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Electromagnetic zonal flow residual responses

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The collisionless axisymmetric zonal flow residual calculation for a tokamak plasma is generalized to include electromagnetic perturbations. We formulate and solve the complete initial value zonal flow problem by retaining the fully self-consistent axisymmetric spatial perturbations in the electric and magnetic fields. Simple expressions for the electrostatic, shear and compressional magnetic residual responses are derived that provide a fully electromagnetic test of the zonal flow residual in gyrokinetic codes. Unlike the electrostatic potential, the parallel vector potential and the parallel magnetic field perturbations need not relax to flux functions for all possible initial conditions.

Key words: fusion plasma, magnetized plasmas, plasma dynamics

1. Introduction

A zonal flow is a sheared flow generated by turbulence that has small scale structure compared to the system size in the radial direction and is global in extent in the other directions. In a tokamak an electrostatic zonal flow appears as a radially varying electric field drift due to a radial electric field with rapid radial variation, but with no toroidal variation. It helps reduce and regulate the turbulent transport level in tokamaks through shear-enhanced decorrelation of turbulent structures (Biglari, Diamond & Terry 1990; Terry 2000). Its importance was discovered when a discrepancy between gyrokinetic and gyrofluid descriptions of ion temperature gradient (ITG) turbulence was observed in the earliest nonlinear, electrostatic, δf, flux-tube, particle-in-cell code (now called PG3EQ) (Dimits *et al.* 1996), where δf is the perturbation away from a Maxwellian. The key role of zonal flow in controlling and reducing ITG turbulent transport, especially near marginal stability, soon became apparent. Insights into zonal flow behaviour (missed in early gyrofluid codes) came from code simulations and comparisons (Dimits *et al.* 2000), leading to an understanding that there was a nonlinear Dimits shift away from the ITG linear stability threshold (Dimits *et al.* 1996). Rosenbluth & Hinton (1998) developed an electrostatic analytic check to show that the zonal flow damps to a non-zero residual level in a collisionless, axisymmetric plasma due to polarization effects associated with the magnetic drifts.

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The standard zonal flow residual calculations are electrostatic and assume axisymmetry is maintained. An initial value problem is solved to find the residual zonal flow level once any initial poloidal angle dependence in the electrostatic potential is temporally damped away via the geodesic acoustic mode (GAM) (Winsor, Johnson & Dawson 1968). The initial condition must normally be chosen to depend on poloidal angle to generate a transiently evolving zonal flow or else it will be a homogeneous solution to the non-transit averaged drift kinetic or gyrokinetic equation. The initial distribution function is not allowed to depend on gyrophase since it is assumed that any initial transient response associated with the fast gyromotion has already damped to its classical polarization level. The residual zonal flow level has proven to be an important electrostatic test of gyrokinetic codes in general (Dimits *et al.* 2000) and the GS2 code in particular (Xiao, Catto & Dorland 2007*a*), including collisional damping (Hinton & Rosenbluth 1999; Xiao, Catto & Molvig 2007*b*), short wavelength effects (Jenko *et al.* 2000; Xiao & Catto 2006*b*) and the effect of shaping (Belli 2006; Xiao & Catto 2006*a*).

In subsequent sections we generalize this axisymmetric electrostatic model to its fully electromagnetic counterpart for a tokamak. Unlike the procedure of Terry *et al.* (2013), which treats the effect of an externally imposed, stationary (non-evolving) and non-axisymmetric radial magnetic field perturbation on an equilibrium, we formulate and solve a description retaining the fully self-consistent axisymmetric spatial perturbations in the magnetic field. The poloidal dependence of the perturbed shear and compressional magnetic field perturbations are retained as drives in the kinetic equation, quasineutrality, and the parallel and perpendicular components of Ampère's law. The system of equations are then solved to obtain the complete self-consistent response within a Vlasov–Maxwell description. The expressions obtained by solving this initial value problem provide 15 fully electromagnetic tests of the zonal flow residual in gyrokinetic codes. The poloidally dependent initial conditions for the fields and the distribution function are chosen to satisfy quasineutrality and Ampère's law at $t = 0$. We assume any GAM behaviour due to poloidal variation has damped away so that only the residual zonal flow levels are obtained. Importantly, the residual zonal flow levels must allow for poloidal variation of the parallel vector potential and parallel magnetic field perturbations for all initial conditions. The description is general enough that even in the absence of any initial electrostatic perturbation, a magnetic perturbation is able to generate a zonal flow response.

The subsequent sections are organized as follows. First, in § 2 we specify the representations of the perturbed and unperturbed fields. The kinetic equation is given in § 3, then a suitable initial condition in terms of perturbed fields and distribution function is chosen in § 4. The system is closed with Maxwell's equations and solved in § 5. Approximate expressions for the zonal flow responses in the various fields are given in § 6, before we briefly summarize our results in § 7.

2. Potentials, fields and currents

The standard electrostatic zonal flow residual calculation (Rosenbluth & Hinton 1998) assumes axisymmetry is preserved during the time evolution of the zonal flow. We seek to generalize this model to its fully electromagnetic counterpart in a tokamak, assuming that the magnetic field remains axisymmetric at all times. Unperturbed quantities are assumed to evolve slowly compared to the zonal flow relaxation.

The total magnetic field is

$$\boldsymbol{B} = \boldsymbol{B}_0 + \boldsymbol{B}_1, \tag{2.1}$$

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where

$$\boldsymbol{B}_0 = I(\psi)\nabla\zeta + \nabla\zeta \times \nabla\psi = B_0\boldsymbol{b} \tag{2.2}$$

is the background axisymmetric magnetic field and $\boldsymbol{B}_1$ is the perturbed magnetic field. Here $I = I(\psi)$ must be a flux function to make the unperturbed radial current density vanish, $2\pi\psi$ is the unperturbed poloidal flux, $R\nabla\zeta = \hat{\zeta}$ is the toroidal unit vector, with R the major radius, $B_0 = |\boldsymbol{B}_0|$ is the magnitude of the unperturbed magnetic field, and $\boldsymbol{b} = \boldsymbol{B}_0/B_0$ is the unit vector in the direction of the unperturbed magnetic field.

We start by representing $\boldsymbol{B}_1$ in a form convenient to derive the gyroaveraged kinetic equation,

$$\boldsymbol{B}_1 = \nabla \times \boldsymbol{A}, \tag{2.3}$$

where

$$\boldsymbol{A} = A_\parallel \boldsymbol{b} + A_\perp \boldsymbol{e}_\perp + A_\times \boldsymbol{e}_\times. \tag{2.4}$$

Here $\boldsymbol{e}_\perp = \nabla\psi/RB_p$, $\boldsymbol{e}_\times = \boldsymbol{b} \times \boldsymbol{e}_\perp$, $|\nabla\psi| = RB_p$ and B_p is the unperturbed poloidal magnetic field. We assume that the characteristic length scale of $\boldsymbol{B}_1$ perpendicular to the background magnetic field $\boldsymbol{B}_0$ is small compared to the characteristic size of the device. Since we are considering zonal components, the perpendicular gradient is mainly in the radial direction. To describe this rapid radial variation of the perturbed fields, we use the eikonal form

$$\{A_\parallel, A_\perp, A_\times\} = \{\tilde{A}_\parallel, \tilde{A}_\perp, \tilde{A}_\times\} \exp[\mathrm{i}S(\psi)], \tag{2.5}$$

with $\nabla S = S'\nabla\psi = k_\perp \boldsymbol{e}_\perp$. The coefficients with tilde are functions of time and are only allowed to be slow functions of ψ and θ, for example, varying as $\cos\theta$. Using (2.3), (2.4) and (2.5), we obtain

$$\boldsymbol{B}_1 \simeq B_\parallel \boldsymbol{b} - \mathrm{i}k_\perp A_\parallel \boldsymbol{e}_\times, \tag{2.6}$$

where $B_\parallel = \mathrm{i}k_\perp A_\times$ is the parallel component of $\boldsymbol{B}_1$. The form for $\boldsymbol{B}_1$ in (2.6) is the most common way to express the perturbed magnetic field in gyrokinetic simulations. Note that the component $A_\perp$ of the vector potential never appears in the final expression for $\boldsymbol{B}_1$, and can be safely ignored.

The form for $\boldsymbol{B}_1$ in (2.6) ensures that the magnetic field $\boldsymbol{B}_1$ is axisymmetric. We can make this more explicit by showing that (2.6) is equivalent to the axisymmetric form

$$\boldsymbol{B}_1 = \delta\nabla\zeta - \nabla\zeta \times \nabla A, \tag{2.7}$$

where $\delta(\psi, \theta)$, which need not be a flux function, is the perturbation to $I(\psi)$, and $-A(\psi, \theta)$ is the perturbation to ψ. To match (2.6) and (2.7), we must realize that A has an eikonal form similar to those in (2.5). Thus,

$$\nabla A \simeq \frac{\mathrm{i}k_\perp A}{RB_p}\nabla\psi, \tag{2.8}$$

and as a result, equation (2.7) gives

$$\boldsymbol{B}_1 \simeq \delta\nabla\zeta - \frac{\mathrm{i}k_\perp A}{RB_p}\nabla\zeta \times \nabla\psi. \tag{2.9}$$

Equation (2.9) proves that $\boldsymbol{B}_1$ is, to lowest order in $(k_\perp a)^{-1} \ll 1$, parallel to the flux surface. Then, it is easy to obtain the relations between δ, A, $B_\parallel$ and $A_\parallel$. From (2.6),

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we deduce that $B_\parallel = \boldsymbol{B}_1 \cdot \boldsymbol{b}$ and $A_\parallel = -\boldsymbol{B}_1 \cdot \boldsymbol{e}_\times / \mathrm{i}k_\perp$. Substituting into these two equations the form for $\boldsymbol{B}_1$ given in (2.9), and using

$$\boldsymbol{e}_\times \cdot \nabla \zeta = -\frac{B_p}{RB_0} \tag{2.10}$$

and

$$\boldsymbol{e}_\times \cdot (\nabla \zeta \times \nabla \psi) = \frac{IB_p}{RB_0}, \tag{2.11}$$

we obtain

$$B_\parallel = \boldsymbol{B}_1 \cdot \boldsymbol{b} = \frac{I}{R^2 B_0}\delta - \frac{\mathrm{i}k_\perp B_p}{RB_0}A \tag{2.12}$$

and

$$A_\parallel = -\frac{1}{\mathrm{i}k_\perp}\boldsymbol{B}_1 \cdot \boldsymbol{e}_\times = \frac{B_p}{\mathrm{i}k_\perp RB_0}\delta + \frac{I}{R^2 B_0}A. \tag{2.13}$$

Similarly, from (2.9), we find $\delta = R^2 \boldsymbol{B}_1 \cdot \nabla \zeta$ and $A = -(R/\mathrm{i}k_\perp B_p)\boldsymbol{B}_1 \cdot (\nabla \zeta \times \nabla \psi)$. Substituting into these equations the form for $\boldsymbol{B}_1$ given in (2.6), and using (2.10) and (2.11), we obtain

$$\delta = R^2 \boldsymbol{B}_1 \cdot \nabla \zeta = \frac{I}{B_0} B_\parallel + \frac{\mathrm{i}k_\perp RB_p}{B_0} A_\parallel \tag{2.14}$$

and

$$A = -\frac{R}{\mathrm{i}k_\perp B_p}\boldsymbol{B}_1 \cdot (\nabla \zeta \times \nabla \psi) = -\frac{RB_p}{\mathrm{i}k_\perp B_0} B_\parallel + \frac{I}{B_0} A_\parallel. \tag{2.15}$$

Expressions (2.12)–(2.15) allow us to change from the form most convenient for gyrokinetics, in (2.6), to the axisymmetric form in (2.9).

It is of interest to discuss the possible changes that the magnetic field can undergo. It is possible to change the direction of the magnetic field lines without changing the magnitude of $\boldsymbol{B}$ if $B_\parallel = 0 = I\delta - \mathrm{i}k_\perp RB_p A$. To avoid changing the direction of the field lines, the perturbation must satisfy $\boldsymbol{B}_1 \times \boldsymbol{B}_0 = 0$, or $A_\parallel = 0 = RB_p\delta + \mathrm{i}k_\perp IA$. Changing the local theta dependent direction of the magnetic field line does not necessarily imply a change in the safety factor $q(\psi) = (2\pi)^{-1}\int_0^{2\pi}(\boldsymbol{B} \cdot \nabla \zeta / \boldsymbol{B} \cdot \nabla \theta)\,\mathrm{d}\theta$. To verify this we write the safety factor as

$$q = q_0 + q_1, \tag{2.16}$$

where

$$q_0 = \frac{1}{2\pi}\int_0^{2\pi} \frac{\boldsymbol{B}_0 \cdot \nabla \zeta}{\boldsymbol{B}_0 \cdot \nabla \theta}\,\mathrm{d}\theta \tag{2.17}$$

is the unperturbed safety factor and

$$q_1 = \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{\boldsymbol{B}_1 \cdot \nabla \zeta}{\boldsymbol{B}_0 \cdot \nabla \zeta} - \frac{\boldsymbol{B}_1 \cdot \nabla \theta}{\boldsymbol{B}_0 \cdot \nabla \theta}\right)\frac{\boldsymbol{B}_0 \cdot \nabla \zeta}{\boldsymbol{B}_0 \cdot \nabla \theta}\,\mathrm{d}\theta \tag{2.18}$$

is the result of the perturbation. Note that our poloidal angle-like variable θ is not changed by the perturbations. Using (2.6) and (2.9), and recalling that $k_\perp / RB_p$ is independent of θ, the perturbation q_1 becomes

$$q_1 = \frac{1}{2\pi}\frac{\mathrm{i}k_\perp}{RB_p}\int_0^{2\pi} \frac{A_\parallel}{\boldsymbol{b} \cdot \nabla \theta}\,\mathrm{d}\theta = \frac{1}{2\pi}\frac{\mathrm{i}k_\perp}{RB_p}\int_0^{2\pi} \left(\frac{B_p\delta}{\mathrm{i}k_\perp R} + \frac{IA}{R^2}\right)\frac{\mathrm{d}\theta}{\boldsymbol{B}_0 \cdot \nabla \theta}. \tag{2.19}$$

When q_1 changes, the lines in the flux surface change topology by switching between rational and irrational – the only form of reconnection allowed for axisymmetric perturbations.

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3. Kinetic equation and solution

We need to solve the linearized gyrokinetic equation in which the unperturbed quantities are time independent or evolve slowly compared to the zonal flow relaxation. The unperturbed ion distribution function f_0 is assumed to be Maxwellian:

$$f_0 = n\left(\frac{M}{2\pi T}\right)^{3/2}\exp\left(-\frac{Mv^2}{2T}\right), \tag{3.1}$$

with n, M and T the ion density, mass and temperature, respectively. Then, as we shall consider a collisionless plasma, the linearized distribution function f_1 satisfies the Vlasov equation

$$\dot{f}_1 = \frac{\partial f_1}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla} f_1 + \Omega\boldsymbol{v}\times\boldsymbol{b}\cdot\boldsymbol{\nabla}_v f_1 = \frac{Ze}{T}\boldsymbol{E}_1\cdot\boldsymbol{v} f_0, \tag{3.2}$$

where $\Omega = ZeB_0/Mc$ with Ze the ion charge and c the speed of light. We have neglected the unperturbed electric field. For the perturbed electric field we use $\boldsymbol{E}_1 = -\boldsymbol{\nabla}\Phi - c^{-1}\partial\boldsymbol{A}/\partial t$, with Φ the perturbed electrostatic potential. To remove the adiabatic piece we let $f_1 = h - (Ze\Phi/T)f_0$ to obtain

$$\dot{h} = \frac{\partial h}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla} h + \Omega\boldsymbol{v}\times\boldsymbol{b}\cdot\boldsymbol{\nabla}_v h = \frac{Ze}{T}\left[\frac{\partial\Phi}{\partial t} - \frac{\partial}{\partial t}\left(\frac{\boldsymbol{v}\cdot\boldsymbol{A}}{c}\right)\right] f_0. \tag{3.3}$$

This is the form that we will use to obtain the desired gyrokinetic equation.

Rather than perform a conventional gyrokinetic treatment (Catto 1978) of (3.3), we use the canonical angular momentum ψ_* for our radial variable when we change variables (Kagan & Catto 2008). Then using $\boldsymbol{v} = \boldsymbol{v}_\perp + v_\parallel\boldsymbol{b}$ gives

$$\psi_* = \psi - \frac{Mc}{Ze}R\hat{\boldsymbol{\zeta}}\cdot\boldsymbol{v} = \psi + \frac{1}{\Omega}\boldsymbol{v}_\perp\times\boldsymbol{b}\cdot\boldsymbol{\nabla}\psi - \frac{Iv_\parallel}{\Omega}. \tag{3.4}$$

As for the vector potential in (2.5), we describe the rapid radial variation of the perturbed electrostatic potential Φ by the eikonal expression

$$\Phi = \tilde{\Phi}\exp[\mathrm{i}S(\psi)], \tag{3.5}$$

where the coefficient $\tilde{\Phi}$ is a function of time and is only allowed to be a slow function of θ and ψ. Changing from ψ, θ, ζ and $\boldsymbol{v}$ variables to ψ_*, θ, ζ, v, $\mu = v_\perp^2/2B_0$, and gyrophase φ, with $\boldsymbol{v}_\perp = v_\perp(\boldsymbol{e}_\perp\cos\varphi + \boldsymbol{e}_\times\sin\varphi)$, and using $\dot{\psi}_* = 0$ to remove the ψ_* derivative (Kagan & Catto 2009) yields the lowest-order gyrokinetic equation

$$\frac{\partial h}{\partial t} + v_\parallel\boldsymbol{b}\cdot\boldsymbol{\nabla} h = \frac{Zef_0}{cT}\left\langle\exp[\mathrm{i}S(\psi)]\frac{\partial}{\partial t}[c\tilde{\Phi} - v_\parallel\tilde{A}_\parallel - \boldsymbol{v}_\perp\cdot(\boldsymbol{e}_\perp\tilde{A}_\perp + \boldsymbol{e}_\times\tilde{A}_\times)]\right\rangle_\varphi, \tag{3.6}$$

where the θ dependence of h is assumed slow, drift corrections to parallel streaming are neglected as small, and the gyroaverage $\langle\cdots\rangle_\varphi = (2\pi)^{-1}\oint\mathrm{d}\varphi(\cdots)$ is performed at fixed ψ_*.

Next we write h in the eikonal form

$$h = h(\psi_*, \theta, v, \mu, t) = \tilde{h}_*(\psi_*, \theta, v, \mu, t)\exp[\mathrm{i}S(\psi_*)], \tag{3.7}$$

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where only a weak ψ_* dependence of $\tilde{h}_*$ not captured by $S(\psi_*)$ is allowed, $v = |\boldsymbol{v}|$, and $\mu = v_\perp^2/2B_0$. Then, we Taylor expand to obtain $S(\psi) - S(\psi_*) = Q - L + \cdots$, where $L = (k_\perp v_\perp/\Omega)\sin\varphi$, $Q = S' I v_\parallel/\Omega = k_\perp v_\parallel/\Omega_p$ and $\Omega_p = ZeB_p/Mc = \Omega B_p/B_0$. We retain the order ϵ Shafranov shift Δ_S of the flux surfaces by writing $R = R_0(\psi) + r(\psi)\cos\theta$ with $R_0(\psi = 0) = R_0(0)$ the location of the magnetic axis, $R_0(\psi) = R_0(0) - \Delta_S$, and r the minor radius for circular flux surfaces, then $\epsilon = r/R_0(\psi)$ is a flux function and the ratio of the poloidal over the toroidal magnetic field is $B_p/B_t = (\epsilon/q)[1 - \Delta_S\cos\theta + O(\epsilon^2)]$ with $I = RB_t$. As a result, the lowest order gyrokinetic equation becomes

$$\frac{\partial \tilde{h}_*}{\partial t} + v_\parallel \boldsymbol{b}\cdot\nabla\tilde{h}_* = \frac{Zef_0}{cT}\left\langle \exp[\mathrm{i}Q - \mathrm{i}L]\frac{\partial}{\partial t}[c\tilde{\Phi} - v_\parallel\tilde{A}_\parallel - \boldsymbol{v}_\perp\cdot(\boldsymbol{e}_\perp\tilde{A}_\perp + \boldsymbol{e}_\times\tilde{A}_\times)]\right\rangle_\varphi . \quad (3.8)$$

Performing the gyroaverages we obtain the desired form of the ion gyrokinetic equation

$$\frac{\partial \tilde{h}_*}{\partial t} + v_\parallel \boldsymbol{b}\cdot\nabla\tilde{h}_* = \frac{Zef_0}{cT}\exp(\mathrm{i}Q)\frac{\partial}{\partial t}\left[\mathrm{J}_0\left(\frac{k_\perp v_\perp}{\Omega}\right)(c\tilde{\Phi} - v_\parallel\tilde{A}_\parallel) + \frac{2\mathrm{J}_1(k_\perp v_\perp/\Omega)}{k_\perp v_\perp/\Omega}\frac{v_\perp^2}{2\Omega}\tilde{B}_\parallel\right], \quad (3.9)$$

where $\tilde{B}_\parallel = \mathrm{i}k_\perp\tilde{A}_\times$, $\boldsymbol{b}\cdot\nabla = (qR)^{-1}\partial/\partial\theta$, the coefficient of the $\tilde{A}_\perp$ term has gyroaveraged to zero and J_0 and J_1 denote Bessel functions of the first kind.

To lowest order the streaming term dominates for a weakly collisional plasma. The damping away of any initial θ dependence leads to the GAM behaviour observed during the early evolution to the final residual zonal flow steady state. We are not interested in the GAM behaviour so we annihilate the θ derivative in (3.9) to obtain the transit averaged gyrokinetic equation

$$\frac{\partial\overline{\tilde{h}_*}}{\partial t} = \frac{Zef_0}{cT}\frac{\partial}{\partial t}\overline{\left[\mathrm{J}_0\left(\frac{k_\perp v_\perp}{\Omega}\right)(c\tilde{\Phi} - v_\parallel\tilde{A}_\parallel) + \frac{2\mathrm{J}_1(k_\perp v_\perp/\Omega)}{k_\perp v_\perp/\Omega}\frac{v_\perp^2}{2\Omega}\tilde{B}_\parallel\right]\exp(\mathrm{i}Q)}, \quad (3.10)$$

where $\overline{\tilde{h}_*}$ is the θ independent long time solution, while $\tilde{h}_*(t=0)$ is allowed to depend on θ so we can relate an initial θ dependent perturbation to the final steady state θ independent solution. The transit average of any quantity X is defined as $\overline{X} = \oint \mathrm{d}\tau X/\oint\mathrm{d}\tau$, with $\mathrm{d}\tau = \mathrm{d}\theta/(v_\parallel\boldsymbol{b}\cdot\nabla\theta) \approx qR\,\mathrm{d}\theta/v_\parallel$. The transit average is over a full bounce for trapped ions and over a complete poloidal circuit for the passing ones. The sign of $v_\parallel$ changes at turning points, while $d\tau > 0$, giving $\overline{v_\parallel} = 0$ for the trapped. For the passing $\overline{v_\parallel} \to \langle 1/v_\parallel\rangle^{-1}$ in the large aspect ratio limit, where $\langle X\rangle = [\oint X\,\mathrm{d}\theta/(\boldsymbol{B}_0\cdot\nabla\theta)][\oint \mathrm{d}\theta/(\boldsymbol{B}_0\cdot\nabla\theta)]^{-1}$ denotes a flux surface average. It is easy to show that $\langle\int \mathrm{d}^3v f_0 X\rangle = \langle\int \mathrm{d}^3v f_0\overline{X}\rangle$. More details are given in appendix A.

Equation (3.10) generalizes the usual electrostatic result to include electromagnetic effects through $A_\parallel$ and $B_\parallel$. It is important to realize that Φ, $A_\parallel$ and $B_\parallel$ are allowed to be slow functions of θ and ψ. Solving (3.10) by integrating from $t = 0$ gives

$$\overline{\tilde{h}_*} = \overline{\tilde{h}_*(t=0)} + \frac{Zef_0}{cT}\overline{\left[\mathrm{J}_0(c\tilde{\Phi} - v_\parallel\tilde{A}_\parallel) + \frac{2\mathrm{J}_1}{z}\frac{v_\perp^2}{2\Omega}\tilde{B}_\parallel\right]\exp(\mathrm{i}Q)}$$
$$- \frac{Zef_0}{cT}\overline{\left\{\mathrm{J}_0[c\tilde{\Phi}^{(0)} - v_\parallel\tilde{A}_\parallel^{(0)}] + \frac{2\mathrm{J}_1}{z}\frac{v_\perp^2}{2\Omega}\tilde{B}_\parallel^{(0)}\right\}\exp(\mathrm{i}Q)}, \quad (3.11)$$

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where $\tilde{\Phi}^{(0)} = \tilde{\Phi}(t=0)$, $\tilde{A}_\parallel^{(0)} = \tilde{A}_\parallel(t=0)$ and $\tilde{B}_\parallel^{(0)} = \tilde{B}_\parallel(t=0)$ are allowed to be flux functions, $\mathrm{J}_0 \equiv \mathrm{J}_0(z)$ and $\mathrm{J}_1 \equiv \mathrm{J}_1(z)$, and $z = k_\perp v_\perp/\Omega$. The strong poloidal variation in (3.11) is due to $v_\parallel$ and $Q \propto v_\parallel$, while the poloidal variation of B_0 is weaker and sometimes unimportant. The electron response $\overline{\tilde{h}_e}$ is given by an equation similar to (3.11), but with $Z \to -1$, $M \to m$ and $f_0 \to f_{0e}$. We will omit species subscripts to streamline notation except where they are needed to avoid confusion.

Next we form the perturbed quasineutrality equation

$$0 = \sum Ze \int \mathrm{d}^3 v f_1, \tag{3.12}$$

and the two components of Ampère's law

$$k_\perp^2 A_\parallel \simeq (4\pi/c)\boldsymbol{J}_1 \cdot \boldsymbol{b} = (4\pi/c) \sum Ze \int \mathrm{d}^3 v f_1 v_\parallel \tag{3.13}$$

and

$$k_\perp^2 A_\times \simeq (4\pi/c)\boldsymbol{J}_1 \cdot \boldsymbol{e}_\times = (4\pi/c) \sum Ze \int \mathrm{d}^3 v f_1 v_\perp \sin\varphi, \tag{3.14}$$

where the integrals are taken at fixed ψ, $\sum$ denotes a sum over ions and electrons. Notice that $\boldsymbol{J}_1 \cdot \boldsymbol{e}_\perp \simeq 0$ since $\boldsymbol{\nabla} \cdot \boldsymbol{J}_1 = 0 = \boldsymbol{\nabla} \cdot \boldsymbol{A}$. Using $f_1 = \tilde{h}_* \exp[\mathrm{i}S(\psi_*)] - (Zef_0/T)\tilde{\Phi} \exp[\mathrm{i}S(\psi)]$ and $\int \mathrm{d}^3 v h\{1, v_\parallel, v_\perp \sin\varphi\} = \int \mathrm{d}^3 v \tilde{h}_*\{1, v_\parallel, v_\perp \sin\varphi\} \exp[\mathrm{i}(L - Q + S)] = \int \mathrm{d}^3 v \tilde{h}_*\{\mathrm{J}_0, v_\parallel \mathrm{J}_0, \mathrm{i}v_\perp \mathrm{J}_1\} \exp[\mathrm{i}(S-Q)]$, equations (3.12)–(3.14) give

$$\sum Ze \int \mathrm{d}^3 v\, \tilde{h}_* \mathrm{J}_0 \mathrm{e}^{-\mathrm{i}Q} - \sum \frac{Z^2 \mathrm{e}^2 n}{T} \tilde{\Phi} = 0, \tag{3.15}$$

$$k_\perp^2 \tilde{A}_\parallel = \frac{4\pi}{c} \sum Ze \int \mathrm{d}^3 v\, \tilde{h}_* v_\parallel \mathrm{J}_0 \mathrm{e}^{-\mathrm{i}Q} \tag{3.16}$$

and

$$\frac{B_0 \tilde{B}_\parallel}{4\pi} + \sum \int \mathrm{d}^3 v\, \tilde{h}_* \frac{M v_\perp^2}{2} \frac{2\mathrm{J}_1}{z} \mathrm{e}^{-\mathrm{i}Q} = 0. \tag{3.17}$$

Note that the perpendicular Ampère's law (3.14) has become perpendicular pressure balance (3.17).

Equation (3.11) represents the formal solution of an initial value problem for a time scale long compared to the periodic gyromotion about guiding centres. It depends on the initial value $\tilde{h}_*(0)$, which we are free to choose arbitrarily as long as it satisfies Maxwell's equations. Our particular choice for $\tilde{h}_*(0)$ will be motivated in the next section.

4. Choice of initial condition

We must pick $\tilde{h}_*$ to satisfy the Maxwell equations at $t = 0$, while also obtaining a convenient and sensible form for $\overline{\tilde{h}_*}$ to evaluate the long time relaxation behaviour of the zonal flow response. The non-transit averaged initial condition for $\tilde{h}_*(t=0)$ must depend on θ as well as ψ_*, v, μ to generate a transiently evolving zonal flow (the GAM) or else it will be a homogeneous solution to the non-transit averaged Vlasov equation. However, we do not allow $\tilde{h}_*(t=0)$ to depend on gyrophase since we

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assume any initial transient response associated with gyromotion has already damped to its classical polarization level.

We require Maxwell's equations to be satisfied at $t = 0$, which we take to mean after many gyrations, but much less than the time for a poloidal bounce or transit to be completed. Consequently, at $t = 0$ we must satisfy (3.15)–(3.17).

A GAM develops on a time scale of the order of a transit or bounce time and much longer than a gyration period. It oscillates as it damps away to the residual zonal flow level. The initial conditions are sometimes viewed as approximating the turbulent sources (Rosenbluth & Hinton 1998; Sugama & Watanabe 2005) of charge and current densities on a time much less than a transit or bounce time, but after many gyrations. When the long time behaviour of the system is studied electrostatically we may use the transit average result

$$f_1 = \overline{\tilde{h}_*(t)}\mathrm{e}^{\mathrm{i}S(\psi_*)} - \frac{Zef_0}{T}\tilde{\Phi}\mathrm{e}^{\mathrm{i}S(\psi)}, \tag{4.1}$$

where $\overline{\tilde{h}_*(t)}$ is given by the electrostatic limit of (3.11).

Next we explain how we choose $\tilde{h}_*(0)$ to include magnetic perturbations. For our results to have the required generality we do not flux surface average quasineutrality and the two components of Ampère's law. Moreover, we desire forms at $t = 0$ in which $\tilde{\Phi}^{(0)}$, $\tilde{A}_\parallel^{(0)}$ and $\tilde{B}_\parallel^{(0)}$ terms only contribute to quasineutrality, and the parallel and perpendicular components of Ampère's law, respectively. To avoid the need for complicated velocity space structure, we do not consider arbitrary wavenumbers (Xiao & Catto 2006*b*; Xiao *et al.* 2007*a*). We do manipulations consistent with an expansion in $k_\perp v_\perp/\Omega \ll 1$, but to avoid lengthy expressions, we prefer to keep the finite Larmor radius terms in the form of Bessel functions for a while and make the expansions explicit later. To simplify our treatment and properly recover the $k_\perp v_\perp/\Omega \ll Q \ll 1$ limit electrostatically we can use the simple form

$$\tilde{h}_*(0) = \frac{Zef_0\mathrm{e}^{\mathrm{i}Q}}{T\mathrm{J}_0}\tilde{\Phi}^{(0)}, \tag{4.2}$$

where since we are only interested in $k_\perp v_\perp/\Omega \ll 1$ the zeroes of the Bessel function are of no concern. A nice discussion of the difference between treating electrostatic zonal flows as an initial value problem rather than as a turbulent source in quasineutrality is presented in § 3 of Monreal *et al.* (2016). Moreover, their paper and the references therein should be consulted to understand the differences between the residual zonal flow behaviour in stellarators and tokamaks.

The preceding expression motivates us to assume

$$\tilde{h}_*(0) = \frac{Zef_0\mathrm{e}^{\mathrm{i}Q}}{T\mathrm{J}_0}\left[H\tilde{\Phi}^{(0)} + \frac{k_\perp^2 c}{\omega_p^2}v_\parallel\tilde{A}_\parallel^{(0)} - \frac{2Tu}{ZeB_0\beta}\tilde{B}_\parallel^{(0)}\right], \tag{4.3}$$

where we have introduced the species dependent quantities

$$H = 1 - \frac{3\alpha}{2}\left(\frac{Mv^2}{3T} - 1\right) \tag{4.4}$$

and

$$\alpha = \frac{2\int \mathrm{d}^3v f_0 v_\perp^2\left[\dfrac{2\mathrm{J}_1}{z\mathrm{J}_0} - 1\right]}{\displaystyle\int \mathrm{d}^3v f_0 v_\perp^2 \frac{2\mathrm{J}_1}{z\mathrm{J}_0}\left(\frac{Mv^2}{T} - 3\right)}, \tag{4.5}$$

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with $z = k_\perp v_\perp/\Omega$, $2\mathrm{J}_1/z\mathrm{J}_0 \approx 1 + z^2/8$ giving $\alpha \approx k_\perp^2 T/2M\Omega^2 \ll 1$ for $z \ll 1$, and the species independent quantity

$$u = \frac{\sum nT}{\sum \int \mathrm{d}^3 v f_0 M v_\perp^2 \dfrac{\mathrm{J}_1}{z\mathrm{J}_0}}. \tag{4.6}$$

In the preceding $\beta = 8\pi \sum nT/B_0^2 = 8\pi(n_i T_i + n_e T_e)/B_0^2 = \beta_i + \beta_e$ and $\omega_p^2 = 4\pi \sum Z^2 \mathrm{e}^2 n/M = 4\pi \mathrm{e}^2[(Z^2 n_i/M) + (n_e/m)] = \omega_{pi}^2 + \omega_{pe}^2$. We will also make use of the definitions $v_i = (2T_i/M)^{1/2}$, $\rho_i = v_i/\Omega_i$, $\Omega_i = ZeB_0/(Mc)$ and $\rho_{pi} = q\rho_i/\epsilon$.

The functional form of H in (4.3) is chosen so at $t = 0$ it does not alter quasi-neutrality (3.15) since

$$\tilde{\Phi}^{(0)} \sum \frac{Z^2 \mathrm{e}^2}{T} n = \tilde{\Phi}^{(0)} \sum \frac{Z^2 \mathrm{e}^2}{T} \int \mathrm{d}^3 v f_0 H = \tilde{\Phi}^{(0)} \sum \frac{Z^2 \mathrm{e}^2}{T} \int \mathrm{d}^3 v f_0. \tag{4.7}$$

Moreover, the α in H is chosen so $\tilde{\Phi}^{(0)}$ will not enter perpendicular Ampère's law by taking

$$\tilde{\Phi}^{(0)} \sum \frac{ZeM}{2T} \int \mathrm{d}^3 v f_0 v_\perp^2 \left[H \frac{2\mathrm{J}_1}{z\mathrm{J}_0} - 1 \right] = 0. \tag{4.8}$$

The $\tilde{A}_\parallel^{(0)}$ term does not enter quasineutrality because its integral is odd in $v_\parallel$, and $\tilde{B}_\parallel^{(0)}$ does not enter because u and β are species independent as we can use unperturbed quasineutrality. We can also see that parallel Ampère's law, equation (3.16), is satisfied at $t = 0$ since the $\tilde{\Phi}^{(0)}$ and $\tilde{B}_\parallel^{(0)}$ integrals are odd in $v_\parallel$ leaving

$$\tilde{A}_\parallel^{(0)} = \frac{4\pi}{\omega_p^2} \tilde{A}_\parallel^{(0)} \sum \frac{Z^2 \mathrm{e}^2}{T} \int \mathrm{d}^3 v f_0 v_\parallel^2 = \frac{\tilde{A}_\parallel^{(0)}}{\omega_p^2} \sum \frac{4\pi Z^2 \mathrm{e}^2 n}{M}. \tag{4.9}$$

To satisfy the perpendicular Ampère's law, equation (3.17), at $t = 0$ we require (4.4)–(4.6) to be satisfied as can be seen from

$$B_0 \tilde{B}_\parallel^{(0)} = -4\pi \sum \int \mathrm{d}^3 v f_0 M v_\perp^2 (\mathrm{J}_1/z\mathrm{J}_0) \left[\frac{Ze}{T} H \tilde{\Phi}^{(0)} - \frac{2u}{B_0 \beta} \tilde{B}_\parallel^{(0)} \right] = \tilde{B}_\parallel^{(0)} \frac{u}{\beta} \frac{8\pi \sum nT}{B_0 u}, \tag{4.10}$$

where we use unperturbed quasineutrality as well as $\int \mathrm{d}^3 v f_0 v_\perp^2 [(2\mathrm{J}_1 H/z\mathrm{J}_0) - 1] = 0$.

Apart from the need to satisfy Maxwell's equations, $\tilde{h}_*(t = 0)$ is arbitrary. Other choices for $\tilde{h}_*(t = 0)$ may be used, but (4.3) is sufficient for our purpose. It has the transit average

$$\overline{\tilde{h}_*(0)} = \frac{Zef_0}{T} \left[H \Phi^{(0)} \overline{\mathrm{e}^{\mathrm{i}Q}/\mathrm{J}_0} - \frac{2Tu}{ZeB_0^2 \beta} \tilde{B}_\parallel^{(0)} \overline{B_0 \mathrm{e}^{\mathrm{i}Q}/\mathrm{J}_0} + \frac{k_\perp^2 c}{\omega_p^2} \tilde{A}_\parallel^{(0)} \overline{v_\parallel \mathrm{e}^{\mathrm{i}Q}/\mathrm{J}_0} \right], \tag{4.11}$$

where the weak θ dependences of u and α are neglected.

Before closing this section we prove that $\tilde{\Phi}$ is a lowest order flux function for fully electromagnetic initial conditions. For this demonstration we will only be interested in the $L \sim k_\perp v_\perp/\Omega \ll Q \sim k_\perp v_\parallel/\Omega_p \ll 1$ limit. Consequently, to lowest order we let $k_\perp \to 0$ to make $\mathrm{e}^{\mathrm{i}Q} \to 1$, $\mathrm{J}_0 \to 1$, $\mathrm{J}_1 \to k_\perp v_\perp/2\Omega$, $\alpha \to 0$, $H \to 1$ and $u \to 1$ in

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(3.15)–(3.17), and then use $\tilde{h}_* \approx \overline{\tilde{h}_*}$ to obtain quasineutrality in the form

$$\tilde{\Phi} \sum \frac{Z^2 e^2}{T} n = \sum \frac{Ze}{T} \int d^3 v \overline{\tilde{h}_*}, \tag{4.12}$$

with

$$\overline{\tilde{h}_*(t)} = \overline{\tilde{h}_*(0)} + \frac{Ze}{cT} f_0 [(c\overline{\tilde{\Phi}} - \overline{v_\parallel \tilde{A}_\parallel} + v_\perp^2 \overline{\tilde{B}_\parallel}/2\Omega) - (c\tilde{\Phi}^{(0)} - \overline{v_\parallel}\tilde{A}_\parallel^{(0)} + \tilde{B}_\parallel^{(0)} v_\perp^2/2\Omega)], \tag{4.13}$$

and

$$\overline{\tilde{h}_*(0)} = \frac{Ze}{T} f_0 \left[\tilde{\Phi}^{(0)} + \frac{k_\perp^2 c}{\omega_p^2} \overline{v_\parallel} \tilde{A}_\parallel^{(0)} - \frac{2T\overline{B_0}}{ZeB_0^2 \beta} \tilde{B}_\parallel^{(0)} \right], \tag{4.14}$$

where we have set $\alpha = 0$ and $u = 1$ since we let $k_\perp \to 0$ and used $\overline{v_\perp^2/2B_0} = \mu = v_\perp^2/2B_0$. The integrals odd in $v_\parallel$ do not contribute to quasineutrality, of course. In addition, unperturbed quasineutrality prevents the $\tilde{B}_\parallel^{(0)}$ and $\tilde{B}_\parallel$ from contributing to perturbed quasineutrality as well since they result in the integrals $\sum Ze \int d^3 v f_0 \overline{B_0} = (B_0/2\underline{B}_0)(\int_0^{\underline{B}_0/B_0} d\lambda \overline{B_0}/\xi) \sum Zen = 0$ and

$$\begin{aligned} \sum \frac{ZeM}{T} \int d^3 v f_0 \mu \overline{\tilde{B}_\parallel} &= \frac{3B_0}{2\overline{B_0}^2} \sum Zen \left(\int_0^{\underline{B}_0/B_0} d\lambda \lambda \frac{\overline{\tilde{B}_\parallel}}{\xi} \right) \\ &= \frac{3B_0}{2\overline{B_0}^2} \left(\int_0^{\underline{B}_0/B_0} d\lambda \lambda \frac{\overline{\tilde{B}_\parallel}}{\xi} \right) \sum Zen = 0, \end{aligned} \tag{4.15}$$

where $B_0/\underline{B}_0 \simeq R_0(\psi)/R = 1 - \epsilon(\psi) \cos\theta + O(\epsilon^2)$, $\underline{B}_0 = \langle B_0 \rangle$, $\epsilon = r(\psi)/R_0(\psi)$, $\lambda = 2\mu \underline{B}_0/v^2$, $\xi = \sqrt{1 - \lambda B_0/\underline{B}_0}$ and $d^3 v \to \sum_{sgn\, v_\parallel} \pi B_0 v^3 \, dv \, d\lambda/\underline{B}_0 |v_\parallel|$ (where $\sum_{sgn\, v_\parallel}$ can be replaced by a factor 2 when the integrand is even in $\mathrm{sgn}\, v_\parallel$). Consequently, assuming $k_\perp v_\perp/\Omega \ll Q \ll 1$ the lowest order perturbed quasineutrality equation becomes

$$\tilde{\Phi} \sum \frac{Z^2 e^2}{T} n = \sum \frac{Ze}{T} \int d^3 v \overline{\tilde{h}_*} = \sum \frac{Z^2 e^2}{T} \int d^3 v f_0 \overline{\tilde{\Phi}}, \tag{4.16}$$

which requires

$$\tilde{\Phi} = \frac{\int d^3 v \, \overline{\tilde{\Phi}} f_0}{\int d^3 v f_0}, \tag{4.17}$$

and is satisfied if $\tilde{\Phi} = \langle \tilde{\Phi} \rangle$. Therefore, we may safely assume $\tilde{\Phi}$ is a flux function to lowest order. To see this more rigorously, we multiply (4.16) by $\tilde{\Phi}$ to form $\tilde{\Phi}^2 \int d^3 v f_0 = \int d^3 v f_0 \tilde{\Phi}^2 = \tilde{\Phi} \int d^3 v f_0 \overline{\tilde{\Phi}} = \int d^3 v f_0 \overline{\tilde{\Phi}} \tilde{\Phi}$, then flux surface average to obtain

$$\left\langle \int d^3 v f_0 \tilde{\Phi}^2 \right\rangle = \left\langle \int d^3 v f_0 \overline{\tilde{\Phi}} \tilde{\Phi} \right\rangle. \tag{4.18}$$

As a result, using $\langle \int d^3 v f_0 \tilde{\Phi}^2 \rangle = \langle \int d^3 v f_0 \overline{\tilde{\Phi}^2} \rangle$ and $\langle \int d^3 v f_0 \overline{\tilde{\Phi}} \tilde{\Phi} \rangle = \langle \int d^3 v f_0 \overline{\tilde{\Phi}}^2 \rangle$ we find

$$0 = \left\langle \int d^3 v f_0 (\overline{\tilde{\Phi}^2} - \overline{\tilde{\Phi}}^2) \right\rangle = \left\langle \int d^3 v f_0 \overline{(\tilde{\Phi} - \overline{\tilde{\Phi}})^2} \right\rangle. \tag{4.19}$$

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Therefore, we need

$$\tilde{\Phi} = \overline{\tilde{\Phi}}, \tag{4.20}$$

but $\tilde{\Phi} = \tilde{\Phi}(\psi, \theta)$, while $\overline{\tilde{\Phi}} = \overline{\tilde{\Phi}}(\psi, \lambda)$, so it must be that

$$\tilde{\Phi} = \langle \tilde{\Phi} \rangle \tag{4.21}$$

to lowest order.

To generalize our results for quasineutrality and to treat both components of Ampère's law, we extend our initial condition (4.14) to include finite orbit effects by employing (4.11) in (3.11).

In the calculation of the electrostatic zonal flow residual (Rosenbluth & Hinton 1998) the non-adiabatic electron response could be neglected as small in $(m/M)^{1/2}$, where m is the electron mass, however here it sometimes needs to be retained, since the electromagnetic terms in the non-adiabatic response are proportional to the thermal speed of the species. In the next section we will use the preceding results to form quasineutrality and Ampère's law.

5. Quasineutrality and Ampère's law

In this section we form and consider the non-flux surface averaged components of Ampère's law to demonstrate that poloidal variation of the parallel vector potential and the parallel magnetic field must be retained. Indeed, we will find that there are cases for which these field responses have strong poloidal variation. In addition, we will perform a more complete evaluation of quasineutrality once we have examined the two components of Ampère's law.

To perform the derivation of the two components of Ampère's law we must realize that $\tilde{A}_\parallel$ and $\tilde{B}_\parallel$ are not normally flux functions. Indeed, even when $\tilde{B}^{(0)}_\parallel = 0 = \tilde{A}^{(0)}_\parallel$ we will find they both have poloidal variation. Retaining $\mathrm{e}^{\mathrm{i}Q}$ modifications, but ignoring $L \sim k_\perp v_\perp/\Omega$ corrections as unimportant except in the $\tilde{\Phi}$ and $\tilde{\Phi}^{(0)}$ terms, we use

$$\begin{aligned}\overline{\tilde{h}_*(t)} \;=\; & \overline{\tilde{h}_*(0)} + \frac{Ze}{cT} f_0 \left[c\overline{\tilde{\Phi} \mathrm{J}_0 \mathrm{e}^{\mathrm{i}Q}} + \frac{v_\perp^2}{2\Omega} \overline{\tilde{B}_\parallel \mathrm{e}^{\mathrm{i}Q}} - \overline{\tilde{A}_\parallel v_\parallel \mathrm{e}^{\mathrm{i}Q}} - c\tilde{\Phi}^{(0)} \overline{\mathrm{J}_0 \mathrm{e}^{\mathrm{i}Q}} \right. \\ & \left. - \frac{v_\perp^2}{2\Omega} \tilde{B}^{(0)}_\parallel \overline{\mathrm{e}^{\mathrm{i}Q}} + \tilde{A}^{(0)}_\parallel \overline{v_\parallel \mathrm{e}^{\mathrm{i}Q}} \right],\end{aligned} \tag{5.1}$$

with

$$\overline{\tilde{h}_*(0)} = \frac{Ze}{T} f_0 \left[\tilde{\Phi}^{(0)} \overline{(\mathrm{e}^{\mathrm{i}Q} H/\mathrm{J}_0)} + \frac{k_\perp^2 c}{\omega_p^2} \overline{v_\parallel \mathrm{e}^{\mathrm{i}Q}} \tilde{A}^{(0)}_\parallel - \frac{2T}{ZeB_0^2\beta} \tilde{B}^{(0)}_\parallel \overline{B_0 \mathrm{e}^{\mathrm{i}Q}} \right]. \tag{5.2}$$

5.1. *Ampère's law*

Inserting the preceding into

$$\tilde{B}_\parallel = -\sum \frac{2\pi M}{B_0} \int \mathrm{d}^3 v \overline{\tilde{h}_*} v_\perp^2 \mathrm{e}^{-\mathrm{i}Q} \tag{5.3}$$

and

$$k_\perp^2 \tilde{A}_\parallel = \frac{4\pi}{c} \sum Ze \int \mathrm{d}^3 v \overline{\tilde{h}_*} v_\parallel \mathrm{e}^{-\mathrm{i}Q}, \tag{5.4}$$

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we obtain

$$\tilde{B}_\parallel = -\sum \frac{4\pi Ze}{B_0}\int \mathrm{d}^3v f_0 \frac{Mv_\perp^2}{2T}\frac{2\mathrm{J}_1}{z}\mathrm{e}^{-\mathrm{i}Q}[\overline{\tilde{\Phi}\mathrm{J}_0\mathrm{e}^{\mathrm{i}Q}} + \tilde{\Phi}^{(0)}(\overline{\mathrm{e}^{\mathrm{i}Q}H/\mathrm{J}_0} - \overline{\mathrm{J}_0\mathrm{e}^{\mathrm{i}Q}})]$$
$$+\sum \frac{4\pi Ze}{cB_0}\int \mathrm{d}^3v f_0 \frac{Mv_\perp^2}{2T}\mathrm{e}^{-\mathrm{i}Q}\left[\overline{v_\parallel\tilde{A}_\parallel\mathrm{e}^{\mathrm{i}Q}} - \tilde{A}_\parallel^{(0)}\left(1+\frac{k_\perp^2c^2}{\omega_p^2}\right)\overline{v_\parallel\mathrm{e}^{\mathrm{i}Q}}\right]$$
$$-\sum \frac{4\pi T}{B_0^2}\int \mathrm{d}^3v f_0 \frac{Mv_\perp^2}{2T}\mathrm{e}^{-\mathrm{i}Q}\left[\frac{Mv_\perp^2}{2T}(\overline{\tilde{B}_\parallel\mathrm{e}^{\mathrm{i}Q}} - \tilde{B}_\parallel^{(0)}\overline{\mathrm{e}^{\mathrm{i}Q}}) - \frac{2\tilde{B}_\parallel^{(0)}}{B_0\beta}\overline{B_0\mathrm{e}^{\mathrm{i}Q}}\right] \quad (5.5)$$

and

$$\tilde{A}_\parallel = -\sum \frac{4\pi Z^2\mathrm{e}^2}{k_\perp^2c^2T}\int \mathrm{d}^3v f_0 v_\parallel \mathrm{e}^{-\mathrm{i}Q}\left[\overline{v_\parallel\tilde{A}_\parallel\mathrm{e}^{\mathrm{i}Q}} - \tilde{A}_\parallel^{(0)}\left(1+\frac{k_\perp^2c^2}{\omega_p^2}\right)\overline{v_\parallel\mathrm{e}^{\mathrm{i}Q}}\right]$$
$$+\sum \frac{4\pi Z^2\mathrm{e}^2}{k_\perp^2cT}\int \mathrm{d}^3v f_0 v_\parallel \mathrm{e}^{-\mathrm{i}Q}\mathrm{J}_0[\overline{\tilde{\Phi}\mathrm{J}_0\mathrm{e}^{\mathrm{i}Q}} + \tilde{\Phi}^{(0)}(\overline{\mathrm{e}^{\mathrm{i}Q}H/\mathrm{J}_0} - \overline{\mathrm{J}_0\mathrm{e}^{\mathrm{i}Q}})]$$
$$+\sum \frac{4\pi Ze}{k_\perp^2cB_0}\int \mathrm{d}^3v f_0 v_\parallel \mathrm{e}^{-\mathrm{i}Q}\left[\frac{Mv_\perp^2}{2T}(\overline{\tilde{B}_\parallel\mathrm{e}^{\mathrm{i}Q}} - \tilde{B}_\parallel^{(0)}\overline{\mathrm{e}^{\mathrm{i}Q}}) - \frac{2\tilde{B}_\parallel^{(0)}}{B_0\beta}\overline{B_0\mathrm{e}^{\mathrm{i}Q}}\right]. \quad (5.6)$$

Recalling that $\tilde{A}_\parallel$ and $\tilde{B}_\parallel$ need not be flux functions and that the unperturbed magnetic field gives rise to $\cos\theta$ dependence, we Fourier decompose and retain only the leading poloidal dependence by writing

$$\tilde{A}_\parallel = \langle\tilde{A}\rangle + a\cos\theta + \cdots \simeq \left[\langle\tilde{A}_\parallel\rangle - \frac{a}{2}(\epsilon - \Delta_S')\right] + a\cos\theta \quad (5.7)$$

and

$$\tilde{B}_\parallel = \langle\tilde{B}\rangle + b\cos\theta + \cdots \simeq \left[\langle\tilde{B}_\parallel\rangle - \frac{b}{2}(\epsilon - \Delta_S')\right] + b\cos\theta, \quad (5.8)$$

with the coefficients $\langle\tilde{A}_\parallel\rangle$, $\langle\tilde{B}_\parallel\rangle$, $\langle\tilde{A}\rangle$, $\langle\tilde{B}\rangle$, a, and b flux functions and $\boldsymbol{B}_0\cdot\boldsymbol{\nabla}\theta \propto 1-(\epsilon - \Delta_S')\cos\theta$ in the flux surface averages for our Shafranov shifted circular flux surface model, where $\Delta_S' = \mathrm{d}\Delta_S/\mathrm{d}r \sim \epsilon$. We do not assume an ordering of a and b relative to $\langle\tilde{A}_\parallel\rangle$ and $\langle\tilde{B}_\parallel\rangle$ as we will calculate all four of these coefficients. In addition, we recall the electrostatic result of Rosenbluth & Hinton (1998),

$$\langle\tilde{\Phi}\rangle = \frac{\tilde{\Phi}^{(0)}}{1+\gamma q^2/\epsilon^{1/2}}, \quad (5.9)$$

with $\gamma \approx 1.64$. The preceding allows us to assume $q^2\epsilon^{-1/2}\langle\tilde{\Phi}\rangle \sim \tilde{\Phi}^{(0)}$ for $\beta \ll 1$ and $\epsilon \ll 1$.

Expanding for $Q = k_\perp v_\parallel/\Omega_p \ll 1$, inserting H, neglecting ϵ corrections to $k_\perp^2\rho_i^2$ terms, ignoring all Q^2, β and $\epsilon^{3/2}$ corrections to the $\tilde{B}_\parallel$ and $\tilde{B}_\parallel^{(0)}$ terms in (5.5), recalling

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$\tilde{\Phi} = \langle\tilde{\Phi}\rangle$, and using unperturbed quasineutrality, perpendicular Ampère's law becomes

$$\tilde{B}_\parallel - \tilde{B}_\parallel^{(0)}\frac{B_0}{B_0} = -\langle\tilde{\Phi}\rangle \sum \frac{2\pi ZeM}{TB_0}\int \mathrm{d}^3 v f_0 v_\perp^2 \left[\overline{Q}Q - \frac{(Q^2+\overline{Q^2})}{2}\right] + (\langle\tilde{\Phi}\rangle - \tilde{\Phi}^{(0)})\frac{3c\beta_i k_\perp^2}{4\Omega_i}$$
$$+ \mathrm{i}k_\perp \sum \frac{2\pi qM^2}{\epsilon TB_0}\int \mathrm{d}^3 v f_0 v_\perp^2 (\overline{B_0^{-1}v_\parallel^2\tilde{A}_\parallel} - B_0^{-1}v_\parallel\overline{v_\parallel\tilde{A}_\parallel})$$
$$- \mathrm{i}k_\perp\tilde{A}_\parallel^{(0)}\left(1+\frac{k_\perp^2c^2}{\omega_p^2}\right)\sum\frac{2\pi qM^2}{\epsilon TB_0}\int \mathrm{d}^3 v f_0 v_\perp^2(\overline{B_0^{-1}v_\parallel^2} - B_0^{-1}v_\parallel\overline{v_\parallel}). \quad (5.10)$$

Here and hereafter, we use $Q \propto v_\parallel/B_0$ and then treat $k_\perp$ as a flux function, except in $k_\perp^2\tilde{A}_\parallel^{(0)}$ terms from (5.2) and $k_\perp^2\tilde{A}_\parallel$ from (5.4).

In (5.6) we neglect Q corrections for $\tilde{A}_\parallel$ and $\tilde{A}_\parallel^{(0)}$ terms, insert H, and continue to use $\tilde{\Phi} = \langle\tilde{\Phi}\rangle$, to find that parallel Ampère's law is

$$\tilde{A}_\parallel + \sum\frac{4\pi Z^2\mathrm{e}^2}{k_\perp^2c^2T}\int \mathrm{d}^3 v f_0 v_\parallel\overline{v_\parallel\tilde{A}_\parallel} - \tilde{A}_\parallel^{(0)}\left(1+\frac{k_\perp^2c^2}{\omega_p^2}\right)\sum\frac{4\pi Z^2\mathrm{e}^2}{k_\perp^2c^2T}\int \mathrm{d}^3 v f_0 v_\parallel\overline{v_\parallel}$$
$$= \mathrm{i}\sum\frac{4\pi qM}{k_\perp\epsilon B_0}\int \mathrm{d}^3 v f_0\frac{Mv_\perp^2}{2T}v_\parallel[(\overline{B_0^{-1}v_\parallel\tilde{B}_\parallel} - B_0^{-1}v_\parallel\overline{\tilde{B}_\parallel}) - \tilde{B}_\parallel^{(0)}(\overline{B_0^{-1}v_\parallel} - B_0^{-1}v_\parallel)]$$
$$- \mathrm{i}\frac{2}{\beta}\tilde{B}_\parallel^{(0)}\sum\frac{4\pi qM}{k_\perp\epsilon B_0^2}\int \mathrm{d}^3 v f_0 v_\parallel(\overline{v_\parallel} - v_\parallel)$$
$$+ \mathrm{i}\langle\tilde{\Phi}\rangle\sum\frac{2\pi Z^2\mathrm{e}^2}{k_\perp^2cT}\int \mathrm{d}^3 v f_0 v_\parallel\left[Q\overline{Q^2} - Q^2\overline{Q} + \frac{Q^3 - \overline{Q^3}}{3}\right]$$
$$+ \mathrm{i}\sum\frac{2\pi cM^2}{TB_0^2}\left[(\langle\tilde{\Phi}\rangle - \tilde{\Phi}^{(0)})\int \mathrm{d}^3 v f_0 v_\parallel(Q-\overline{Q})v_\perp^2\right.$$
$$\left. + \tilde{\Phi}^{(0)}\frac{T}{M}\int \mathrm{d}^3 v f_0 v_\parallel(Q-\overline{Q})\right], \quad (5.11)$$

where we used (A 22), (A 30), and (A 37) and similar integrals for $\langle\tilde{\Phi}\rangle$ and $\tilde{\Phi}^{(0)}$ terms to see that due to unperturbed quasineutrality $\sum(Z^2\mathrm{e}^2/T)\int \mathrm{d}^3 v f_0 v_\parallel(Q-\overline{Q}) = 0$. The zonal flow responses of the perturbed parallel magnetic field and vector potential in (5.10) and (5.11) are generated by the polarizations terms associated with poloidally varying departures from flux surfaces.

Next, we flux surface average using $\langle\int \mathrm{d}^3 v f_0 \overline{X}Y\rangle = \langle\int \mathrm{d}^3 v f_0 X\overline{Y}\rangle$, to obtain the flux surface averaged perpendicular Ampère's law

$$\langle\tilde{B}_\parallel\rangle - \tilde{B}_\parallel^{(0)} \simeq \frac{c\beta_i k_\perp^2\rho_i}{4v_i}[(5\gamma q^2\epsilon^{-1/2}+3)\langle\tilde{\Phi}\rangle - 3\tilde{\Phi}^{(0)}]$$
$$+ \frac{\mathrm{i}q\beta k_\perp}{2}\left\{\frac{5\gamma\epsilon^{1/2}}{2}\left[\langle\tilde{A}_\parallel\rangle - \tilde{A}_\parallel^{(0)}\left(1+\frac{k_\perp^2c^2}{\omega_p^2}\right)\right] + a\frac{3\epsilon}{2}\right\}, \quad (5.12)$$

where we used (A 44), (A 63), (A 66), (A 67), and (A 72). The B_0 factors in (5.10) must be treated with care, even though $\overline{v_\parallel\cos\theta} = 0$ for the trapped and $\overline{v_\parallel\cos\theta} \sim \epsilon v$ for the passing, since they alter the coefficient. For now we retain the $\epsilon^{1/2}$ smaller $\langle\tilde{\Phi}\rangle$ and $\tilde{\Phi}^{(0)}$ terms compared to the $\gamma\epsilon^{-1/2}q^2\langle\tilde{\Phi}\rangle$ term.

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In the parallel Ampère's law for now we keep $O(\epsilon^{3/2})$ corrections to the typically large $\omega_p^2/k_\perp^2c^2$ terms obtained by using (A 30). By keeping the $\epsilon^{3/2}$ correction to $\tilde{A}_\parallel$ terms we are making the easily satisfied assumption that $\epsilon^{3/2}\omega_p^2/k_\perp^2c^2 \gg q^2\beta$, or roughly $M/m \gg \epsilon^{1/2}k_\perp^2\rho_{pi}^2$, since we already neglected the more complicated Q^2 corrections to the same terms. The flux surface averaged parallel Ampère's law then becomes

$$\langle\tilde{A}_\parallel\rangle\left[1+\frac{\omega_p^2}{k_\perp^2c^2}(1-\gamma\epsilon^{3/2})\right]-\tilde{A}_\parallel^{(0)}\left[\left(1+\frac{\omega_p^2}{k_\perp^2c^2}\right)(1-\gamma\epsilon^{3/2})\right]$$
$$-a\left[\frac{\epsilon\omega_p^2}{2k_\perp^2c^2}(1-\gamma\epsilon^{3/2})+\Delta_S'\right]$$
$$\simeq\frac{\mathrm{i}q\gamma\epsilon^{1/2}\tilde{B}_\parallel^{(0)}}{k_\perp}+\mathrm{i}\frac{c\beta_iqk_\perp\rho_i}{4v_i}[(\sigma\epsilon^{-1/2}q^2+5\gamma\epsilon^{1/2}+4\epsilon)\langle\tilde{\Phi}\rangle-4(\gamma\epsilon^{1/2}+\epsilon)\tilde{\Phi}^{(0)}], \quad (5.13)$$

where the constant $\sigma\approx 5.3$ is evaluated in appendix A, and we make use of (A 30), (A 35), (A 37), (A 44), (A 73) and (A 80). We use (5.7) and keep the Shafranov shift, Δ_S, terms from $k_\perp \propto RB_p \propto 1-\Delta_S'\cos\theta$ in the $\langle\tilde{A}_\parallel\rangle$ and $\tilde{A}_\parallel^{(0)}$ terms that arise from (5.2) and the left hand side of (5.4). To simplify the flux surface averaged parallel Ampère's law we ignored $\beta\tilde{B}_\parallel$ and $\beta\tilde{B}_\parallel^{(0)}$ terms. Also, we have neglected all b terms because they are multiplied by β. Based on (5.9) we must retain the $\sigma q^2\epsilon^{-1/2}\langle\tilde{\Phi}\rangle$ term in (5.13), but will ignore the ϵ and $\epsilon^{3/2}$ smaller term $(5\gamma\epsilon^{1/2}+4\epsilon)\langle\tilde{\Phi}\rangle$, as well as the $(\gamma\epsilon^{1/2}+\epsilon)\tilde{\Phi}^{(0)}$ term, all associated with the Bessel function corrections.

The terms γ and σ in (5.12) and (5.13) arise from the trapped particle responses to any initial perturbation, while the $a\epsilon$ and $a\Delta_S'$ terms are passing responses. We estimate the different behaviour of the passing and trapped (and barely passing) particles by using $\overline{v_\parallel}=0$ and estimating $v_\parallel\sim\epsilon^{1/2}v$ and $\int \mathrm{d}^3v\propto\epsilon^{1/2}$ for the trapped particles, while using $\overline{v_\parallel}\simeq v_\parallel[1+O(\epsilon)]$ for the passing ones.

Next, we subtract the flux surface averaged equations from the full equations. For the perpendicular Ampère's law we use (A 63), (A 66), (A 67), and (A 72) to find

$$b\simeq\epsilon\tilde{B}_\parallel^{(0)}-\frac{\mathrm{i}q\beta k_\perp}{2}(3-\varsigma\epsilon^{1/2})\left[\langle\tilde{A}_\parallel\rangle-\tilde{A}_\parallel^{(0)}\left(1+\frac{k_\perp^2c^2}{\omega_p^2}\right)\right]$$
$$-\frac{\mathrm{i}\chi\epsilon^{3/2}q\beta k_\perp a}{4}+\frac{c\varsigma\beta_iq^2k_\perp^2\rho_i}{2\epsilon^{1/2}v_i}\langle\tilde{\Phi}\rangle, \quad (5.14)$$

where Bessel function terms are ignored since any $\cos\theta$ dependence is smaller by ϵ and the constants $\chi\approx 0.11$ and $\varsigma\approx 5.3$ are evaluated in appendix A. The $\langle\tilde{A}_\parallel\rangle$, $\tilde{A}_\parallel^{(0)}$, and $\langle\tilde{\Phi}\rangle$ terms are from passing contributions, while the $\epsilon^{3/2}a$ term is a trapped contribution.

To form the difference equation for the parallel Ampère's law we ignore $\beta\langle\tilde{B}_\parallel\rangle$, $\beta\tilde{B}_\parallel^{(0)}$ and βb terms. As a result, the poloidally varying parallel Ampère's law reduces to

$$a\left(1+\frac{\epsilon^2\omega_p^2}{2k_\perp^2c^2}\right)\simeq\left(\frac{\epsilon\omega_p^2}{k_\perp^2c^2}+2\Delta_S'\right)(\langle\tilde{A}_\parallel\rangle-\tilde{A}_\parallel^{(0)})-\epsilon\tilde{A}_\parallel^{(0)}$$
$$+\frac{\mathrm{i}q\tilde{B}_\parallel^{(0)}}{k_\perp}+\mathrm{i}\frac{c\beta_iqk_\perp\rho_i}{4v_i}[(5\gamma\epsilon^{-1/2}q^2+4)\langle\tilde{\Phi}\rangle-3\tilde{\Phi}^{(0)}], \quad (5.15)$$

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where we make use of use (A 30), (A 35), (A 37), (A 44), (A 73), and (A 80), and account for $k_\perp \propto 1 - \Delta'_S \cos\theta$. Only passing contributions enter for $\langle \tilde{A}_\parallel \rangle$, $\tilde{A}^{(0)}_\parallel$, a, and $\tilde{B}^{(0)}_\parallel$ terms, and only Bessel terms enter for $\tilde{\Phi}^{(0)}$, while for $\langle \tilde{\Phi} \rangle$ both trapped and Bessel contributions enter.

5.2. *Parallel Ampère's law limit*

Simulations are sometimes run ignoring the perpendicular Ampère's law. If we do the same by dropping all b and $\tilde{B}^{(0)}_\parallel$ terms in parallel Ampère's law, then (5.15) becomes

$$a\left(1+\frac{\epsilon^2\omega_p^2}{2k_\perp^2c^2}\right) \simeq \left(\frac{\epsilon\omega_p^2}{k_\perp^2c^2}+2\Delta'_S\right)(\langle\tilde{A}_\parallel\rangle-\tilde{A}^{(0)}_\parallel)-\epsilon\tilde{A}^{(0)}_\parallel + \mathrm{i}\frac{c\beta_i q k_\perp \rho_i}{2v_i}[1-(\epsilon^{1/2}/2\gamma q^2)]\tilde{\Phi}^{(0)}, \tag{5.16}$$

and (5.13) reduces to

$$\langle\tilde{A}_\parallel\rangle\left[1+\frac{\omega_p^2}{k_\perp^2c^2}(1-\gamma\epsilon^{3/2})\right]-\tilde{A}^{(0)}_\parallel\left[\left(1+\frac{\omega_p^2}{k_\perp^2c^2}\right)(1-\gamma\epsilon^{3/2})\right] - a\left[\frac{\epsilon\omega_p^2}{2k_\perp^2c^2}(1-\gamma\epsilon^{3/2})+\Delta'_S\right] \simeq \mathrm{i}\frac{c\sigma\beta_i q k_\perp\rho_i}{4\gamma v_i}[1-(\epsilon^{1/2}/\gamma q^2)-(4\gamma^2\epsilon^{1/2}/\sigma)]\tilde{\Phi}^{(0)}, \tag{5.17}$$

where we have used $\beta \ll 1$ so we can use (5.9) to eliminate $\langle\tilde{\Phi}\rangle$.

If $\beta \to 0$ at finite $k_\perp$ so that $\omega_p^2/k_\perp^2c^2 \to 0$ then (5.16) and (5.17) reduce to the large skin depth results

$$\frac{v_i}{c}\langle\tilde{A}_\parallel\rangle \to \frac{v_i}{c}\tilde{A}^{(0)}_\parallel(1-\gamma\epsilon^{3/2})+\mathrm{i}\frac{\sigma\beta_i q k_\perp\rho_i}{4\gamma}\tilde{\Phi}^{(0)} \tag{5.18}$$

and

$$\frac{v_i}{c}a \to -\frac{v_i}{c}\epsilon\tilde{A}^{(0)}_\parallel+\mathrm{i}\frac{\beta_i q k_\perp\rho_i}{2}\tilde{\Phi}^{(0)}, \tag{5.19}$$

where we neglect $\epsilon^{1/2}$ corrections to $\tilde{\Phi}^{(0)}$ terms. Interestingly, for $\tilde{A}^{(0)}_\parallel = 0$ we see that $\langle\tilde{A}_\parallel\rangle/a \simeq \sigma/2\gamma \simeq O(1)$, meaning that substantial poloidal variation occurs when $\tilde{\Phi}^{(0)} \neq 0$. When $\tilde{\Phi}^{(0)} = 0$, $\langle\tilde{A}_\parallel\rangle \simeq \tilde{A}^{(0)}_\parallel(1-\gamma\epsilon^{3/2})$ and $a \simeq -\epsilon\langle\tilde{A}_\parallel\rangle$, and poloidal variation is somewhat weak.

More interestingly, we consider finite $\beta \ll 1$ by allowing $\epsilon^{3/2} \gg k_\perp^2c^2/\omega_p^2$ or $q^2\beta_i\epsilon^{1/2} \gg Zmk_\perp^2\rho_{pi}^2/M$. Continuing to neglect $\epsilon^{1/2}$ corrections in this small skin depth limit we find

$$\frac{v_i}{c}\langle\tilde{A}_\parallel\rangle \simeq \frac{v_i}{c}\tilde{A}^{(0)}_\parallel\left(1-\frac{\epsilon^2}{2}\right)+\frac{\mathrm{i}\epsilon\beta_i k_\perp\rho_{pi}}{4}\tilde{\Phi}^{(0)}, \tag{5.20}$$

and

$$\frac{v_i}{c}a \simeq -\frac{v_i}{c}\epsilon\tilde{A}^{(0)}_\parallel+\frac{\mathrm{i}\epsilon\beta_i k_\perp\rho_{pi}}{2}\tilde{\Phi}^{(0)}. \tag{5.21}$$

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When $\tilde{\Phi}^{(0)}=0$, $\langle\tilde{A}_\parallel\rangle\simeq\tilde{A}_\parallel^{(0)}\simeq -a/\epsilon$ so only weak poloidal variation occurs. However, for $\tilde{A}_\parallel^{(0)}\simeq 0$, we find that $a/\langle\tilde{A}_\parallel\rangle\simeq 2$, which results in strong poloidal variation.

In both the small and large skin depth limits, when $\tilde{\Phi}^{(0)}=0$, only small changes in $\tilde{A}_\parallel$ occur since $\tilde{A}_\parallel\simeq\langle\tilde{A}_\parallel\rangle\simeq\tilde{A}_\parallel^{(0)}\gg a$. However, for $\tilde{A}_\parallel^{(0)}\simeq 0$, we see that poloidal variation in $\tilde{A}_\parallel$ arises due to $\tilde{\Phi}^{(0)}$ with $a/\langle\tilde{A}_\parallel\rangle\sim 1$ in both limits.

5.3. *Full Ampère's law*

The general case retains the perpendicular Ampère's law. Using (5.13) and (5.15) with (5.9) inserted and ignoring $\epsilon^{1/2}$ corrections we find

$$a\left(1+\frac{\epsilon^2\omega_p^2}{2k_\perp^2c^2}\right)\simeq\left(\frac{\epsilon\omega_p^2}{k_\perp^2c^2}+2\Delta_S'\right)(\langle\tilde{A}_\parallel\rangle-\tilde{A}_\parallel^{(0)})-\epsilon\tilde{A}_\parallel^{(0)}+\mathrm{i}\frac{c\beta_iqk_\perp\rho_i}{2v_i}\tilde{\Phi}^{(0)}+\mathrm{i}qk_\perp^{-1}\tilde{B}_\parallel^{(0)} \tag{5.22}$$

and

$$(\langle\tilde{A}_\parallel\rangle-\tilde{A}_\parallel^{(0)})\left(1+\frac{\omega_p^2}{k_\perp^2c^2}\right)-a\left[\frac{\epsilon\omega_p^2}{2k_\perp^2c^2}(1-\gamma\epsilon^{3/2})+\Delta_S'\right]$$
$$\simeq\mathrm{i}\frac{c\sigma\beta_iqk_\perp\rho_i}{4\gamma v_i}\tilde{\Phi}^{(0)}+\mathrm{i}q\gamma\epsilon^{1/2}k_\perp^{-1}\tilde{B}_\parallel^{(0)}. \tag{5.23}$$

In addition, equations (5.12) and (5.14) reduce to

$$\langle\tilde{B}_\parallel\rangle-\tilde{B}_\parallel^{(0)}\simeq\frac{c\beta_ik_\perp^2\rho_i}{2v_i}\tilde{\Phi}^{(0)}$$
$$+\frac{\mathrm{i}q\beta k_\perp}{2}\left\{\frac{5\gamma\epsilon^{1/2}}{2}\left[\langle\tilde{A}_\parallel\rangle-\tilde{A}_\parallel^{(0)}\left(1+\frac{k_\perp^2c^2}{\omega_p^2}\right)\right]+a\frac{3\epsilon}{2}\right\} \tag{5.24}$$

and

$$b\simeq\epsilon\tilde{B}_\parallel^{(0)}-\frac{\mathrm{i}q\beta k_\perp}{2}(3-\varsigma\epsilon^{1/2})\left[\langle\tilde{A}_\parallel\rangle-\tilde{A}_\parallel^{(0)}\left(1+\frac{k_\perp^2c^2}{\omega_p^2}\right)\right]$$
$$-\frac{\mathrm{i}\chi\epsilon^{3/2}q\beta k_\perp a}{4}+\frac{c\varsigma\beta_ik_\perp^2\rho_i}{2\gamma v_i}\tilde{\Phi}^{(0)}, \tag{5.25}$$

where we used the lowest order version of (5.9),

$$\langle\tilde{\Phi}\rangle\simeq\frac{\epsilon^{1/2}\tilde{\Phi}^{(0)}}{\gamma q^2}. \tag{5.26}$$

When $\tilde{B}_\parallel^{(0)}=0$ we may employ (5.18)–(5.21) to find $\langle\tilde{B}_\parallel\rangle$ and b from (5.24) and (5.25). In the $\omega_p^2/k_\perp^2c^2\to 0$ limit (5.18) and (5.19) are used along with (5.24) and (5.25) to find

$$\frac{v_i\langle\tilde{B}_\parallel\rangle}{k_\perp c}\to\frac{\beta_ik_\perp\rho_i}{2}\tilde{\Phi}^{(0)}-\mathrm{i}\frac{5\gamma\epsilon^{1/2}q\beta k_\perp^2c^2}{4\omega_p^2}\frac{v_i}{c}\tilde{A}_\parallel^{(0)} \tag{5.27}$$

and

$$\frac{v_ib}{k_\perp c}\to\mathrm{i}q\beta\frac{3k_\perp^2c^2}{2\omega_p^2}\frac{v_i}{c}\tilde{A}_\parallel^{(0)}+\frac{\varsigma\beta_ik_\perp\rho_i}{2\gamma}\tilde{\Phi}^{(0)}. \tag{5.28}$$

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Consequently, when $\tilde{\Phi}^{(0)} = 0$ we see that $\langle \tilde{B}_\parallel \rangle / b \simeq -5\gamma\epsilon^{1/2}/6$ so strong poloidal variation occurs in $\tilde{B}_\parallel$. For $\tilde{A}_\parallel^{(0)} = 0$ the poloidal variation of $\tilde{B}_\parallel$ is strong since $b/\langle \tilde{B}_\parallel \rangle \simeq \varsigma/\gamma \approx 3.2$.

For $\tilde{B}_\parallel^{(0)} = 0$, $\beta \ll 1$, and $\epsilon^{3/2} \gg k_\perp^2 c^2/\omega_p^2$ we use (5.20) and (5.21) to find the small skin depth forms

$$\frac{v_i \langle \tilde{B}_\parallel \rangle}{k_\perp c} \simeq \frac{\epsilon \beta_i k_\perp \rho_{pi}}{2q} \tilde{\Phi}^{(0)} - \frac{\mathrm{i}3\epsilon^2 \beta q v_i}{4c} \tilde{A}_\parallel^{(0)} \tag{5.29}$$

and

$$\frac{v_i b}{k_\perp c} \simeq \frac{\mathrm{i}3\beta q v_i}{4c} \left(\epsilon^2 + \frac{2k_\perp^2 c^2}{\omega_p^2} \right) \tilde{A}_\parallel^{(0)} + \frac{\varsigma \epsilon \beta_i k_\perp \rho_{pi}}{2\gamma q} \tilde{\Phi}^{(0)}. \tag{5.30}$$

When $\tilde{A}_\parallel^{(0)} = 0$ the poloidal variation of $\tilde{B}_\parallel$ is strong since $b/\langle \tilde{B}_\parallel \rangle \simeq \varsigma/\gamma \approx 3.2$. For $\tilde{\Phi}^{(0)} = 0$ it is also strong with $b/\langle \tilde{B}_\parallel \rangle \simeq -[1 + (2k_\perp^2 c^2/\epsilon^2 \omega_p^2)]$. Consequently, in the small skin depth limit, for $\tilde{B}_\parallel^{(0)} = 0$ the poloidal variation of $\tilde{B}_\parallel$ is strong when $\tilde{A}_\parallel^{(0)} = 0$ and weak when $\tilde{\Phi}^{(0)} = 0$.

Next we consider the case $\tilde{\Phi}^{(0)} = 0$ and $\tilde{A}_\parallel^{(0)} = 0$ when $\tilde{B}_\parallel^{(0)} \neq 0$ by using (5.22)–(5.25). The $\omega_p^2/k_\perp^2 c^2 \to 0$ limit gives

$$k_\perp a \to \mathrm{i}q\tilde{B}_\parallel^{(0)}, \tag{5.31}$$

$$k_\perp \langle \tilde{A}_\parallel \rangle \to \mathrm{i}q\gamma\epsilon^{1/2}\tilde{B}_\parallel^{(0)} \tag{5.32}$$

$$\langle \tilde{B}_\parallel \rangle \to [1 - (3 + 5\gamma^2)(\epsilon q^2 \beta/4)]\tilde{B}_\parallel^{(0)}, \tag{5.33}$$

and

$$b \to (\epsilon + 3\gamma\epsilon^{1/2} q^2 \beta/2)\tilde{B}_\parallel^{(0)}. \tag{5.34}$$

Consequently, the poloidal variation of $\tilde{B}_\parallel$ is weak (with $b/\langle \tilde{B}_\parallel \rangle \simeq \epsilon$), while the poloidal variation of $\tilde{A}_\parallel$ is strong (with $\langle \tilde{A}_\parallel \rangle / a \simeq \gamma\epsilon^{1/2}$) in this large skin depth limit.

In the small skin depth limit $\epsilon^{3/2} \gg k_\perp^2 c^2/\omega_p^2$ for finite $\beta \ll 1$ and $\tilde{\Phi}^{(0)} = 0 = \tilde{A}_\parallel^{(0)}$ we use (5.22) and (5.23) to find $\epsilon a \simeq 2(1 + \gamma\epsilon^{3/2})\langle \tilde{A}_\parallel \rangle$. Then (5.22) to (5.25) give

$$k_\perp a \simeq \frac{\mathrm{i}q\tilde{B}_\parallel^{(0)}}{1 + (\gamma\epsilon^{7/2}\omega_p^2/2k_\perp^2 c^2)}, \tag{5.35}$$

$$k_\perp \langle \tilde{A}_\parallel \rangle \simeq \frac{\mathrm{i}\epsilon q\tilde{B}_\parallel^{(0)}}{2 + (\gamma\epsilon^{7/2}\omega_p^2/k_\perp^2 c^2)} \tag{5.36}$$

$$\langle \tilde{B}_\parallel \rangle \simeq \tilde{B}_\parallel^{(0)} \left\{ 1 - \frac{3\epsilon\beta q^2}{2[2 + (\gamma\epsilon^{7/2}\omega_p^2/k_\perp^2 c^2)]} \right\}, \tag{5.37}$$

and

$$b \simeq \tilde{B}_\parallel^{(0)} \left\{ \epsilon + \frac{3\epsilon\beta q^2}{2[2 + (\gamma\epsilon^{7/2}\omega_p^2/k_\perp^2 c^2)]} \right\}. \tag{5.38}$$

As a result, for this $\tilde{B}_\parallel^{(0)} \neq 0$ case with $\tilde{\Phi}^{(0)} = 0$ and $\tilde{A}_\parallel^{(0)} = 0$, the poloidal variation of $\tilde{A}_\parallel$ is strong with $\langle \tilde{A}_\parallel \rangle / a \simeq \epsilon/2$, while the poloidal variation of $\tilde{B}_\parallel$ varies from weak to strong with $b/\langle \tilde{B}_\parallel \rangle \simeq \epsilon$ to 1.

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5.4. *Quasineutrality*

To complete our treatment of the Maxwell equations we need to form quasineutrality with its finite β effects retained using

$$\tilde{\Phi}\sum\frac{Z^2\mathrm{e}^2n}{T}=\sum Ze\int \mathrm{d}^3v\overline{\tilde{h}_*(t)\mathrm{e}^{\mathrm{i}(L-Q)}}=\sum Ze\int \mathrm{d}^3v\overline{\tilde{h}_*(t)}\mathrm{J}_0\mathrm{e}^{-\mathrm{i}Q}. \tag{5.39}$$

After inserting (3.11) and (4.11), we only retain Bessel function modifications to $\tilde{\Phi}\simeq\langle\tilde{\Phi}\rangle$ and $\tilde{\Phi}^{(0)}$ terms. We then find

$$\begin{aligned}\tilde{\Phi}\sum\frac{Z^2\mathrm{e}^2n}{T}&=(\langle\tilde{\Phi}\rangle-\tilde{\Phi}^{(0)})\sum\frac{Z^2\mathrm{e}^2}{T}\int \mathrm{d}^3vf_0\mathrm{J}_0\mathrm{e}^{-\mathrm{i}Q}\overline{\mathrm{J}_0\mathrm{e}^{\mathrm{i}Q}}\\&+\tilde{\Phi}^{(0)}\sum\frac{Z^2\mathrm{e}^2}{T}\int \mathrm{d}^3vf_0H\mathrm{J}_0\mathrm{e}^{-\mathrm{i}Q}\overline{(\mathrm{e}^{\mathrm{i}Q}/\mathrm{J}_0)}\\&-\sum\frac{Z^2\mathrm{e}^2}{cT}\int \mathrm{d}^3vf_0\mathrm{e}^{-\mathrm{i}Q}[\overline{\tilde{A}_\parallel v_\parallel\mathrm{e}^{\mathrm{i}Q}}-\tilde{A}_\parallel^{(0)}(1+k_\perp^2c^2/\omega_p^2)\overline{v_\parallel\mathrm{e}^{\mathrm{i}Q}}]\\&-\tilde{B}_\parallel^{(0)}\frac{2B_0}{\beta B_0^2}\sum Ze\int \mathrm{d}^3vf_0\mathrm{e}^{-\mathrm{i}Q}\overline{\mathrm{e}^{\mathrm{i}Q}}+\sum\frac{ZeM}{2TB_0}\int \mathrm{d}^3vf_0v_\perp^2\mathrm{e}^{-\mathrm{i}Q}(\overline{\tilde{B}_\parallel\mathrm{e}^{\mathrm{i}Q}}-\tilde{B}_\parallel^{(0)}\overline{\mathrm{e}^{\mathrm{i}Q}}).\end{aligned} \tag{5.40}$$

Expanding the Bessel functions and the exponentials in Q, inserting H with $\alpha=k_\perp^2T/2M\Omega^2$, and using unperturbed quasineutrality leaves

$$\begin{aligned}0\;=\;&\langle\tilde{\Phi}\rangle\sum\frac{Z^2\mathrm{e}^2}{T}\int \mathrm{d}^3vf_0\left[Q\bar{Q}-\frac{Q^2+\overline{Q^2}}{2}\right]-(\langle\tilde{\Phi}\rangle-\tilde{\Phi}^{(0)})\sum\frac{Z^2\mathrm{e}^2nk_\perp^2}{M\Omega^2}\\&-\mathrm{i}\sum\frac{Z^2\mathrm{e}^2}{2cT}\int \mathrm{d}^3vf_0\left[\overline{\tilde{A}_\parallel v_\parallel}Q^2-\overline{\tilde{A}_\parallel v_\parallel Q^2}Q-\frac{\overline{\tilde{A}_\parallel v_\parallel}Q^3-\overline{\tilde{A}_\parallel v_\parallel Q^3}}{3}\right]\\&+\tilde{A}_\parallel^{(0)}\left(1+\frac{k_\perp^2c^2}{\omega_p^2}\right)\mathrm{i}\sum\frac{Z^2\mathrm{e}^2}{2cT}\int \mathrm{d}^3vf_0\left[\overline{v_\parallel Q}Q^2-\overline{v_\parallel Q^2}Q-\frac{\overline{v_\parallel}Q^3-\overline{v_\parallel Q^3}}{3}\right]\\&-\tilde{B}_\parallel^{(0)}\frac{2B_0}{\beta B_0^2}\sum Ze\int \mathrm{d}^3vf_0[Q\bar{Q}-(Q^2+\overline{Q^2})/2]\\&+\sum\frac{ZeM}{2TB_0}\int \mathrm{d}^3vf_0v_\perp^2\left\{\left[Q\overline{\tilde{B}_\parallel Q}-\frac{Q^2\overline{\tilde{B}_\parallel}+\overline{Q^2\tilde{B}_\parallel}}{2}\right]-\tilde{B}_\parallel^{(0)}\left[Q\bar{Q}-\frac{Q^2+\overline{Q^2}}{2}\right]\right\}.\end{aligned} \tag{5.41}$$

We have already shown that the poloidal variation of $\tilde{\Phi}$ is negligible so we only require the flux surface average of (5.41), which is

$$\begin{aligned}&\langle\tilde{\Phi}\rangle\left[\sum\frac{Z^2\mathrm{e}^2}{T}\left\langle\int \mathrm{d}^3vf_0Q(Q-\bar{Q})\right\rangle+\sum\frac{Z^2\mathrm{e}^2nk_\perp^2}{M\Omega^2}\right]=\tilde{\Phi}^{(0)}\sum\frac{Z^2\mathrm{e}^2nk_\perp^2}{M\Omega^2}\\&-\mathrm{i}\sum\frac{Z^2\mathrm{e}^2}{2cT}\left\langle\left[\tilde{A}_\parallel-\tilde{A}_\parallel^{(0)}\left(1+\frac{k_\perp^2c^2}{\omega_p^2}\right)\right]\int \mathrm{d}^3vf_0v_\parallel\left[\overline{Q^2}Q-\bar{Q}Q^2+\frac{Q^3-\overline{Q^3}}{3}\right]\right\rangle\\&+\tilde{B}_\parallel^{(0)}\frac{2B_0}{\beta B_0^2}\Sigma Ze\left\langle\int \mathrm{d}^3vf_0Q(Q-\bar{Q})\right\rangle,\end{aligned} \tag{5.42}$$

where we neglect $\beta\langle\tilde{B}_\parallel\rangle$, $\beta\tilde{B}_\parallel^{(0)}$ and βb corrections.

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Performing the integrals using (A 30), (A 76), (A 79), (A 82), and (A 83), and noting that only the trapped ion and Bessel contributions matter, we obtain

$$(\gamma q^2\epsilon^{-1/2}+1)\langle\tilde{\Phi}\rangle = \tilde{\Phi}^{(0)} + \gamma\epsilon^{1/2}q\rho_{pi}\frac{v_i}{c\beta}\tilde{B}_\parallel^{(0)} - \mathrm{i}\frac{v_i}{4c}\epsilon^{1/2}q^2k_\perp\rho_{pi}\left\{\sigma\left[\langle\tilde{A}_\parallel\rangle - \tilde{A}_\parallel^{(0)}\left(1+\frac{k_\perp^2c^2}{\omega_p^2}\right)\right] + \frac{5\gamma a}{2}\right\}. \tag{5.43}$$

In the limit in which we take $\beta\to 0$ at finite $k_\perp$ (such that $\omega_p^2/k_\perp^2c^2\to 0$), equation (5.43) with (5.19) and (5.20) inserted gives the large skin depth expression

$$\langle\tilde{\Phi}\rangle \to \left(\frac{\gamma q^2}{\epsilon^{1/2}}+1\right)^{-1}\left[\tilde{\Phi}^{(0)} + \mathrm{i}\frac{v_i\tilde{A}_\parallel^{(0)}}{4c}\frac{k_\perp^2c^2}{\omega_p^2}(\sigma\epsilon^{1/2}q^2k_\perp\rho_{pi}) + \frac{\gamma\epsilon^{1/2}}{\beta}\frac{q\rho_{pi}v_i\tilde{B}_\parallel^{(0)}}{c}\right]. \tag{5.44}$$

In the more interesting finite limit, for which $\epsilon^{3/2}\gg k_\perp^2c^2/\omega_p^2$, using (5.20), (5.21), (5.35), and (5.36) we find the small skin depth result

$$\langle\tilde{\Phi}\rangle \simeq \frac{\tilde{\Phi}^{(0)}}{(\gamma q^2\epsilon^{-1/2}+1)}\left[1+\frac{\sigma+5\gamma}{16}\beta_i\epsilon^{3/2}q^2k_\perp^2\rho_{pi}^2\right] + \frac{\mathrm{i}5\gamma\epsilon^{3/2}q^2k_\perp\rho_{pi}v_i\tilde{A}_\parallel^{(0)}}{8(\gamma q^2\epsilon^{-1/2}+1)c} + \frac{\gamma\epsilon^{1/2}qk_\perp\rho_{pi}v_i\tilde{B}_\parallel^{(0)}}{(\gamma q^2\epsilon^{-1/2}+1)\beta k_\perp c}. \tag{5.45}$$

Clearly, the modification of the electrostatic limit is small when $\tilde{B}_\parallel^{(0)}=0=\tilde{A}_\parallel^{(0)}$. From (5.45), we see that initial magnetic perturbations as large as $\epsilon^{1/2}q\rho_{pi}v_i\tilde{B}_\parallel^{(0)}/c\sim\beta\tilde{\Phi}^{(0)}$ and/or $\epsilon^{3/2}q^2k_\perp\rho_{pi}v_i\tilde{A}_\parallel^{(0)}/c\sim\tilde{\Phi}^{(0)}$ are required to obtain order unity corrections to the electrostatic response. For the same size initial perturbations in the expression (5.44) the response to $\tilde{B}_\parallel^{(0)}$ remains the same, but that due to $\tilde{A}_\parallel^{(0)}$ is very small.

6. Zonal flow responses in terms of initial field values: 15 test cases

In this section we present a summary of our results in the small skin depth limit. Before doing so we note that

$$\frac{k_\perp^2c^2}{\omega_p^2} = \frac{Z(1+\tau)\epsilon^2m}{q^2\beta M}k_\perp^2\rho_{pi}^2 = \frac{Z(1+\tau)m}{\beta M}k_\perp^2\rho_i^2 = \frac{(1+\tau)}{\tau\beta}k_\perp^2\rho_e^2, \tag{6.1}$$

with $\beta=\beta_i(1+\tau)$ and $\tau=ZT_e/T_i$. We, of course, neglect mass ratio corrections in the plasma frequency.

The large skin depth limit of $\beta\to 0$ requires $q^2\beta\ll Zmk_\perp^2\rho_{pi}^2/M\ll Zm/M$ since we must keep $k_\perp^2\rho_{pi}^2\ll 1$ for our analysis to hold. Consequently, the more interesting limit is that of small skin depth $\epsilon^2\sim\epsilon\Delta_S'\gg k_\perp^2c^2/\omega_p^2$ and small β_i ($1\gg\beta_i\gtrsim Zmk_\perp^2\rho_{pi}^2/q^2M$), which will be our focus. In the following we summarize our results for this limit. We will find that electromagnetic perturbations give an electrostatic potential response comparable to the electrostatic limit of $\langle\tilde{\Phi}\rangle\simeq\epsilon^{1/2}\tilde{\Phi}^{(0)}/\gamma q^2$ when $v_i\tilde{A}_\parallel^{(0)}\epsilon^{3/2}q^2k_\perp\rho_{pi}/c\sim\tilde{\Phi}^{(0)}$ or $\epsilon^{1/2}q\rho_{pi}v_i\tilde{B}_\parallel^{(0)}/c\sim\beta\tilde{\Phi}^{(0)}$.

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6.1. *Electromagnetic response to an electrostatic initial perturbation* ($\tilde{B}_\parallel^{(0)} = 0 = \tilde{A}_\parallel^{(0)}$)

In this case $\langle\tilde{\Phi}\rangle/\tilde{\Phi}^{(0)}$ has a very small, linear in β, correction to the electrostatic Rosenbluth and Hinton form as seen from (5.45),

$$\left(\frac{\gamma q^2}{\epsilon^{1/2}} + 1\right)\frac{\langle\tilde{\Phi}\rangle}{\tilde{\Phi}^{(0)}} \simeq 1 + \frac{\sigma + 5\gamma}{16}\beta_i \epsilon^{3/2} q^2 k_\perp^2 \rho_{pi}^2, \tag{6.2}$$

where $\sigma \approx 5.3$. Moreover, $\langle\tilde{B}_\parallel\rangle\rho_i v_i/c\tilde{\Phi}^{(0)}$ and $b\rho_i v_i/c\tilde{\Phi}^{(0)}$ are linear in β and are seen from (5.29) and (5.30) to be given by

$$\frac{v_i\langle\tilde{B}_\parallel\rangle}{k_\perp c\tilde{\Phi}^{(0)}} \simeq \frac{\epsilon\beta_i k_\perp \rho_{pi}}{2q} \tag{6.3}$$

and

$$\frac{v_i b}{k_\perp c\tilde{\Phi}^{(0)}} \simeq \frac{\varsigma\epsilon\beta_i k_\perp \rho_{pi}}{2\gamma q}. \tag{6.4}$$

From (5.20) and (5.21) we find

$$\frac{v_i\langle\tilde{A}_\parallel\rangle}{\mathrm{i}c\tilde{\Phi}^{(0)}} \simeq \frac{\epsilon\beta_i k_\perp \rho_{pi}}{4}, \tag{6.5}$$

and

$$\frac{v_i a}{\mathrm{i}c\tilde{\Phi}^{(0)}} \simeq \frac{\epsilon\beta_i k_\perp \rho_{pi}}{2}, \tag{6.6}$$

where all electromagnetic responses are proportional to β. The correction to the electrostatic result is very small, but all electromagnetic responses are larger and of the same order so strong poloidal variation occurs.

6.2. *Electromagnetic response to a plucked field line initial condition* ($\tilde{B}_\parallel^{(0)} = 0 = \tilde{\Phi}^{(0)}$)

Using (5.45) gives the electrostatic potential response for this $\tilde{A}_\parallel^{(0)} \neq 0$ case to be

$$\frac{c\langle\tilde{\Phi}\rangle}{\mathrm{i}v_i\tilde{A}_\parallel^{(0)}} \simeq \frac{5\gamma\epsilon^{3/2}q^2 k_\perp \rho_{pi}}{8(\gamma q^2\epsilon^{-1/2} + 1)} \simeq \frac{5\epsilon^2 k_\perp \rho_{pi}}{8}. \tag{6.7}$$

For this case, the parallel vector potential responses from (5.20) and (5.21) are given by

$$\frac{\langle\tilde{A}_\parallel\rangle}{\tilde{A}_\parallel^{(0)}} \simeq 1 - \frac{\epsilon^2}{2} \tag{6.8}$$

and

$$\frac{a}{\tilde{A}_\parallel^{(0)}} \simeq -\epsilon, \tag{6.9}$$

showing that $\tilde{A}_\parallel$ will only be very slightly perturbed from $\tilde{A}_\parallel^{(0)}$ with weak poloidal variation due to field line plucking satisfying $R\tilde{A}_\parallel \simeq R_0(\psi)\tilde{A}_\parallel^{(0)}$. The β proportional $\tilde{B}_\parallel$ responses from (5.29) and (5.30) give b and $\langle\tilde{B}_\parallel\rangle$ as comparable, as seen from

$$\frac{\mathrm{i}\langle\tilde{B}_\parallel\rangle}{k_\perp\tilde{A}_\parallel^{(0)}} \simeq \frac{3\epsilon^2\beta q}{4} \tag{6.10}$$

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and

$$\frac{b}{\mathrm{i}k_\perp \tilde{A}_\parallel^{(0)}} \simeq \frac{3\beta q}{4}\left(\epsilon^2 + \frac{2k_\perp^2 c^2}{\omega_p^2}\right). \tag{6.11}$$

6.3. *Electromagnetic response to a compressed field line initial condition* $(\tilde{A}_\parallel^{(0)} = 0 = \tilde{\Phi}^{(0)})$

The response $ck_\perp\langle\tilde{\Phi}\rangle/v_i\tilde{B}_\parallel^{(0)}$ is proportional to $1/\beta$ as seen from (5.45) so in this $\tilde{B}_\parallel^{(0)} \neq 0$ case we multiply through by β to form

$$\frac{c\beta k_\perp\langle\tilde{\Phi}\rangle}{v_i\tilde{B}_\parallel^{(0)}} \simeq \frac{\gamma\epsilon^{1/2}qk_\perp\rho_{pi}}{\gamma q^2\epsilon^{-1/2}+1} \simeq \frac{\epsilon k_\perp\rho_{pi}}{q}. \tag{6.12}$$

The responses $\langle\tilde{B}_\parallel\rangle/\tilde{B}_\parallel^{(0)}$ and $b/\tilde{B}_\parallel^{(0)}$ are given by (5.37) and (5.38) for this field line stretching case:

$$\frac{\langle\tilde{B}_\parallel\rangle}{\tilde{B}_\parallel^{(0)}} \simeq 1 - \frac{3\epsilon q^2\beta}{2[2+(\gamma\epsilon^{7/2}\omega_p^2/k_\perp^2c^2)]} \simeq 1 \tag{6.13}$$

and

$$\frac{b}{\tilde{B}_\parallel^{(0)}} \simeq \epsilon + \frac{3\epsilon q^2\beta}{2[2+(\gamma\epsilon^{7/2}\omega_p^2/k_\perp^2c^2)]} \simeq \epsilon. \tag{6.14}$$

The responses $k_\perp\langle\tilde{A}_\parallel\rangle/\mathrm{i}\tilde{B}_\parallel^{(0)}$ and $k_\perp a/\mathrm{i}\tilde{B}_\parallel^{(0)}$ are obtained from (5.35) and (5.36)

$$\frac{k_\perp\langle\tilde{A}_\parallel\rangle}{\mathrm{i}\tilde{B}_\parallel^{(0)}} \simeq \frac{\epsilon q}{2+(\gamma\epsilon^{7/2}\omega_p^2/k_\perp^2c^2)} \tag{6.15}$$

and

$$\frac{k_\perp a}{\mathrm{i}\tilde{B}_\parallel^{(0)}} \simeq \frac{q}{1+(\gamma\epsilon^{7/2}\omega_p^2/2k_\perp^2c^2)}. \tag{6.16}$$

Stretching or compressing field lines causes very strong poloidal variation in $\tilde{A}_\parallel$ but weak poloidal variation in $\tilde{B}_\parallel$ satisfying $R_0(\psi)\tilde{B}_\parallel \simeq R\tilde{B}_\parallel^{(0)}$.

7. Conclusions

We have derived approximate analytical expressions for long wavelength, collisionless zonal flow residual responses at low β in the Shafranov shifted, circular cross section, large aspect ratio limit of a tokamak. To do so, we formulate and solve a Maxwell–Vlasov description in the form of an initial value problem, retaining the fully self-consistent spatial perturbations in the electric and magnetic fields. As zonal flow perturbations are axisymmetric, the only magnetic field topology change allowed is the switch between rational and irrational field lines within a flux surface – no magnetic islands can be formed. The choice of the initial condition in the non-adiabatic part of the distribution function must be consistent with Maxwell's equations, but it is otherwise arbitrary. The specific choice we make is motivated by the desire to recover the usual long wavelength result in the electrostatic limit that has the residual proportional to the ratio of the classical polarization over the classical

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plus neoclassical polarization. Also, our choice of initial conditions is such that at t = 0 the initial electrostatic, and shear and compressional magnetic perturbations contribute only to quasineutrality, the parallel and the perpendicular components of Ampère's law, respectively. This form is convenient since then the initial conditions in the amplitude of the fields can be chosen independently.

The results we obtain are expected to prove useful for testing turbulent electromagnetic gyrokinetic codes just as Rosenbluth and Hinton has proven useful in the electrostatic limit. The electromagnetic case is of course far more complex, and further complicated by the fact that the parallel vector potential and the parallel magnetic field responses are no longer simply flux functions. Our electromagnetic zonal flow responses provide 15 meaningful tests of fully electromagnetic gyrokinetic turbulence codes. We focus on the small skin depth limit in this section and §§ 5 and 6, but § 5 also gives large skin depth results.

For a pure electrostatic initial condition ($\tilde{\Phi}^{(0)} \neq 0$, $\tilde{B}_\parallel^{(0)} = 0 = \tilde{A}_\parallel^{(0)}$), the usual long wavelength Rosenbluth and Hinton electrostatic zonal flow result is recovered with only a small β correction as seen from (6.2). However, the responses of the parallel vector potential and parallel magnetic fields will have $\cos\theta$ dependence as well flux surface averaged responses and are given by (6.3)–(6.6). Indeed the poloidally varying and flux surface averaged responses are comparable for $\tilde{B}_\parallel$ and $\tilde{A}_\parallel$.

The shear Alfvén field line plucking initial condition ($\tilde{A}_\parallel^{(0)} \neq 0$, $\tilde{B}_\parallel^{(0)} = 0 = \tilde{\Phi}^{(0)}$), also gives stronger flux surface averaged $\tilde{A}_\parallel$ and $\tilde{B}_\parallel$ responses as can be seen from (6.8)–(6.11). Because the initial perturbation is electromagnetic, the $\langle\tilde{\Phi}\rangle$, as well as the $\tilde{A}_\parallel$ responses, are free of any β multipliers as can be seen from (6.7)–(6.9). The compressional Alfvén responses of (6.10) and (6.11) contain β multipliers. Only weak poloidal variation is found in this case for $\tilde{A}_\parallel$. However, the poloidal variation of $\tilde{B}_\parallel$ is important with $b \sim \langle\tilde{B}_\parallel\rangle$.

When the initial perturbation is pure field compression ($\tilde{B}_\parallel^{(0)} \neq 0$, $\tilde{A}_\parallel^{(0)} = 0 = \tilde{\Phi}^{(0)}$) the response of $\langle\tilde{\Phi}\rangle$ appears very large since it is proportional to $1/\beta$ as can be seen from (6.12). In this case $\langle\tilde{\Phi}\rangle$ is better viewed as an order unity response to a $\tilde{B}_\parallel^{(0)} \propto \beta$ initial condition. Then the smallness of $\tilde{B}_\parallel^{(0)}$ cancels the $1/\beta$ dependence making $\beta\tilde{B}_\parallel^{(0)}$ independent of β. The $\tilde{B}_\parallel$ response is given by (6.13) and (6.14) with poloidal variation again weak. However, the poloidal variation of $\tilde{A}_\parallel$ response is $1/\epsilon$ stronger than the flux surface averaged response as seen from (6.15) and (6.16).

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Appendix A. Endless integrals

For any quantity X we define the flux surface average as

$$\langle X\rangle = \frac{\oint \dfrac{\mathrm{d}\theta X}{\boldsymbol{B}_0\cdot\nabla\theta}}{\oint \dfrac{\mathrm{d}\theta}{\boldsymbol{B}_0\cdot\nabla\theta}}, \tag{A 1}$$

and the transit average as

$$\overline{X} = \frac{\oint \mathrm{d}\tau X}{\oint \mathrm{d}\tau} = \frac{\oint \dfrac{\mathrm{d}\theta X B_0}{v_\parallel \boldsymbol{B}_0\cdot\nabla\theta}}{\oint \dfrac{\mathrm{d}\theta B_0}{v_\parallel \boldsymbol{B}_0\cdot\nabla\theta}}, \tag{A 2}$$

with $\mathrm{d}\tau = \mathrm{d}\theta B_0/(v_\parallel \boldsymbol{B}_0\cdot\nabla\theta)$. We allow X to depend on $v_\parallel$ and take the θ integrations in both averages (numerators and denominators) to be over a full poloidal circuit following a charged particle. In this way $v_\parallel$ and θ change signs together at a turning point for trapped particles, and odd functions of $v_\parallel$, such as $\overline{v_\parallel} = 0$ and $\overline{v_\parallel/B_0} = 0$, result in a vanishing transit average. Using $R/R_0(\psi) \simeq 1 + \epsilon(\psi)\cos\theta$, $\epsilon = \epsilon(\psi) = r(\psi)/R_0$, $\underline{B}_0 \equiv \langle B_0\rangle$, $B_0/\underline{B}_0 \simeq R_0/R \simeq 1 - \epsilon\cos\theta + O(\epsilon^2)$, $\lambda = 2\mu\underline{B}_0/v^2$, and

$$\xi = \sqrt{1 - \lambda B_0/\underline{B}_0}, \tag{A 3}$$

we have for trapped particles that

$$\overline{\xi} = 0, \tag{A 4}$$

and

$$\underline{B}_0\overline{\xi/B_0} = 0. \tag{A 5}$$

For the passing particles we see from (A 1) and (A 2) that

$$\underline{B}_0\overline{\xi/B_0} = \underline{B}_0/\langle B_0/\xi\rangle \tag{A 6}$$

and

$$\overline{\xi W} = \langle B_0 W\rangle/\langle B_0/\xi\rangle, \tag{A 7}$$

when $W(\psi,\theta)$. For trapped particles $\overline{\xi W} = 0$, while $W = 1$ gives the passing result

$$\overline{\xi} = \langle B_0\rangle/\langle B_0/\xi\rangle \simeq \frac{2\pi}{\oint \mathrm{d}\theta/\xi}[1 + O(\epsilon^2)]. \tag{A 8}$$

We employ a Shafranov shifted circular flux surface model (Shafranov 1966; Helander & Sigmar 2005). We retain the Shafranov shift Δ_S by taking

$$\boldsymbol{B}_0\cdot\nabla\theta \simeq \frac{\underline{B}_0}{qR_0[1 + (\epsilon - \Delta_S')\cos\theta]}, \tag{A 9}$$

with $\Delta_S' = \mathrm{d}\Delta_S/\mathrm{d}r \sim \epsilon$. Then $B_0/\boldsymbol{B}_0\cdot\nabla\theta \simeq qR_0(\psi)[1 - \Delta_S'\cos\theta + O(\epsilon^2)]$.

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There are many integrals that need to be performed. The simple ones involve combinations of powers of $v_\perp$ and $v_\parallel$ multiplied by a Maxwellian without a transit average. More complicated ones involve transit and/or flux surface averages, for example, integrals of the form $\int \mathrm{d}^3 v f_0 v_\parallel \overline{v_\parallel}$. The integrals involving transit average are most conveniently performed using v and λ variables so that

$$\mathrm{d}^3 v \to \sum_\sigma \frac{\sigma \pi B_0 v^3 \,\mathrm{d}v \,\mathrm{d}\lambda}{\underline{B}_0 |v_\parallel|} \to \frac{2\pi B_0 v^3 \,\mathrm{d}v \,\mathrm{d}\lambda}{\underline{B}_0 |v_\parallel|} = \frac{2\pi B_0 v^2 \,\mathrm{d}v \,\mathrm{d}\lambda}{\underline{B}_0 |\xi|}, \tag{A 10}$$

with $\sum_\sigma \sigma |v_\parallel| \to 2|v_\parallel|$ to account for both directions of $v_\parallel$ for integrals over even functions of $v_\parallel$.

We also see from (A 1) and (A 2) that

$$B_0^{-1} \int \mathrm{d}^3 v f_0 v_\parallel \overline{X} = \left\langle \int \mathrm{d}^3 v \frac{f_0 X}{\langle B_0/v_\parallel \rangle} \right\rangle, \tag{A 11}$$

and

$$B_0^{-1} \int \mathrm{d}^3 v f_0 v_\parallel \overline{X} = \left\langle \int \mathrm{d}^3 v f_0 X \overline{v_\parallel/B_0} \right\rangle = \left\langle B_0^{-1} \int \mathrm{d}^3 v f_0 v_\parallel \overline{X} \right\rangle, \tag{A 12}$$

where we use (A 6). The preceding gives, for example,

$$\int \mathrm{d}^3 v f_0 v_\parallel \overline{v_\parallel} = B_0 \left\langle B_0^{-1} \int \mathrm{d}^3 v f_0 v_\parallel \overline{v_\parallel} \right\rangle. \tag{A 13}$$

In the preceding evaluations and hereafter we will often make use of

$$\left\langle \int \mathrm{d}^3 v f_0 \overline{X} \right\rangle = \left\langle \int \mathrm{d}^3 v f_0 X \right\rangle \tag{A 14}$$

and

$$\left\langle \int \mathrm{d}^3 v f_0 Y \overline{X} \right\rangle = \left\langle \int \mathrm{d}^3 v f_0 X \overline{Y} \right\rangle, \tag{A 15}$$

for arbitrary gyrophase independent functions X and Y.

Rosenbluth & Hinton (1998) analytically evaluated $\langle \int \mathrm{d}^3 v f_0 v_\parallel (v_\parallel - \overline{v_\parallel}) \rangle$ to find

$$(M/nT) \left\langle \int \mathrm{d}^3 v f_0 v_\parallel (v_\parallel - \overline{v_\parallel}) \right\rangle = \gamma \epsilon^{3/2} + O(\epsilon^2), \tag{A 16}$$

where the numerical constant $\gamma \approx 1.64$ comes from $3 \int_0^{1-\epsilon} \mathrm{d}\lambda \bar{\xi} = 2[1 - \gamma \epsilon^{3/2} + O(\epsilon^2)]$. It is instructive to obtain their result by first using

$$\left\langle \int \mathrm{d}^3 v f_0 v_\parallel \overline{v_\parallel} \right\rangle = 2\pi \langle B_0/\underline{B}_0 \rangle \int_0^\infty \mathrm{d}v f_0 v^4 \int_0^{1-\epsilon} \mathrm{d}\lambda \bar{\xi} = (3nT/2M) \int_0^{1-\epsilon} \mathrm{d}\lambda \bar{\xi} [1 + O(\epsilon^2)], \tag{A 17}$$

where

$$4\pi M \int_0^\infty \mathrm{d}v f_0 v^4 = 3nT. \tag{A 18}$$

Then using (A 8) for the passing, with $\bar{\xi} = 0$ for the trapped, and $\cos\theta = 1 - 2\sin^2(\theta/2)$ gives

$$\xi \simeq [1 - \lambda(1 - \epsilon \cos\theta)]^{1/2} = \sqrt{[1 - (1-\epsilon)\lambda] - 2\epsilon\lambda \sin^2(\theta/2)}. \tag{A 19}$$

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We then let $\alpha = \theta/2$ and introduce

$$k^2 = 2\epsilon\lambda/[1-(1-\epsilon)\lambda], \tag{A 20}$$

with

$$\lambda = k^2/[(1-\epsilon)k^2 + 2\epsilon], \tag{A 21}$$

to obtain the passing result in terms of a complete elliptic integral of the first kind

$$\begin{aligned}\oint \frac{\mathrm{d}\theta}{2\pi\xi} &= \frac{\sqrt{2}k}{\pi\sqrt{\epsilon\lambda}}\int_0^{\pi/2}\frac{\mathrm{d}\alpha}{\sqrt{1-k^2\sin^2\alpha}} \\ &= \frac{2\sqrt{(1-\epsilon)k^2+2\epsilon}}{\pi\sqrt{2\epsilon}}K(k) \to \begin{cases} 1+k^2/4 & k\to 0 \\ \dfrac{\sqrt{1+\epsilon}}{\pi\sqrt{2\epsilon}}\ln\left(\dfrac{16}{1-k^2}\right) & k\to 1. \end{cases}\end{aligned} \tag{A 22}$$

Using $\mathrm{d}\lambda = 4\epsilon k\,\mathrm{d}k/[(1-\epsilon)k^2+2\epsilon]^2$ then yields

$$\int_0^{1-\epsilon}\mathrm{d}\lambda\overline{\xi} = 4\epsilon\int_0^1\frac{\mathrm{d}k k\overline{\xi}}{[(1-\epsilon)k^2+2\epsilon]^2} = 8\pi\epsilon\int_0^1\frac{\mathrm{d}k k}{[(1-\epsilon)k^2+2\epsilon]^2\oint\mathrm{d}\theta/\xi}. \tag{A 23}$$

Inserting $\oint \mathrm{d}\theta/\xi$ from (A 22) gives

$$\begin{aligned}\int_0^{1-\epsilon}\mathrm{d}\lambda\overline{\xi} &= 2\pi(2\epsilon)^{3/2}\int_0^1\frac{\mathrm{d}k k}{[(1-\epsilon)k^2+2\epsilon]^{5/2}2K(k)} \\ &= 2(2\epsilon)^{3/2}\int_0^1\frac{\mathrm{d}k k}{[(1-\epsilon)k^2+2\epsilon]^{5/2}}\left\{\left(1-\frac{k^2}{4}\right)+\left[\frac{\pi}{2K(k)}-\left(1-\frac{k^2}{4}\right)\right]\right\}.\end{aligned} \tag{A 24}$$

Using

$$\begin{aligned}\int_0^1\frac{\mathrm{d}k k}{[(1-\epsilon)k^2+2\epsilon]^{5/2}} &= \frac{-1}{3(1-\epsilon)}\int_0^1\mathrm{d}k\frac{\mathrm{d}}{\mathrm{d}k}\frac{1}{[(1-\epsilon)k^2+2\epsilon]^{3/2}} \\ &= \frac{1}{3(1-\epsilon)}\left(\frac{1}{(2\epsilon)^{3/2}}-\frac{1}{(1+\epsilon)^{3/2}}\right),\end{aligned} \tag{A 25}$$

$$\begin{aligned}\int_0^1\frac{\mathrm{d}k k}{[(1-\epsilon)k^2+2\epsilon]^{3/2}} &= \frac{-1}{1-\epsilon}\int_0^1\mathrm{d}k\frac{\mathrm{d}}{\mathrm{d}k}\frac{1}{[(1-\epsilon)k^2+2\epsilon]^{1/2}} \\ &= \frac{1}{1-\epsilon}\left(\frac{1}{\sqrt{2\epsilon}}-\frac{1}{\sqrt{1+\epsilon}}\right),\end{aligned} \tag{A 26}$$

$$\begin{aligned}\int_0^1\frac{\mathrm{d}k k^3}{[(1-\epsilon)k^2+2\epsilon]^{5/2}} &= \frac{1}{(1-\epsilon)} \\ &\times\left[\int_0^1\frac{\mathrm{d}k k}{[(1-\epsilon)k^2+2\epsilon]^{3/2}}-2\epsilon\int_0^1\frac{\mathrm{d}k k}{[(1-\epsilon)k^2+2\epsilon]^{5/2}}\right]\simeq\frac{2(1+2\epsilon)}{3(2\epsilon)^{1/2}}-1,\end{aligned} \tag{A 27}$$

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and

$$\int_0^1 \frac{\mathrm{d}kk}{[(1-\epsilon)k^2+2\epsilon]^{5/2}}\left[\frac{\pi}{2K(k)}-\left(1-\frac{k^2}{4}\right)\right]$$
$$\simeq \int_0^1 \frac{\mathrm{d}k}{k^4}\left[\frac{\pi}{2K(k)}-\left(1-\frac{k^2}{4}\right)\right]+O(\epsilon)\simeq -0.10953, \tag{A 28}$$

gives the result needed to recover (A 16)

$$\int_0^{1-\epsilon} \mathrm{d}\lambda\overline{\xi}=\frac{2}{3}[1-\gamma\epsilon^{3/2}+O(\epsilon^2)]. \tag{A 29}$$

Using (A 13) and (A 29) allows us to generalize the Rosenbluth and Hinton result to find

$$(M/nT)\int \mathrm{d}^3v f_0 v_\parallel(v_\parallel-\overline{v_\parallel})=1-(B_0/\underline{B}_0)[1-\gamma\epsilon^{3/2}+O(\epsilon^2)]=\epsilon\cos\theta+\gamma\epsilon^{3/2}+O(\epsilon^2). \tag{A 30}$$

This θ dependence can also be checked by using $v_\parallel^2=v^2(1-\lambda B_0/\underline{B}_0)$ and

$$v_\parallel\partial v_\parallel/\partial\theta=-(v^2\lambda/2\underline{B}_0)\partial B_0/\partial\theta\simeq-\epsilon(v^2\lambda/2)\sin\theta \tag{A 31}$$

to see that

$$(\partial/\partial\theta)\left[(\underline{B}_0/B_0)\int \mathrm{d}^3v f_0 v_\parallel(v_\parallel-\overline{v_\parallel})\right]\simeq\int \mathrm{d}^3v f_0 v_\parallel\partial v_\parallel/\partial\theta\simeq-\epsilon(nT/M)\sin\theta, \tag{A 32}$$

which when integrated agrees with (A 29).

We can account for the different behaviour of the trapped (and barely passing) and passing particles by using $\overline{v_\parallel}=0$ and estimating $v_\parallel\sim\epsilon^{1/2}v$ and $\int \mathrm{d}^3v\propto\epsilon^{1/2}$ for the trapped particles, while using $\overline{v_\parallel}\simeq v_\parallel[1+O(\epsilon)]$ for the passing ones. In (A 30) these estimates give the order ϵ poloidal variation as coming from the passing particles while the order $\epsilon^{3/2}$ behaviour is due to the trapped particles.

However, we have to be more careful with a related integral since the Shafranov shift will enter. Using

$$\overline{\xi\cos\theta}=0, \tag{A 33}$$

for the trapped particles, while noting that the passing particle result depends on the Shafranov shift we find

$$\overline{\xi\cos\theta}\simeq\left[\oint \mathrm{d}\theta(1-\Delta_S'\cos\theta)\cos\theta\right]\Big/\left(\oint \mathrm{d}\theta/\xi\right)\simeq-(\Delta_S'\overline{\xi}/2)[1+O(\epsilon^2)], \tag{A 34}$$

where we have used (A 8). Consequently, when integrated over λ we obtain

$$(2M/3nT)\int \mathrm{d}^3v f_0 v_\parallel\overline{v_\parallel\cos\theta}\simeq(B_0/\underline{B}_0)\int_0^{1-\epsilon}\mathrm{d}\lambda\overline{\xi\cos\theta}\simeq-\pi\Delta_S'(B_0/\underline{B}_0)\int_0^{1-\epsilon}\mathrm{d}\lambda$$
$$\times\left(\oint \mathrm{d}\theta/\xi\right)^{-1}$$
$$\simeq -(\Delta_S'B_0/2\underline{B}_0)\int_0^{1-\epsilon}\mathrm{d}\lambda\overline{\xi}\simeq-(\Delta_S'/3)(1-\epsilon\cos\theta-\gamma\epsilon^{3/2}+\cdots)$$
$$\simeq O(\epsilon). \tag{A 35}$$

Notice that (A 35) is $O(\epsilon)$ since our shifted circle model requires $\Delta_S'=\mathrm{d}\Delta_S/\mathrm{d}r\sim\epsilon$. Consequently, we see that $\overline{\xi\cos\theta}\sim\epsilon$ for the passing particles, as expected.

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A related integral involves $\underline{B_0^{-1}\overline{B_0\xi}}$, with $\underline{B_0^{-1}\overline{B_0\xi}} = 0$ for the trapped, and

$$\underline{B_0^{-1}\overline{B_0\xi}} \simeq \bar{\xi}[1 + O(\epsilon^2)] \simeq \frac{2\pi}{\oint \mathrm{d}\theta/\xi}, \tag{A 36}$$

for the passing particles, where we can use (A 8) and (A 22). As a result, expanding and neglecting ϵ^2 corrections yields

$$\begin{aligned}(M/nTB_0) &\int \mathrm{d}^3 v f_0 v_\parallel (\overline{B_0 v_\parallel} - \overline{B_0 v_\parallel}) \\ &\simeq \gamma\epsilon^{3/2} + \epsilon\cos\theta - \epsilon(M/nT)\int \mathrm{d}^3 v f_0 (v_\parallel^2 \overline{\cos\theta} - v_\parallel \overline{v_\parallel \cos\theta}) \\ &\simeq \gamma\epsilon^{3/2} + \epsilon\cos\theta + O(\epsilon^2),\end{aligned} \tag{A 37}$$

where we use $\overline{\cos\theta} \sim \epsilon$ for the passing and $\overline{\cos\theta} \sim 1$ for the trapped particles.

Another integral of interest is

$$\underline{B_0^2 B_0^{-2}} \int \mathrm{d}^3 v f_0 v_\perp^2 v_\parallel \overline{v_\parallel} = 2\pi \int_0^\infty \mathrm{d}v f_0 v^6 \int_0^{1-\epsilon} \mathrm{d}\lambda\lambda\bar{\xi} = \frac{15nT^2}{2M^2}\int_0^{1-\epsilon}\mathrm{d}\lambda\lambda\bar{\xi}, \tag{A 38}$$

with $\bar{\xi} = 0$ for the trapped particles, $\bar{\xi}$ given by (A 8) and (A 22) for the passing particles, and

$$4\pi M^2 \int_0^\infty \mathrm{d}v f_0 v^6 = 15nT^2. \tag{A 39}$$

Hence, we need

$$\int_0^{1-\epsilon}\mathrm{d}\lambda\lambda\bar{\xi} = 2(2\epsilon)^{3/2}\int_0^1 \frac{\mathrm{d}k k^3}{[(1-\epsilon)k^2 + 2\epsilon]^{7/2}}\left\{\left(1 - \frac{k^2}{4}\right) + \left[\frac{\pi}{2K(k)} - \left(1 - \frac{k^2}{4}\right)\right]\right\}, \tag{A 40}$$

along with

$$\begin{aligned}\int_0^1 \frac{\mathrm{d}k k^3}{[(1-\epsilon)k^2 + 2\epsilon]^{7/2}} &= \frac{-1}{5(1-\epsilon)}\int_0^1 \mathrm{d}k k^2 \frac{\mathrm{d}}{\mathrm{d}k}\frac{1}{\left[(1-\epsilon)k^2 + 2\epsilon\right]^{5/2}} \\ &= \frac{-1}{5(1-\epsilon)}\left[\frac{1}{(1+\epsilon)^{5/2}} - 2\int_0^1 \frac{\mathrm{d}k k}{[(1-\epsilon)k^2 + 2\epsilon]^{5/2}}\right] \\ &\simeq \frac{2(1 + 2\epsilon + 3\epsilon^2)}{15(2\epsilon)^{3/2}} - \frac{1}{3},\end{aligned} \tag{A 41}$$

and

$$\begin{aligned}\int_0^1 \frac{\mathrm{d}k k^5}{[(1-\epsilon)k^2 + 2\epsilon]^{7/2}} &= \frac{1}{(1-\epsilon)} \\ &\quad \times \left[\int_0^1 \frac{\mathrm{d}k k^3}{[(1-\epsilon)k^2 + 2\epsilon]^{5/2}} - 2\epsilon\int_0^1 \frac{\mathrm{d}k k^3}{[(1-\epsilon)k^2 + 2\epsilon]^{7/2}}\right] \\ &\simeq \frac{8(1 + 3\epsilon)}{15(2\epsilon)^{1/2}} - 1.\end{aligned} \tag{A 42}$$

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Therefore, we find

$$\int_0^{1-\epsilon} \mathrm{d}\lambda \lambda \overline{\xi} \simeq \frac{4}{15}\left[1-\frac{5}{2}\gamma\epsilon^{3/2}+O(\epsilon^2)\right]. \tag{A 43}$$

As a result,

$$\underline{B}_0^2 B_0^{-2}\int \mathrm{d}^3 v f_0 v_\perp^2 v_\parallel \overline{v_\parallel} = (2nT^2/M^2)[1-(5\gamma\epsilon^{3/2}/2)+O(\epsilon^2)]. \tag{A 44}$$

To perform some of the more complicated integrals we need to evaluate some transit averages. We have already evaluated $\oint \mathrm{d}\theta \xi^{-1}$ for the passing particles, but now we also need it for the trapped particles. For the trapped we let

$$\sin(\theta/2) = \kappa \sin\alpha, \tag{A 45}$$

with

$$\kappa^2 = k^{-2} = [1-(1-\epsilon)\lambda]/2\epsilon\lambda, \tag{A 46}$$

and

$$\lambda = 1/[(1-\epsilon)+2\epsilon\kappa^2], \tag{A 47}$$

then $\sqrt{\kappa^2-\sin^2(\theta/2)} = \kappa\cos\alpha$ and $\cos(\theta/2)(\mathrm{d}\theta/2) = \kappa\cos\alpha\,\mathrm{d}\alpha = \sqrt{1-\kappa^2\sin^2\alpha}\,\mathrm{d}\theta/2$, give the half bounce result

$$\oint \mathrm{d}\theta/\xi = 4(2\epsilon\lambda)^{-1/2}\int_0^{\pi/2} \mathrm{d}\alpha/\sqrt{1-\kappa^2\sin^2\alpha} = 4(2\epsilon)^{-1/2}\sqrt{(1-\epsilon)+2\epsilon\kappa^2}K(\kappa). \tag{A 48}$$

We also need to evaluate $\overline{\cos\theta}$ for the trapped particles, but we also evaluate it for the passing particles to check our estimates. For the passing we use $k^2\cos\theta = k^2-2+2[1-k^2\sin^2(\theta/2)]$ to find

$$\begin{aligned}\overline{\cos\theta} &\simeq \frac{\int_{-\pi}^{\pi}\mathrm{d}\theta\cos\theta/\xi}{\int_{-\pi}^{\pi}\mathrm{d}\theta/\xi}+O(\epsilon) \simeq 1-\frac{2}{k^2}+\frac{2\int_0^{\pi/2}\mathrm{d}\theta\sqrt{1-k^2\sin^2\alpha}}{k^2\int_0^{\pi/2}\mathrm{d}\theta/\sqrt{1-k^2\sin^2\alpha}} \\ &= 1-\frac{2}{k^2}+\frac{2E(k)}{k^2K(k)} \to \begin{cases} 7k^2/16 & k\to 0 \\ -1 & k\to 1, \end{cases}\end{aligned} \tag{A 49}$$

where E is a complete elliptic integral of the second kind. For the freely passing particles $\lambda \to 0$ giving $k^2 \to 2\epsilon\lambda \to 0$ so we recover the estimate $\overline{\cos\theta} \sim \epsilon$. For the trapped particles we again use (A 45)–(A 47) to find

$$\begin{aligned}\overline{\cos\theta} &\simeq \frac{\oint \mathrm{d}\theta\cos\theta/\xi}{\oint \mathrm{d}\theta/\xi}+O(\epsilon) \simeq -1+\frac{2\int_0^{\pi/2}\mathrm{d}\alpha\sqrt{1-\kappa^2\sin^2\alpha}}{\int_0^{\pi/2}\mathrm{d}\alpha/\sqrt{1-\kappa^2\sin^2\alpha}} \\ &= \frac{2E(\kappa)}{K(\kappa)}-1 \to \begin{cases} 1-\kappa^2 & \kappa\to 0 \\ -1+\frac{1}{2}\ln\left(\frac{16}{1-\kappa^2}\right) & \kappa\to 1, \end{cases}\end{aligned} \tag{A 50}$$

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where we recover the estimate $\overline{\cos\theta} \sim 1$. The preceding gives the passing particle result

$$\oint \mathrm{d}\theta \xi^{-1} \cos\theta = \frac{4\sqrt{(1-\epsilon)k^2+2\epsilon}}{\sqrt{2\epsilon}k^2}[2E(k)-(2-k^2)K(k)], \tag{A 51}$$

and the half-bounce trapped particle result

$$\oint \mathrm{d}\theta \xi^{-1} \cos\theta = 4(2\epsilon)^{-1/2}\sqrt{(1-\epsilon)+2\epsilon\kappa^2}[2E(\kappa)-K(\kappa)], \tag{A 52}$$

In addition, we will need $\oint \mathrm{d}\theta \xi^{-1}\sin^2\theta$ for the trapped and passing particle. Using

$$\sin^2\theta = 1-[1-2\sin^2(\theta/2)]^2 = 1-[2(1-\kappa^2\sin^2\alpha)-1]^2 = -4(1-\kappa^2\sin^2\alpha)\kappa^2\sin^2\alpha, \tag{A 53}$$

gives the half-bounce trapped particle result

$$\begin{aligned}\oint \mathrm{d}\theta \xi^{-1}\sin^2\theta &= -\frac{16\kappa^2}{\sqrt{2\epsilon}}\sqrt{1-\epsilon+2\epsilon\kappa^2}\int_0^{\pi/2}\mathrm{d}\alpha\sin^2\alpha\sqrt{1-\kappa^2\sin^2\alpha} \\ &= -\frac{16}{3\sqrt{2\epsilon}}\sqrt{(1-\epsilon)+2\epsilon\kappa^2}[(1-\kappa^2)K(\kappa)+(2\kappa^2-1)E(\kappa)],\end{aligned} \tag{A 54}$$

where we use no. 2.583.4 on p. 182 of Gradshteyn & Ryzhik (2007). We will also need to evaluate $\oint \mathrm{d}\theta \xi^{-1}\sin^2\theta$ for the passing particles (because of the barely passing particle contribution). Using

$$\sin^2\theta = 1-[1-2\sin^2(\theta/2)]^2 = 4k^{-2}\sin^2\alpha[(1-k^2\sin^2\alpha)+k^2-1], \tag{A 55}$$

gives the passing particle result

$$\begin{aligned}\oint \mathrm{d}\theta \xi^{-1}\sin^2\theta &= \frac{16\sqrt{(1-\epsilon)k^2+2\epsilon}}{k^2\sqrt{2\epsilon}} \\ &\quad\times\int_0^{\pi/2}\mathrm{d}\alpha\sin^2\alpha\left[\sqrt{1-k^2\sin^2\alpha}-\frac{1-k^2}{\sqrt{1-k^2\sin^2\alpha}}\right] \\ &= \frac{16\sqrt{(1-\epsilon)k^2+2\epsilon}}{k^2\sqrt{2\epsilon}} \\ &\quad\times\int_0^{\pi/2}\mathrm{d}\alpha\left[\left(\sin^2\alpha+\frac{1-k^2}{k^2}\right)\sqrt{1-k^2\sin^2\alpha}-\frac{1-k^2}{k^2\sqrt{1-k^2\sin^2\alpha}}\right] \\ &= \frac{16\sqrt{(1-\epsilon)k^2+2\epsilon}}{3k^4\sqrt{2\epsilon}}[(2-k^2)E(k)-2(1-k^2)K(k)].\end{aligned} \tag{A 56}$$

Using the preceding we can evaluate more complicated integrals like

$$\begin{aligned}(M^2\underline{B}_0^2/2nT^2B_0^2)\int \mathrm{d}^3v f_0 v_\perp^2\overline{v_\parallel^2} &= (M^2\underline{B}_0/2nT^2B_0)\int \mathrm{d}^3v f_0 v^4\lambda(1-\lambda+\epsilon\lambda\overline{\cos\theta}) \\ &= 1-2\epsilon\cos\theta+(15\epsilon/4)\int_0^{\underline{B}_0/B_0}\mathrm{d}\lambda\lambda^2\xi^{-1}\overline{\cos\theta},\end{aligned} \tag{A 57}$$

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where we note that

$$\left\langle (M^2\underline{B}_0^2/2nT^2B_0^2)\int \mathrm{d}^3v f_0 v_\perp^2 \overline{v_\parallel^2}\right\rangle = 1 + O(\epsilon^2). \tag{A 58}$$

To evaluate any $O(\epsilon^{3/2})$ terms we Fourier decompose keeping only the leading harmonic,

$$\varsigma \cos\theta \simeq (15/4\sqrt{\epsilon})\int_0^{\underline{B}_0/B_0} \mathrm{d}\lambda \lambda^2 \xi^{-1}\overline{\cos\theta}. \tag{A 59}$$

Then we can determine ς from

$$\begin{aligned}\varsigma &\simeq (15/4\pi\sqrt{\epsilon})\oint \mathrm{d}\theta \cos\theta \int_0^{\underline{B}_0/B_0} \mathrm{d}\lambda \lambda^2 \xi^{-1}\overline{\cos\theta}\\ &= (15/4\pi\sqrt{\epsilon})\int_0^{1+\epsilon} \mathrm{d}\lambda \lambda^2 \overline{\cos\theta}\oint \mathrm{d}\theta \xi^{-1}\cos\theta.\end{aligned} \tag{A 60}$$

Using the preceding results for $\oint \mathrm{d}\theta \xi^{-1}\cos\theta$ and $\overline{\cos\theta}$ yields

$$\begin{aligned}\varsigma &\simeq \frac{30\sqrt{2}}{\pi}\left\{\int_0^1 \mathrm{d}\kappa \frac{\kappa[2E(\kappa)-K(\kappa)]^2}{(1-\epsilon+2\epsilon\kappa^2)^{5/2}K(\kappa)} + \int_0^1 \mathrm{d}k \frac{[2E(k)-(2-k^2)K(k)]^2}{[(1-\epsilon)k^2+2\epsilon]^{5/2}kE(k)}\right\}\\ &\simeq \frac{30\sqrt{2}}{\pi}\left\{\int_0^1 \frac{\mathrm{d}\kappa\kappa}{K(\kappa)}[2E(\kappa)-K(\kappa)]^2 + \int_0^1 \frac{\mathrm{d}k}{k^6E(k)}[2E(k)-(2-k^2)K(k)]^2\right\}\\ &\approx 5.294,\end{aligned} \tag{A 61}$$

where the last integral is well behaved since $[2E(k)-(2-k^2)K(k)]^2 \propto k^8$ at small k, and the k integral is the passing particle contribution and the κ integral the trapped particle contribution. Consequently,

$$(M^2\underline{B}_0^2/2nT^2B_0^2)\int \mathrm{d}^3v f_0 v_\perp^2 \overline{v_\parallel^2} = 1 - (2\epsilon - \varsigma\epsilon^{3/2})\cos\theta + O(\epsilon^2). \tag{A 62}$$

Combining (A 44) and (A 56) gives

$$\underline{B}_0^2 B_0^{-2}\int \mathrm{d}^3v f_0 v_\perp^2 (v_\parallel \overline{v_\parallel} - \overline{v_\parallel^2}) = (2nT^2/M^2)[(2\epsilon - \varsigma\epsilon^{3/2})\cos\theta - (5\gamma\epsilon^{3/2}/2) + O(\epsilon^2)]. \tag{A 63}$$

Another related integral that we will need is

$$\int \mathrm{d}^3v f_0 v_\perp^2[(v_\parallel^2 + \overline{v_\parallel^2}) - 2v_\parallel\overline{v_\parallel}] = \int \mathrm{d}^3v f_0 v_\perp^2[(v_\parallel - \overline{v_\parallel})^2 + \overline{(v_\parallel - \overline{v_\parallel})^2}] = O(\epsilon^{3/2}). \tag{A 64}$$

Using $(v_\parallel^2 + \overline{v_\parallel^2}) - 2v_\parallel\overline{v_\parallel} = 2(\overline{v_\parallel^2} - v_\parallel\overline{v_\parallel}) + (v_\parallel^2 - \overline{v_\parallel^2})$ and recalling that to evaluate (A 62) we used

$$(M^2\underline{B}_0^2/2nT^2B_0^2)\int \mathrm{d}^3v f_0 v_\perp^2 (v_\parallel^2 - \overline{v_\parallel^2}) \simeq 4\epsilon\cos\theta + \cdots, \tag{A 65}$$

then we see that

$$(M^2\underline{B}_0^2/2nT^2B_0^2)\int \mathrm{d}^3v f_0 v_\perp^2[v_\parallel\overline{v_\parallel} - (v_\parallel^2 + \overline{v_\parallel^2})/2] \simeq -5\gamma\epsilon^{3/2}/2 - \varsigma\epsilon^{3/2}\cos\theta + \cdots. \tag{A 66}$$

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We also require the $O(\epsilon)$ flux function

$$\begin{aligned}\underline{B}_0^2 B_0^{-2} \int \mathrm{d}^3 v f_0 v_\perp^2 v_\parallel \overline{v_\parallel \cos\theta} &= (15nT^2/2M^2) \int_0^{1-\epsilon} \mathrm{d}\lambda \lambda \overline{\xi \cos\theta} \\ &\simeq -(15\Delta_S' nT^2/4M^2) \int_0^{1-\epsilon} \mathrm{d}\lambda\lambda\overline{\xi} \simeq -(\Delta_S' nT^2/M^2)[1 - (5\gamma\epsilon^{3/2}/2) + \cdots] \\ &\simeq -(\Delta_S' nT^2/M^2), \end{aligned} \tag{A 67}$$

where we have used (A 18), (A 34), and (A 43).

The more complicated, poloidally dependent integral $\int \mathrm{d}^3 v f_0 v_\perp^2 \overline{v_\parallel^2 \cos\theta}$ is required to $O(\epsilon^{3/2})$. Flux surface averaging gives

$$\begin{aligned}\langle \underline{B}_0^2 B_0^{-2} \int \mathrm{d}^3 v f_0 v_\perp^2 \overline{v_\parallel^2 \cos\theta} \rangle &= \underline{B}_0 \left\langle \cos\theta \int \mathrm{d}^3 v f_0 v^2 \lambda v_\parallel^2 \overline{B_0^{-1}} \right\rangle \\ &\simeq \left\langle \cos\theta \int \mathrm{d}^3 v f_0 v^2 \lambda v_\parallel^2 (1 + \epsilon \overline{\cos\theta}) \right\rangle \\ &\simeq \underline{B}_0 \left\langle B_0^{-1} \cos\theta \right\rangle \int \mathrm{d}^3 v f_0 v_\perp^2 v_\parallel^2 \\ &\quad + \epsilon \left\langle \cos\theta \int \mathrm{d}^3 v f_0 v^2 \lambda v_\parallel^2 \overline{\cos\theta} \right\rangle \\ &\simeq (nT^2/M^2)[2\epsilon - \Delta_S' + O(\epsilon^2)]. \end{aligned} \tag{A 68}$$

We also need the $\cos\theta$ dependent $O(\epsilon^{3/2})$ portion of this poloidally dependent integral. Fourier decomposing by keeping only the fundamental

$$\begin{aligned}\chi\epsilon^{3/2}\cos\theta &\simeq -(M^2\underline{B}_0^2/nT^2B_0^2) \int \mathrm{d}^3 v f_0 v_\perp^2 \overline{v_\parallel^2 \cos\theta} \\ &\quad + \left\langle (M^2\underline{B}_0^2/nT^2B_0^2) \int \mathrm{d}^3 v f_0 v_\perp^2 \overline{v_\parallel^2 \cos\theta} \right\rangle, \end{aligned} \tag{A 69}$$

we evaluate the Fourier coefficient by multiplying by $qR_0\underline{B}_0^{-1}\cos\theta \boldsymbol{B}_0 \cdot \nabla\theta$ and flux surface averaging to find

$$\begin{aligned}\chi\epsilon^{3/2}/2 &\simeq -(qR_0M^2/nT^2) \left\langle B_0^{-1}\cos\theta \boldsymbol{B}_0 \cdot \nabla\theta \int \mathrm{d}^3 v f_0 v^2 \lambda \overline{v_\parallel^2 \cos\theta} \right\rangle \simeq O(\epsilon^{3/2}) \\ &\simeq -(qR_0M^2/nT^2) \left\langle \cos\theta \int \mathrm{d}^3 v f_0 v^2 \lambda v_\parallel^2 \overline{B_0^{-1}\cos\theta \boldsymbol{B}_0 \cdot \nabla\theta} \right\rangle \\ &\simeq -(M^2/nT^2) \left\langle \cos\theta \int \mathrm{d}^3 v f_0 v^2 \lambda v_\parallel^2 \overline{\cos\theta} \right\rangle + O(\epsilon^2) \\ &\simeq -(15/4\pi) \oint \mathrm{d}\theta \cos\theta \int_0^{\underline{B}_0/B_0} \mathrm{d}\lambda\lambda\xi\overline{\cos\theta} \\ &\simeq -(15/4\pi) \int_0^{1+\epsilon} \mathrm{d}\lambda\lambda\overline{\cos\theta} \oint \mathrm{d}\theta\xi\cos\theta \\ &\simeq -(15/4\pi) \int_0^{1+\epsilon} \mathrm{d}\lambda\lambda\overline{\cos\theta} \oint \mathrm{d}\theta\xi\partial(\sin\theta)/\partial\theta \end{aligned}$$

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$$\simeq +(15/4\pi)\int_0^{1+\epsilon} d\lambda\lambda\overline{\cos\theta}\oint d\theta\,\sin\theta\partial\xi/\partial\theta$$
$$\simeq -(15\epsilon/8\pi)\int_0^{1+\epsilon} d\lambda\lambda^2\overline{\cos\theta}\oint d\theta\xi^{-1}\sin^2\theta. \tag{A 70}$$

Inserting $\oint d\theta\xi^{-1}\sin^2\theta$ and $\overline{\cos\theta}$ for the trapped and passing particles gives

$$\chi \simeq -\frac{20\sqrt{2}}{\pi}\left\{\int_0^1 dk\frac{\kappa\,[2E(\kappa)-K(\kappa)]}{(1-\epsilon+2\epsilon\kappa^2)^{7/2}K(\kappa)}[(1-\kappa^2)K(\kappa)+(2\kappa^2-1)E(\kappa)]\right.$$
$$\left.+\int_0^1 dk\frac{[2E(k)-(2-k^2)K(k)]}{[(1-\epsilon)k^2+2\epsilon]^{7/2}kK(k)}[(2-k^2)E(k)-2(1-k^2)K(k)]\right\}$$
$$\simeq -\frac{20\sqrt{2}}{\pi}\left\{\int_0^1\frac{d\kappa\kappa}{K(\kappa)}[2E(\kappa)-K(\kappa)][(1-\kappa^2)K(\kappa)+(2\kappa^2-1)E(\kappa)]\right.$$
$$\left.+\int_0^1\frac{dk}{k^8K(k)}[2E(k)-(2-k^2)K(k)][(2-k^2)E(k)-2(1-k^2)K(k)]\right\}\approx 0.1131, \tag{A 71}$$

where in the last integral $[2E(k)-(2-k^2)K(k)][(2-k^2)E(k)-2(1-k^2)K(k)]\propto k^8$ at small k to keep it well behaved, and the k integral is the passing particle contribution and the κ integral is the trapped particle contribution (a notation used from here on). Then the preceding gives

$$(M^2\underline{B}_0^2/nT^2B_0^2)\int d^3v f_0 v_\perp^2\overline{v_\parallel^2\cos\theta}\simeq(2\epsilon-\Delta_S')-\chi\epsilon^{3/2}\cos\theta+O(\epsilon^2). \tag{A 72}$$

We next define the more involved integral I and approximate it by

$$I\equiv(M^2/3nT^2)\int d^3v f_0 v_\parallel[3v_\parallel(\overline{v_\parallel^2}-v_\parallel\overline{v_\parallel})+(v_\parallel^3-\overline{v_\parallel^3})]\simeq\epsilon^{5/2}(\sigma+\upsilon\cos\theta), \tag{A 73}$$

with σ and υ order unity constants. The form I is then rewritten using

$$\int d^3v f_0 v_\parallel^2(\overline{v_\parallel^2}-v_\parallel\bar{v}_\parallel) = \int d^3v f_0 v_\parallel^2\{\overline{[\bar{v}_\parallel+(v_\parallel-\bar{v}_\parallel)]^2}-v_\parallel\bar{v}_\parallel\}$$
$$= \int d^3v f_0 v_\parallel^2[\overline{(v_\parallel-\bar{v}_\parallel)^2}-\bar{v}_\parallel(v_\parallel-\bar{v}_\parallel)] \tag{A 74}$$

and

$$\int d^3v f_0 v_\parallel\frac{v_\parallel^3-\overline{v_\parallel^3}}{3} = \int d^3v f_0 v_\parallel\left\{\frac{(v_\parallel-\bar{v}_\parallel)^3-\overline{(v_\parallel-\bar{v}_\parallel)^3}}{3}\right.$$
$$\left.+\,v_\parallel\bar{v}_\parallel(v_\parallel-\bar{v}_\parallel)-\bar{v}_\parallel\overline{(v_\parallel-\bar{v}_\parallel)^2}\right\}, \tag{A 75}$$

to find that

$$I=(M^2/3nT^2)\int d^3v f_0 v_\parallel\{[(v_\parallel-\bar{v}_\parallel)^3-\overline{(v_\parallel-\bar{v}_\parallel)^3}]+3(v_\parallel-\bar{v}_\parallel)\overline{(v_\parallel-\bar{v}_\parallel)^2}\}\simeq O(\epsilon^{5/2}) \tag{A 76}$$

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and

$$\begin{aligned}\langle I\rangle &= (M^2/3nT^2)\left\langle \int \mathrm{d}^3v f_0 v_\parallel [3v_\parallel(\overline{v_\parallel^2} - v_\parallel \bar{v}_\parallel) + (v_\parallel^3 - \overline{v_\parallel^3})]\right\rangle \\ &= (M^2/3nT^2)\left\langle \int \mathrm{d}^3v f_0 [(v_\parallel - \bar{v}_\parallel)^4 + 3(v_\parallel - \bar{v}_\parallel)^2 \overline{(v_\parallel - \bar{v}_\parallel)^2}]\right\rangle \\ &= (M^2/3nT^2)\left\langle \int \mathrm{d}^3v f_0 [\overline{(v_\parallel - \bar{v}_\parallel)^4} + 3(v_\parallel - \bar{v}_\parallel)^2 \overline{(v_\parallel - \bar{v}_\parallel)^2}]\right\rangle \simeq \sigma\epsilon^{5/2}, \quad \text{(A 77)}\end{aligned}$$

with the domain of integration of order $\epsilon^{1/2}$ for the trapped and order unity for the passing. We use $\overline{v_\parallel} = 0$ for the trapped particles, estimate $v_\parallel \sim \epsilon^{1/2} v$ for the trapped (and barely passing) particles, and use $\overline{v_\parallel} \simeq v_\parallel[1 + O(\epsilon)]$ for the freely passing particles. Therefore, we anticipate that the trapped particle ($\sim \epsilon^{5/2}$) contributions will give the final form of (A 72), with passing particle contributions $\sim \epsilon^3$.

To verify this more completely we next form

$$B_0 \partial(B_0^{-1} I)/\partial\theta = (M^2 B_0/nT^2 \underline{B}_0)\int \mathrm{d}^3v f_0 v_\parallel (\partial v_\parallel/\partial\theta)[(v_\parallel - \bar{v}_\parallel)^2 + \overline{(v_\parallel - \bar{v}_\parallel)^2}] \simeq O(\epsilon^{5/2}), \quad \text{(A 78)}$$

where θ derivatives of the λ limits do not contribute because terms containing transit averages have θ independent λ limits and those not containing transit averages are multiples of $v_\parallel$ that will vanish at the upper limit of $\lambda = \underline{B}_0/B_0$. Using $v_\parallel^2 = v^2(1 - \lambda B_0/\underline{B}_0)$ and $v_\parallel \partial v_\parallel/\partial\theta = -(v^2\lambda/2\underline{B}_0)\partial B_0/\partial\theta \simeq -\epsilon(v^2\lambda/2)\sin\theta$ gives

$$\begin{aligned}B_0 \partial(B_0^{-1} I)/\partial\theta &\simeq -\epsilon(M^2 B_0/2nT^2\underline{B}_0)\sin\theta \int \mathrm{d}^3v f_0 \lambda v^2 [(v_\parallel - \bar{v}_\parallel)^2 + \overline{(v_\parallel - \bar{v}_\parallel)^2}] \\ &\simeq O(\epsilon^{5/2}) \\ &\simeq -\epsilon(M^2/2nT^2)\sin\theta \int \mathrm{d}^3v f_0 v_\perp^2 (v_\parallel^2 - 2v_\parallel \bar{v}_\parallel + \overline{v_\parallel^2}) \\ &\simeq -(5\gamma + \varsigma\cos\theta)\epsilon^{5/2}\sin\theta. \quad \text{(A 79)}\end{aligned}$$

where we have used (A 44) and (A 62). Integrating gives to lowest order

$$I \simeq \epsilon^{5/2}[\sigma + 5\gamma\cos\theta + (\varsigma/4)\cos 2\theta], \quad \text{(A 80)}$$

giving $\upsilon = 5\gamma$ and σ given by (A 77) to be the constant found by evaluating

$$\begin{aligned}\sigma &\simeq (5/2\epsilon^{5/2})\left\langle \int_0^{\underline{B}_0/B_0} \mathrm{d}\lambda \xi^{-1}[(\xi - \bar{\xi})^4 + 3(\xi - \bar{\xi})^2 \overline{(\xi - \bar{\xi})^2}]\right\rangle \\ &\simeq (5/2\epsilon^{5/2})\left\langle \int_0^{\underline{B}_0/B_0} \mathrm{d}\lambda \xi^{-1}[\overline{(\xi - \bar{\xi})^4} + 3(\xi - \bar{\xi})^2 \overline{(\xi - \bar{\xi})^2}]\right\rangle \\ &\simeq (5/2\epsilon^{5/2})\left\{ \int_0^{1+\epsilon} \mathrm{d}\lambda \overline{(\xi - \bar{\xi})^4} \oint \frac{\mathrm{d}\theta}{2\pi\xi} + 3\int_0^{1+\epsilon} \mathrm{d}\lambda \overline{(\xi - \bar{\xi})^2} \right. \\ &\quad \left. \times \left[\oint \frac{\mathrm{d}\theta\xi}{2\pi} - 2\bar{\xi} + \bar{\xi}^2 \oint \frac{\mathrm{d}\theta}{2\pi\xi}\right]\right\}. \quad \text{(A 81)}\end{aligned}$$

The $\varsigma\cos 2\theta$ term in (A 80) is of no consequent for our purposes since we only keep $\cos\theta$ terms.

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We also need

$$I_h \equiv (M^2/3nT^2)\int \mathrm{d}^3v f_0 v_\parallel [3v_\parallel h(\overline{v_\parallel^2 h^2} - v_\parallel h\overline{v_\parallel h}) + (v_\parallel^3 h^3 - \overline{v_\parallel^3 h^3})]$$
$$= (M^2/3nT^2)\int \mathrm{d}^3v f_0 v_\parallel \{[(v_\parallel h - \overline{v_\parallel h})^3 - \overline{(v_\parallel h - \overline{v_\parallel h})^3}] + 3(v_\parallel h - \overline{v_\parallel h})\overline{(v_\parallel h - \overline{v_\parallel h})^2}\}$$
$$\simeq O(\epsilon^{5/2}), \tag{A 82}$$

where we define $h = \underline{B}_0/B_0 \simeq 1 + \epsilon\cos\theta$ and note that for the trapped particles $\overline{v_\parallel h} = 0$. Using $v_\parallel h - \overline{v_\parallel h} \simeq v_\parallel - \overline{v_\parallel} + \epsilon v_\parallel \cos\theta - \epsilon\overline{v_\parallel \cos\theta} \simeq v_\parallel - \overline{v_\parallel} + \epsilon v_\parallel \cos\theta + O(\epsilon^2)$ we expand for $\epsilon \ll 1$ to find

$$I_h - I \simeq \epsilon(M^2/nT^2)\int \mathrm{d}^3v f_0 v_\parallel [v_\parallel (v_\parallel - \overline{v_\parallel})^2 \cos\theta + v_\parallel \overline{(v_\parallel - \overline{v_\parallel})^2}\cos\theta] \simeq O(\epsilon^3), \tag{A 83}$$

where we use $v_\parallel - \overline{v_\parallel} \sim v\epsilon$ for the passing particles and $v_\parallel \sim v\epsilon^{1/2}$ for the trapped particles.

Finally, we can simplify (A 81) further by first forming $\oint \mathrm{d}\theta\xi$. For the passing particles we let $\alpha = \theta/2$ to find

$$\oint \mathrm{d}\theta\xi = \frac{4\sqrt{2\epsilon}}{\sqrt{(1-\epsilon)k^2 + 2\epsilon}}\int_0^{\pi/2} \mathrm{d}\alpha \sqrt{1 - k^2\sin^2\alpha} = \frac{4\sqrt{2\epsilon}E(k)}{\sqrt{(1-\epsilon)k^2 + 2\epsilon}}, \tag{A 84}$$

while for the trapped particles we let $\sin(\theta/2) = \kappa\sin\alpha$ to obtain the half bounce result

$$\oint \mathrm{d}\theta\xi = \frac{4\sqrt{2\epsilon}\kappa^2}{\sqrt{1-\epsilon+2\epsilon\kappa^2}}\int_0^{\pi/2} \frac{\mathrm{d}\alpha\cos^2\alpha}{\sqrt{1-\kappa^2\sin^2\alpha}} = \frac{4\sqrt{2\epsilon}[E(\kappa) - (1-\kappa^2)K(\kappa)]}{\sqrt{1-\epsilon+2\epsilon\kappa^2}}, \tag{A 85}$$

where we use

$$\int_0^{\pi/2} \frac{\mathrm{d}\alpha\cos^2\alpha}{\sqrt{1-\kappa^2\sin^2\alpha}} = \frac{1}{\kappa^2}[E(\kappa) - (1-\kappa^2)K(\kappa)] \tag{A 86}$$

from no. 2.584.6 on p. 186 of Gradshteyn & Ryzhik (2007) or use our previous results.

Inserting (A 22), (A 48), (A 84) and (A 85) into (A 81) and using $\bar{\xi} = 0$ for the trapped and (A 8) for the passing particles gives

$$\sigma \simeq \frac{10\sqrt{2}}{\pi\epsilon^2}\left\{\int_0^1 \frac{\mathrm{d}k k\overline{(\xi - \bar{\xi})^4}K(k)}{[(1-\epsilon)k^2 + 2\epsilon]^{3/2}} + \int_0^1 \frac{\mathrm{d}\kappa\kappa\overline{\xi^4}K(\kappa)}{[1-\epsilon+2\epsilon\kappa^2]^{3/2}}\right.$$
$$+ 6\epsilon\int_0^1 \frac{\mathrm{d}k k\overline{(\xi - \bar{\xi})^2}}{[(1-\epsilon)k^2 + 2\epsilon]^{5/2}}\left[E(k) - \frac{\pi^2}{4K(k)}\right]$$
$$\left. + 6\epsilon\int_0^1 \frac{\mathrm{d}\kappa\kappa\overline{\xi^2}}{[1-\epsilon+2\epsilon\kappa^2]^{5/2}}[E(\kappa) - (1-\kappa^2)K(\kappa)]\right\}. \tag{A 87}$$

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Simplifying by noting that $\overline{(\xi-\bar{\xi})^2}\propto k^2$,

$$\sigma \simeq \frac{10\sqrt{2}}{\pi\epsilon^2}\left\{\int_0^1 \mathrm{d}k k^{-2}\overline{(\xi-\bar{\xi})^4}K(k)+\int_0^1 \mathrm{d}\kappa\kappa\overline{\xi^4}K(\kappa)\right.$$
$$+6\epsilon\int_0^1 \mathrm{d}k k^{-4}(\overline{\xi^2}-\overline{\xi}^2)\left[E(k)-\frac{\pi^2}{4K(k)}\right]$$
$$\left.+6\epsilon\int_0^1 \mathrm{d}\kappa\kappa\overline{\xi^2}[E(\kappa)-(1-\kappa^2)K(\kappa)]\right\}. \tag{A 88}$$

We can simplify the trapped particle contributions further by using no. 2.584.15 on p. 187 of Gradshteyn & Ryzhik (2007),

$$\int_0^{\pi/2}\frac{\mathrm{d}\alpha\cos^4\alpha}{\sqrt{1-\kappa^2\sin^2\alpha}}=\frac{4\kappa^2-2}{3\kappa^4}E(\kappa)+\frac{3\kappa^4-5\kappa^2+2}{3\kappa^4}K(\kappa), \tag{A 89}$$

to write

$$\overline{\xi^4}\simeq\frac{\oint \mathrm{d}\theta\xi^3}{\oint \mathrm{d}\theta\xi^{-1}}\simeq\frac{4\epsilon^2\kappa^4\displaystyle\int_0^{\pi/2}\frac{\mathrm{d}\alpha\cos^4\alpha}{\sqrt{1-\kappa^2\sin^2\alpha}}}{\displaystyle\int_0^{\pi/2}\mathrm{d}\alpha/\sqrt{1-\kappa^2\sin^2\alpha}}\simeq\frac{4\epsilon^2}{3}\left[(4\kappa^2-2)\frac{E(\kappa)}{K(\kappa)}+(3\kappa^4-5\kappa^2+2)\right]. \tag{A 90}$$

Then the first trapped particle term becomes

$$\int_0^1 \mathrm{d}\kappa\kappa\overline{\xi^4}K(\kappa)\simeq\frac{4\epsilon^2}{3}\int_0^1 \mathrm{d}\kappa\kappa[(4\kappa^2-2)E(\kappa)+(3\kappa^4-5\kappa^2+2)K(\kappa)]=\frac{32\epsilon^2}{75}, \tag{A 91}$$

upon using $\int_0^1 \mathrm{d}\kappa\kappa E(\kappa)=2/3$, $\int_0^1 \mathrm{d}\kappa\kappa^3E(\kappa)=14/45$, $\int_0^1 \mathrm{d}\kappa\kappa K(\kappa)=1$, $\int_0^1 \mathrm{d}\kappa\kappa^3K(\kappa)=5/9$ and $\int_0^1 \mathrm{d}\kappa\kappa^5K(\kappa)=89/225$, from pp. 615–616, no. 5.112.3–7 of Gradshteyn & Ryzhik (2007). To simplify the second trapped particle term we use

$$\overline{\xi^2}\simeq\frac{\oint \mathrm{d}\theta\xi}{\oint \mathrm{d}\theta\xi^{-1}}\simeq 2\epsilon\left[\frac{E(\kappa)}{K(\kappa)}-(1-\kappa^2)\right], \tag{A 92}$$

to find

$$\int_0^1 \mathrm{d}\kappa\kappa\overline{\xi^2}[E(\kappa)-(1-\kappa^2)K(\kappa)]\simeq 2\epsilon\int_0^1 \mathrm{d}\kappa\kappa[E(\kappa)-(1-\kappa^2)K(\kappa)]^2/K(\kappa),$$
$$\simeq 2\epsilon\left\{\left[\int_0^1 \mathrm{d}\kappa\kappa\frac{E^2(\kappa)}{K(\kappa)}\right]-\frac{32}{75}\right\}, \tag{A 93}$$

where we use the preceding results to perform some of the integrals.

We can also simplify the simpler of the passing particle terms using

$$\overline{\xi^2}\simeq\frac{\oint \mathrm{d}\theta\xi}{\oint \mathrm{d}\theta\xi^{-1}}\simeq\frac{2\epsilon E(k)}{[(1-\epsilon)k^2+2\epsilon]K(k)}\simeq 2\epsilon\frac{E(k)}{k^2K(k)} \tag{A 94}$$

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and

$$\overline{\xi}^2 \simeq \frac{4\pi^2}{\left(\oint \mathrm{d}\theta \xi^{-1}\right)^2} \simeq \frac{\pi^2\epsilon}{2[(1-\epsilon)k^2+2\epsilon]K^2(k)} \simeq \frac{\pi^2\epsilon}{2k^2K^2(k)}, \tag{A 95}$$

giving

$$\int_0^1 \mathrm{d}k k^{-4}(\overline{\xi^2}-\overline{\xi}^2)\left[E(k)-\frac{\pi^2}{4K(k)}\right] \simeq 2\epsilon\int_0^1 \frac{\mathrm{d}k}{k^6K(k)}\left[E(k)-\frac{\pi^2}{4K(k)}\right]^2. \tag{A 96}$$

The remaining passing particle term can be rewritten as

$$\begin{aligned}
&\int_0^1 \mathrm{d}k k^{-2}\overline{(\xi-\bar{\xi})^4}K(k) = \int_0^1 \mathrm{d}k k^{-2}(\overline{\xi^4}-4\overline{\xi^3}\overline{\xi}+6\overline{\xi^2}\overline{\xi}^2-3\overline{\xi}^4)K(k)\\
&\quad= \int_0^1 \mathrm{d}k k^{-2}[(\overline{\xi^4}-4\overline{\xi^3}\overline{\xi}+3\overline{\xi^2}\overline{\xi}^2)+3\overline{\xi}^2(\overline{\xi^2}-\overline{\xi}^2)]K(k)\\
&\quad= 4\epsilon^2\int_0^1 \frac{\mathrm{d}k}{k^6}\left\{\frac{2(2-k^2)E(k)-(1-k^2)K(k)}{3}+\frac{\pi^2}{4K^2(k)}[3E(k)-2(2-k^2)K(k)]\right\}\\
&\qquad+3\pi^2\epsilon^2\int_0^1 \frac{\mathrm{d}k}{k^6K(k)}\left[\frac{E(k)}{K(k)}-\frac{\pi^2}{4K^2(k)}\right]\\
&\quad\simeq 3\pi^2\epsilon^2\int_0^1 \frac{\mathrm{d}k}{k^6K(k)}\left[\frac{E(k)}{K(k)}-\frac{\pi^2}{4K^2(k)}\right]+3\pi^2\epsilon^2\int_0^1 \frac{\mathrm{d}k}{k^6K(k)}\left[\frac{E(k)}{K(k)}-1\right]\\
&+\frac{4\epsilon^2}{3}\int_0^1 \frac{\mathrm{d}kK(k)}{k^6}\left[2(2-k^2)\frac{E(k)}{K(k)}-(1-k^2)-\frac{3\pi^2}{4K^2(k)}(1-2k^2)\right],
\end{aligned} \tag{A 97}$$

by using the passing particle results

$$\bar{\xi} \simeq \frac{2\pi}{\oint \mathrm{d}\theta\xi^{-1}} \simeq \frac{\pi\sqrt{2\epsilon}}{2\sqrt{(1-\epsilon)k^2+2\epsilon}K(k)} \simeq \frac{\pi\sqrt{2\epsilon}}{2kK(k)}, \tag{A 98}$$

$$\overline{\xi^3} \simeq \frac{\oint \mathrm{d}\theta\xi^2}{\oint \mathrm{d}\theta\xi^{-1}} \simeq \frac{\pi\epsilon\sqrt{2\epsilon}(2-k^2)}{2\left[(1-\epsilon)k^2+2\epsilon\right]^{3/2}K(k)} \simeq \frac{\pi\epsilon\sqrt{2\epsilon}(2-k^2)}{2k^3K(k)}, \tag{A 99}$$

and

$$\begin{aligned}
\overline{\xi^4} &\simeq \frac{\oint \mathrm{d}\theta\xi^3}{\oint \mathrm{d}\theta\xi^{-1}} \simeq \frac{4\epsilon^2}{[(1-\epsilon)k^2+2\epsilon]^2K(k)}\\
&\quad\times\int_0^{\pi/2}\frac{\mathrm{d}\alpha[(1-k^2)^2+2k^2(1-k^2)\cos^2\alpha+k^4\cos^4\alpha]}{\sqrt{1-k^2\sin^2\alpha}}\\
&\simeq \frac{4\epsilon^2}{k^4}\left\{(1-k^2)^2+\frac{2(1-k^2)}{K(k)}[E(k)-(1-k^2)K(k)]\right.
\end{aligned}$$

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$$+\frac{[(4k^2-2)E(k)+(3k^4-5k^2+2)K(k)]}{3K(k)}\Bigg\}$$
$$\simeq \frac{4\epsilon^2}{3k^2}\left[2(2-k^2)\frac{E(k)}{K(k)}-(1-k^2)\right]. \tag{A 100}$$

Putting all this together (A 88) becomes

$$\sigma = \frac{40\sqrt{2}}{\pi}\left\{3\left[\int_0^1 \mathrm{d}\kappa\,\kappa\frac{E^2(\kappa)}{K(\kappa)}\right]-\frac{88}{75}+3\int_0^1\frac{\mathrm{d}k}{k^6K(k)}\left[E(k)-\frac{\pi^2}{4K(k)}\right]^2\right.$$
$$+\int_0^1\frac{\mathrm{d}k}{k^6}\left[\frac{2(2-k^2)E(k)}{3}-\frac{(1-k^2)K(k)}{3}\right.$$
$$\left.\left.+\frac{3\pi^2}{4K(k)}\left(\frac{2E(k)}{K(k)}-\frac{\pi^2}{4K^2(k)}-\frac{2(2-k^2)}{3}\right)\right]\right\}. \tag{A 101}$$

Numerically evaluating the integrals yields the constant σ to be

$$\sigma \approx 5.325 \tag{A 102}$$

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