsymbol{u}})), \boldsymbol{\varepsilon}(\Delta_{\boldsymbol{h}}(\chi_1(\boldsymbol{u}-\boldsymbol{u}_0)))\rangle_{O,N\times N}}{|\boldsymbol{h}|^N}\, dt\, d\boldsymbol{h} \\
&\quad \leqslant 2\big(\mathfrak{A}_1^{1/2}(L) + c_1(L)\big)\mathfrak{A}_3^{1/2}(L).
\end{aligned} \tag{4.12}$$

Moreover, the mapping $L \mapsto \mathfrak{A}_3(L)$ is non-decreasing on $[0, T]$ and $\mathfrak{A}_3(0) = 0$. Hence, the integration of (4.12) over $[0, L]$ yields

$$\mathfrak{A}_3(L) \leqslant c_1 \int_0^L (\mathfrak{A}_1(\ell) + \mathfrak{A}_3(\ell))\, d\ell + c_2$$

for all $0 < L < T$. Analogously we obtain

$$\mathfrak{X}_2(L) \leqslant c_3 \int_0^L (\mathfrak{X}_1(\ell) + \mathfrak{X}_2(\ell))\, d\ell + c_2,$$

for all $0 < L < T$. With these relations and the Gronwall lemma we obtain from (4.11)

$$\mathfrak{A}_1 + \mathfrak{A}_2 + \mathfrak{X}_1 \leqslant c.$$

The term $\mathfrak{A}_1 + \mathfrak{A}_2$ is equivalent to the norm $\|\chi \dot{\boldsymbol{u}}\|_{H^{3/4,3/2,1}(J\times\Omega_{\mathfrak{F}})}$, where the first, second and third index corresponds to the regularity with respect to time, tangential space variable, and normal space variable, respectively. Analogously, $\mathfrak{X}_2$ is equivalent to the $H^{1/4,1/2}(J \times \Omega_{\mathfrak{F}})$–norm of $\chi\boldsymbol{\theta}$. Therefore, we have proved the required time regularity and the tangential space regularity for both $\dot{\boldsymbol{u}}$ and $\boldsymbol{\theta}$ on $Q_{\mathfrak{F}} := J \times \Omega_{\mathfrak{F}}$, with some possibly small neighbourhood $\Omega_{\mathfrak{F}}$ of the boundary Γ. The proof is finished by an application of the trace theorem.

5 Main existence result

In this section we perform the limit procedure as the smoothing parameter η and the penalty parameter λ converge to zero. We assume that the assumptions of Propositions 1 and 5 are still valid. From the *a priori* estimates of Propositions 3 and 5 we conclude that there exists a sequence of solution $\boldsymbol{u}_m = \boldsymbol{u}_{\lambda_m\eta_m}$ of Problem $\mathscr{P}^V_{\lambda_m,\eta_m}$ with approximation parameters $\lambda_m \to 0$, $\eta_m \to 0$ as $m \to +\infty$, such that $\boldsymbol{u}_m$ has the following convergence properties:

$$\begin{aligned}
\boldsymbol{u}_m &\rightharpoonup \boldsymbol{u} && \text{in } H^1(J; \boldsymbol{H}^1(\Omega)), \\
\boldsymbol{u}_m(T) &\rightharpoonup \boldsymbol{u}(T) && \text{in } \boldsymbol{H}^1(\Omega), \\
\dot{\boldsymbol{u}}_m(T) &\rightharpoonup \dot{\boldsymbol{u}}(T) && \text{in } \boldsymbol{L}_2(\Omega), \\
\dot{\boldsymbol{u}}_m &\rightharpoonup \dot{\boldsymbol{u}} && \text{in } \boldsymbol{H}^{1/2,1}(S_{\mathfrak{F},T}) \text{ and strongly in } \boldsymbol{L}_p(S_{\mathfrak{F},T}) \text{ for } p < \tfrac{2(N+1)}{N-1} \text{ and} \\
\ddot{\boldsymbol{u}}_m &\rightharpoonup \ddot{\boldsymbol{u}} && \text{in } L_2\big(J; \boldsymbol{H}^1_0(\Omega)^*\big).
\end{aligned}$$

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By standard techniques of convex analysis, these convergence properties imply

$$\liminf_{m\to+\infty}\langle\ddot{\boldsymbol{u}}_m,\dot{\boldsymbol{u}}_m\rangle_{Q_T,N}\geqslant\langle\ddot{\boldsymbol{u}},\dot{\boldsymbol{u}}\rangle_{Q_T,N},$$
$$\liminf_{m\to+\infty}\langle\mathscr{A}_1\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_m),\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_m)\rangle_{Q_T,N\times N}\geqslant\langle\mathscr{A}_1\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}),\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}})\rangle_{Q_T,N\times N},$$
$$\liminf_{m\to+\infty}\langle\mathscr{A}_0\boldsymbol{\varepsilon}(\boldsymbol{u}_m),\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_m)\rangle_{Q_T,N\times N}\geqslant\langle\mathscr{A}_0\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}})\rangle_{Q_T,N\times N}.$$

From the definition of p_λ and the a priori estimate $\|p_{\lambda_m}(\dot{u}_{mv})\dot{u}_{mv}\|_{L_1(S_{3,T})}\leqslant c_1$ we conclude that $\dot{u}_v-g\leqslant 0$ almost everywhere and, therefore, $\dot{\boldsymbol{u}}\in\mathscr{K}$. The strong convergence $\dot{\boldsymbol{u}}_m\to\dot{\boldsymbol{u}}$ in $\boldsymbol{L}_2(S_{3,T})$ implies $p(u_{mv})\to p(u_v)$ in $L_2(S_{3,T})$ and $\Phi_{\eta_m}((\dot{\boldsymbol{u}}_m)_\tau)\to|\dot{\boldsymbol{u}}_\tau|$ in $L_r(S_{\mathfrak{F},T})$ for any $r<+\infty$. Using also the monotonicity of p_λ we obtain

$$\liminf_{m\to+\infty}\langle p_\lambda(\dot{u}_{mv})-p(\dot{u}_v),\dot{u}_{mv}-\dot{u}_v\rangle_{S_{3,T}}\geqslant 0.$$

We now concentrate on the limit of the viscoplastic stress. Let $\boldsymbol{\zeta}_m$, $\boldsymbol{\zeta}$ be defined as the solutions of the integral equations

$$\boldsymbol{\zeta}_m(t)=\mathscr{A}_0\boldsymbol{\varepsilon}(\boldsymbol{u}_m(t))+\int_0^t\mathscr{G}(\boldsymbol{\zeta}_m(s),\boldsymbol{\varepsilon}(\boldsymbol{u}_m(s)))\,ds\quad\text{and}$$
$$\boldsymbol{\zeta}(t)=\mathscr{A}_0\boldsymbol{\varepsilon}(\boldsymbol{u}(t))+\int_0^t\mathscr{G}(\boldsymbol{\zeta}(s),\boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\,ds.$$

The existence of these solutions and the property $\boldsymbol{\zeta}_m,\boldsymbol{\zeta}\in L_2\big(Q_T;\mathbb{S}^N\big)$ follows from Proposition 2 (in fact $\boldsymbol{\zeta}=\boldsymbol{\theta}+\mathscr{A}_0\boldsymbol{\varepsilon}(\boldsymbol{u})$). We obtain

$$\begin{aligned}\|(\boldsymbol{\zeta}_m-\boldsymbol{\zeta})(t)\|_{L_2(\Omega;\mathbb{S}^N)}&\leqslant c_1\|\boldsymbol{\varepsilon}(\boldsymbol{u}_m-\boldsymbol{u})(t)\|_{L_2(\Omega;\mathbb{S}^N)}\\&\quad+c_2\int_0^t\big(\|(\boldsymbol{\zeta}_m-\boldsymbol{\zeta})(s)\|_{L_2(\Omega;\mathbb{S}^N)}+\|\boldsymbol{\varepsilon}(\boldsymbol{u}_m-\boldsymbol{u})(s)\|_{L_2(\Omega;\mathbb{S}^N)}\big)\,ds\end{aligned}$$

and, by using the Gronwall lemma, we find that

$$\|(\boldsymbol{\zeta}_m-\boldsymbol{\zeta})(t)\|_{L_2(\Omega;\mathbb{S}^N)}\leqslant c_3\|\boldsymbol{\varepsilon}(\boldsymbol{u}_m-\boldsymbol{u})(t)\|_{L_2(\Omega;\mathbb{S}^N)}+c_4\|\boldsymbol{\varepsilon}(\boldsymbol{u}_m-\boldsymbol{u})\|_{L_2(Q_t;\mathbb{S}^N)}.\tag{5.1}$$

We write now the variational equation (3.14) as a variational inequality

$$\begin{aligned}&\langle\ddot{\boldsymbol{u}}_m,\boldsymbol{v}-\dot{\boldsymbol{u}}_m\rangle_{Q_T,N}+\langle\mathscr{A}_1\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_m)+\boldsymbol{\zeta}_m,\boldsymbol{\varepsilon}(\boldsymbol{v}-\dot{\boldsymbol{u}}_m)\rangle_{Q_T,N\times N}\\&\quad+\langle p_{\lambda_m}(\dot{u}_{mv}),v_v-\dot{u}_{mv}\rangle_{S_{3,T}}+\langle\mathfrak{F}(\dot{\boldsymbol{u}}_m)p_{\lambda_m}(\dot{u}_{mv}),\Phi_{\eta_m}(\boldsymbol{v}_\tau)-\Phi_{\eta_m}(\dot{\boldsymbol{u}}_{m\tau})\rangle_{S_{3,T}}\\&\geqslant\langle\boldsymbol{f},\boldsymbol{v}-\dot{\boldsymbol{u}}_m\rangle_{Q_T,N}\quad\forall\,\boldsymbol{v}\in\mathscr{K}.\end{aligned}\tag{5.2}$$

Using there the test function $\boldsymbol{v}=\chi_t\dot{\boldsymbol{u}}$ with the indicator function χ_t of the time interval $(0,t)$ and rearranging some terms we obtain

$$\begin{aligned}\langle\mathscr{A}_1\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_m-\dot{\boldsymbol{u}}),\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_m-\dot{\boldsymbol{u}})\rangle_{Q_t,N\times N}&\leqslant\langle\mathscr{A}_1\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}),\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}-\dot{\boldsymbol{u}}_m)\rangle_{Q_t,N\times N}\\&\quad+\langle\ddot{\boldsymbol{u}}_m,\dot{\boldsymbol{u}}-\dot{\boldsymbol{u}}_m\rangle_{Q_t,N}+\langle\boldsymbol{\zeta}_m-\boldsymbol{\zeta},\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}-\dot{\boldsymbol{u}}_m)\rangle_{Q_t,N\times N}+\langle\boldsymbol{\zeta},\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}-\dot{\boldsymbol{u}}_m)\rangle_{Q_t,N\times N}\\&\quad+\langle p_{\lambda_m}(\dot{u}_{mv}),\dot{u}_v-\dot{u}_{mv}\rangle_{S_{3,t}}+\langle\mathfrak{F}(\dot{\boldsymbol{u}}_m)p_{\lambda_m}(\dot{u}_{mv}),\Phi_{\eta_m}(\dot{\boldsymbol{u}}_\tau)-\Phi_{\eta_m}(\dot{\boldsymbol{u}}_{m\tau})\rangle_{S_{3,t}}\\&\quad+\langle\boldsymbol{f},\dot{\boldsymbol{u}}_m-\dot{\boldsymbol{u}}\rangle_{Q_t,N}.\end{aligned}$$

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According to the convergence properties of $\boldsymbol{u}_m$, the upper limit of every term on the right-hand side is smaller than or equal to zero, with the only exception of $\langle \boldsymbol{\zeta}_m - \boldsymbol{\zeta}, \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}} - \dot{\boldsymbol{u}}_m)\rangle_{Q_t,N\times N}$. Using the estimate

$$\langle \boldsymbol{\zeta}_m - \boldsymbol{\zeta}, \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_m - \dot{\boldsymbol{u}})\rangle_{Q_t,N\times N} \leqslant \kappa \|\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}} - \dot{\boldsymbol{u}})\|_{L_2(Q_t;\mathbb{S}^N)} + C(\kappa)\|\boldsymbol{\zeta}_m - \boldsymbol{\zeta}\|_{L_2(Q_t;\mathbb{S}^N)}$$

and relation (5.1) we derive

$$\|\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_m - \dot{\boldsymbol{u}})\|^2_{L_2(Q_t;\mathbb{S}^N)} \leqslant C_m(t) + c_1\|\boldsymbol{\varepsilon}(\boldsymbol{u}_m - \boldsymbol{u})\|^2_{L_2(Q_t;\mathbb{S}^N)},$$

with $C_m(t) \to 0$ as $m \to +\infty$. Due to equality $\boldsymbol{u}(t) = \boldsymbol{u}_0 + \int_0^t \dot{\boldsymbol{u}}(s)\,ds$ we find that

$$\|\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_m - \dot{\boldsymbol{u}})\|^2_{L_2(Q_t;\mathbb{S}^N)} \leqslant C_m(t) + c_1 t\|\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_m - \dot{\boldsymbol{u}})\|^2_{L_2(Q_t;\mathbb{S}^N)}.$$

Therefore, for a sufficiently small t we get

$$\|\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_m - \dot{\boldsymbol{u}})\|_{L_2(Q_t;\mathbb{S}^N)} \to 0 \quad \text{as } t \to 0.$$

This consideration can be repeated iteratively to prove convergence $\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_m) \to \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}})$ in $L_2((nt,(n+1)t)\times\Omega;\mathbb{S}^N)$ for $n = 1,2,\ldots$ and, therefore, in $L_2(Q_T;\mathbb{S}^N)$. This result also implies that $\boldsymbol{\varepsilon}(\boldsymbol{u}_m) \to \boldsymbol{\varepsilon}(\boldsymbol{u})$ and $\boldsymbol{\zeta}_m \to \boldsymbol{\zeta}$ in $L_2(Q_T;\mathbb{S}^N)$. Moreover, we have

$$\langle \ddot{\boldsymbol{u}}_m, \boldsymbol{v}\rangle_{Q_T,N} + \langle \mathscr{A}_1\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_m) + \boldsymbol{\zeta}_m, \boldsymbol{\varepsilon}(\boldsymbol{v})\rangle_{Q_T,N\times N} + \langle p_\lambda(\dot{u}_{mv}), v_v\rangle_{S_{3,T}} = \langle \boldsymbol{f}, \boldsymbol{v}\rangle_{Q_T,N}$$

for an arbitrary test function $\boldsymbol{v}$ with $\boldsymbol{v}_\tau = \boldsymbol{0}$ on S. Passing to the limit in this equation as $m \to +\infty$, we obtain $\lim_{m\to+\infty} p_\lambda(\dot{u}_{mv}) = -\sigma_v(\boldsymbol{u})$, where $\sigma_v(\boldsymbol{u}) = \boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{v}\cdot\boldsymbol{v}$ and $\boldsymbol{\sigma}(\boldsymbol{u}) = \mathscr{A}_1\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + \boldsymbol{\zeta}$. Finally, passing to the limit as $m \to +\infty$ in (5.2), we find that $\boldsymbol{u}$ is the solution to the variational inequality

$$\begin{aligned}\langle \ddot{\boldsymbol{u}}, \boldsymbol{v} - \dot{\boldsymbol{u}}\rangle_{Q_T,N} + \langle \boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v} - \dot{\boldsymbol{u}})\rangle_{Q_T,N\times N} + \langle p(\dot{u}_v), v_v - \dot{u}_v\rangle_{S_{3,T}} \\ + \langle \mathfrak{F}(\dot{\boldsymbol{u}})|\sigma_v|, |\boldsymbol{v}_\tau| - |\dot{\boldsymbol{u}}_\tau|\rangle_{S_{3,T}} = \langle \boldsymbol{f}, \boldsymbol{v} - \dot{\boldsymbol{u}}\rangle_{Q_T,N} \quad \forall\, \boldsymbol{v} \in L_2(J;K).\end{aligned} \tag{5.3}$$

In conclusion, we have established the following theorem.

Theorem 6 *Assume that conditions* (2.8)–(2.13), (2.17) *and Assumption 4 are satisfied. Then Problem* $\mathscr{P}^V$ *has at least one solution. Moreover, the solution satisfies*

$$\dot{\boldsymbol{u}} \in \boldsymbol{H}^{1/2,1}(Q_T) \quad \text{and} \quad \boldsymbol{\sigma} \in L_2\big(Q_T;\mathbb{S}^N\big). \tag{5.4}$$

6 Conclusion

We have proved the existence of a weak solution to a contact problem with Coulomb friction for an elastic-visco-plastic body with a contact law that combines a normal damped response condition and a unilateral condition for the velocities. The proof of this theorem was based on a fixed-point technique for the computation of the plastic stress and a regularity result that was proved by using the local rectification of the boundary and

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the translation technique. The theorem is valid, if the coefficient of friction is smaller than a certain bound that depends on the *viscous* properties of the material, i.e. on the viscosity tensor $\mathscr{A}_1$. This upper bound has been calculated in [4] for general viscoelastic materials and for isotropic viscoelastic materials. The bound for general viscoelastic materials is given there in terms of the lower and upper bound of the viscosity tensor and has values around 1/4 at most. The bound for isotropic viscoelastic materials is much bigger, with values slightly below 1/2, in the best cases. There is also a possibility to compute bounds for orthotropic material, as done for the static problem in [3]. In this case the upper bound can be even larger than one. The question of the uniqueness of the solution is open.

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References

[1] Cristescu, N. & Suliciu, I. (1982) *Viscoplasticity*, Martinus Nijhoff, Editura Tehnica, Bucharest.
[2] Eck, C. (2002) Existence of solutions to a thermo-viscoelastic contact problem with Coulomb friction. *Math. Mod. Methods Appl. Sci.* **12**, 1491–1511.
[3] Eck, C. & Jarušek, J. (2008) Solvability of static contact problems with Coulomb friction for orthotropic material. *J. Elasticity* **93**, 93–104.
[4] Eck, C., Jarušek, J. & Krbec, M. (2005) *Unilateral Contact Problems: Variational Methods and Existence Theorems*, Pure and Applied Mathematics **270**, Chapman/CRC Press, New York.
[5] Han, W. & Sofonea, M. (2002) *Quasistatic Contact Problems in Viscoelasticity and Viscoplasticity*, Studies in Advanced Mathematics **30**, American Mathematical Society, International Press.
[6] Ionescu, I. R. & Sofonea, M. (1993) *Functional and Numerical Methods in Viscoplasticity*, Oxford University Press, Oxford.
[7] Jarušek, J. (1990) Contact problem with given time-dependent friction force in linear viscoelasticity. *Comment. Math. Univ. Carolinae* **31**, 257–262.
[8] Jarušek, J. (1996) Dynamic contact problems with given friction for viscoelastic bodies. *Czech. Math. J.* **46**, 475–487.
[9] Jarušek, J. & Eck, C. (1999) Dynamic contact problems with small Coulomb friction for viscoelastic bodies. Existence of solutions. *Math. Mod. Methods Appl. Sci.* **9**, 11–34.
[10] Jarušek, J. & Sofonea, M. (2008) Dynamic contact problems with small Coulomb friction for viscoelastic bodies. *ZAMM* **88**, 3–22.
[11] Kuttler, K. L. (2007) Dynamic frictional contact for elastic viscoplastic material. *Electron. J. Differ. Equ.* **75**, 11–20.
[12] Shillor, M., Sofonea, M. & Telega, J. J. (2004) *Models and Analysis of Quasistatic Contact*, Lecture Notes in Physics **655**, Springer, Berlin.
[13] Sofonea, M., Renon, N. & Shillor, M. (2004) Stress formulation for frictionless contact of an elastic-perfectly-plastic body. *Appl. Anal.* **83**, 1157–1170.

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