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Diffraction by a spherical cap

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1. *Introduction.* It was first observed by Rayleigh (17) in 1897 that low-frequency approximations to the solutions of the steady-state wave equation could be obtained from the solutions of the corresponding static problems. Rayleigh determined only the first term in the expansion of the solution in powers of the product of the wave number and a typical dimension. Many recent investigations have been concerned with deriving a systematic method of calculating the higher-order terms in the above expansion. Most of the problems which have been solved in this manner have been concerned with scattering by a disk or by a strip: the first systematic approach to these problems was that of Bouwkamp ((3), p. 71).

There are several different methods of setting up a perturbation procedure for these problems (16), but the simplest and most effective procedure appears to be that of reducing the boundary-value problem to a Fredholm integral equation of the second kind whose kernel vanishes for zero wave number. This reduction was first carried out for the disk problem by Magnus (14); alternative simpler integral equations have since been given by Jones (12), Bazer and Brown (2), Collins (7) and Heins and MacCamy (8).

The diffraction problems of disks are particular examples of a general class of boundary-value problems which can be reduced by a routine procedure to Fredholm integral equations of the second kind using a method developed by Williams (20), (21). Problems involving diffraction by a spherical cap are also amenable to reduction by the method of (20) and in the present paper the approach of (20) is applied to the problem of diffraction of an axisymmetric incident scalar wave by a soft spherical cap.

The problem is reduced to finding the solution of a Fredholm integral equation of the second kind. A solution is obtained by iteration for low frequencies when the incident field is (i) a plane wave, and (ii) due to a point source on the polar axis of the cap. Expansions are obtained for the scattering coefficient for both problems.

In the special case when the cap becomes a sphere the integral equation can be solved exactly and the solution obtained is shown to agree with the known separation of variables solution.

The electrostatic problems of the prolate and oblate spheroidal caps can be formulated as Fredholm integral equations of the first kind. The method of reduction of the present problem to a Fredholm integral equation of the second kind is such that the corresponding integral equation of the second kind for each of these two electro-

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static problems can be written down immediately from a knowledge of the appropriate integral equation of the first kind.

2. *Statement of problem.* Spherical polar coordinates, (r, θ, ϕ), are chosen so that the spherical cap is defined by $r = R$, $0 \leqslant \theta \leqslant \alpha$, all ϕ.

It is assumed that a small amplitude acoustic wave whose velocity potential is given by $\psi_0(r, \theta, \phi) \exp(-i\omega t)$ is incident on the cap. The total velocity potential will then be of the form $\{\psi_0(r, \theta, \phi) + \psi_D(r, \theta, \phi)\} \exp(-i\omega t)$, where $\psi_D(r, \theta, \phi)$ satisfies the differential equation

$$(\nabla^2 + k^2)\,\psi(r, \theta, \phi) = 0,$$

and $k^2 = \omega^2/c^2$, where c is the speed of the wave propagation. The diffracted velocity potential, $\psi_D(r, \theta, \phi)$, is required to satisfy the Sommerfeld radiation conditions at infinity ((1), p. 25), i.e.

(1) $\lim_{r\to\infty} r\psi_D(r, \theta, \phi)$ is bounded,

(2) $\lim_{r\to\infty} r\left\{\frac{\partial \psi_D}{\partial r}(r, \theta, \phi) - ik\psi_D(r, \theta, \phi)\right\} = 0.$

The cap is assumed to be soft, i.e. the total velocity potential vanishes on the surface of the cap ((3), p. 38).

3. *Formulation of integral equation.* It can be shown by standard methods (4, 13) that

$$\psi_D(r, \theta, \phi) = -R^2 \int_0^{\alpha} \int_0^{2\pi} \frac{\exp ikD}{4\pi D} g(\theta')\, d\chi\, d\theta', \tag{3·1}$$

where

$$D = (R^2 + r^2 - 2rR\cos\omega)^{\frac{1}{2}}, \tag{3·2}$$

$$\cos\omega = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\chi \tag{3·3}$$

and

$$g(\theta') = \left\{\frac{\partial \psi_D}{\partial r'}(r', \theta', \phi')\bigg|_{r'\to R+} - \frac{\partial \psi_D}{\partial r'}(r', \theta', \phi')\bigg|_{r'\to R-}\right\} \sin\theta' \quad (0 \leqslant \theta' \leqslant \alpha). \tag{3·4}$$

It follows, from equation (3·1) and the boundary condition on the cap, that $g(\theta')$ satisfies the integral equation

$$f(\theta) = \int_0^{\alpha} K_1(\theta', \theta)\, g(\theta')\, d\theta', \tag{3·5}$$

where

$$f(\theta) = \psi_0(r, \theta, \phi) \quad (r = R,\ 0 \leqslant \theta \leqslant \alpha), \tag{3·6}$$

$$K_1(\theta', \theta) = R^2 \int_0^{2\pi} \frac{\exp ikD_0}{4\pi D_0} d\chi$$

and

$$D_0 = R(2 - 2\cos\omega)^{\frac{1}{2}}.$$

The integral equation (3·5) can be written as

$$f(\theta) = \int_0^{\alpha} K_0(\theta', \theta)\, g(\theta')\, d\theta' + \int_0^{\alpha} F(\theta', \theta)\, g(\theta')\, d\theta, \tag{3·7}$$

where

$$K_0(\theta', \theta) = R^2 \int_0^{2\pi} \frac{d\chi}{4\pi D_0}$$

and

$$F(\theta', \theta) = R^2 \int_0^{2\pi} \frac{\exp ikD_0 - 1}{4\pi D_0} d\chi. \tag{3·8}$$

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It has been shown (5) that

$$\int_0^\alpha K_0(\theta',\theta)\,g(\theta')\,d\theta' = \frac{R}{2\pi}\int_0^\theta \frac{1}{(\cos\theta'-\cos\theta)^{\frac{1}{2}}}\int_{\theta'}^\alpha \frac{g(t)\,dt\,d\theta'}{(\cos\theta'-\cos t)^{\frac{1}{2}}} \quad (0\leqslant\theta\leqslant\alpha). \tag{3·9}$$

It now follows immediately from equations (3·7) and (3·9) and from the analysis of Williams (20) that equation (3·7) reduces to

$$S(\theta)+\int_0^\alpha L(\theta,v)\,S(v)\,dv = f^*(\theta) \quad (0\leqslant\theta\leqslant\alpha), \tag{3·10}$$

where

$$S(\theta) = \frac{R}{2\pi}\int_\theta^\alpha \frac{g(t)\,dt}{(\cos\theta-\cos t)^{\frac{1}{2}}}$$

and $f^*(\theta)$ and $L(\theta,v)$ are defined by

$$f(\theta) = \int_0^\theta \frac{f^*(t)\,dt}{(\cos t-\cos\theta)^{\frac{1}{2}}} \tag{3·11}$$

and

$$F(\theta',\theta) = \frac{R}{2\pi}\int_0^\theta \frac{1}{(\cos u-\cos\theta)^{\frac{1}{2}}}\int_0^{\theta'} \frac{L(u,v)\,dv\,du}{(\cos v-\cos\theta')^{\frac{1}{2}}}. \tag{3·12}$$

4. *Determination of the kernel $L(u,v)$.* The function F defined in equation (3·8) is a particular case of a more general class of functions of the type

$$F(\theta',\theta) = \frac{R^2}{4\pi}\int_0^{2\pi} A(\cos\omega)\,d\chi, \tag{4·1}$$

where A is a piecewise continuous function and $\cos\omega$ is defined in equation (3·3). In this section the kernel L corresponding to this more general class of functions will be determined.

The function A can be represented by a series of Legendre polynomials ((15), pp. 726, 752), thus

$$A(\cos\omega) = \frac{1}{R}\sum_{n=0}^{\infty} a_n P_n(\cos\omega), \tag{4·2}$$

where the coefficients a_n can be determined by standard methods ((15), p. 753). Hence ((15), p. 1274)

$$F(\theta',\theta) = \frac{R}{2}\sum_{n=0}^{\infty} a_n P_n(\cos\theta)\,P_n(\cos\theta').$$

It follows from the Mehler–Dirichlet integral for the Legendre function that

$$F(\theta',\theta) = \frac{R}{2\pi}\int_0^\theta \frac{1}{(\cos u-\cos\theta)^{\frac{1}{2}}}\int_0^{\theta'} \frac{\frac{2}{\pi}\sum_{n=0}^{\infty} a_n\cos(n+\frac{1}{2})u\,\cos(n+\frac{1}{2})v\,dv\,du}{(\cos v-\cos\theta')^{\frac{1}{2}}}. \tag{4·3}$$

Comparison of equation (3·12) with equation (4·3) shows that

$$L(u,v) = \frac{2}{\pi}\sum_{n=0}^{\infty} a_n\cos(n+\tfrac{1}{2})u\,\cos(n+\tfrac{1}{2})v. \tag{4·4}$$

$L(u,v)$ can be rewritten as $(1/\pi)\{L_1(u-v)+L_1(u+v)\}$, (4·5)

where

$$L_1(x) = \sum_{n=0}^{\infty} a_n\cos(n+\tfrac{1}{2})x. \tag{4·6}$$

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Inversion of the Mehler–Dirichlet integral ((6), p. 381) gives

$$\cos(n+\tfrac{1}{2})x = \frac{1}{\sqrt{2}}\frac{d}{dx}\int_0^x \frac{P_n(\cos\theta)\sin\theta\,d\theta}{(\cos\theta-\cos x)^{\frac{1}{2}}}; \tag{4·7}$$

hence from equations (4·1), (4·2), (4·6) and (4·7)

$$L_1(x) = \frac{\sqrt{2}}{R}\frac{d}{dx}\int_0^x \frac{F(\tfrac{1}{2}\pi,\theta)\sin\theta\,d\theta}{(\cos\theta-\cos x)^{\frac{1}{2}}} = \frac{R}{\sqrt{2}}\frac{d}{dx}\int_0^x \frac{A(\cos\theta)\sin\theta\,d\theta}{(\cos\theta-\cos x)^{\frac{1}{2}}}. \tag{4·8}$$

The electrostatic boundary-value problems for prolate and oblate spheroidal caps can be formulated as integral equations of the form of equation (3·7), where $F(\theta',\theta)$ is of the form given by equation (4·1), and can then be reduced to integral equations similar to equation (3·10)(20). The above analysis thus enables the kernel L corresponding to these problems to be written down immediately from equations (4·5) and (4·8).

For $F(\theta',\theta)$ defined by equation (3·8), equation (4·8) gives

$$L_1(x) = \frac{d}{dx}\int_0^1 \frac{(\exp i2kR\,|b|\,s-1)\,ds}{(1-s^2)^{\frac{1}{2}}}\operatorname{sgn} b,$$

where $b = \sin\frac{1}{2}x$. The modified kernel, $L_1(x)$, can be written in a closed form as

$$L_1(x) = ikR\cos\tfrac{1}{2}x - \tfrac{1}{2}\pi kR\cos\tfrac{1}{2}x\{\mathrm{J}_1(2kR\,|b|)+i\mathrm{H}_1(2kR\,|b|)\},$$

where $\mathrm{J}_1(\rho)$ and $\mathrm{H}_1(\rho)$ are, respectively, the Bessel and Struve functions of the first order.

5. *Solution of integral equation for an incident plane wave.* The integral equation (3·10) will be solved by an iterative method, when kR is small, for the special case when a plane wave travelling along the polar axis is incident on the convex surface of the spherical cap; i.e. $\psi_0(r,\theta,\phi) = \exp -ikr\cos\theta$. Inversion of equation (3·11) ((6), p. 368) leads to

$$f^*(\theta) = \frac{1}{\pi}\frac{d}{d\theta}\int_0^\theta \frac{f(x)\sin x\,dx}{(\cos x-\cos\theta)^{\frac{1}{2}}}. \tag{5·1}$$

It is known ((15), p. 1466) that

$$\psi_0(r,\theta,\phi) = \sum_{n=0}^{\infty}(-i)^n(2n+1)j_n(kr)\mathrm{P}_n(\cos\theta); \tag{5·2}$$

hence by using equations (3·6), (4·7) and (5·2), equation (5·1) becomes

$$f^*(\theta) = \frac{\sqrt{2}}{\pi}\sum_{n=0}^{\infty}(-i)^n(2n+1)j_n(kR)\cos(n+\tfrac{1}{2})\theta. \tag{5·3}$$

The kernel can be expanded in powers of kR yielding

$0 \leqslant v \leqslant \theta$,

$$L(\theta,v) = ikR\,\mathrm{P}_1(\theta,v)+(kR)^2\mathrm{P}_4(\theta,v)+i(kR)^3\mathrm{P}_2(\theta,v)+(kR)^4\mathrm{P}_5(\theta,v) + i(kR)^5\mathrm{P}_3(\theta,v)+O(kR)^6;$$

$\theta \leqslant v \leqslant \alpha$,

$$L(\theta,v) = ikR\,\mathrm{P}_1(\theta,v)+(kR)^2\mathrm{P}_4(v,\theta)+i(kR)^3\mathrm{P}_2(\theta,v)+(kR)^4\mathrm{P}_5(v,\theta) + i(kR)^5\mathrm{P}_3(\theta,v)+O(kR)^6;$$

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where
$$\mathrm{P}_1(\theta, v) = (2/\pi)\cos\tfrac{1}{2}\theta\cos\tfrac{1}{2}v,$$
$$\mathrm{P}_2(\theta, v) = -(2/3\pi)(\cos\tfrac{1}{2}\theta\cos\tfrac{1}{2}v - \cos\tfrac{3}{2}\theta\cos\tfrac{3}{2}v),$$
$$\mathrm{P}_3(\theta, v) = (2/45\pi)(2\cos\tfrac{1}{2}\theta\cos\tfrac{1}{2}v - 3\cos\tfrac{3}{2}\theta\cos\tfrac{3}{2}v + \cos\tfrac{5}{2}\theta\cos\tfrac{5}{2}v),$$
$$\mathrm{P}_4(\theta, v) = -\tfrac{1}{2}\sin\theta\cos v,$$
$$\mathrm{P}_5(\theta, v) = \tfrac{1}{16}(2\sin\theta\cos v - \sin 2\theta\cos 2v).$$

The iterative solution of the integral equation (3·10) is found to be

$$\begin{aligned} S(\theta) = {} & (\sqrt{2}/\pi)\cos\tfrac{1}{2}\theta \\ & - ikR(\sqrt{2}/\pi)(V_1\cos\tfrac{1}{2}\theta + \cos\tfrac{3}{2}\theta) \\ & - (kR)^2(\sqrt{2}/\pi)(V_2\cos\tfrac{1}{2}\theta + V_3\cos\theta + \tfrac{1}{3}\cos\tfrac{5}{2}\theta) \\ & + i(kR)^3(\sqrt{2}/\pi)(V_4\cos\tfrac{1}{2}\theta + V_5\cos\theta + V_6\cos\tfrac{3}{2}\theta + \tfrac{1}{15}\cos\tfrac{7}{2}\theta) \\ & + (kR)^4(\sqrt{2}/\pi)(V_7\cos\tfrac{1}{2}\theta + V_8\cos\theta + V_9\theta\sin\theta + V_{10}\cos\tfrac{3}{2}\theta \\ & \quad + V_{11}\cos 2\theta + \tfrac{1}{18}\cos\tfrac{5}{2}\theta + \tfrac{1}{105}\cos\tfrac{9}{2}\theta) \\ & - i(kR)^5(\sqrt{2}/\pi)(V_{12}\cos\tfrac{1}{2}\theta + V_{13}\cos\theta + V_{14}\theta\sin\theta + V_{15}\cos\tfrac{3}{2}\theta \\ & \quad + V_{16}\cos 2\theta + V_{17}\cos\tfrac{5}{2}\theta + \tfrac{1}{150}\cos\tfrac{7}{2}\theta + \tfrac{1}{945}\cos\tfrac{11}{2}\theta) \\ & + O(kR)^6, \end{aligned}$$

where the coefficients V_i $(i = 1, \ldots, 17)$ are given in the Appendix.

In the case of $\alpha = \pi$, these coefficients simplify and $S(\theta)$ then takes the form

$$\begin{aligned} S(\theta) = {} & (\sqrt{2}/\pi)\cos\tfrac{1}{2}\theta \\ & - ikR(\sqrt{2}/\pi)(\cos\tfrac{1}{2}\theta + \cos\tfrac{3}{2}\theta) \\ & - (kR)^2(\sqrt{2}/\pi)(\tfrac{1}{2}\cos\tfrac{1}{2}\theta + \tfrac{1}{3}\cos\tfrac{5}{2}\theta) \\ & + i(kR)^3(\sqrt{2}/\pi)(\tfrac{1}{6}\cos\tfrac{1}{2}\theta + \tfrac{1}{2}\cos\tfrac{3}{2}\theta + \tfrac{1}{15}\cos\tfrac{7}{2}\theta) \\ & + (kR)^4(\sqrt{2}/\pi)(\tfrac{1}{24}\cos\tfrac{1}{2}\theta - \tfrac{1}{3}\cos\tfrac{3}{2}\theta + \tfrac{1}{18}\cos\tfrac{5}{2}\theta + \tfrac{1}{105}\cos\tfrac{9}{2}\theta) \\ & - i(kR)^5(\sqrt{2}/\pi)(\tfrac{1}{120}\cos\tfrac{1}{2}\theta + \tfrac{3}{8}\cos\tfrac{3}{2}\theta + \tfrac{1}{150}\cos\tfrac{7}{2}\theta + \tfrac{1}{945}\cos\tfrac{11}{2}\theta) \\ & + O(kR)^6. \end{aligned}$$

6. *Limiting case of a sphere.* In the case of a soft sphere the total velocity potential is zero on and within the sphere. Hence the boundary-value problem for a sphere can be reduced to equation (3·5) provided that $g(\theta')$ is defined as

$$g(\theta') = \left\{\frac{\partial\psi_D}{\partial r'}(r', \theta', \phi')\bigg|_{r'\to R_+} + \frac{\partial\psi_0}{\partial r'}(r', \theta', \phi')\bigg|_{r'=R}\right\}\sin\theta' \quad (0 \leqslant \theta' \leqslant \pi), \tag{6·1}$$

and α is set equal to π. The corresponding Fredholm integral equation is

$$S(\theta) + \int_0^\pi L(\theta, v)\, S(v)\, dv = f^*(\theta), \tag{6·2}$$

where
$$S(\theta) = \frac{R}{2\pi}\int_\theta^\pi \frac{g(x)\, dx}{(\cos\theta - \cos x)^{\frac{1}{2}}} \tag{6·3}$$

and $L(\theta, v)$ and $f^*(\theta)$ are defined by equations (3·12) and (3·11), respectively.

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Equation (6·2) will now be solved when the incident wave is a plane wave. From the series expansions for $\exp ikD_0/D_0$ and $1/D_0$ ((15), pp. 1274, 1466) and equations (3·8), (4·1), (4·2) and (4·5) it can be seen that

$$L(\theta, v) = \sum_{n=0}^{\infty} (A_n - 1) \cos (n+\tfrac{1}{2})\theta \cos (n+\tfrac{1}{2}) v, \tag{6·4}$$

where $A_n = ikR(2n+1) j_n(kR)\, h_n^{(1)}(kR)$. It follows from equations (5·3), (6·2) and (6·4) that S is of the form

$$\sum_{n=0}^{\infty} B_n \cos (n+\tfrac{1}{2})\theta,$$

where the B_n are, as yet, undetermined constants. Substitution of equations (5·3) and (6·4) and the above form for S into equation (6·2) and the use of the orthogonality property of the cosines give

$$S(\theta) = \frac{\sqrt{2}}{\pi} \sum_{n=0}^{\infty} \frac{(-i)^{n+1} \cos (n+\tfrac{1}{2})\theta}{kRh_n^{(1)}(kR)}. \tag{6·5}$$

$S(\theta)$ as defined by equation (5·4) is in agreement with the expansion of equation (6·5) in powers of kR up to the fifth.

The problem of diffraction of an incident plane scalar wave by a soft sphere, radius R, can be solved exactly by the standard separation of variables technique ((19), p. 3). Thus, for the incident wave, $\psi_0(r, \theta, \phi)$, defined in equation (5·2), it is found that

$$\psi_D(r, \theta, \phi) = -\sum_{n=0}^{\infty} (-i)^n (2n+1) \frac{j_n(kR)}{h_n^{(1)}(kR)} h_n^{(1)}(kr)\, \mathrm{P}_n(\cos\theta). \tag{6·6}$$

It follows from equations (5·2), (6·1) and (6·6) after slight manipulation that

$$g(t) = \sum_{n=0}^{\infty} \frac{(-i)^{n+1} (2n+1)\, \mathrm{P}_n(\cos t) \sin t}{kR^2 h_n^{(1)}(kR)}. \tag{6·7}$$

Inversion of the Mehler–Dirichlet integral ((6), p. 372) gives

$$\sin (n+\tfrac{1}{2}) x = -\frac{1}{\sqrt{2}} \frac{d}{dx} \int_x^{\pi} \frac{\mathrm{P}_n(\cos t) \sin t\, dt}{(\cos x - \cos t)^{\frac{1}{2}}} \tag{6·8}$$

and an integration of equation (6·8) with respect to x from $x = \theta$ to $x = \pi$ then yields

$$(2n+1) \int_{\theta}^{\pi} \frac{\mathrm{P}_n(\cos t) \sin t\, dt}{(\cos\theta - \cos t)^{\frac{1}{2}}} = 2\sqrt{2} \cos (n+\tfrac{1}{2})\theta. \tag{6·9}$$

Hence, from equations (6·3), (6·7) and (6·9),

$$S(\theta) = \frac{\sqrt{2}}{\pi} \sum_{n=0}^{\infty} \frac{(-i)^{n+1} \cos (n+\tfrac{1}{2})\theta}{kRh_n^{(1)}(kR)}.$$

This expression for $S(\theta)$ agrees with that already found by solving the integral equation (6·2).

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7. *Scattering cross-section.* A relationship has been shown to exist between the extinction cross-section and the amplitude of the diffracted far field for an obstacle in an incident plane wave ((9),(10),(11)). In the problem under consideration there is no absorption taking place; thus, the scattering cross-section, σ_s, is defined by

$$\sigma_s = \frac{4\pi}{k}\Im\{A(\pi)\}, \tag{7·1}$$

where $$\psi_D(r,\theta) = A(\theta)\frac{\exp ikr}{r} + O\left(\frac{1}{r^2}\right) \quad \text{as} \quad r \to \infty.$$

The diffracted far field will now be determined as a function of $S(\theta)$. When r is large,

$$\frac{\exp ikD}{4\pi D} = \frac{\exp ik(r - R\cos\omega)}{4\pi r} + O\left(\frac{1}{r^2}\right),$$

where D and $\cos\omega$ are defined in equations (3·2) and (3·3), respectively. It can be seen from equation (3·1) that

$$\psi_D(r,\theta) = -R^2\frac{\exp ikr}{4\pi r}\int_0^\alpha\int_0^{2\pi}\exp(-ikR\cos\omega)\,g(\theta')\,d\chi d\theta' + O\left(\frac{1}{r^2}\right).$$

Substituting for $\exp(-ikR\cos\omega)$ from equation (5·2) and carrying out the integration with respect to χ give

$$\psi_D(r,\theta) = -R^2\frac{\exp ikr}{2r}\int_0^\alpha\sum_{n=0}^{\infty}(-i)^n(2n+1)j_n(kR)\,\mathrm{P}_n(\cos\theta)\,\mathrm{P}_n(\cos\theta') \times g(\theta')\,d\theta' + O\left(\frac{1}{r^2}\right). \tag{7·2}$$

The replacement of $\mathrm{P}_n(\cos\theta')$ in equation (7·2) by the Mehler–Dirichlet integral and the interchanging of the order of integration lead to

$$\psi_D(r,\theta) = -R\sqrt{2}\frac{\exp ikr}{r}\sum_{n=0}^{\infty}(-i)^n(2n+1)j_n(kR)\,\mathrm{P}_n(\cos\theta) \times \int_0^\alpha S(x)\cos(n+\tfrac{1}{2})x\,dx + O\left(\frac{1}{r^2}\right).$$

After evaluating the necessary integrals, the diffracted far field can be written as

$$\psi_D(r,\theta) = (\mathbf{N}.\mathbf{M}_\alpha.\mathbf{P}kR + O(kR)^7)\frac{\exp ikr}{kr} + O\left(\frac{1}{r^2}\right), \tag{7·3}$$

where $$\mathbf{N} = (1, ikR, (kR)^2, i(kR)^3, (kR)^4, i(kR)^5),$$

$$\mathbf{M}_\alpha = (m_{ij};\ 0 \leqslant i,j \leqslant 5:\ m_{ij} \equiv 0, j > i),$$

$$\mathbf{P}' = (\mathrm{P}_0(\cos\theta), \mathrm{P}_1(\cos\theta), \mathrm{P}_2(\cos\theta), \mathrm{P}_3(\cos\theta), \mathrm{P}_4(\cos\theta), \mathrm{P}_5(\cos\theta))$$

and $\mathbf{P}'$ is the transpose of $\mathbf{P}$, i.e. $\mathbf{N}$ is a row vector, $\mathbf{M}_\alpha$ is a square matrix and $\mathbf{P}$ is a column vector. The non-zero elements of the matrix $\mathbf{M}_\alpha$ are given in the Appendix.

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When $\alpha = \pi$ the matrix $\mathbf{M}_\alpha$ simplifies to

$$\mathbf{M}_\pi = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{2}{3} & 1 & 0 & 0 & 0 & 0 \\ -\frac{1}{3} & 0 & 0 & 0 & 0 & 0 \\ -\frac{2}{15} & -\frac{3}{5} & -\frac{1}{9} & 0 & 0 & 0 \\ -\frac{2}{45} & -\frac{1}{3} & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then $$\psi_D = (\mathbf{N} \,.\, \mathbf{M}_\pi \,.\, \mathbf{P} kR + O(kR)^7) \frac{\exp ikr}{kr} + O\left(\frac{1}{r^2}\right)$$

agrees with the expression for the diffracted far field given by Senior (18), who treats the problem of diffraction by a prolate spheroid.

It can be seen from equations (7·1) and (7·3) that, for arbitrary values of α,

$$\sigma_s = 4\pi R^2[(m_{10} - m_{11}) + (kR)^2 (m_{30} - m_{31} + m_{32} - m_{33}) + (kR)^4 (m_{50} - m_{51} + m_{52} - m_{53} + m_{54} - m_{55}) + O(kR)^5].$$

When $\alpha = \pi$ the above expression simplifies to the known result for the soft sphere, namely, $$\sigma_s = 4\pi R^2[1 - \tfrac{1}{3}(kR)^2 + \tfrac{17}{45}(kR)^4 + O(kR)^5].$$

8. *Solution of integral equation for point source.* When the incident field is that due to a point source situated at $(z_0, 0, 0)$, $z_0 > R$, the incident velocity potential is

$$\psi_0(r, \theta, \phi) = \frac{z_0 \exp ikD_1}{D_1}, \tag{8·1}$$

where $D_1 = (r^2 + z_0^2 - 2rz_0 \cos\theta)^{\frac{1}{2}}$. From equations (3·6), (4·7) and (5·1) and the series expression for $\psi_0(r, \theta, \phi)$ ((15), p. 1466) it can be shown that

$$f^*(\theta) = \frac{\sqrt{2}}{\pi} \sum_{n=0}^{\infty} (-i)^n (2n+1) j_n(kR) d_n \cos(n + \tfrac{1}{2})\theta \quad (z_0 > R,\ 0 \leqslant \theta \leqslant \alpha),$$

where $$d_n = (i)^{n+1} kz_0 h_n^{(1)}(kz_0) = \exp ikz_0 \sum_{m=0}^{\infty} \frac{(n+m)!}{m!\,(n-m)!} \left(\frac{i}{2kz_0}\right)^m. \tag{8·2}$$

The iterative solution of the integral equation (3·10) is

$$\begin{aligned} S(\theta) = {} & (\sqrt{2}/\pi) d_0 \cos\tfrac{1}{2}\theta \\ & - ikR(\sqrt{2}/\pi)(W_1 \cos\tfrac{1}{2}\theta + d_1 \cos\tfrac{3}{2}\theta) \\ & - (kR)^2 (\sqrt{2}/\pi)(W_2 \cos\tfrac{1}{2}\theta + W_3 \cos\theta + \tfrac{1}{3} d_2 \cos\tfrac{5}{2}\theta) \\ & + i(kR)^3 (\sqrt{2}/\pi)(W_4 \cos\tfrac{1}{2}\theta + W_5 \cos\theta + W_6 \cos\tfrac{3}{2}\theta + \tfrac{1}{15} d_3 \cos\tfrac{7}{2}\theta) + O(kR)^4, \end{aligned}$$

where W_i $(i = 1, \ldots, 6)$ are given in the Appendix.

When $\alpha = \pi$ the above expression simplifies to

$$\begin{aligned} S(\theta) = {} & (\sqrt{2}/\pi) d_0 \cos\tfrac{1}{2}\theta \\ & - ikR(\sqrt{2}/\pi)(d_0 \cos\tfrac{1}{2}\theta + d_1 \cos\tfrac{3}{2}\theta) \\ & - (kR)^2 (\sqrt{2}/\pi)(\tfrac{1}{2} d_0 \cos\tfrac{1}{2}\theta + \tfrac{1}{3} d_2 \cos\tfrac{5}{2}\theta) \\ & + i(kR)^3 (\sqrt{2}/\pi)(\tfrac{1}{6} d_0 \cos\tfrac{1}{2}\theta + \tfrac{1}{2} d_1 \cos\tfrac{3}{2}\theta + \tfrac{1}{15} d_3 \cos\tfrac{7}{2}\theta) + O(kR)^4. \end{aligned}$$

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The limiting case of a sphere will now be considered. The exact solution of integral equation (6·2) when the incident wave is due to a point source is found to be

$$S(\theta) = \frac{\sqrt{2}}{\pi} \sum_{n=0}^{\infty} \frac{(-i)^{n+1} d_n \cos(n+\frac{1}{2})\theta}{kRh_n^{(1)}(kR)}. \tag{8·3}$$

When this expression is expanded in powers of kR up to the third power it is seen that it agrees with the iterative solution of the integral equation (3·10) evaluated at $\alpha = \pi$. The expression for $S(\theta)$ which is given in equation (8·3) is also in agreement with that which can be obtained directly from equations (6·1) and (6·3) and a use of the standard technique of separation of variables to solve the problem of diffraction by a soft sphere of an incident wave that is due to a point source.

When the incident wave due to a point source at $(z_0, 0, 0)$ is defined by equation (8·1), the sum of the scattering and absorption coefficients, $C_s + C_A$, can be written as (11)

$$C_s + C_A = \frac{4\pi}{kz_0\Omega} \Im\{\psi_{DE}(z_0, 0, 0)\}, \tag{8·4}$$

where Ω is the solid angle subtended at $(z_0, 0, 0)$ by the cap,

$$\psi_{DE}(r, \theta, \phi) = -R^2 \int_0^{\alpha} \int_0^{2\pi} \frac{\exp(-ikD)}{4\pi D} g(\theta')\, d\chi\, d\theta' \tag{8·5}$$

and D and $g(\theta')$ are defined in equations (3·2) and (3·4) respectively. From the series expansion for $\exp(-ikD)/4\pi D$ ((15), p. 1466) it can be seen that equation (8·5) becomes

$$\psi_{DE}(r, \theta, \phi) = \frac{ikR^2}{2} \sum_{n=0}^{\infty} (2n+1) j_n(kR)\, h_n^{(2)}(kr)\, \mathrm{P}_n(\cos\theta) \int_0^{\alpha} \mathrm{P}_n(\cos\theta')\, g(\theta')\, d\theta' \quad (r > R). \tag{8·6}$$

It follows from equation (7·2) that equation (8·6) can be rewritten as

$$\psi_{DE}(r, \theta, \phi) = ikR\sqrt{2} \sum_{n=0}^{\infty} (2n+1) j_n(kR)\, h_n^{(2)}(kr)\, \mathrm{P}_n(\cos\theta) \times \int_0^{\alpha} S(x) \cos(n+\tfrac{1}{2}) x\, dx \quad (r > R). \tag{8·7}$$

In the problem that is being considered there is no absorption, therefore the absorption coefficient, C_A, is zero. From equations (8·4) and (8·7) it follows that

$$C_s = \frac{4\pi R}{\Omega z_0} \Re\left\{\sqrt{2} \sum_{n=0}^{\infty} (2n+1) j_n(kR)\, h_n^{(2)}(kz_0) \int_0^{\alpha} S(x) \cos(n+\tfrac{1}{2}) x\, dx\right\}. \tag{8·8}$$

The explicit form of C_s will now be calculated for $z_0 \gg R$. The solid angle subtended at $(z_0, 0, 0)$ by the spherical cap is then

$$0 \leqslant \alpha \leqslant \alpha_0 = \cos^{-1}\left(\frac{R}{z_0}\right),$$

$$\frac{1}{\Omega} = \frac{z_0^2}{\pi R^2 \sin^2\alpha}\left[1 - 2\left(\frac{R}{z_0}\right)\cos\alpha + \left(\frac{R}{z_0}\right)^2 \left(\frac{3+\cos^2\alpha}{4}\right) + O\left(\frac{R}{z_0}\right)^3\right];$$

$$\cos^{-1}\left(\frac{R}{z_0}\right) = \alpha_0 \leqslant \alpha \leqslant \pi,$$

$$\frac{1}{\Omega} = \frac{z_0^2}{\pi R^2}\left[1 - \frac{1}{4}\left(\frac{R}{z_0}\right)^2 + O\left(\frac{R}{z_0}\right)^4\right].$$

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By carrying out the necessary integrations, the following expressions for the scattering coefficient are obtained.

$0 \leqslant \alpha \leqslant \alpha_0$,

$$C_s = \frac{4}{\sin^2\alpha}\left[q_1 + (kR)^2 q_2 + \left(\frac{R}{z_0}\right)(q_3 - 2q_1\cos\alpha) + \left(\frac{R}{z_0}\right)^2\left\{q_4 - 2q_3\cos\alpha + q_1\left(\frac{3+\cos^2\alpha}{4}\right)\right\} + O\left(\frac{R}{z_0}\right)^3\right].$$

$\alpha_0 \leqslant \alpha \leqslant \pi$,

$$C_s = 4\left[q_1 + (kR)^2 q_2 + \left(\frac{R}{z_0}\right) q_3 + \left(\frac{R}{z_0}\right)^2 (q_4 - \tfrac{1}{4}q_1) + O\left(\frac{R}{z_0}\right)^3\right].$$

It is assumed that kR and R/z_0 are of the same order of magnitude. The coefficients q_i $(i = 1, \ldots, 4)$ are given in the Appendix. The first two terms in the above expressions for the scattering coefficients correspond to the plane wave scattering coefficient.

When $\alpha = \pi$ the scattering coefficient simplifies to

$$C_s = 4\left[1 - \tfrac{1}{3}(kR)^2 - \frac{1}{4}\left(\frac{R}{z_0}\right)^2 + O\left(\frac{R}{z_0}\right)^3\right]. \tag{8·9}$$

The scattering coefficient for a soft sphere can be obtained directly from equations (8·2), (8·3) and (8·8): i.e.

$$C_s = \frac{4\pi}{\Omega}\Re\left\{\sum_{n=0}^{\infty}(2n+1)\frac{j_n(kR)}{h_n^{(1)}(kR)}h_n^{(1)}(kz_0)\,h_n^{(2)}(kz_0)\right\}. \tag{8·10}$$

The expression for C_s in equation (8·9) agrees with the expansion of equation (8·10) in powers of kR and R/z_0 up to terms of $O(R/z_0)^3$.

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APPENDIX

Notation:

$$a(n,m) = \frac{1}{\pi}\left\{\frac{\sin(n+m)\alpha}{(n+m)} + \frac{\sin(n-m)\alpha}{(n-m)}\right\},$$

$$a(n,n) = \frac{1}{\pi}\left\{\alpha + \frac{\sin 2n\alpha}{2n}\right\},$$

$$b(n,m) = \frac{\cos(n+m)\alpha}{2(n+m)} + \frac{\cos(n-m)\alpha}{2(n-m)},$$

$$C(n) = \frac{\alpha}{2}\left\{\frac{\cos(n-1)\alpha}{(n-1)} - \frac{\cos(n+1)\alpha}{(n+1)}\right\} + \frac{\sin(n+1)\alpha}{2(n+1)^2} - \frac{\sin(n-1)\alpha}{2(n-1)^2}.$$

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$$V_1 = a(\tfrac{1}{2}, \tfrac{1}{2}),$$

$$V_2 = -\tfrac{1}{2} + a(\tfrac{3}{2}, \tfrac{1}{2}) + a^2(\tfrac{1}{2}, \tfrac{1}{2}),$$

$$V_3 = \tfrac{1}{2} b(1, \tfrac{1}{2}),$$

$$V_4 = -\tfrac{5}{6} a(\tfrac{1}{2}, \tfrac{1}{2}) + \tfrac{1}{3} a(\tfrac{5}{2}, \tfrac{1}{2}) + a(\tfrac{1}{2}, \tfrac{1}{2})\, a(\tfrac{3}{2}, \tfrac{1}{2}) + \tfrac{1}{2} a(1, \tfrac{1}{2})\, b(1, \tfrac{1}{2}) + a^3(\tfrac{1}{2}, \tfrac{1}{2}),$$

$$V_5 = \tfrac{1}{2} b(1, \tfrac{3}{2}) + \tfrac{1}{2} a(\tfrac{1}{2}, \tfrac{1}{2})\, b(1, \tfrac{1}{2}),$$

$$V_6 = \tfrac{1}{2} - \tfrac{1}{3} a(\tfrac{3}{2}, \tfrac{1}{2}),$$

$$V_7 = \tfrac{5}{24} + \tfrac{1}{6} a(\tfrac{3}{2}, \tfrac{1}{2}) + \tfrac{1}{15} a(\tfrac{7}{2}, \tfrac{1}{2}) - \tfrac{7}{6} a^2(\tfrac{1}{2}, \tfrac{1}{2}) + \tfrac{1}{3} a(\tfrac{1}{2}, \tfrac{1}{2})\, a(\tfrac{5}{2}, \tfrac{1}{2}) - \tfrac{1}{3} a^2(\tfrac{3}{2}, \tfrac{1}{2})$$
$$+ \tfrac{1}{2} a(1, \tfrac{1}{2})\, b(1, \tfrac{3}{2}) + a^2(\tfrac{1}{2}, \tfrac{1}{2})\, a(\tfrac{3}{2}, \tfrac{1}{2}) + a(\tfrac{1}{2}, \tfrac{1}{2})\, a(1, \tfrac{1}{2})\, b(1, \tfrac{1}{2}) + a^4(\tfrac{1}{2}, \tfrac{1}{2}),$$

$$V_8 = \tfrac{7}{48} b(1, \tfrac{1}{2}) + \tfrac{1}{6} b(1, \tfrac{5}{2}) + \tfrac{1}{16} b(1, \tfrac{1}{2}) \cos 2\alpha + \tfrac{1}{4} a(\tfrac{1}{2}, \tfrac{1}{2})\, b(1, \tfrac{1}{2}) + \tfrac{1}{6} a(\tfrac{5}{2}, \tfrac{1}{2})\, b(1, \tfrac{1}{2})$$
$$+ \tfrac{1}{2} a^2(\tfrac{1}{2}, \tfrac{1}{2})\, b(1, \tfrac{1}{2}),$$

$$V_9 = -\tfrac{1}{8} b(1, \tfrac{1}{2}),$$

$$V_{10} = -\tfrac{1}{3} a(\tfrac{3}{2}, \tfrac{3}{2}) - \tfrac{1}{3} a(\tfrac{1}{2}, \tfrac{1}{2})\, a(\tfrac{3}{2}, \tfrac{1}{2}),$$

$$V_{11} = -\tfrac{1}{16} b(2, \tfrac{1}{2}),$$

$$V_{12} = \tfrac{61}{120} a(\tfrac{1}{2}, \tfrac{1}{2}) - \tfrac{1}{18} a(\tfrac{5}{2}, \tfrac{1}{2}) + \tfrac{1}{105} a(\tfrac{9}{2}, \tfrac{1}{2}) - \tfrac{1}{6} a(\tfrac{1}{2}, \tfrac{1}{2})\, a(\tfrac{3}{2}, \tfrac{1}{2}) + \tfrac{1}{15} a(\tfrac{1}{2}, \tfrac{1}{2})\, a(\tfrac{7}{2}, \tfrac{1}{2})$$
$$- \tfrac{1}{48} a(1, \tfrac{1}{2})\, b(1, \tfrac{1}{2}) + \tfrac{1}{6} a(1, \tfrac{1}{2})\, b(1, \tfrac{5}{2}) - \tfrac{1}{3} a(\tfrac{3}{2}, \tfrac{1}{2})\, a(\tfrac{3}{2}, \tfrac{3}{2}) - \tfrac{1}{16} a(2, \tfrac{1}{2})\, b(2, \tfrac{1}{2})$$
$$- \tfrac{3}{2} a^3(\tfrac{1}{2}, \tfrac{1}{2}) + \tfrac{1}{3} a^2(\tfrac{1}{2}, \tfrac{1}{2})\, a(\tfrac{5}{2}, \tfrac{1}{2}) + \tfrac{1}{2} a(\tfrac{1}{2}, \tfrac{1}{2})\, a(1, \tfrac{1}{2})\, b(1, \tfrac{3}{2}) + \tfrac{1}{4} a(\tfrac{1}{2}, \tfrac{1}{2})\, a(1, \tfrac{1}{2})\, b(1, \tfrac{1}{2})$$
$$- \tfrac{4}{3} a(\tfrac{1}{2}, \tfrac{1}{2})\, a^2(\tfrac{3}{2}, \tfrac{1}{2}) + \tfrac{1}{6} a(1, \tfrac{1}{2})\, a(\tfrac{5}{2}, \tfrac{1}{2})\, b(1, \tfrac{1}{2}) + \tfrac{1}{16} a(1, \tfrac{1}{2})\, b(1, \tfrac{1}{2}) \cos 2\alpha$$
$$+ a^3(\tfrac{1}{2}, \tfrac{1}{2})\, a(\tfrac{3}{2}, \tfrac{1}{2}) + \tfrac{3}{2} a^2(\tfrac{1}{2}, \tfrac{1}{2})\, a(1, \tfrac{1}{2})\, b(1, \tfrac{1}{2}) + a^5(\tfrac{1}{2}, \tfrac{1}{2}) - \tfrac{1}{4} b(1, \tfrac{1}{2})\, C(\tfrac{1}{2}),$$

$$V_{13} = \tfrac{5}{16} b(1, \tfrac{3}{2}) + \tfrac{1}{30} b(1, \tfrac{7}{2}) - \tfrac{29}{48} a(\tfrac{1}{2}, \tfrac{1}{2})\, b(1, \tfrac{1}{2}) - \tfrac{1}{6} a(\tfrac{3}{2}, \tfrac{1}{2})\, b(1, \tfrac{3}{2})$$
$$+ \tfrac{1}{16} b(1, \tfrac{3}{2}) \cos 2\alpha + \tfrac{1}{2} a(\tfrac{1}{2}, \tfrac{1}{2})\, a(\tfrac{3}{2}, \tfrac{1}{2})\, b(1, \tfrac{1}{2}) + \tfrac{1}{16} a(\tfrac{1}{2}, \tfrac{1}{2})\, b(1, \tfrac{1}{2}) \cos 2\alpha$$
$$+ \tfrac{1}{4} a(1, \tfrac{1}{2})\, b^2(1, \tfrac{1}{2}) + \tfrac{1}{2} a^3(\tfrac{1}{2}, \tfrac{1}{2})\, b(1, \tfrac{1}{2}),$$

$$V_{14} = -\tfrac{1}{8} b(1, \tfrac{3}{2}) - \tfrac{1}{8} a(\tfrac{1}{2}, \tfrac{1}{2})\, b(1, \tfrac{1}{2}),$$

$$V_{15} = \tfrac{3}{8} - \tfrac{1}{30} a(\tfrac{3}{2}, \tfrac{1}{2}) - \tfrac{1}{3} a(\tfrac{3}{2}, 1) - \tfrac{1}{9} a(\tfrac{5}{2}, \tfrac{3}{2}) - \tfrac{1}{3} a^2(\tfrac{3}{2}, \tfrac{1}{2}) - \tfrac{1}{3} a^2(\tfrac{1}{2}, \tfrac{1}{2})\, a(\tfrac{3}{2}, \tfrac{1}{2}),$$

$$V_{16} = -\tfrac{1}{16} b(2, \tfrac{3}{2}) - \tfrac{1}{16} a(\tfrac{1}{2}, \tfrac{1}{2})\, b(2, \tfrac{1}{2}),$$

$$V_{17} = \tfrac{1}{45} a(\tfrac{5}{2}, \tfrac{1}{2}).$$

$$m_{00} = -a(\tfrac{1}{2}, \tfrac{1}{2}),$$

$$m_{10} = \{V_1 a(\tfrac{1}{2}, \tfrac{1}{2}) + a(\tfrac{3}{2}, \tfrac{1}{2})\},$$

$$m_{20} = \{(V_2 + \tfrac{1}{6})\, a(\tfrac{1}{2}, \tfrac{1}{2}) + V_3 a(1, \tfrac{1}{2}) + \tfrac{1}{3} a(\tfrac{5}{2}, \tfrac{1}{2})\},$$

$$m_{30} = -\{(V_4 + \tfrac{1}{6} V_1)\, a(\tfrac{1}{2}, \tfrac{1}{2}) + V_5 a(1, \tfrac{1}{2}) + (V_6 + \tfrac{1}{6})\, a(\tfrac{3}{2}, \tfrac{1}{2}) + \tfrac{1}{15} a(\tfrac{7}{2}, \tfrac{1}{2})\},$$

$$m_{40} = -\{(V_7 + \tfrac{1}{6} V_2 + \tfrac{1}{120})\, a(\tfrac{1}{2}, \tfrac{1}{2}) + (V_8 + \tfrac{1}{6} V_3)\, a(1, \tfrac{1}{2}) - 2V_9 C(\tfrac{1}{2}) - V_{10} a(\tfrac{3}{2}, \tfrac{1}{2})$$
$$- V_{11} a(2, \tfrac{1}{2}) + \tfrac{1}{9} a(\tfrac{5}{2}, \tfrac{1}{2}) + \tfrac{1}{105} a(\tfrac{9}{2}, \tfrac{1}{2})\},$$

$$m_{50} = \{(V_{12} + \tfrac{1}{6} V_4 + \tfrac{1}{120} V_1)\, a(\tfrac{1}{2}, \tfrac{1}{2}) + (V_{13} + \tfrac{1}{6} V_5)\, a(1, \tfrac{1}{2}) + (V_{15} + \tfrac{1}{6} V_6 + \tfrac{1}{120})\, a(\tfrac{3}{2}, \tfrac{1}{2})$$
$$- V_{16} a(2, \tfrac{1}{2}) + 2V_{14} C(\tfrac{1}{2}) + V_{17} a(\tfrac{5}{2}, \tfrac{1}{2}) + \tfrac{4}{225} a(\tfrac{7}{2}, \tfrac{1}{2}) + \tfrac{1}{945} a(\tfrac{11}{2}, \tfrac{1}{2})\},$$

$$m_{11} = a(\tfrac{3}{2}, \tfrac{1}{2}),$$

$$m_{21} = \{V_1 a(\tfrac{3}{2}, \tfrac{1}{2}) + a(\tfrac{3}{2}, \tfrac{3}{2})\},$$

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$$m_{31} = -\{(V_2+\tfrac{1}{10})\,a(\tfrac{3}{2},\tfrac{1}{2})+V_3a(\tfrac{3}{2},1)+\tfrac{1}{3}a(\tfrac{5}{2},\tfrac{3}{2})\},$$

$$m_{41} = -\{(V_4+\tfrac{1}{10}V_1)\,a(\tfrac{3}{2},\tfrac{1}{2})+V_5a(\tfrac{3}{2},1)+(V_6+\tfrac{1}{10})\,a(\tfrac{3}{2},\tfrac{3}{2})+\tfrac{1}{15}a(\tfrac{7}{2},\tfrac{3}{2})\},$$

$$m_{51} = \{(V_7+\tfrac{1}{10}V_2+\tfrac{1}{280})\,a(\tfrac{3}{2},\tfrac{1}{2})+(V_8+\tfrac{1}{10}V_3)\,a(\tfrac{3}{2},1)-2V_9C(\tfrac{3}{2})-V_{10}a(\tfrac{3}{2},\tfrac{3}{2})$$
$$-V_{11}a(2,\tfrac{3}{2})+\tfrac{4}{45}a(\tfrac{5}{2},\tfrac{3}{2})+\tfrac{1}{105}a(\tfrac{9}{2},\tfrac{3}{2})\},$$

$$m_{22} = \tfrac{1}{3}a(\tfrac{5}{2},\tfrac{1}{2}),$$

$$m_{32} = -\tfrac{1}{3}\{V_1a(\tfrac{5}{2},\tfrac{1}{2})+a(\tfrac{5}{2},\tfrac{3}{2})\},$$

$$m_{42} = -\tfrac{1}{3}\{(V_2+\tfrac{1}{14})\,a(\tfrac{5}{2},\tfrac{1}{2})+V_3a(\tfrac{5}{2},1)+\tfrac{1}{3}a(\tfrac{5}{2},\tfrac{5}{2})\},$$

$$m_{52} = \tfrac{1}{3}\{(V_4+\tfrac{1}{14}V_1)\,a(\tfrac{5}{2},\tfrac{1}{2})+V_5a(\tfrac{5}{2},1)+(V_6+\tfrac{1}{14})\,a(\tfrac{5}{2},\tfrac{3}{2})+\tfrac{1}{15}a(\tfrac{7}{2},\tfrac{5}{2})\},$$

$$m_{33} = -\tfrac{1}{15}a(\tfrac{7}{2},\tfrac{1}{2}),$$

$$m_{43} = -\tfrac{1}{15}\{V_1a(\tfrac{7}{2},\tfrac{1}{2})+a(\tfrac{7}{2},\tfrac{3}{2})\},$$

$$m_{53} = \tfrac{1}{15}\{(V_2+\tfrac{1}{18})\,a(\tfrac{7}{2},\tfrac{1}{2})+V_3a(\tfrac{7}{2},1)+\tfrac{1}{3}a(\tfrac{7}{2},\tfrac{5}{2})\},$$

$$m_{44} = -\tfrac{1}{105}a(\tfrac{9}{2},\tfrac{1}{2}),$$

$$m_{54} = \tfrac{1}{105}\{V_1a(\tfrac{9}{2},\tfrac{1}{2})+a(\tfrac{9}{2},\tfrac{3}{2})\},$$

$$m_{55} = \tfrac{1}{945}a(\tfrac{11}{2},\tfrac{1}{2}).$$

$$W_1 = d_0a(\tfrac{1}{2},\tfrac{1}{2}),$$

$$W_2 = -\tfrac{1}{2}d_0+d_1a(\tfrac{3}{2},\tfrac{1}{2})+d_0a^2(\tfrac{1}{2},\tfrac{1}{2}),$$

$$W_3 = \tfrac{1}{2}d_0b(1,\tfrac{1}{2}),$$

$$W_4 = -\tfrac{5}{6}d_0a(\tfrac{1}{2},\tfrac{1}{2})+\tfrac{1}{3}d_2a(\tfrac{5}{2},\tfrac{1}{2})+d_1a(\tfrac{1}{2},\tfrac{1}{2})\,a(\tfrac{3}{2},\tfrac{1}{2})+\tfrac{1}{2}d_0a(1,\tfrac{1}{2})\,b(1,\tfrac{1}{2})+d_0a^3(\tfrac{1}{2},\tfrac{1}{2}),$$

$$W_5 = \tfrac{1}{2}d_1b(1,\tfrac{3}{2})+\tfrac{1}{2}d_0a(\tfrac{1}{2},\tfrac{1}{2})\,b(1,\tfrac{1}{2}),$$

$$W_6 = \tfrac{1}{2}d_1-\tfrac{1}{3}d_0a(\tfrac{3}{2},\tfrac{1}{2}).$$

$$q_1 = m_{10}-m_{11},$$

$$q_2 = m_{30}-m_{31}+m_{32}-m_{33},$$

$$q_3 = 2a(\tfrac{1}{2},\tfrac{1}{2})\,a(\tfrac{3}{2},\tfrac{1}{2}),$$

$$q_4 = 2a(\tfrac{1}{2},\tfrac{1}{2})\,a(\tfrac{5}{2},\tfrac{1}{2})+a^2(\tfrac{3}{2},\tfrac{1}{2}).$$

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