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RECURSIVELY PROVABLE DILATORS

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ABSTRACT. It is shown that $|\mathsf{RCA}_0|_{\Pi^1_2} = \sum_{k\in\mathbb{N}}(\omega+\mathsf{id})^k$. The argument relies almost entirely on the structure theory of dilators and wellorders. The proof motivates the notion of *shallow dilators*, which we also study and relate to earlier work by the second author. We define a version of the Π^1_2-norm for ω-models and show that $|\mathsf{RCA}_0|^{\omega}_{\Pi^1_2} = \sum_{\alpha<\omega_1^{\mathrm{ck}}}(\omega+\mathsf{id})^\alpha$.

1. INTRODUCTION

This article is a contribution to the structure theory of dilators, presenting an application to proof theory. *Dilators* (see Girard [9] for background) are particularly uniform transformations of linear orders which preserve wellfoundedness. More precisely, they are finitary, pullback-preserving, wellfoundedness-preserving functors on the category of linear orders. They themselves form a functor category. The class of countable dilators is Π^1_2-complete and in many ways is a generalization of the category of wellorders, which is Π^1_1-complete, and thus of relevance to computability and descriptive set theory.

Dilators can be used to measure the strength of mathematical theories. The notion of the Π^1_2-*norm* $|T|_{\Pi^1_2}$ of a theory T was introduced in [1] as a generalization of the notion of *proof-theoretic ordinal* $|T|_{\Pi^1_1}$ of a theory to the domain of Π^1_2 formulas and, as shown in [1], captures the Π^1_2 consequences of T when T is sufficiently strong, just like $|T|_{\Pi^1_1}$ does for the Π^1_1 consequences. Here, recall that a sentence is Π^1_2 if it is of the form $\forall X\,\exists Y\,\phi(X,Y)$, where the displayed quantifiers range over sets of integers and all quantifiers in ϕ range over integers. $|T|_{\Pi^1_2}$ is defined as the least dilator bounding all the provable dilators of T, where "least" refers to embeddability (i.e., the existence of natural transformations). Here, a provable dilator of a theory T is a dilator D such that for some Turing machine Φ_e computing a recursive presentation of D, we have $T \vdash$ "the set computed by Φ_e is a dilator."

In [2], the Π^1_2 norms of various notable theories are computed, including $|\mathsf{ACA}_0|_{\Pi^1_2}$, $|\mathsf{ATR}_0|_{\Pi^1_2}$, $|\mathsf{KP}|_{\Pi^1_2}$, and $|\Pi^1_1\text{–}\mathsf{CA}_0|_{\Pi^1_2}$. A notable absence from this list is $|\mathsf{RCA}_0|_{\Pi^1_2}$, which by [7, Theorem 3.1] is equal to $|\mathsf{WKL}_0|_{\Pi^1_2}$. The ostentive purpose of this article is to compute the Π^1_2-norm of RCA_0.

Theorem 1.1. $|\mathsf{RCA}_0|_{\Pi^1_2} = \sum_{k\in\mathbb{N}}(\omega+\mathsf{id})^k$.

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Here, recall that ω denotes the order-type of $\mathbb{N}$. While the methods of [2] are proof-theoretic and involve functorial proofs, cut-elimination, conservativity results, and forcing, the proof of Theorem 1.1 is purely algebraic and order-theoretic.

Nonetheless, we recall the missing definitions and provide some context: RCA_0 is the theory of recursive comprehension, which deals with natural numbers and sets of natural numbers. Its axioms are the defining axioms for addition and multiplication on $\mathbb{N}$, together with the schema of induction for $\mathbf{\Sigma}^0_1$ formulas and the comprehension axiom for $\mathbf{\Delta}^0_1$ formulas. Essentially, RCA_0 embodies mathematics where all reasoning is computable. Theorem 1.1 thus characterizes the finitary, pullback-preserving functors on linear orders which can be proved to be wellfoundedness-preserving using computable methods.

Related to Theorem 1.1 is a theorem of Carlucci, Mainardi, and Rathjen [4] according to which for each recursive wellorder α we have:

$$|\mathsf{RCA}_0 + \mathsf{wo}(\alpha)|_{\Pi^1_1} = |\mathsf{RCA}_0|_{\Pi^1_2}(\alpha) = \alpha^\omega.$$

These theories of the form $\mathsf{RCA}_0 + \mathsf{wo}(\alpha)$ arise naturally in various contexts. For instance, we mention the recent work of Le Houérou, Patey, and Yokoyama [10] on Ramsey's theorem for pairs which strengthened earlier results by Chong, Slaman, and Yang [5]; as well as Simpson's [11] work concerning Hilbert's Basis Theorem and the work of Carlucci, Dehornoy, and Weiermann [3] on braid theory.

Motivated by the proof of Theorem 1.1 we present what we call the *Cell Decomposition Theorem* for dilators. This result had been previously obtained by Girard [8] and an extension thereof to higher types will appear in a forthcoming book by H. Jeon. The proof of the theorem leads to the notion of a *shallow dilator*, which is a particularly interesting class of "simple" dilators consisting of those with wellfounded cell decomposition trees (introduced below). Shallow dilators play a central role in the proof of our second main theorem, which once more concerns proof theory.

Definition 1.2. Let T be a theory extending RCA_0. We denote by $|T|^\omega_{\Pi^1_2}$ the least dilator with respect to embeddability such that D embeds into $|T|^\omega_{\Pi^1_2}$ whenever D is a recursive dilator in every ω-model of T.

The analog of $|T|^\omega_{\Pi^1_2}$ for Π^1_1 is trivial, as all true Π^1_1 sentences hold in all ω-models, but the notion becomes interesting for Π^1_2. The proof of [1] adapts to show that $|T|^\omega_{\Pi^1_2}$ always exists when T is sound, but it will no longer be recursive. Making use of the connection between shallow dilators and the notion of a "thread" through a dilator, introduced by the second author [6], we are able to apply his dichotomy theorem [6, Corollary 4.2] to adapt the proof of Theorem 1.1 in order to obtain the following result, once more through mainly algebraic and order-theoretic methods:

Theorem 1.3. $|\mathsf{RCA}_0|^\omega_{\Pi^1_2} = \sum_{\alpha<\omega_1^{\mathrm{ck}}}(\omega + \mathsf{id})^\alpha$.

Before moving on, let us recall some definitions. An *ordinal denotation system* D consists of a set of terms $t(x_1, \dots, x_n)$, each of some arity $n = \mathrm{arity}(t)$, and comparison rules between these. Given a linear order A, $D(A)$ is a linear order whose domain is the set of all expressions $t(a_1, \dots, a_n)$ where $a_1, \dots, a_n$ is a sequence of elements of A and $a_n <_A a_{n-1} <_A \cdots <_A a_1$. Here and throughout we write $a <_A b$ to emphasize that a and b are elements of some linear order A and that $<_A$ is its ordering.

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Expressions $t(\vec{a})$ and $s(\vec{b})$ are compared according to the comparison rules given by D, which must satisfy the following properties:

(1) whether $t(\vec{a}) <_{D(A)} s(\vec{b})$ holds or not depends only on t and s and on the isomorphism class of the structure $(\vec{a}, \vec{b}, <_A)$, and not on the precise values of $\vec{a}$ and $\vec{b}$;
(2) $t(\vec{\alpha}) \leq_{D(A)} t(\vec{\beta})$ whenever $\vec{\alpha} \leq_A \vec{\beta}$ pointwise; and
(3) $t(\vec{\alpha}) = s(\vec{\beta})$ only if $s = t$ and $\vec{\alpha} = \vec{\beta}$.

The last condition asserts that denotations of ordinals according to $D(A)$ are unique. We sometimes call ordinal denotation systems (i.e., terms systems with comparison rules obeying the three conditions above) *pre-dilators*, and we say that a pre-dilator D is a *dilator* if $D(A)$ is wellfounded whenever A is. (Dilators are technically a more general kind of structure, but according to Girard's representation theorem [9], every dilator can be represented as an ordinal denotation system. Below, all dilators are assumed to be ordinal denotation systems.) In the context of subsystems of second-order arithmetic, one speaks only of dilators with a countable set of terms, as these are those which can be coded by sets of numbers. Thus, we may speak of recursive, arithmetical, and other kinds of dilators.

2. The cell decomposition of a dilator

Throughout the following discussion, fix a dilator D and (to simplify notation, also) a wellorder A of length at least ω^2. A will provide a large enough reservoir of points to use in the definitions below. The definitions do not depend on A itself. We typically use variables t, s, and variants of these to denote terms; variables such as l, m, n, k typically denote natural numbers. Let $t(x_1, \ldots, x_k)$ be a term in D. Recall that by convention, the notation $t(\alpha_1, \ldots, \alpha_k)$ indicates that $\alpha_k <_A \cdots <_A \alpha_1$.

2.1. **The priority order on terms.** There is a natural ordering $<_t$ on the finite set of variables $\{x_1, \ldots, x_k\}$ associated to t (equivalently, on the set $\{1, \ldots, k\}$). This order is defined as follows: x_i is the largest element of $<_t$ if

$$t(\alpha_1, \alpha_2, \ldots, \alpha_i, \ldots, \alpha_k) <_{D(A)} t(\alpha'_1, \alpha'_2, \ldots, \alpha'_i, \ldots, \alpha'_k)$$

holds whenever $\alpha_i <_A \alpha'_i$, regardless of the value of the other parameters. Having such an i fixed, x_j is the second largest element of $<_t$ if either $j < i$ and for (every A and) every $\alpha^*_i \in A$,

$$t(\alpha_1, \alpha_2, \ldots, \alpha_j, \ldots, \alpha^*_i, \ldots, \alpha_k) <_{D(A)} t(\alpha'_1, \alpha'_2, \ldots, \alpha'_j, \ldots, \alpha^*_i, \ldots, \alpha'_k)$$

holds whenever $\alpha_j <_A \alpha'_j$, regardless of the value of the other parameters; or if $i < j$ and for (every A and) every $\alpha^*_i \in A$,

$$t(\alpha_1, \alpha_2, \ldots, \alpha^*_i, \ldots, \alpha_j, \ldots, \alpha_k) <_{D(A)} t(\alpha'_1, \alpha'_2, \ldots, \alpha^*_i, \ldots, \alpha'_j, \ldots, \alpha'_k)$$

holds whenever $\alpha_j <_A \alpha'_j$, regardless of the value of the other parameters. The definition is extended the natural way to a total order on $\{1, \ldots, k\}$.

Theorem 2.1 (Girard; Theorem 3.2.4 of [9])**.** *For every term t of arity k in D, $<_t$ is a linear ordering on $\{1, 2, \ldots, k\}$.*

As mentioned before, we shall simultaneously think of $<_t$ as an order on the arity of t or on the variables, as convenient. Motivated by the theorem, we define:

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Definition 2.2. Let $t(x_1, \dots, x_k)$ be a term in D. We call the order $<_t$ the *priority order* of t. If x_i is maximum in this order, we call x_i the *leading argument* of t.

Definition 2.3. Let $t(x_1, \dots, x_k)$ be a term in D. We denote by $x_{\pi(n)}$ the nth argument of $t(x_1, \dots, x_k)$, ordered by priority. Thus, $x_{\pi(1)}$ is the leading argument of t, $x_{\pi(2)}$ is the second-leading argument of t, etc. We may also write $x_{\pi(i,t)}$ to emphasize that the ordering of variables is taken according to $<_t$, if necessary.

2.2. **The cell-decomposition theorem.** In this section, we present the Cell Decomposition Theorem for dilators. More important than the theorem itself are the definitions needed to state it, as well as the notation.

By induction on n, we define a preordering $\leq_n$ on D-terms and an associated equivalence relation $\equiv_n$. We shall maintain the following induction hypothesis

$(*)_n$: Suppose that $t \equiv_n s$. Then,

$$<_t\restriction (\pi(1,t), \pi(2,t), \dots, \pi(n,t)) \cong <_s\restriction (\pi(1,s), \pi(2,s), \dots, \pi(n,s)).$$

In other words, the n leading arguments of t and s are ordered the same way within the lists of arguments of t and s.

Definition 2.4. We define the pre-ordering $\leq_n$ on D-terms, and the associated equivalence relation $\equiv_n$, by induction. First, $t \leq_0 s$ holds always, so all terms are $\leq_0$-equivalent. We put $t \leq_{n+1} s$ if the following hold:

(1) t and s are terms of arity at least n.
(2) $t \equiv_n s$.
(3) There exist a linear order A and some possible arguments $\vec{\alpha}$ of t and $\vec{\beta}$ of s in A such that the following hold:
   (a) $\alpha_{\pi(i,t)} = \beta_{\pi(i,s)}$ for $1 \leq i \leq n$; and
   (b) $t(\vec{\alpha}) <_{D(A)} s(\vec{\beta})$.

   In other words, t can possibly take on a lower value than s even assuming that their n leading arguments are the same.

The definition of $\leq_{n+1}$ is meaningful assuming that hypothesis $(*)_n$ holds; without it, it is not clear that there can even exist assignments $t(\vec{\alpha})$ and $s(\vec{\beta})$ satisfying (3a). Let us continue the induction by proving that $(*)_{n+1}$ holds.

Lemma 2.5. *Hypothesis $(*)_{n+1}$ holds.*

Proof. Suppose that t and s are D-terms such that $t \equiv_{n+1} s$. By hypothesis $(*)_n$ we have

$$<_t\restriction (\pi(1,t), \dots, \pi(n,t)) \cong <_s\restriction (\pi(1,s), \dots, \pi(n,s)).$$

Suppose towards a contradiction that $(*)_{n+1}$ fails for t and s, so that

$$<_t\restriction (\pi(1,t), \dots, \pi(n,t), \pi(n+1,t)) \text{ and } <_s\restriction (\pi(1,s), \dots, \pi(n,s), \pi(n+1,s))$$

are no longer order-isomorphic. Choose indices $1 \leq i, j \leq n$ so that $\pi(n+1,t)$ occurs between $\pi(i,t)$ and $\pi(j,t)$ and no other index of the form $\pi(l,t)$ for $1 \leq l \leq n$ occurs between $\pi(i,t)$ and $\pi(j,t)$ (as natural numbers; other indices $\pi(l,t)$ for $n+1 < l \leq \text{arity}(t)$ might occur between $\pi(i,t)$ and $\pi(j,t)$). Let us proceed assuming that such i and j exist; if not, i.e., if $\pi(n+1,t)$ is greater than (or smaller than) all $\pi(l,t)$ for $1 \leq l \leq n$, then the proof is similar.

Since $t \equiv_{n+1} s$, there exist a linear order A and evaluations $t(\vec{\alpha})$, $t(\vec{\xi})$, and $s(\vec{\beta})$ such that the following hold:

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(1) $t(\vec{\alpha}) <_{D(A)} s(\vec{\beta}) <_{D(A)} t(\vec{\xi})$, and
(2) $\alpha_{\pi(l,t)} = \xi_{\pi(l,t)} = \beta_{\pi(l,s)}$ for $1 \leq l \leq n$.

Our first task is to modify $\vec{\alpha}, \vec{\beta}$, and $\vec{\xi}$ to ensure that the values of $\alpha_{\pi(n+1,t)}, \xi_{\pi(n+1,t)}$, and $\beta_{\pi(n+1,s)}$ do not overlap with the other parameters.

First, without loss of generality (by expanding A to a larger order if necessary), assume that all values α_l, ξ_l, β_l are limit ordinals and without loss of generality assume that $\pi(i,t) <_{\mathbb{N}} \pi(j,t)$, so that

$$\alpha_{\pi(j,t)} <_A \alpha_{\pi(n+1,t)} <_A \alpha_{\pi(i,t)}.$$

The case $\pi(j,t) <_{\mathbb{N}} \pi(i,t)$ is treated essentially the same way. By the requirements for tuples to be arguments of a term, we must also have

$$\xi_{\pi(j,t)} <_A \xi_{\pi(n+1,t)} <_A \xi_{\pi(i,t)}.$$

Let $\vec{\beta}_*$ be the modification of $\vec{\beta}$ obtained as follows:

(1) $\beta_{\pi(n+1,s)}$ is replaced by $\beta_{\pi(n+1,s)} + 1$;
(2) every argument β_l in $(\alpha_{\pi(j,t)}, \alpha_{\pi(i,t)})_A$ is replaced by $\alpha_{\pi(j,s)} + 1 + \mathrm{arity}(s) - l$.

Note that these two conditions are compatible with each other precisely due to our hypothesis on the disagreement between the orders $<_t$ and $<_s$: as $\pi(n+1,s)$ is not an index between $\pi(i,s)$ and $\pi(j,s)$, it follows that $\beta_{\pi(n+1,s)}$ is not an ordinal between $\alpha_{\pi(i,t)} = \beta_{\pi(i,s)}$ and $\alpha_{\pi(j,t)} = \beta_{\pi(j,s)}$. See the picture below.

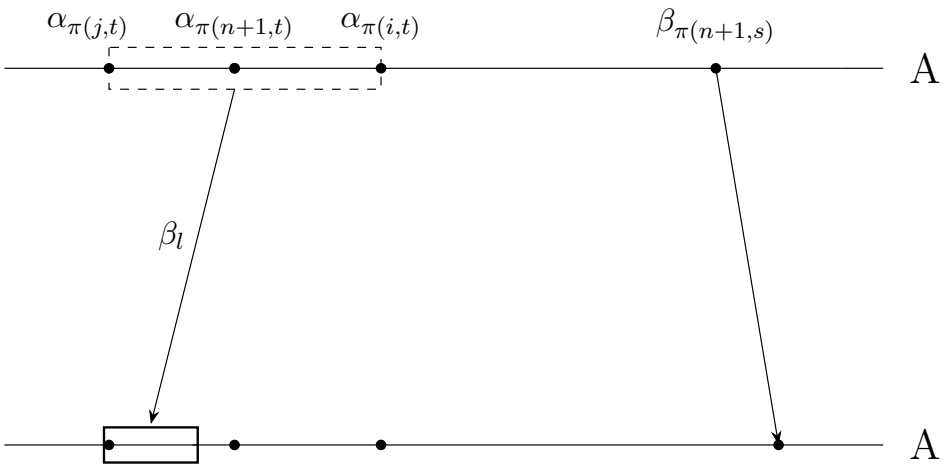


FIGURE 1. Definition of $\vec{\beta}_*$.

Since by assumption every element of $\vec{\beta}$ was a limit ordinal, it follows that $\vec{\beta}_*$ is a decreasing sequence of ordinals, so it makes sense to consider $s(\vec{\beta}_*)$. By hypothesis $(*)_n$, we see that $\vec{\beta}$ and $\vec{\beta}_*$ agree on the value of the n leading arguments of s, so since $\beta_{\pi(n+1,s)}$ was replaced by $\beta_{\pi(n+1,s)} + 1$ in $\vec{\beta}_*$, we have

$$t(\vec{\alpha}) <_{D(A)} s(\vec{\beta}) <_{D(A)} s(\vec{\beta}_*) \tag{2.1}$$

according to the definition of $<_s$. The point now is that there are no elements of $\vec{\beta}_*$ in the interval $[\alpha_{\pi(j,t)} + \omega, \alpha_{\pi(i,t)})_A$.

Now let $\vec{\alpha}_* = (\alpha_{*,1}, \ldots, \alpha_{*,\mathrm{arity}(t)})$ be the modification of $\vec{\alpha}$ obtained as follows:

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(1) $\alpha_{\pi(n+1,t)}$ is replaced by $\alpha_{\pi(n+1,t)+1} + \mathrm{arity}(s) + 1$;
(2) if l is such that $\alpha_l = \beta_{\pi(n+1,s)}$, then α_l is replaced by $\alpha_l + 2$.

Again these two conditions do not contradict each other, as by hypothesis $\alpha_{\pi(n+1,t)}$ belongs to the interval $(\alpha_{\pi(j,t)}, \alpha_{\pi(i,t)})_A$ and $\beta_{\pi(n+1,s)}$ does not belong to the interval $[\alpha_{\pi(j,t)}, \alpha_{\pi(i,t)}]_A$. With these modifications, $\beta_{\pi(n+1,s)}$ does not occur in $\vec{\alpha}_*$ and all elements of $\vec{\alpha}_*$ in the interval $(\alpha_{\pi(j,t)}, \alpha_{\pi(i,t)})_A$ occur strictly above all elements of $\vec{\beta}_*$ in that interval. Observe that $\vec{\alpha}_*$ is again an $<_A$-decreasing sequence, so it makes sense to consider $t(\vec{\alpha}_*)$. By applying (2.1) and the definition of $<_t$, we have

$$t(\vec{\alpha}_*) <_{D(A)} t(\vec{\alpha}) <_{D(A)} s(\vec{\beta}_*). \tag{2.2}$$

Since the comparison of terms in $D(A)$ does not depend on the exact values of the arguments and only on their relative ordering, (2.2) remains true if we modify $\vec{\alpha}_*$ by increasing the values of

$$\alpha_{*,\pi(i,t)+1}, \alpha_{*,\pi(i,t)+2}, \ldots, \alpha_{*,\pi(j,t)-2}, \alpha_{*,\pi(j,t)-1} \tag{2.3}$$

arbitrarily below the value of $\alpha_{\pi(i,t)} = \beta_{\pi(i,t)}$. Note that $\alpha_{*,\pi(n+1,t)}$ is among the numbers occuring in (2.3).

Let $\vec{\alpha}_{**}$ be the result of modifying $\vec{\alpha}_*$ by increasing the values of (2.3) so that they remain below $\alpha_{\pi(i,t)}$ but are larger than all arguments of $\vec{\xi}$ and $\vec{\beta}$ in the interval $(\alpha_{\pi(j,t)}, \alpha_{\pi(i,t)})_A$. Once more this is possible as all these arguments are limit ordinals. As (2.2) remains true for $t(\vec{\alpha}_{**})$, we have

$$t(\vec{\alpha}_{**}) <_{D(A)} s(\vec{\beta}_*).$$

However, we also have

$$s(\vec{\beta}) <_{D(A)} t(\vec{\xi}) <_{D(A)} t(\vec{\alpha}_{**}), \tag{2.4}$$

and since the structures $(\vec{\alpha}_{**}, \vec{\beta}, <_A)$ and $(\vec{\alpha}_{**}, \vec{\beta}_*, <_A)$ are order-isomorphic, (2.4) implies

$$s(\vec{\beta}_*) <_{D(A)} t(\vec{\alpha}_{**}),$$

which is a contradiction. □

Definition 2.6. Let $n \in \mathbb{N}$. An *n-cell* in D is an $\equiv_n$-equivalence class. Given two distinct n-cells c and d, we put $c <_n d$ if $t \leq_n s$ for some $t \in c$ and some $s \in d$.

Theorem 2.7 (Girard)**.** *Let D be a dilator and c be an n-cell in D. Then, the collection of $(n+1)$-cells contained in c is wellordered under $\leq_{n+1}$.*

Moreover, D is completely determined by the orders $<_t$ and $\leq_n$ in the following sense: suppose that A is a linear order and

(1) $t \equiv_n s$,
(2) $\alpha_{\pi(i,t)} = \beta_{\pi(i,s)}$ for $1 \leq i < n$; and
(3) $\alpha_{\pi(n,t)} <_A \beta_{\pi(n,s)}$.

Then, $t(\vec{\alpha}) <_{D(A)} s(\vec{\beta})$.

Proof. We first prove the first part of the statement. The collection of cells contained in each n-cell c is linearly ordered. Suppose towards a contradiction that it is illfounded for some c. We claim that $D(\omega^2)$ is illfounded. Choose a term from each $(n+1)$-cell in the infinite descending sequence. Use parameters of the form $\omega \cdot (1 + l)$ for $l < n$ to evaluate the leading n arguments of these terms. Then, the fact that ordinals up to $\omega \cdot (n+1)$ are available guarantees that we have enough

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parameters to serve as the remaining arguments for the terms, yielding an infinite descending sequence in $D(\omega \cdot (n+2))$, contradicting the fact that D is a dilator.

For the second part, suppose towards a contradiction that $s(\vec{\beta}) <_{D(A)} t(\vec{\alpha})$. Since $t \equiv_n s$, we can find elements $\vec{\xi}$ of A such that $t(\vec{\xi}) <_{D(A)} s(\vec{\beta})$ and the values of $\vec{\xi}$ and $\vec{\beta}$ (and thus also $\vec{\alpha}$) agree on the leading $n-1$ arguments of the terms. Since $t(\vec{\xi}) <_{D(A)} t(\vec{\alpha})$, we must have $\xi_{\pi(n,t)} \leq_A \alpha_{\pi(n,t)} <_A \beta_{\pi(n,s)}$. An argument as in Lemma 2.5 then leads to a contradiction. □

The import of Theorem 2.7 is that it provides an algorithm for comparing terms in $D(A)$: first compare the 1-cells of the terms, then the leading argument of each term, then the 2-cells, and so on.

2.3. **Shallow dilators.** The purpose of 2.3 is to study a particular class of simple dilators which we call *shallow dilators*. The definition is given in terms of its cell tree, defined below. Later, in §2.4 we will give an alternative characterization in terms of the notion of a *thread* from [6]. The results in 2.3, which are proved directly here, can equivalently be established by making use of this characterization; alternatively, these proofs provide new proofs of the results from [6].

Definition 2.8. Let D be a dilator. We associate to D a tree T_D as follows: at level 0, T_D has a single root labelled with the unique 0-cell. Given a node c at level n in T_D, immediate successors of c are points labelled by each $(n+1)$-cell contained in c. A dilator D is called *shallow* if T_D is wellfounded.

The following result asserts that shallow pre-dilators which are not dilators become illfounded quickly.

Theorem 2.9. *Suppose that D is a shallow pre-dilator. The following are equivalent:*

(1) D is a dilator;
(2) $D(\alpha)$ is wellfounded for all $\alpha < \omega^2$.

Proof. Consider a shallow pre-dilator D which is not a dilator. If the set of 1-cells of D is wellordered under $\leq_1$, then there must be a 1-cell c in D which is not a dilator. Let c be such a cell. If the set of 2-cells within c is not wellordered under $\leq_2$, then there must be a 2-cell c' in c which is not a dilator. Similarly for k-cells. By the shallowness of D, one must eventually reach some n-cell c such that the set of $(n+1)$-cells within c are illfounded. In the first part of the proof of Theorem 2.7 we verified that arguments $< \omega^2$ suffice to find an infinite descending sequence through D in this case. □

The following two results establish the complexity of shallow dilators and pre-dilators.

Theorem 2.10. *The class of shallow dilators is a Π^1_1-complete subset of $\mathbb{R}$. The set of indices of shallow recursive dilators is a Π^1_1-complete subset of $\mathbb{N}$.*

Proof. The proof of both claims is the same. By focusing on the class of constant pre-dilators, we see immediately that the classes are Π^1_1-hard. Clearly, whether a pre-dilator is shallow is Π^1_1. Whether a shallow pre-dilator is a dilator is also Π^1_1, by Theorem 2.9. □

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Theorem 2.11. *The class of shallow pre-dilators is a Π^1_1-complete subset of $\mathbb{R}$. The set of indices of shallow recursive pre-dilators is a Π^1_1-complete subset of $\mathbb{N}$.*

Proof. Clearly, being a shallow pre-dilator is Π^1_1. It remains to prove that it is Π^1_1-hard. Let T be a tree, i.e., a subset of $\mathbb{N}^{<\mathbb{N}}$ closed under initial segments. We define a pre-dilator D_T as follows (actually, D_T will be a dilator): terms in D_T are nonempty sequences $s \in T$, and the arity of each $s \in T$ is equal to its length. Terms $s(x_1, \dots, x_n)$ are ordered according to the lexicographic order of the sequence

$$\big(s(0), x_1, s(1), x_2, \dots, x_{n-1}, s(n-1), x_n\big).$$

Here, $s(i)$ is an element of $\mathbb{N}$ and is compared using the usual order on $\mathbb{N}$, while x_i is a variable compared according to the order of the argument of D. Thus, for each k, there is a k-cell in D for each element of T of depth k, and so D is shallow if and only if T is wellfounded. □

2.4. **Threadable dilators.** We recall the definition of a *thread* from [6]. Given a linear ordering L of $\mathbb{N}$, we define a pre-dilator D_L. For each n, there is one term $t(x_1, \dots, x_n)$ in D_L with priority order equal to L restricted to $\{1, 2, \dots, n\}$.

Definition 2.12. A pre-dilator D is *threadable* if there is a natural transformation from some D_L to $D \circ H$, where $H(\alpha) = \omega \cdot (1 + \alpha)$.

Theorem 2.13. *Let D be a pre-dilator. The following are equivalent:*

(1) D is shallow; and
(2) D is not threadable.

Proof. Clearly if D is threadable then it is not shallow. The proof of the converse implication is implicit in the argument of [6, Theorem 3.6] □

We observe that some results from [6] can be alternatively established from Theorem 2.13 (e.g., [6, Theorem 3.6] follows by an application of Theorem 2.10). Similarly, Theorem 2.9 follows from [6, Corollary 3.7].

3. The Π^1_2-norm of RCA_0

In this section, we prove:

Theorem 3.1. *Suppose that D is a recursive pre-dilator such that $\mathsf{RCA}_0 \vdash$ "D is a dilator." Then, there is a natural transformation*

$$T : D \to (\omega + \mathsf{id})^k$$

for some $k \in \mathbb{N}$.

Here, $(\omega + \mathsf{id})^k$ is the dilator which to each β assigns the order of all tuples $(\beta_1, \beta_2, \dots, \beta_k)$, with each $\beta_i < \omega + \beta$ ordered lexicographically. To recover Theorem 1.1 from Theorem 3.1, it remains to verify the "easy" direction that, provably in RCA_0, $(\omega+\alpha)^k$ is wellfounded whenever α is, for each (standard) $k \in \mathbb{N}$. This is essentially immediate, as RCA_0 proves that if A and B are wellorders, then so too are $A + B$ and $A \times B$.

The proof of Theorem 3.1 will take up the entirety of this section. The first observation is that RCA_0 cannot prove the existence of any non-shallow dilator. Indeed, one can do much better:

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Lemma 3.2. *Suppose that* $\mathsf{RCA}_0 \vdash$ "D *is a dilator." Then all terms of* D *have arity bounded by some* $k \in \mathbb{N}$.

Proof. Suppose that $\mathsf{RCA}_0 \vdash$ "D is a dilator" and let $t(x_1, \dots, x_k)$ be a term in D of arity k. Let D_t be the subdilator of D whose only term is t. Define an embedding $e : \omega^k \to D_t(\omega \cdot k)$ by

$$e\Big(\omega^{k-1} \cdot n_1 + \omega^{k-2} \cdot n_2 + \cdots + \omega \cdot n_{k-1} + n_k\Big) = t(\alpha_1, \alpha_2, \dots, \alpha_k),$$

where

$$\alpha_{\pi(i)} = \omega \cdot (k - \pi(i)) + n_i$$

and $\pi(i)$ denotes the index of the argument with ith priority, as in Definition 2.3. This embedding is order-preserving, so we see that $\omega^k \leq |D_t(\omega \cdot k)|$.

Suppose now towards a a contradiction that D is an RCA_0-provable dilator with terms of arbitrarily large arity. For each $k \in \mathbb{N}$, let t_k be a term in D of arity at least k. Then, for each k we have $\omega^k \leq |D_{t_k}(\omega \cdot k)| \leq |D(\omega^2)|$, where $|A|$ here denotes the order-type of A, so $D(\omega^2)$ is an RCA_0-provable wellorder of length $\geq \omega^\omega$, contradicting the fact that $|\mathsf{RCA}_0|_{\Pi^1_1} = \omega^\omega$. □

The following definition will be useful.

Definition 3.3. Let α be an ordinal. A pre-dilator D is *α-branching* if for all n-cells c of D, the order $\leq_{n+1}$ restricted to c has length at most α.

Lemma 3.4. *Suppose that* $\mathsf{RCA}_0 \vdash$ "D *is a dilator." Then, there is* m *such that* D *is* ω^m*-branching.*

Proof. Let k bound the arity of terms in D, as ensured by Lemma 3.2. Let $m \in \mathbb{N}$ and let c be an n-cell of D and suppose that the the order $\leq_{n+1}$ restricted to c has length greater than ω^m. Let D_c be the subdilator of D given by c. Then, $\omega^m < |D_c((2k)^n)|$. To see this, it suffices to note that $(2k)^n$ provides enough constants to serve as arguments for all terms in D_c in such a way that the n leading terms all have the same argument, by ensuring that the ith leading term takes an argument of the form $(2k)^{n+1-i}$ for $i \leq n$. Thus, there is a subset of $D_c((2k)^n)$ containing only one evaluation of each term and which has length ω^m. Therefore, if no m as in the statement of the lemma existed, then we would have $\omega^\omega \leq |D(\omega)|$, contradicting the fact that $|\mathsf{RCA}_0|_{\Pi^1_1} = \omega^\omega$ □

Lemma 3.5. *Let* α *be an ordinal. Suppose* D *is* α*-branching and contains only terms of arity* $\leq k$*. Then, there is a natural transformation*

$$T : D \to (\alpha + \mathsf{id})^{2k}.$$

Proof. We make use of the cell tree of D given by Definition 2.8. Since D is α-branching, this tree can be regarded as a subset of $\alpha^{<\mathbb{N}}$.

The natural transformation is obtained as follows: given an evaluated term $t(\beta_1, \dots, \beta_m)$ of $D(\beta)$, where $m \leq k$, T_β maps

$$t(\beta_1, \dots, \beta_m) \mapsto (\alpha_1, \alpha + \beta_{\pi(1)}, \alpha_2, \alpha + \beta_{\pi(2)}, \dots, \alpha_m, \alpha + \beta_{\pi(m)}, \alpha_{m+1}, 0, \dots, 0).$$

Let us explain this. Recall that elements of $(\alpha + \beta)^{2k}$ are $2k$-sequences of ordinals $\leq \alpha + \beta$ ordered lexicographically. Above, α_i denotes the order-type of the i-cell of t within the $(i-1)$-cell of t, in the sense of $\leq_i$. As in Definition 2.3, $\pi(i)$ denotes the index of the argument with priority i in t. By Theorem 2.7, this mapping precisely describes the ordering of D. Clearly it is natural in β, so it is as desired. □

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Lemma 3.6. *There is a natural transformation*

$$T : (\omega^n + \mathsf{id})^{2k} \to (\omega + \mathsf{id})^{2kn}.$$

Proof. Given a wellorder β, T_β maps $\{\gamma_i : i < 2k\} \in (\omega^n + \beta)^{2k}$ to the sequence $\{\delta_i : i < 2kn\}$, where for each $i < 2k$, the following holds:

(1) if $\gamma_i = \omega^{n-1} \cdot l_1 + \ldots + \omega \cdot l_{n-1} \cdot l_n < \omega^n$, then

$$(\delta_{in}, \delta_{in+1}, \ldots, \delta_{in+n-1}) = (l_1, l_2, \ldots, l_n);$$

(2) if $\gamma_i = \omega^n + \xi$ with $\xi < \beta$, then

$$(\delta_{in}, \delta_{in+1}, \ldots, \delta_{in+n-1}) = (\omega + \xi, 0, 0, 0, \ldots, 0).$$

In other words, for each γ_i, T_β sets aside a block of n elements of the formal sum $\omega+\beta$ and assigns either an n-sequence of natural numbers coding the Cantor normal form of $\gamma_i < \omega^n$, or else a sequence where the first element is $\omega + \xi$. Clearly this mapping is natural in β and is order-preserving. □

4. Dilators in ω-models of RCA_0

Theorem 4.1. *Suppose that D is a recursive dilator and $M \models$ "D is a dilator" for every ω-model M of RCA_0. Then, there is a natural transformation*

$$T : D \to (\omega + \mathsf{id})^\alpha$$

for some $\alpha < \omega_1^{ck}$.

Here, $(\omega + \mathsf{id})^\alpha$ is the dilator which to each wellorder β assigns the order of all Cantor normal forms

$$(\omega + \beta)^{\alpha_1} \cdot \beta_1 + \cdots + (\omega + \beta)^{\alpha_k} \cdot \beta_k, \tag{4.1}$$

where $k \in \mathbb{N}$, $\alpha_1 > \cdots > \alpha_k$, and $\beta_i < \omega + \beta$ for all $1 \leq i \leq k$.

By Uftring [12, Theorem 5], $(\omega + \mathsf{id})^\alpha$ is a dilator in every ω-model of RCA_0, so Theorem 1.3 follows from Theorem 4.1, which we prove now. Suppose D is a recursive dilator and that D is a dilator in every ω-model of RCA_0. By the dichotomy theorem of [6] and Theorem 2.13, D is shallow.

Definition 4.2. Let D be a dilator. We define a Kleene-Brouwer order on the cell tree T_D from Definition 2.8 as follows: given an n-cell c and an m-cell d, we put $c <_{\mathrm{KB}(T_D)} d$ if $m < n$ and c is contained in d (i.e., if $d <_{T_D} c$), or else if $n = m$, $c \equiv_{n-1} m$, and $c \leq_n d$.

By Theorem 2.7, each order $\leq_{n+1}$ is wellfounded when restricted to an n-cell, so the Kleene-Brouwer order of T_D is a wellorder whenever D is a shallow dilator.

Lemma 4.3. *Suppose D is a shallow, recursive dilator. Then, there is a natural transformation*

$$T : D \to (\omega + \mathsf{id})^\alpha,$$

where $\alpha = KB(T_D)$.

Proof. This argument is similar to that of Lemma 3.5. Given a wellorder β, the mapping T_β embeds $D(\beta)$ into $(\omega + \beta)^\alpha$ by

$$t(\beta_1, \ldots, \beta_m) \mapsto \sum_{i=1}^{m} \left[(\omega + \beta)^{c_i} \cdot (\omega + \beta_{\pi(i)})\right] + (\omega + \beta)^{c_{m+1}}.$$

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Let us explain this. Above, c_i denotes the i-cell of t, which is an element of T_D, which in turn is linearly ordered by its Kleene-Brouwer order. Thus, c_i represents an ordinal $< \alpha$. As in Definition 2.3, $\pi(i)$ denotes the index of the argument with priority i in t. This is defined for $1 \leq i \leq \mathrm{arity}(t)$. As the Kleene-Brouwer order of trees descends along branches, $T_\beta(t(\beta_1, \ldots, \beta_m))$ is indeed always a legal Cantor normal form. By Theorem 2.7, this mapping describes the ordering of D and is natural in β, as desired. □

Thus to complete the proof it remains to show:

Lemma 4.4. *Suppose D is a shallow, recursive dilator. Then,* $\mathrm{KB}(T_D)$ *has a recursive order-type.*

Proof. The statement of the lemma is not immediately obvious as $\mathrm{KB}(T_D)$ is a wellorder on cells of D, and cells can potentially consist of infinitely many terms.

For each term t of arity n and each $k \leq n+1$ in D, we consider a pair (k, t) regarded as a name for the k-cell of t; thus, each k-cell might have many names. Then, there is a recursive prewellorder on such pairs induced by Definition 4.2. This prewellorder can then be easily refined into a wellorder by putting $(k, t) < (k, s)$ if $t \equiv_k s$ and $t <_{\mathbb{N}} s$ (as Gödel numbers). Since determining whether $t \leq_n s$ requires finitely many appeals to the comparison rules of D, it can be done recursively, so the order on pairs (k, t) just described is recursive. □

This completes the proof of Theorem 4.1.

We finish with a strengthening of Theorem 2.10 as an application of Theorem 4.1 and its proof. When combined with the dichotomy theorem of [6], this result illustrates that dilators cannot exploit the full strength of ω-models locally, but rather merely a fragment of transfinite induction of bounded length. Below, $\mathbf{\Sigma}^0_1\text{-}\mathsf{TI}(\alpha)$ denotes the schema of transfinite induction along α for $\mathbf{\Sigma}^0_1$ formulas. Note that this schema is sensitive to the presentation of α.

Theorem 4.5. *Suppose that D is a shallow, recursive dilator. Then, there is a recursive wellordering α of $\mathbb{N}$ such that*

$$\mathsf{RCA}_0 + \mathbf{\Sigma}^0_1\text{-}\mathsf{TI}(\alpha) \vdash \text{“}D \text{ is a dilator.”}$$

Proof. First, observe that Theorem 2.7 is provable in RCA_0, as all the arguments in §2.2 are purely combinatorial and only require access to arguments $< \omega^2$ (in particular, the order A from Definition 2.4 can always taken to be $< \omega^2$).

Let $\hat{\alpha}$ be the recursive wellordering constructed in the proof of Lemma 4.4, so $\hat{\alpha}$ bounds the order-type of $\mathrm{KB}(T_D)$. Observe that for each wellorder β, the construction of Lemma 4.3 yields an embedding of $D(\beta)$ into $(\omega + \beta)^{\hat{\alpha}}$. Thus, working in RCA_0, D is a dilator if the mapping

$$\beta \mapsto (\omega + \beta)^{\hat{\alpha}}$$

preserves wellfoundedness. By Uftring [12, Theorem 5], this statement follows from $\mathbf{\Pi}^0_2\text{-}\mathsf{TI}(\hat{\alpha})$ and thus from $\mathbf{\Sigma}^0_1\text{-}\mathsf{TI}(\alpha)$, where $\alpha = \omega \cdot \hat{\alpha}$. □

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