Math. Proc. Camb. Phil. Soc. (2007), **143**, 221 © 2007 *Cambridge Philosophical Society*
doi:10.1017/S0305004107000084 *Printed in the United Kingdom*

Initial-boundary value problems for linear PDEs with variable coefficients

BY P. A. TREHARNE

School of Mathematics and Statistics, University of Sydney, Sydney 2006*, Australia.*
e-mail: P.Treharne@maths.usyd.edu.au

AND A. S. FOKAS

*Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge CB*3 0*WA.*
e-mail: T.Fokas@damtp.cam.ac.uk

(*Received* 17 *March* 2006; *revised* 19 *June* 2006)

Abstract

A new approach for studying initial-boundary value problems for linear partial differential equations (PDEs) with variable coefficients was introduced recently by the second author, and was applied to PDEs involving second order derivatives. Here, we extend this approach further to solve an initial-boundary value problem for a third-order evolution PDE with a space-dependent coefficient. The analysis is presented in such a way that it can be applied to PDEs with higher derivatives, and thus provides a method for solving initial-boundary value problems for a certain class of linear evolution equations with variable coefficients of *arbitrary* order.

1. *Introduction*

A new method for solving boundary value problems for linear and for integrable nonlinear PDEs in two dimensions was introduced in [**1**] and further extended in a series of papers (see for example [**2**–**9**]). This method was used in [**10**] for studying boundary value problems for a class of PDEs with variable coefficients; in particular, it was used to solve a Poincaré problem on the quarter plane for a variable coefficient generalization of the Laplace equation, as well as a Dirichlet and a Neumann problem for the time-dependent Schrödinger equation with a space-dependent potential on the half-line,

$$iq_t + q_{xx} + u(x)q = 0, \quad 0 < x < \infty, \quad t > 0. \tag{1·1}$$

Here we will extend this method to higher-order linear evolution equations with variable coefficients. As an illustrative example we will solve the following Dirichlet problem

$$q_t + q_{xxx} + u(x)q = 0, \tag{1·2}$$

$$q(x, 0) = q_0(x), \quad 0 < x < \infty; \quad q(0, t) = g_0(t), \quad t > 0. \tag{1·3}$$

We assume that $u(x)$, $q_0(x)$, $g_0(t)$ have sufficient smoothness, that $u(x)$, $q_0(x)$ have sufficient decay as $x \to \infty$, and that $q_0(x)$ and $g_0(t)$ are compatible at $x = t = 0$.

https://doi.org/10.1017/S0305004107000084 Published online by Cambridge University Press

The method introduced in [**1**] involves two basic steps:

(i) construct an integral representation for the solution $q(x,t)$ in the complex k-plane (the complex "Fourier plane"). This representation involves appropriate transforms of the initial and boundary values. Some of these boundary values are *not* prescribed as boundary conditions, and therefore this integral representation is not yet effective;

(ii) use certain algebraic global relations to determine the transforms of the unknown boundary values in terms of transforms of the known data.

For simple boundary value problems, such as for linear evolution PDEs on the half-line, step (i) can be implemented using one of several different approaches, including the application of the usual Fourier transform, see [**3**]. For more complicated problems, however, step (i) relies on the existence of a Lax pair formulation for the given PDE. Such a Lax pair for equation (1·2) is the following pair of equations

$$\mu_{xxx} + (u(x) + ik^3)\mu = q(x,t) \tag{1·4a}$$

$$\mu_t - ik^3\mu = -q(x,t), \tag{1·4b}$$

where $\mu(x,t,k)$ is a scalar function. Equations (1·4) are compatible if and only if $q(x,t)$ satisfies (1·2): applying the operator $\partial_t - ik^3$ to (1·4a), noting that this operator commutes with $\partial_x^3 + (u(x) + ik^3)$, and using (1·4$b$), we find (1·2).

For evolution PDEs with variable coefficients the relevant integral representations can be constructed by using certain completeness relations derived from the x-part of the associated Lax pair, and then using certain contour deformations in the complex k-plane (see [**10**]). Alternatively, one can use the *simultaneous* spectral analysis of the associated Lax pair [**10**]. These two approaches provide the generalization to PDEs with variable coefficients of the approaches used in [**3**] and [**1**], respectively. In this paper, we adopt the latter approach to solve the initial-boundary value problem specified by equations (1·2) and (1·3).

The remainder of the paper is organized as follows. In Section 2 we summarize our main results. In particular, we present an integral representation for $q(x,t)$ which is uniquely defined in terms of appropriate transforms of the given functions $q_0(x)$ and $g_0(t)$, see Theorem 2·8. In Sections 3 and 4 we derive the main results: in Section 3 we perform the simultaneous spectral analysis of the associated Lax pair to derive the solution formula for $q(x,t)$ in terms of transforms of the boundary values. In Section 4 we first derive certain algebraic global relations which are valid in the complex k-plane and which relate all boundary values. We then use these global relations to eliminate the unknown boundary values and express $q(x,t)$ in terms of $q_0(x)$ and $g_0(t)$. In Section 5 we conclude with a further discussion of these results, and comment on how they can be generalized to variable coefficient evolution equations of arbitrary order.

2. *The linearized KdV equation with variable coefficient*

We begin by considering solutions of the homogeneous version of equation (1·4a),

$$\psi_{xxx} + (u(x) - (ik)^3)\psi = 0, \quad x > 0, \quad k \in \mathbb{C}. \tag{2·1}$$

Particular solutions of this ODE satisfy the linear integral equation

$$\psi(x,k) = f(x,k) + \frac{1}{3k^2}\int_0^x \left[e^{ik(x-\xi)} + \alpha e^{i\alpha k(x-\xi)} + \alpha^2 e^{i\alpha^2 k(x-\xi)}\right] u(\xi)\psi(\xi,k)\,d\xi, \tag{2·2a}$$

https://doi.org/10.1017/S0305004107000084 Published online by Cambridge University Press

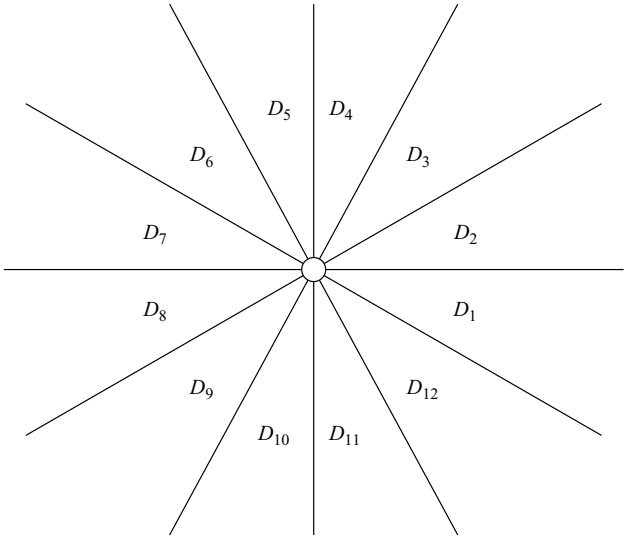


Fig. 1. The domains D_1–D_{12} in the punctured complex k-plane.

where the forcing $f(x,k)$ is any of the following three exponentials,

$$e^{ikx}, \quad e^{i\alpha kx}, \quad e^{i\alpha^2 kx}, \quad \alpha = e^{2\pi i/3}. \tag{2·2b}$$

Additional solutions can be obtained by replacing $\int_0^x$ with $-\int_x^\infty$ in any of the three integrals on the right-hand side of (2·2*a*).

PROPOSITION 2·1 (The base functions). *Define the space-dependent eigenfunctions* ψ_1, ψ_2, ψ_3 *by the following linear integral equations*:

$$\psi_1(x,k) = e^{ikx} + \frac{1}{3k^2}\int_0^x \left[e^{ik(x-\xi)} + \alpha e^{i\alpha k(x-\xi)}\right] u(\xi)\psi_1(\xi,k)\,d\xi - \frac{1}{3k^2}\int_x^\infty \alpha^2 e^{i\alpha^2 k(x-\xi)} u(\xi)\psi_1(\xi,k)\,d\xi, \tag{2·3a}$$

$$\psi_2(x,k) = e^{i\alpha kx} - \frac{1}{3k^2}\int_x^\infty e^{ik(x-\xi)} u(\xi)\psi_2(\xi,k)d\xi + \frac{1}{3k^2}\int_0^x \alpha e^{i\alpha k(x-\xi)} u(\xi)\psi_2(\xi,k)\,d\xi - \frac{1}{3k^2}\int_x^\infty \alpha^2 e^{i\alpha^2 k(x-\xi)} u(\xi)\psi_2(\xi,k)\,d\xi, \tag{2·3b}$$

$$\psi_3(x,k) = e^{i\alpha^2 kx} + \frac{1}{3k^2}\int_0^x \left[e^{ik(x-\xi)} + \alpha e^{i\alpha k(x-\xi)}\right] u(\xi)\psi_3(\xi,k)\,d\xi + \frac{1}{3k^2}\int_0^x \alpha^2 e^{i\alpha^2 k(x-\xi)} u(\xi)\psi_3(\xi,k)\,d\xi. \tag{2·3c}$$

Define the sectors $\{D_j\}_{j=1}^{12}$ *in the complex* k*-plane as follows* (*see Figure* 1)

$$D_j = \left\{k \in \mathbb{C}\backslash\{0\} \mid \arg k \in \left[(j-2)\frac{\pi}{6}, (j-1)\frac{\pi}{6}\right]\right\}, \quad j = 1,\ldots,12. \tag{2·4}$$

The functions $\psi_1(x,k)$, $\psi_2(x,k)$, *and* $\psi_3(x,k)$ *are defined in the following domains of the*

https://doi.org/10.1017/S0305004107000084 Published online by Cambridge University Press

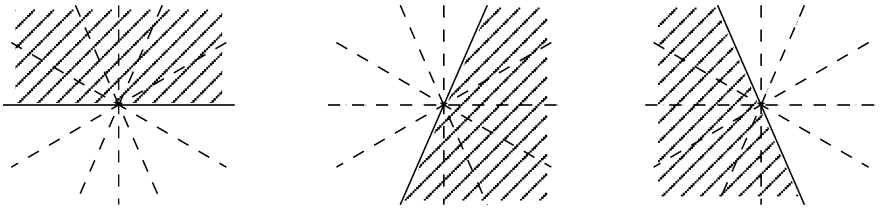


Fig. 2. The domains of boundedness of e^{ikx}, $e^{i\alpha kx}$, $e^{i\alpha^2 kx}$.

punctured complex k-plane, $k \in \mathbb{C}\backslash\{0\}$, (see Figure 2):

$$\left.\begin{aligned} &\psi_1(x,k) : \{k \in \mathbb{C}\backslash\{0\} \mid \operatorname{Im}\alpha^2 k \leqslant 0\} \equiv \{k \in D_{12} \cup D_1 \cup D_2 \cup D_3 \cup D_4 \cup D_5\},\\ &\psi_2(x,k) : \{k \in \mathbb{C}\backslash\{0\} \mid \operatorname{Im} k \leqslant 0 \cap \operatorname{Im}\alpha^2 k \leqslant 0\} \equiv \{k \in D_{12} \cup D_1\},\\ &\psi_3(x,k) : \forall\ k \in \mathbb{C}\backslash\{0\}. \end{aligned}\right\} \tag{2·5}$$

The functions $\psi_1(x,k)e^{-ikx}$, $\psi_2(x,k)e^{-i\alpha kx}$, and $\psi_3(x,k)e^{-i\alpha^2 kx}$ are bounded and analytic in the following domains of the punctured complex k-plane, $k \in \mathbb{C}\backslash\{0\}$:

$$\left.\begin{aligned} &\psi_1(x,k)e^{-ikx} &&: k \in D_1 \cup D_2,\\ &\psi_2(x,k)e^{-i\alpha kx} &&: k \in D_{11} \cup D_{12} \cup D_1 \cup D_2,\\ &\psi_3(x,k)e^{-i\alpha^2 kx} &&: k \in D_1 \cup D_2 \cup D_3 \cup D_4. \end{aligned}\right\} \tag{2·6}$$

Similarly, define the space-dependent eigenfunctions ϕ_1 and ϕ_2 by the following linear integral equations:

$$\begin{aligned} \phi_1(x,k) = e^{ikx} &+ \frac{1}{3k^2}\int_0^x e^{ik(x-\xi)}u(\xi)\phi_1(\xi,k)\,d\xi - \frac{1}{3k^2}\int_x^\infty + \alpha e^{i\alpha k(x-\xi)}u(\xi)\phi_1(\xi,k)\,d\xi\\ &- \frac{1}{3k^2}\int_x^\infty \alpha^2 e^{i\alpha^2 k(x-\xi)}u(\xi)\phi_1(\xi,k)\,d\xi, \end{aligned} \tag{2·7a}$$

$$\begin{aligned} \phi_2(x,k) = e^{i\alpha kx} &+ \frac{1}{3k^2}\int_0^x \left[e^{ik(x-\xi)} + \alpha e^{i\alpha k(x-\xi)}\right]u(\xi)\phi_2(\xi,k)\,d\xi\\ &- \frac{1}{3k^2}\int_x^\infty \alpha^2 e^{i\alpha^2 k(x-\xi)}u(\xi)\phi_2(\xi,k)\,d\xi. \end{aligned} \tag{2·7b}$$

The functions $\phi_1(x,k)$ and $\phi_2(x,k)$ are defined in the following domains of the punctured complex k-plane, $k \in \mathbb{C}\backslash\{0\}$:

$$\left.\begin{aligned} &\phi_1(x,k) : \{k \in \mathbb{C}\backslash\{0\} \mid \operatorname{Im}\alpha k \leqslant 0 \cap \operatorname{Im}\alpha^2 k \leqslant 0\} \equiv \{k \in D_4 \cup D_5\},\\ &\phi_2(x,k) : \{k \in \mathbb{C}\backslash\{0\} \mid \operatorname{Im}\alpha^2 k \leqslant 0\} \equiv \{k \in D_{12} \cup D_1 \cup D_2 \cup D_3 \cup D_4 \cup D_5\}. \end{aligned}\right\} \tag{2·8}$$

Furthermore, the functions $\phi_1(x,k)e^{-ikx}$ and $\phi_2(x,k)e^{-i\alpha kx}$ are bounded and analytic in the following domains:

$$\begin{aligned} &\phi_1(x,k)e^{-ikx} &&: D_3 \cup D_4 \cup D_5 \cup D_6,\\ &\phi_2(x,k)e^{-i\alpha kx} &&: D_3 \cup D_4. \end{aligned} \tag{2·9}$$

Proof. The proof of this result is given in Appendix A·1.

Remark 2·2. The integral equations for $\psi_1(x,k)$, $\psi_2(x,k)$ and $\phi_1(x,k)$, $\phi_2(x,k)$, given above in (2·3) and (2·7), respectively, are Fredholm as opposed to Volterra integral equations,

https://doi.org/10.1017/S0305004107000084 Published online by Cambridge University Press

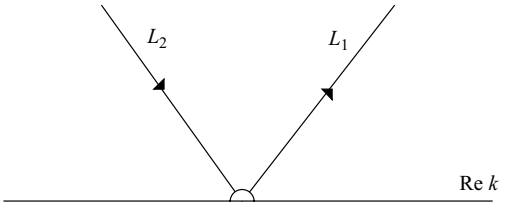


Fig. 3. The contours L_1 and L_2.

and therefore these functions may have poles. This issue was considered in [**11**] by Beals, Deift and Zhou (see also [**12**]), where appropriate eigenfunctions were defined in terms of Volterra integral equations. By expressing the "Fredholm" eigenfunctions in terms of the "Volterra" eigenfunctions, possible singularities can be analysed effectively. In what follows we *assume* that $u(x)$ is such that there exist *no* such singularities.

Also, in order to avoid the analysis of the singularity at $k = 0$ we consider the punctured complex k-plane, $k \in \mathbb{C}\backslash\{0\}$. We recall that in equation (1·1) the relevant singularity is removable. The best analysis for the corresponding problem for the Schrödinger equation can be found in [**13**].

In Section 3 we will use the analyticity properties of these eigenfunctions to prove the following proposition.

PROPOSITION 2·3 (A transform pair). *Assume that there exists a real-valued function $q(x,t)$ where $0 < x < \infty$ and $0 < t < T$, with T a positive constant, which satisfies equation* (1·2). *Assume that $q(x,t)$ has sufficient decay as $x \to \infty$ and sufficient smoothness all the way to the boundary. Furthermore, assume that $u(x)$ has sufficient smoothness and sufficient decay as $x \to \infty$. Then $q(x,t)$ admits the integral representation*

$$\begin{aligned} q(x,t) = \frac{1}{2\pi}\left[\int_0^\infty \frac{-1}{\Delta(k)} e^{ik^3t}\psi_1(x,k)\hat{q}_1(k)dk + \int_{-\infty}^0 \frac{-1}{\Delta(\alpha^2 k)} e^{ik^3t}\phi_2(x,\alpha^2 k)\tilde{q}_2(\alpha^2 k)dk\right] \\ \times \frac{1}{2\pi}\left[\int_{L_1} \frac{-1}{\Delta(k)} e^{ik^3t}\phi_1(x,k)\tilde{G}_1(k)dk \right. \\ \left. + \int_{L_2} \frac{-1}{\Delta(\alpha^2 k)} e^{ik^3t}\psi_2(x,\alpha^2 k)\hat{G}_2(\alpha^2 k)dk\right], \end{aligned} \tag{2·10}$$

where

$$\Delta(k) = ik(\alpha - \alpha^2), \quad \alpha = e^{2\pi i/3}, \tag{2·11}$$

the spectral functions $\{\hat{q}_1(k), \tilde{q}_2(k), \tilde{G}_1(k), \hat{G}_2(k)\}$ are defined below, and the contours L_1 and L_2 are the rays $\arg k = \pi/3$ and $\arg k = 2\pi/3$, respectively, indented to avoid $k = 0$ and with the orientations shown in Figure 3.

The functions $\hat{q}_1(k)$ and $\tilde{q}_2(k)$ are defined by

$$\hat{q}_1(k) = \int_0^\infty q(x,0)[\psi_2(x,k)\psi_3'(x,k) - \psi_2'(x,k)\psi_3(x,k)]\,dx, \quad k \in D_1, \tag{2·12a}$$

$$\tilde{q}_2(k) = \int_0^\infty q(x,0)[\psi_3(x,k)\phi_1'(x,k) - \psi_3'(x,k)\phi_1(x,k)]\,dx, \quad k \in D_4, \tag{2·12b}$$

where prime denotes differentiation with respect to the spatial variable. The functions $\tilde{G}_1(k)$

https://doi.org/10.1017/S0305004107000084 Published online by Cambridge University Press

and $\hat{G}_2(k)$ *are defined by*

$$\tilde{G}_1(k) = \hat{g}_2(k)N_1(0,k) - \hat{g}_1(k)N_1'(0,k) + \hat{g}_0(k)N_1''(0,k), \tag{2·13a}$$

$$\hat{G}_2(k) = \hat{g}_2(k)M_2(0,k) - \hat{g}_1(k)M_2'(0,k) + \hat{g}_0(k)M_2''(0,k), \tag{2·13b}$$

where the functions $\{\hat{g}_j(k)\}_{j=0}^2$ *are defined by*

$$\hat{g}_j(k) = \int_0^T e^{-ik^3\tau}\partial_x^j q(0,\tau)\,d\tau, \quad j = 0,1,2, \quad k \in \mathbb{C}\backslash\{0\}, \tag{2·14}$$

and the functions $M_2(x,k)$, $N_1(x,k)$ *are defined by*

$$M_2(x,k) = \psi_3(x,k)\psi_1'(x,k) - \psi_3'(x,k)\psi_1(x,k), \quad k \in \{\mathbb{C}\backslash\{0\} \mid \operatorname{Im}\alpha^2 k \leqslant 0\}, \tag{2·15a}$$

$$N_1(x,k) = \phi_2(x,k)\psi_3'(x,k) - \phi_2'(x,k)\psi_3(x,k), \quad k \in \{\mathbb{C}\backslash\{0\} \mid \operatorname{Im}\alpha^2 k \leqslant 0\}. \tag{2·15b}$$

Proof. This result is derived in Section 3.

Remark 2·4. The functions $\psi_1(x,k)$ and $\hat{q}_1(k)$ are defined for k in $\{\mathbb{C}\backslash\{0\} \mid \operatorname{Im}\alpha^2 k \leqslant 0\}$ and in D_1, respectively, and hence these functions are well defined for k on the positive real axis. The functions $\phi_2(x,k)$ and $\tilde{q}_2(k)$ are defined for k in $\{\mathbb{C}\backslash\{0\} \mid \operatorname{Im}\alpha^2 k \leqslant 0\}$ and in D_4, respectively. Thus these functions are well defined for k in D_4, and hence $\phi_2(x,\alpha^2 k)$ and $\tilde{q}_2(\alpha^2 k)$ are well defined for k in D_8, which contains the negative real axis. Furthermore, it is shown in Appendix A·2 that the function $\phi_1(x,k)\tilde{G}_1(k)$ is bounded and analytic for k in $D_3 \cup D_4$, which contains the ray L_1, and that the function $\psi_2(x,\alpha^2 k)\hat{G}_2(\alpha^2 k)$ is bounded and analytic for k in $D_5 \cup D_6$, which contains L_2. Thus $q(x,t)$ is well defined.

Remark 2·5. The integral representation of $q(x,t)$ given in (2·10) and the functions $\{\hat{q}_1(k), \tilde{q}_2(k), \tilde{G}_1(k), \hat{G}_2(k)\}$ defined in (2·12)–(2·13) constitute a transform pair for the function $q(x,t)$. We note that the t-dependence in (2·10) is explicit; this is due to the fact that this integral representation is derived by analysing both equations of (1·4) *simultaneously*.

PROPOSITION 2·6 (The global relations). *Let the real-valued function* $q(x,t)$ *satisfy the conditions specified in Proposition* 2·3 *and let the functions* $\hat{q}_1(t,k)$ *and* $\tilde{q}_2(t,k)$ *be defined by equations identical to* (2·12)*, but with* $q(x,0)$ *replaced by* $q(x,t)$*. Then the following global relations are valid for* $k \in D_1$ *and* $k \in D_4$*, respectively,*

$$e^{-ik^3T}\hat{q}_1(T,k) = \hat{q}_1(k) + \hat{g}_2(k)M_1(0,k) - \hat{g}_1(k)M_1'(0,k) + \hat{g}_0(k)M_1''(0,k), \tag{2·16a}$$

$$e^{-ik^3T}\tilde{q}_2(T,k) = \tilde{q}_2(k) + \hat{g}_2(k)N_2(0,k) - \hat{g}_1(k)N_2'(0,k) + \hat{g}_0(k)N_2''(0,k), \tag{2·16b}$$

where the functions $\{\hat{g}_j(k)\}_{j=0}^2$ *are defined in* (2·14)*, and the functions* $M_1(x,k)$, $N_2(x,k)$ *are defined by*

$$M_1(x,k) = \psi_2(x,k)\psi_3'(x,k) - \psi_2'(x,k)\psi_3(x,k), \quad k \in D_{12} \cup D_1, \tag{2·17a}$$

$$N_2(x,k) = \psi_3(x,k)\phi_1'(x,k) - \psi_3'(x,k)\phi_1(x,k), \quad k \in D_4 \cup D_5. \tag{2·17b}$$

Let the functions $\hat{q}_3(k)$ *and* $\tilde{q}_3(k)$ *be defined by*

$$\hat{q}_3(k) = \int_0^\infty q(x,0)[\psi_1(x,k)\psi_2'(x,k) - \psi_1'(x,k)\psi_2(x,k)]\,dx, \quad k \in D_1 \cup D_2, \tag{2·18a}$$

$$\tilde{q}_3(k) = \int_0^\infty q(x,0)[\phi_1(x,k)\phi_2'(x,k) - \phi_1'(x,k)\phi_2(x,k)]\,dx, \quad k \in D_3 \cup D_4, \tag{2·18b}$$

and let the functions $\hat{q}_3(t,k)$ *and* $\tilde{q}_3(t,k)$ *be defined by identical equations but with* $q(x,0)$

https://doi.org/10.1017/S0305004107000084 Published online by Cambridge University Press

replaced by $q(x,t)$. *Then the following global relations are valid fpr* $k \in D_1 \cup D_2$ *and* $k \in D_3 \cup D_4$, *respectively,*

$$e^{-ik^3T}\hat{q}_3(T,k) = \hat{q}_3(k) + \hat{g}_2(k)M_3(0,k) - \hat{g}_1(k)M_3'(0,k) + \hat{g}_0(k)M_3''(0,k), \qquad (2{\cdot}19a)$$

$$e^{-ik^3T}\tilde{q}_3(T,k) = \tilde{q}_3(k) + \hat{g}_2(k)N_3(0,k) - \hat{g}_1(k)N_3'(0,k) + \hat{g}_0(k)N_3''(0,k), \qquad (2{\cdot}19b)$$

where the functions $M_3(x,k)$, $N_3(x,k)$ *are defined by*

$$M_3(x,k) = \psi_1(x,k)\psi_2'(x,k) - \psi_1'(x,k)\psi_2(x,k), \quad k \in D_1 \cup D_2, \qquad (2{\cdot}20a)$$

$$N_3(x,k) = \phi_1(x,k)\phi_2'(x,k) - \phi_1'(x,k)\phi_2(x,k), \quad k \in D_3 \cup D_4. \qquad (2{\cdot}20b)$$

Proof. The proof of this Proposition is given in Section 4.

Remark 2·7. The spectral functions $\hat{q}_1(t,k)$ and $\tilde{q}_2(t,k)$ are bounded and analytic for $k \in D_1$ and $k \in D_4$, respectively. Similarly, the spectral functions $\hat{q}_3(t,k)$ and $\tilde{q}_3(t,k)$ are bounded and analytic for $k \in D_1 \cup D_2$ and $k \in D_3 \cup D_4$ respectively. The proof of this result is presented in Appendix A·2. The proof that the functions $M_1(x,k)$, $M_2(x,k)$ and $N_1(x,k)$, $N_2(x,k)$ are well defined in the domains indicated in the definitions is also given in Appendix A·2.

The integral representation (2·10), together with the global relations (2·16) and (2·19), can be used to solve equation (1·2) subject to a variety of boundary value problems. As an illustrative example, in the next theorem we solve equation (1·2) for the particular boundary conditions given in (1·3).

THEOREM 2·8 (The solution of the Dirichlet problem). *Let* $q(x,t)$ *satisfy equation* (1·2) *in* $0 < x < \infty$, $0 < t < T$, *where* T *is a positive constant, with the boundary conditions given in* (1·3). *Then the solution* $q(x,t)$ *is given by* (2·10) *with the functions* $\tilde{G}_1(k)$ *and* $\hat{G}_2(k)$ *given by the following expressions*

$$\tilde{G}_1(k) = \frac{\phi_2(0,k)}{\phi_1(0,k)}\tilde{q}_2(k) + \frac{\psi_3(0,k)}{\phi_1(0,k)}\tilde{q}_3(k) + \frac{3k^2\Delta(k)}{\phi_1(0,k)}\hat{g}_0(k), \qquad (2{\cdot}21a)$$

$$\hat{G}_2(k) = \frac{\psi_1(0,k)}{\psi_2(0,k)}\hat{q}_1(k) + \frac{\psi_3(0,k)}{\psi_2(0,k)}\hat{q}_3(k) + \frac{3k^2\Delta(k)}{\psi_2(0,k)}\hat{g}_0(k). \qquad (2{\cdot}21b)$$

For the sake of simplicity we have assumed that $\phi_1(0,k)$ *and* $\psi_2(0,\alpha^2 k)$ *do not have any zeros in* D_4 *and* D_5, *respectively.*

Proof. The proof of this Theorem is given in Section 4.

Remark 2·9. The definitions of $\tilde{G}_1(k)$ and $\hat{G}_2(k)$ given above involve the following *known* functions: The function $\hat{g}_0(k)$, which is the t-transform of $g_0(t)$, see equation (2·14); the eigenfunctions $\{\psi_1(x,k), \psi_2(x,k), \psi_3(x,k), \phi_1(x,k), \phi_2(x,k)\}$ evaluated at $x = 0$, which are defined in terms of the function $u(x)$, see equations (2·3) and (2·7); and the functions $\{\hat{q}_1(k), \tilde{q}_2(k), \hat{q}_3(k), \tilde{q}_3(k)\}$, which are defined in terms of the above eigenfunctions and the initial condition $q_0(x)$.

Remark 2·10. The solution formulas (2·10) and (2·21) are consistent with the case of equation (1·2) with $u(x) = 0$, ie. the case of constant coefficients. Indeed, it is shown in [**3**] that the following integral representation is valid for equation (1·2) with $u(x) = 0$:

$$q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx+ik^3t}\hat{q}_0(k)\,dk + \frac{1}{2\pi}\int_L e^{ikx+ik^3t}\hat{g}(k)\,dk, \qquad (2{\cdot}22)$$

https://doi.org/10.1017/S0305004107000084 Published online by Cambridge University Press

where L is the union of L_1 and L_2, and

$$\hat{q}_0(k) = \int_0^\infty e^{-ikx} q_0(x)\, dx, \quad \operatorname{Im} k \leqslant 0, \tag{2·23a}$$

$$\hat{g}(k) = \hat{g}_2(k) + ik\hat{g}_1(k) - k^2\hat{g}_0(k), \quad k \in \mathbb{C}, \tag{2·23b}$$

with the functions $\{\hat{g}_j(k)\}_{j=0}^2$ defined in (2·14). If $q(0,t) = g_0(t)$ is the specified boundary condition, then the above solution formula involves the unknown boundary values $q_x(0,t)$ and $q_{xx}(0,t)$. These unknown function can be eliminated from this solution formula by using certain invariance properties of the following algebraic global relation

$$\hat{q}_0(k) + \hat{g}(k) = e^{-ik^3T} \int_0^\infty e^{-ikx} q(x,T) dx, \quad \operatorname{Im} k \leqslant 0. \tag{2·24}$$

Equation (2·24) can be mapped to two distinct algebraic relations that are both valid on the contour L in equation (2·22). These two relations can then be used to derive the following expression for $\hat{g}(k)$, which depends only on the specified initial and boundary conditions:

$$\hat{g}(k) = -3k^2\hat{g}_0(k) - \frac{\alpha^2-1}{\alpha^2-\alpha}\hat{q}_0(\alpha k) - \frac{\alpha-1}{\alpha-\alpha^2}\hat{q}_0(\alpha^2 k), \quad \alpha = e^{2\pi i/3}. \tag{2·25}$$

The solution formula for $q(x,t)$ in (2·22), and the spectral functions $\{\hat{q}_0(k), \hat{g}_0(k)\}$ in (2·23), correspond to the integral representation for $q(x,t)$ in (2·10), and to the functions $\{\hat{q}_1(k), \tilde{q}_2(k), \tilde{G}_1(k), \hat{G}_2(k)\}$ in (2·12)–(2·13), respectively, in the case $u(x) \neq 0$. Similarly, equation (2·25) corresponds to equations (2·21) in Theorem 2·8.

3. *The construction of the solution formula*

In this section we derive the integral representation for $q(x,t)$ given in Proposition 2·3. The main step in this derivation is the construction of a sectionally analytic solution $\mu(x,t,k)$ of the associated Lax pair (1·4). This is achieved by dividing the complex k-plane into the sectors shown in Figure 1 and solving equations (1·4) within each of these sectors. The construction of $\mu(x,t,k)$ constitutes the solution of the so-called direct problem. The solution of the so-called inverse problem involves relating the representations of $\mu(x,t,k)$ in the different sectors and then formulating a (scalar) Riemann–Hilbert problem.

3·1. *The general form of* $\mu(x,t,k)$

Let $Q(x,t)$ be a solution of the formal adjoint of equation (1·2), ie. let $Q(x,t)$ satisfy

$$-Q_t - Q_{xxx} + u(x)Q = 0. \tag{3·1}$$

Multiplying equations (1·2) and (3·1) by Q and q respectively, and then subtracting the resulting equations we find

$$\big(Qq\big)_t + \big(Qq_{xx} - q_x Q_x + q Q_{xx}\big)_x = 0. \tag{3·2}$$

A particular solution of equation (3·1) is $Q(x,t) = e^{-ik^3t}M(x,k)$ where $M(x,k)$ satisfies

$$M_{xxx} - ik^3 M - u(x)M = 0. \tag{3·3}$$

It can be verified that if $M(x,k)$ is defined by

$$M(x,k) = \psi(x,k)\phi'(x,k) - \psi'(x,k)\phi(x,k), \tag{3·4}$$

https://doi.org/10.1017/S0305004107000084 Published online by Cambridge University Press

where $\psi(x,k)$ and $\phi(x,k)$ are any two solutions of equation (2·1), then M satisfies equation (3·3). Thus, if $q(x,t)$ satisfies equation (1·2), then the following identity is valid,

$$\left(e^{-ik^3t}qM\right)_t + e^{-ik^3t}\left(q_{xx}M - q_x\partial_x M + q\partial_x^2 M\right)_x = 0, \tag{3·5}$$

where M is any of the functions $\{M_j\}_{j=1}^3$ or $\{N_j\}_{j=1}^3$ defined by equations (2·15), (2·17) and (2·20).

A particular solution of equations (1·4) is given by the following expression:

$$\begin{aligned} &3k^2\Delta(k)\mu_p(x,t,k) \\ &\quad = \psi_1(x,k)\int^{(x,t)} e^{ik^3(t-\tau)}[M_1(\xi,k)q(\xi,\tau)\,d\xi - X_1(\xi,\tau,k)\,d\tau] \\ &\qquad + \psi_2(x,k)\int^{(x,t)} e^{ik^3(t-\tau)}[M_2(\xi,k)q(\xi,\tau)d\xi - X_2(\xi,\tau,k)\,d\tau] \\ &\qquad + \psi_3(x,k)\int^{(x,t)} e^{ik^3(t-\tau)}[M_3(\xi,k)q(\xi,\tau)\,d\xi - X_3(\xi,\tau,k)\,d\tau], \end{aligned} \tag{3·6}$$

where $3k^2\Delta(k)$ is the Wronskian of the three solutions of the ODE (2·1) defined by equations (2·3), i.e.

$$3k^2\Delta(k) = \psi_1(\psi_2'\psi_3'' - \psi_2''\psi_3') + \psi_2(\psi_3'\psi_1'' - \psi_3''\psi_1') + \psi_3(\psi_1'\psi_2'' - \psi_1''\psi_2'), \tag{3·7}$$

the functions $\{M_j\}_{j=1}^3$ are defined by (2·17a), (2·15a), (2·20a), and the functions $\{X_j\}_{j=1}^3$ are defined by

$$X_j(x,t,k) = q_{xx}(x,t)M_j(x,k) - q_x(x,t)M_j'(x,k) + q(x,t)M_j''(x,k), \quad j=1,2,3. \tag{3·8}$$

Indeed, we will first show that $\mu_p(x,t,k)$ can be expressed in the form

$$\mu_p(x,t,k) = \frac{1}{3k^2\Delta(k)}\sum_{j=1}^3 \psi_j(x,k)v_j(x,t,k), \tag{3·9}$$

where $\{v_j\}_{j=1}^3$ satisfy

$$\left(e^{-ik^3t}v_j\right)_x = e^{-ik^3t}qM_j, \tag{3·10a}$$

$$\left(e^{-ik^3t}v_j\right)_t = -e^{-ik^3t}X_j. \tag{3·10b}$$

The functions $\{v_j\}_{j=1}^3$ are well defined since the compatibility condition of equations (3·10) is equation (3·5) with M replaced by M_j, $j=1,2,3$.

Substituting (3·9) in the left-hand side of equation (1·4a) and using (3·10a) we find

$$\frac{1}{3k^2\Delta(k)}\sum_{j=1}^3[q_{xx}\psi_jM_j + q_x(2\psi_jM_j' + 3\psi_j'M_j) + q\psi_jM_j'' + 3q\partial_x(\psi_j'M_j)],$$

which equals $q(x,t)$, in lieu of the following identities

$$\begin{aligned} &\sum_{j=1}^3 \psi_j(x,k)M_j(x,k) = 0, \quad \sum_{j=1}^3 \psi_j'(x,k)M_j(x,k) = 0, \\ &\sum_{j=1}^3 \psi_j(x,k)M_j'(x,k) = 0, \quad \sum_{j=1}^3 \psi_j(x,k)M_j''(x,k) = 3k^2\Delta(k). \end{aligned} \tag{3·11}$$

https://doi.org/10.1017/S0305004107000084 Published online by Cambridge University Press

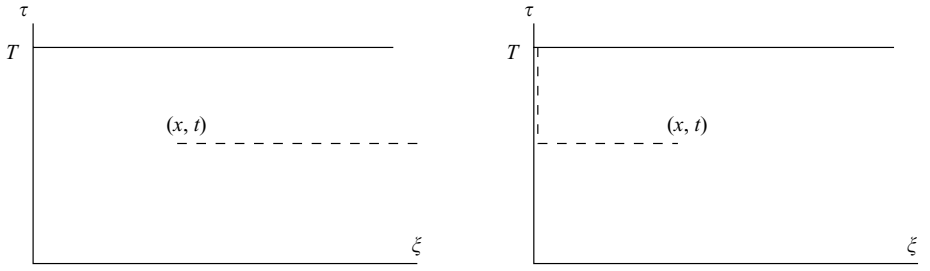


Fig. 4. The contours used in equation (3·13).

Similarly, substituting (3·9) in the left-hand side of equation (1·4*b*) and using (3·10*b*) we find

$$-\frac{1}{3k^2\Delta(k)}\sum_{j=1}^{3}[q_{xx}\psi_j M_j - q_x\psi_j M_j' + q\psi_j M_j''],$$

which equals $-q(x,t)$, in lieu of the identities in (3·11). We note that the right-hand side of the last identity in (3·11) is equal to the Wronskian $W[\psi_1, \psi_2, \psi_3]$, given by the right-hand side of equation (3·7), and is independent of x, i.e. $W = W(k)$ (evaluating $W(k)$ at $x = \infty$, it follows that $W = 3k^2\Delta(k)$, where $\Delta(k)$ is defined by equation (2·11)).

Equations (3·10) can be written in the form

$$d\left[e^{-ik^3t}v_j\right] = e^{-ik^3t}[q(x,t)M_j(x,k)dt + X_j(x,t,k)\,dx], \quad j = 1,2,3. \tag{3·12}$$

Integrating this equation we find

$$v_j(x,t) = \int^{(x,t)} e^{ik^3(t-\tau)}[q(\xi,\tau)M_j(\xi,k)\,d\tau + X_j(\xi,\tau,k)\,d\xi], \quad j = 1,2,3.$$

Substituting these expressions for $\{v_j\}_{j=1}^3$ into equation (3·9) we find equation (3·6).

By choosing the limits of the integrals in (3·6) appropriately, namely as the corners of the polygonal domain $\{0 < x < \infty, 0 < t < T\}$, we can construct a solution $\mu(x,t,k)$ of (1·4) which is sectionally analytic in the entire complex k-plane.

3·2. *The construction of a sectionally analytic* $\mu(x,t,k)$

We first consider the domain D_1. A solution of equations (1·4) which is bounded and analytic in this domain is given by

$$\begin{aligned}3k^2\Delta(k)\mu_1(x,t,k) = &-\psi_1(x,k)\int_x^\infty q(\xi,t)M_1(\xi,k)\,d\xi + \psi_2(x,k)\int_0^x q(\xi,t)M_2(\xi,k)\,d\xi\\ &+\psi_2(x,k)\int_t^T e^{ik^3(t-\tau)}X_2(0,\tau,k)\,d\tau\\ &-\psi_3(x,k)\int_x^\infty q(\xi,t)M_3(\xi,k)\,d\xi,\end{aligned} \tag{3·13}$$

where $\Delta(k)$ is given in (2·11), the base functions ψ_1, ψ_2, ψ_3 are defined in (2·3), the functions M_1, M_2, M_3 are defined in (2·17*a*), (2·15*a*), (2·20*a*), respectively, and the function X_2 is defined by equation (3·8) with $j = 2$. Equation (3·13) follows from equation (3·6) with the following choice of contours: for the first and third integrals in (3·6) the contour is $\{x < \xi < \infty, \tau = t\}$; for the second integral the contour is $\{0 < \xi < x, \tau = t\} \cup \{\xi = \infty, t < \tau < T\}$, see Figure 4.

https://doi.org/10.1017/S0305004107000084 Published online by Cambridge University Press

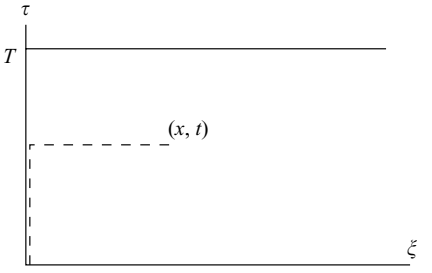


Fig. 5. The contours used in equation (3·14).

Using the identity

$$1 + \alpha + \alpha^2 = 0, \quad \alpha = e^{2\pi i/3},$$

it follows that the first term on the right-hand side of (3·13) equals

$$-\psi_1 e^{-ikx} \int_x^\infty e^{ik(x-\xi)} q(\xi, t) \left[\psi_2 e^{-i\alpha k\xi} \psi_3' e^{-i\alpha^2 k\xi} - \psi_2' e^{-i\alpha k\xi} \psi_3 e^{-i\alpha^2 k\xi} \right] d\xi.$$

The term $\exp[ik(x-\xi)]$ is bounded for $\operatorname{Im} k \leqslant 0$, and $\psi_1(x,k)e^{-ikx}$, $\psi_2(x,k)e^{-i\alpha kx}$ and $\psi_3(x,k)e^{-i\alpha^2 kx}$ are all bounded and analytic in D_1 (see Proposition 2·1). Similar arguments are valid for the other terms on the right-hand side of (3·13). Also, $1/\Delta(k)$ is analytic and bounded for all $k \neq 0$. Thus, $3k^2\mu_1(x,t,k)$ is bounded and analytic in D_1.

A solution of (1·4) which is bounded and analytic in the region D_2 is given by

$$\begin{aligned} &3k^2\Delta(k)\mu_2(x,t,k) \\ &\quad = \psi_1(x,k) \int_0^x q(\xi,t) M_1(\xi,k)\, d\xi - \psi_1(x,k) \int_0^t e^{ik^3(t-\tau)} X_1(0,\tau,k)\, d\tau \\ &\qquad + \psi_2(x,k) \int_0^x q(\xi,t) M_2(\xi,k)\, d\xi - \psi_2(x,k) \int_0^t e^{ik^3(t-\tau)} X_2(0,\tau,k)\, d\tau \\ &\qquad - \psi_3(x,k) \int_x^\infty q(\xi,t) M_3(\xi,k)\, d\xi, \end{aligned} \tag{3·14}$$

where the function X_1 is defined by equation (3·8) with $j = 1$. Equation (3·14) follows from equation (3·6) with the following choice of contours: for the first two integrals in (3·6) the contour is $\{0 < x < \xi, \tau = t\} \cup \{\xi = 0, 0 < \tau < t\}$, see Figure 5. The proof that $3k^2\mu_2$ is bounded and analytic in D_2 is similar to the analogous proof for $3k^2\mu_1$.

In order to construct solutions of (1·4) which are bounded and analytic in the sectors D_3 and D_4 we must use a different set of eigenfunctions from those defined in (2·3). Using the eigenfunctions $\{\phi_j\}_{j=1}^2$ defined in (2·7), we can construct the following sectionally analytic solutions:

$$\begin{aligned} &3k^2\Delta(k)\mu_3(x,t,k) \\ &\quad = \phi_1(x,k) \int_0^x q(\xi,t) N_1(\xi,k)\, d\xi - \phi_1(x,k) \int_0^t e^{ik^3(t-\tau)} X_1^*(0,\tau,k)\, d\tau \\ &\qquad + \phi_2(x,k) \int_0^x q(\xi,t) N_2(\xi,k)\, d\xi - \phi_2(x,k) \int_0^t e^{ik^3(t-\tau)} X_2^*(0,\tau,k)\, d\tau \\ &\qquad - \psi_3(x,k) \int_x^\infty q(\xi,t) N_3(\xi,k)\, d\xi \end{aligned} \tag{3·15}$$

https://doi.org/10.1017/S0305004107000084 Published online by Cambridge University Press

and

$$3k^2\Delta(k)\mu_4(x,t,k) = \phi_1(x,k)\int_0^x q(\xi,t)N_1(\xi,k)\,d\xi + \phi_1(x,k) \times \int_t^T e^{ik^3(t-\tau)}X_1^*(0,\tau,k)\,d\tau - \phi_2(x,k)\int_x^\infty q(\xi,t)N_2(\xi,k)\,d\xi - \psi_3(x,k)\int_x^\infty q(\xi,t)N_3(\xi,k)\,d\xi, \tag{3·16}$$

where the functions N_1, N_2, N_3 are defined in (2·15b), (2·17b), (2·20b), respectively, and the functions X_1^* and X_2^* are defined by

$$X_1^*(x,t,k) = q_{xx}(x,t)N_1(x,k) - q_x(x,t)N_1'(x,k) + q(x,t)N_1''(x,k) \tag{3·17a}$$

$$X_2^*(x,t,k) = q_{xx}(x,t)N_2(x,k) - q_x(x,t)N_2'(x,k) + q(x,t)N_2''(x,k). \tag{3·17b}$$

Starting with the functions $\mu_1(x,t,k)$, $\mu_2(x,t,k)$, $\mu_3(x,t,k)$, $\mu_4(x,t,k)$ defined in (3·13)–(3·17) and using symmetry considerations, it is possible to define solutions of (1·4) which are bounded and analytic in the remaining sectors D_4–D_{12}. Indeed, equations (1·4) are invariant under the transformations $k \mapsto \alpha k$ and $k \mapsto \alpha^2 k$. Thus, if $\mu(x,t,k)$ is a solution of equations (1·4), then $\mu(x,t,\alpha k)$ and $\mu(x,t,\alpha^2 k)$ are also solutions. For example, since $\mu_1(x,t,\alpha k)$ and $\mu_1(x,t,\alpha^2 k)$ are solutions for $\alpha k \in D_1$ and $\alpha^2 k \in D_1$, respectively, it follows that $\mu_9(x,t,k) \doteqdot \mu_1(x,t,\alpha k)$ and $\mu_5(x,t,k) \doteqdot \mu_1(x,t,\alpha^2 k)$ are solutions for $k \in D_9$ and $k \in D_5$, respectively. Hence,

$$\mu_{8+j}(x,t,k) \doteqdot \mu_j(x,t,\alpha k), \quad \mu_{4+j}(x,t,k) \doteqdot \mu_j(x,t,\alpha^2 k), \quad j = 1,2,3,4. \tag{3·18}$$

3·3. *The formulation of a Riemann–Hilbert problem*

We denote the sectionally analytic function $\mu(x,t,k)$ by

$$\mu(x,t,k) = \mu_j(x,t,k), \quad k \in D_j, \quad j = 1,\ldots,12.$$

Using the definitions of $\{\psi_j(x,k)\}_{j=1}^3$ and $\{\phi_j(x,k)\}_{j=1}^2$ in (2·3) and (2·7), respectively, it follows that $3k^2\mu(x,t,k) = O(1/k)$ as $k \to \infty$. Together, these facts define a Riemann–Hilbert problem for the function $3k^2\mu(x,t,k)$. To solve this Riemann–Hilbert problem we must determine the "jumps" across the rays Σ_j, $j = 1,\ldots,12$, shown in Figure 6: the difference of any two solutions satisfy the homogeneous version of equations (1·4), thus

$$\mu_j(x,t,k) - \mu_{j+1}(x,t,k) = e^{ik^3t}\sum_{l=1}^3 \rho_l(k)\Phi_l(x,k), \tag{3·19}$$

where $\{\Phi_l(x,k)\}_{l=1}^3$ are three linearly independent solutions of equation (2·1).

The functions μ_1 and μ_2 are defined using the same space-dependent eigenfunctions $\{\psi_j(x,k)\}_{j=1}^3$, thus the jump across the ray Σ_1 is relatively straightforward to compute: in equation (3·19) we let $j = 1$ and set $\{\Phi_j(x,k)\}_{j=1}^3 = \{\psi_j(x,k)\}_{j=1}^3$, we evaluate the resulting equation at $t = 0$, and we use the definitions of μ_1 and μ_2 (equations (3·13) and (3·14)); this yields

$$3k^2(\mu_2(x,t,k) - \mu_1(x,t,k)) = \frac{1}{\Delta(k)}e^{ik^3t}[\hat{q}_1(k)\psi_1(x,k) - \hat{G}_2(k)\psi_2(x,k)], \tag{3·20}$$

for $k \in \Sigma_1$, where $\psi_1(x,k)$ and $\psi_2(x,k)$ are defined in (2·3), $\hat{q}_1(k)$ is defined in (2·12a) and $\hat{G}_2(k)$ is defined in (2·13b). Similarly, for the jump across the ray Σ_3, in equation (3·19) we

https://doi.org/10.1017/S0305004107000084 Published online by Cambridge University Press

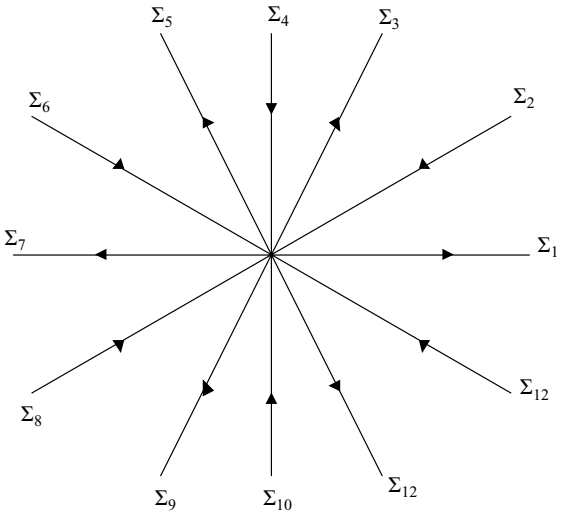


Fig. 6. The contours Σ_j, $j = 1, \ldots, 12$ for the Riemann–Hilbert problem.

let $j = 3$ and set $\Phi_1 = \phi_1$, $\Phi_2 = \phi_2$, $\Phi_3 = \psi_3$, we evaluate the resulting equation at $t = 0$, and we use the definitions of μ_3 and μ_4 (equations (3·15) and (3·16)); this yields

$$3k^2(\mu_4(x,t,k) - \mu_3(x,t,k)) = \frac{1}{\Delta(k)} e^{ik^3t}[\tilde{G}_1(k)\phi_1(x,k) - \tilde{q}_2(k)\phi_2(x,k)], \qquad (3\cdot21)$$

for $k \in \Sigma_3$, where $\phi_1(x,k)$ and $\phi_2(x,k)$ are defined in (2·7), $\tilde{q}_2(k)$ is defined in (2·12*b*) and $\tilde{G}_1(k)$ is defined in (2·13*a*). It is shown in Appendix A·4 that the jumps across the rays Σ_2 and Σ_4 vanish.

Having computed the jumps across the contours Σ_1, Σ_2, Σ_3 and Σ_4, we can now use the α-symmetry to determine the jumps across the remaining contours.

3·4. *An integral representation*

The unique solution of the associated Riemann–Hilbert problem is [**15**]

$$\begin{aligned} k^2\mu(x,t,k) = \frac{1}{2\pi i}\Bigg[& \int_{\Sigma_1} \frac{l^2(\mu_2 - \mu_1)(l)}{l-k}\,dl + \int_{\Sigma_3} \frac{l^2(\mu_4 - \mu_3)(l)}{l-k}\,dl \\ & + \int_{\Sigma_5} \frac{l^2(\mu_2 - \mu_1)(\alpha l)}{l-k}\,dl + \int_{\Sigma_7} \frac{l^2(\mu_4 - \mu_3)(\alpha l)}{l-k}\,dl \\ & + \int_{\Sigma_9} \frac{l^2(\mu_2 - \mu_1)(\alpha^2 l)}{l-k}\,dl + \int_{\Sigma_{11}} \frac{l^2(\mu_4 - \mu_3)(\alpha^2 l)}{l-k}\,dl\Bigg], \end{aligned} \qquad (3\cdot22)$$

where we have suppressed the x, t dependence of the functions $\{\mu_j\}_{j=1}^4$.

The Lax pair (1·4) yields

$$q(x,t) = \lim_{k\to\infty}(ik^3\mu).$$

This equation, together with equation (3·22) implies the following integral representation

https://doi.org/10.1017/S0305004107000084 Published online by Cambridge University Press

for $q(x,t)$:

$$q(x,t) = -\frac{1}{2\pi}\Bigg[\int_{\Sigma_1} k^2(\mu_2-\mu_1)(k)\,dk + \int_{\Sigma_3} k^2(\mu_4-\mu_3)(k)\,dk + \int_{\Sigma_5} k^2(\mu_2-\mu_1)(\alpha k)\,dk + \int_{\Sigma_7} k^2(\mu_4-\mu_3)(\alpha k)\,dk + \int_{\Sigma_9} k^2(\mu_2-\mu_1)(\alpha^2 k)\,dk + \int_{\Sigma_{11}} k^2(\mu_4-\mu_3)(\alpha^2 k)\,dk\Bigg]. \tag{3·23}$$

We note that if k is on Σ_7 then $\alpha^2 k$ is on Σ_3, and thus the integral along Σ_7 is equal to the integral along Σ_3. Using similar arguments for the other integrals we find

$$q(x,t) = -\frac{1}{2\pi}\left[\int_{\Sigma_3} 3k^2(\mu_4(k)-\mu_3(k))\,dk + \int_{\Sigma_5} 3k^2(\mu_2(\alpha k)-\mu_1(\alpha k))\,dk\right]. \tag{3·24}$$

Alternatively, we can find a similar representation involving the contours Σ_1 and Σ_7,

$$q(x,t) = -\frac{1}{2\pi}\left[\int_{\Sigma_1} 3k^2(\mu_2(k)-\mu_1(k))\,dk + \int_{\Sigma_7} 3k^2(\mu_4(\alpha^2 k)-\mu_3(\alpha^2 k))\,dk\right]. \tag{3·25}$$

For the utilisation of the global relations given in Proposition 2·6, it is more convenient to place the part of the jumps involving $\hat{q}_1(k)$ and $\tilde{q}_2(k)$ on the contours Σ_1 and Σ_7, and the part of the jumps which involve $\hat{G}_1(k)$ and $\tilde{G}_2(k)$ on the contours Σ_3 and Σ_5. This yields the representation (2·10).

4. *The analysis of the global relation*

The definitions of $\hat{q}_1(t,k)$ (equation (2·12*a*) with $q(x,0)$ replaced by $q(x,t)$) and of $M_1(x,k)$ (equation (2·17*a*)) imply

$$\hat{q}_1(t,k) = \int_0^\infty q(x,t)M_1(x,k)\,dx.$$

Multiplying this equation by e^{-ik^3t}, differentiating with respect to t and using equation (3·5) with $M = M_1$ we find

$$\left(e^{-ik^3t}\hat{q}_1(t,k)\right)_t = e^{-ik^3t}\left[q_{xx}(0,t)M_1(0,k) - q_x(0,t)\partial_x M_1(0,k) + q(0,t)\partial_x^2 M_1(0,k)\right].$$

Integrating this equation with respect to t from $t=0$ to $t=T$ we find equation (2·16*a*).

A similar analysis, using the functions $N_2(x,k)$, $M_3(x,k)$, $N_3(x,k)$, yields analogous equations for $\tilde{q}_2(T,k)$, $\hat{q}_3(T,k)$ and $\tilde{q}_3(T,k)$, respectively, see equations (2·16*b*), (2·19*a*) and (2·19*b*).

4·1. *The elimination of the unknown boundary values*

The integral representation for $q(x,t)$ in (2·10) indicates that we must calculate $\tilde{G}_1(k)$ and $\hat{G}_2(\alpha^2 k)$ on the rays $\arg k = \pi/3$ and $\arg k = 2\pi/3$, respectively. The functions $\tilde{G}_1$ and $\hat{G}_2$, defined in (2·13), involve the *unknown* functions $\hat{g}_1(k)$ and $\hat{g}_2(k)$. Thus, in order to make the integral representation in (2·10) effective, we must express these unknown functions in terms of the known boundary conditions.

We first analyse $\tilde{G}_1(k)$ and, for this purpose, we use the two global relations (2·16*b*) and (2·19*b*). Both of these expressions are valid in D_4, which contains the ray $\arg k = \pi/3$; they are two independent equations relating the three functions $\hat{g}_0$, $\hat{g}_1$, $\hat{g}_2$. This implies that the problem is well posed if *one* boundary condition is specified. Thus, for instance, if

https://doi.org/10.1017/S0305004107000084 Published online by Cambridge University Press

$q(0,t) = g_0(t)$ is given as a boundary condition, we can use equations (2·16*b*) and (2·19*b*) to express the unknown functions $\hat{g}_1(k)$ and $\hat{g}_2(k)$ in terms of known functions. Indeed, solving these two equations for the two unknown functions $\hat{g}_1(k)$ and $\hat{g}_2(k)$ we find the following expressions:

$$\begin{aligned}\hat{g}_2(k) = &\frac{1}{a(k)}[\tilde{q}_2(k)N_3'(0,k) - \tilde{q}_3(k)N_2'(0,k)] \\ &+ \frac{\hat{g}_0(k)}{a(k)}[N_2''(0,k)N_3'(0,k) - N_2'(0,k)N_3''(0,k)] \\ &- \frac{e^{-ik^3T}}{a(k)}[\tilde{q}_2(T,k)N_3'(0,k) - \tilde{q}_3(T,k)N_2'(0,k)], \end{aligned} \tag{4·1a}$$

$$\begin{aligned}\hat{g}_1(k) = &\frac{1}{a(k)}[\tilde{q}_2(k)N_3(0,k) - \tilde{q}_3(k)N_2(0,k)] \\ &+ \frac{\hat{g}_0(k)}{a(k)}[N_2''(0,k)N_3(0,k) - N_2(0,k)N_3''(0,k)] \\ &- \frac{e^{-ik^3T}}{a(k)}[\tilde{q}_2(T,k)N_3(0,k) - \tilde{q}_3(T,k)N_2(0,k)], \end{aligned} \tag{4·1b}$$

where

$$\begin{aligned}a(k) &= N_2'(0,k)N_3(0,k) - N_2(0,k)N_3'(0,k) \\ &= \phi_1(0,k)[\phi_1(\phi_2'\psi_3'' - \phi_2''\psi_3') + \phi_2(\psi_3'\phi_1'' - \psi_3''\phi_1') + \psi_3(\phi_1'\phi_2'' - \phi_1''\phi_2')](0,k) \\ &= \phi_1(0,k)W[\phi_1,\phi_2,\psi_3].\end{aligned}$$

In the last equation above, we have used the fact that the expression in the middle equation is the Wronskian of the solutions $\{\phi_1, \phi_2, \psi_3\}$ which equals $3k^2\Delta(k)$.

Substituting these expressions into equation (2·13*a*) and using the identities

$$\begin{aligned}N_2'(0,k)N_3(0,k) - N_2(0,k)N_3'(0,k) &= \phi_1(0,k)W[\phi_1,\phi_2,\psi_3], \\ N_3'(0,k)N_1(0,k) - N_3(0,k)N_1'(0,k) &= \phi_2(0,k)W[\phi_1,\phi_2,\psi_3], \\ N_1'(0,k)N_2(0,k) - N_1(0,k)N_2'(0,k) &= \psi_3(0,k)W[\phi_1,\phi_2,\psi_3]\end{aligned}$$

and

$$\begin{aligned}&N_1''(N_2'(0,k)N_3(0,k) - N_2(0,k)N_3'(0,k)) + N_2''(N_3'(0,k)N_1(0,k) - N_3(0,k)N_1'(0,k)) \\ &\quad + N_3''(N_1'(0,k)N_2(0,k) - N_1(0,k)N_2'(0,k)) = W^2[\phi_1,\phi_2,\psi_3],\end{aligned}$$

which follow from the definitions of N_1, N_2, N_3 and $W[\phi_1,\phi_2,\psi_3]$, we find

$$\begin{aligned}\tilde{G}_1(k) = &\frac{\phi_2(0,k)}{\phi_1(0,k)}\tilde{q}_2(k) + \frac{\phi_3(0,k)}{\phi_1(0,k)}\tilde{q}_3(k) + \frac{3k^2\Delta(k)}{\phi_1(0,k)}\hat{g}_0(k) \\ &- e^{-ik^3T}\left[\frac{\phi_2(0,k)}{\phi_1(0,k)}\tilde{q}_2(T,k) + \frac{\phi_3(0,k)}{\phi_1(0,k)}\tilde{q}_3(T,k)\right], \quad k \in D_4. \end{aligned} \tag{4·2}$$

An analogous result can be derived for $\hat{G}_2(\alpha^2 k)$ on the ray $\arg k = 2\pi/3$. Both equations (2·16*a*) and (2·19*a*) are valid for $k \in D_1$, thus for $\alpha k \in D_5$, which contains the ray arg

https://doi.org/10.1017/S0305004107000084 Published online by Cambridge University Press

$k = 2\pi/3$. Solving these two equations for the two unknown functions $\hat{g}_1(k)$ and $\hat{g}_2(k)$, and then substituting the resulting expressions in the definition of $\hat{G}_2(\alpha^2 k)$ we find

$$\hat{G}_2(\alpha^2 k) = \frac{\psi_1(0,\alpha^2 k)}{\psi_2(0,\alpha^2 k)}\hat{q}_1(\alpha^2 k) + \frac{\psi_3(0,\alpha^2 k)}{\psi_2(0,\alpha^2 k)}\hat{q}_3(\alpha^2 k) + \frac{3(\alpha^2 k)^2\Delta(\alpha^2 k)}{\psi_2(0,\alpha^2 k)}\hat{g}_0(k) - e^{-ik^3T}\left[\frac{\psi_1(0,\alpha^2 k)}{\psi_2(0,\alpha^2 k)}\hat{q}_1(T,\alpha^2 k) + \frac{\psi_3(0,\alpha^2 k)}{\psi_2(0,\alpha^2 k)}\hat{q}_3(T,\alpha^2 k)\right], \quad k \in D_5. \quad (4{\cdot}3)$$

This result is derived in Appendix A·5.

We will assume that $\phi_1(0,k)$ does not have any zeros for $k \in D_4$ and that $\psi_2(0,\alpha^2 k)$ does not have any zeros for $k \in D_5$, see Section 4·2 below. In this case the terms involving $\hat{q}_1(T,\alpha^2 k)$, $\tilde{q}_2(T,k)$, $\hat{q}_3(T,\alpha^2 k)$ and $\tilde{q}_3(T,k)$ do *not* contribute to the integral representation for $q(x,t)$ in (2·10), and hence (4·2) and (4·3) reduce to (2·21 *a*) and (2·21 *b*), respectively. The proof that the above terms do not contribute to equation (2·10) relies upon the application of Jordan's Lemma in the sectors D_4 and D_5. Indeed, substituting (4·2) and (4·3) into (2·10) we find that the above terms yield the following contribution

$$\frac{1}{2\pi}\int_{L_1}\frac{1}{\Delta(k)}e^{ik^3(t-T)}\phi_1(x,k)\left[\frac{\phi_2(0,k)}{\phi_1(0,k)}\tilde{q}_1(T,k) + \frac{\phi_3(0,k)}{\phi_1(0,k)}\tilde{q}_3(T,k)\right]dk + \frac{1}{2\pi}\int_{L_2}\frac{1}{\Delta(k)}e^{ik^3(t-T)}\psi_2(x,k)\left[\frac{\psi_1(0,k)}{\psi_2(0,k)}\hat{q}_1(T,k) + \frac{\psi_3(0,k)}{\psi_1(0,k)}\hat{q}_3(T,k)\right]dk.$$

We can rewrite this expression in the form

$$\frac{1}{2\pi}\int_{\partial D_4}\frac{1}{\Delta(k)}e^{ik^3(t-T)}\phi_1(x,k)\left[\frac{\phi_2(0,k)}{\phi_1(0,k)}\tilde{q}_1(T,k) + \frac{\phi_3(0,k)}{\phi_1(0,k)}\tilde{q}_3(T,k)\right]dk + \frac{1}{2\pi}\int_{\partial D_5}\frac{1}{\Delta(k)}e^{ik^3(t-T)}\psi_2(x,k)\left[\frac{\psi_1(0,k)}{\psi_2(0,k)}\hat{q}_1(T,k) + \frac{\psi_3(0,k)}{\psi_1(0,k)}\hat{q}_3(T,k)\right]dk,$$

where ∂D_4 and ∂D_5 are the oriented boundaries of the domains D_4 and D_5, respectively. Here we have added and subtracted appropriate integral terms along the imaginary axis which, by utilising the identities proved in Appendix A·4, can be shown to add to zero. The function $\phi_1(x,k)$ is analytic for $k \in D_4$ and, since $\tilde{q}_2(T,k)$ and $\tilde{q}_3(T,k)$ are both analytic and bounded for $k \in D_4$, by applying Jordan's Lemma in D_4 it follows that the contribution from this term vanishes. Similarly, by applying Jordan's Lemma in D_5, it follows that the contribution from the second term also vanishes.

4·2. *The discrete spectrum*

For brevity of presentation we have assumed that the functions $\phi_1(0,k)$ and $\psi_2(0,\alpha^2 k)$ appearing in (2·21 *a*) and (2·21 *b*) do not have any zeros in the domains D_4 and D_5, respectively. If such zeros do exist then the contours L_1 and L_2 must be deformed to encircle the relevant poles, see [**14**]. However, in general it is nontrivial to identify the relevant zeros. In the case of equation (1·1) it can be shown that there only exist a finite number of zeros [**16**, **17**].

If the functions $\phi_1(0,k)$ and $\psi_2(0,\alpha^2 k)$ do have zeros in D_4 and D_5, respectively, then for the boundary conditions (1·3) the resulting discrete contribution to the formula (2·10) arises exclusively from the potential $u(x)$.

https://doi.org/10.1017/S0305004107000084 Published online by Cambridge University Press

5. *Conclusions*

In this paper we have extended the method of [**10**] for solving initial-boundary value problems for a class of linear evolution PDEs with variable coefficients. The method is based on two basic steps:

(i) construct an integral representation for the solution $q(x,t)$ in the complex k-plane (see for example (2·10)). This representation involves appropriate transforms of *all* boundary values (for example, equation (2·10) involves the x-transform of $q(x,0)$ and the t-transform of $q(0,t)$, $q_x(0,t)$, $q_{xx}(0,t)$);

(ii) use certain algebraic global relations to determine the transforms of the unknown boundary values. For evolution equations this step involves only algebra (see for example equations (4·1)).

For economy of presentation we have *assumed* that the solution of the given initial-boundary value problem exists. However, it is possible to present rigorous theorems without the *a priori* assumption of existence. For the case of constant coefficients this is discussed in [**18**].

The remarkable fact about the new method is that it yields the solution of linear evolution PDEs with variable coefficients with the same level of efficiency as the corresponding PDEs with constant coefficients. The only difference, which however does *not* affect the efficiency of the method, is that one now uses appropriate base functions instead of the exponential function (for instance, $\{\psi_j(x,k)\}_{j=1}^3$ and $\{\phi_j(x,k)\}_{j=1}^2$ defined in (2·3) and (2·7), respectively). For particular cases of variable coefficients, these base functions can be constructed explicitly; such explicit base functions for appropriately chosen $u(x)$ for equation (1·1) are presented in [**10**].

An alternative approach to analysing equations such as (1·1) and (1·2) is to use a Laplace transform in t. However, since t is in general finite, this does not appear to be an appropriate technique unless one is interested in the large t behaviour. Furthermore, even for second order problems where the analysis is much easier than for third order problems it yields formulae which are both complicated and do *not* involve explicit t-dependence (see [**17**] where the Laplace transform approach was used to show that the large t behaviour of the solution of (1·1) is dominated by the bound states of the potential $u(x)$). An advantage of the method described here is that t-dependence in the solution formula is in the form of an exponential. This allows one to compute the large t behaviour of the solution in a straightforward manner using standard asymptotic techniques (such as the steepest descent method).

The analysis presented here generalizes naturally to evolution equations with variable coefficients of the following form:

$$(\partial_t + i\omega(x, -i\partial_x))q(x,t) = 0, \tag{5·1a}$$

where $\omega(x,l)$ is a polynomial of the form

$$\pm l^n + u_{n-1}(x)l^{n-1} + \cdots + u_1(x)l + u_0(x), \quad n \geqslant 3. \tag{5·1b}$$

A Lax pair for equation (5·1) is

$$\left[\partial_x^n - (ik)^n + i^n\left(u_{n-1}(x)(-i\partial_x)^{n-1} + \cdots + u_1(x)(-i\partial_x) + u_0(x)\right)\right]\mu = q(x,t), \tag{5·2a}$$

$$\left[\partial_t \pm (-i)^{n-1}(ik)^n\right]\mu = \mp(-i)^{n-1}q(x,t), \quad k \in \mathbb{C}, \tag{5·2b}$$

where $\mu(x,t,k)$ is a scalar function.

https://doi.org/10.1017/S0305004107000084 Published online by Cambridge University Press

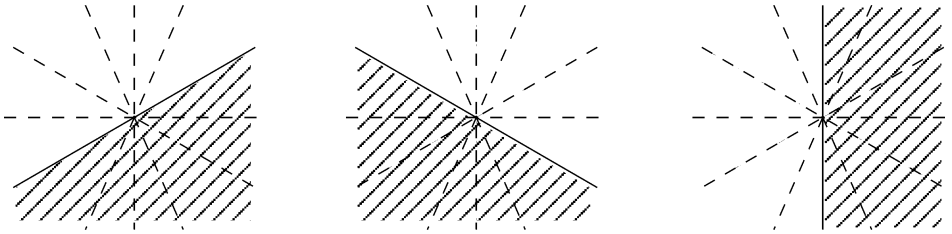


Fig. 7. The domains of boundedness of $e^{i(\alpha-1)kx}$, $e^{i(\alpha^2-1)kx}$, $e^{i(\alpha-\alpha^2)kx}$.

The analysis of equations (5·2) involves dividing the complex k-plane into N sectors, where $N = 2n$ if n is even, and $N = 4n$ if n is odd, and constructing solutions of (5·2) which are analytic in each of these sectors. This construction uses variation of parameters, as well as base functions $\{\psi_j(x,k)\}_{j=1}^n$ analogous to the eigenfunctions defined by (2·3) and (2·7). After an appropriate integral representation of the solution $q(x,t)$ is constructed, the analysis of the associated global relations (analogous to equations (2·16) and (2·19)) allows one to eliminate the unknown boundary values.

Appendices

A·1. *Analyticity properties of the eigenfunctions*

Substituting the representation given by equation (2·2*a*) into equation (2·1), and using the fact that

$$1 + \alpha + \alpha^2 = 0, \quad \alpha = e^{2\pi i/3},$$

we find that equation (2·1) is satisfied identically.

The function $\psi_1(x,k)$ is defined in equation (2·3*a*). We note that the right-hand side of this equation contains one integral involving ∞; taking into consideration the fact that $x - \xi \leqslant 0$, it follows that this integral is well defined in the punctured complex k-plane provided that $\operatorname{Im}\alpha^2 k \leqslant 0$. Similar considerations imply that the function $\psi_2(x,k)$ is well defined provided that $\operatorname{Im} k \leqslant 0$ and $\operatorname{Im}\alpha^2 k \leqslant 0$. The function $\psi_3(x,k)$ defined in (2·3*c*) is an entire function in $k \in \mathbb{C}\backslash\{0\}$and is defined for all $k \in \mathbb{C}$.

The definition of $\psi_1(x,k)$ in (2·3) implies

$$\begin{aligned}\{\psi_1(x,k)e^{-ikx}\} = 1 &+ \frac{1}{3k^2}\int_0^x u(\xi)\{\psi_1(\xi,k)e^{-ik\xi}\}\,d\xi \\ &+ \frac{1}{3k^2}\int_0^x \alpha e^{i(\alpha-1)k(x-\xi)}u(\xi)\{\psi_1(\xi,k)e^{-ik\xi}\}\,d\xi \\ &- \frac{1}{3k^2}\int_x^\infty \alpha^2 e^{i(\alpha^2-1)k(x-\xi)}u(\xi)\{\psi_1(\xi,k)e^{-ik\xi}\}\,d\xi.\end{aligned}$$

Hence $\psi_1(x,k)e^{-ikx}$ is bounded and analytic in the domain $\{k \in \mathbb{C}\backslash\{0\} \mid \operatorname{Im}(\alpha-1)k > 0 \cap \operatorname{Im}(\alpha^2-1)k < 0\}$, which is the domain $D_1 \cup D_2$, see Figure 7. Similarly for $\psi_2(x,k)e^{-i\alpha kx}$ and $\psi_3(x,k)e^{-i\alpha^2 kx}$.

The proof of the analogous statements involving $\phi_1(x,k)$ and $\phi_2(x,k)$ is similar.

A·2. *Analyticity properties of the spectral functions*

The functions $\psi_1'(x,k)e^{-ikx}$, $\psi_2'(x,k)e^{-i\alpha kx}$, $\psi_3'(x,k)e^{-i\alpha^2 kx}$ satisfy integral equations similar to the equations satisfied by $\psi_1(x,k)e^{-ikx}$, $\psi_2(x,k)e^{-i\alpha kx}$, $\psi_3(x,k)e^{-i\alpha^2 kx}$. Thus,

https://doi.org/10.1017/S0305004107000084 Published online by Cambridge University Press

they have identical domains of boundedness and analyticity with these latter functions. Similarly, the functions $\phi_1'(x,k)e^{-ikx}$, $\phi_2'(x,k)e^{-i\alpha kx}$ satisfy integral equations similar to the equations satisfied by $\phi_1(x,k)e^{-ikx}$, $\phi_2(x,k)e^{-i\alpha kx}$, and will therefore have domains of boundedness and analyticity identical with the latter functions.

The spectral function $\hat{q}_1(t,k)$, which is defined by (2·12*a*) with $q(x,0)$ replaced by $q(x,t)$, is bounded and analytic for $k \in D_1$. Indeed, the equation defining $\hat{q}_1(t,k)$ can be rewritten in the form

$$\hat{q}_1(t,k) = \int_0^\infty e^{-ikx} q(x,t)\psi_2(x,k)e^{-i\alpha kx}\psi_3(x,k)'e^{-i\alpha^2 kx}dx - \int_0^\infty e^{-ikx} q(x,t)\psi_2(x,k)'e^{-i\alpha kx}\psi_3(x,k)e^{-i\alpha^2 kx}dx.$$

The functions e^{-ikx}, $\psi_2(x,k)e^{-i\alpha kx}$ and $\psi_3(x,k)e^{-i\alpha^2 kx}$ are bounded and analytic for k in the lower half punctured complex k-plane, for k in $D_{11} \cup D_{12} \cup D_1 \cup D_2$, and for k in $D_1 \cup D_2 \cup D_3 \cup D_4$, respectively. Hence, $\hat{q}_1(t,k)$ is bounded and analytic for k in the intersection of these domains, which is the domain D_1.

Similarly, $\tilde{q}_2(t,k)$, $\hat{q}_3(t,k)$ and $\tilde{q}_3(t,k)$ satisfy the following equations

$$\tilde{q}_2(t,k) = \int_0^\infty e^{-i\alpha kx} q(x,t)\psi_3(x,k)e^{-i\alpha^2 kx}\phi_1(x,k)'e^{-ikx}\,dx - \int_0^\infty e^{-i\alpha kx} q(x,t)\psi_3(x,k)'e^{-i\alpha^2 kx}\phi_1(x,k)e^{-ikx}\,dx,$$

$$\hat{q}_3(t,k) = \int_0^\infty e^{-i\alpha^2 kx} q(x,t)\psi_1(x,k)e^{-ikx}\psi_2(x,k)'e^{-i\alpha kx}\,dx - \int_0^\infty e^{-i\alpha^2 kx} q(x,t)\psi_1(x,k)'e^{-ikx}\psi_2(x,k)e^{-i\alpha kx}\,dx,$$

$$\tilde{q}_3(t,k) = \int_0^\infty e^{-i\alpha^2 kx} q(x,t)\phi_1(x,k)e^{-ikx}\phi_2(x,k)'e^{-i\alpha kx}dx - \int_0^\infty e^{-i\alpha^2 kx} q(x,t)\phi_1(x,k)'e^{-ikx}\phi_2(x,k)e^{-i\alpha kx}\,dx.$$

The function $\tilde{q}_2(t,k)$ is therefore bounded and analytic in the intersection of the domains $\arg k \in [\pi/3, 4\pi/3]$ (associated with $\operatorname{Im}\alpha k \leqslant 0$), $D_1 \cup D_2 \cup D_3 \cup D_4$ and $D_3 \cup D_4 \cup D_5 \cup D_6$, which is the domain D_4. Similarly, the function $\hat{q}_3(t,k)$ is bounded and analytic in the intersection of the domains $\arg k \in [-\pi/3, 2\pi/3]$, $D_1 \cup D_2$ and $D_{11} \cup D_{12} \cup D_1 \cup D_2$, which is the domain $D_1 \cup D_2$. Finally, the function $\tilde{q}_3(t,k)$ is bounded and analytic in the intersection of the domains $\arg k \in [-\pi/3, 2\pi/3]$, $D_3 \cup D_4 \cup D_5 \cup D_6$ and $D_3 \cup D_4$, which is the domain $D_3 \cup D_4$.

We now show that the functions $N_1(x,k)$ and $M_2(x,k)$, defined in (2·15), exist for k in $\{k \,|\, \operatorname{Im}\alpha^2 k \leqslant 0\}$. The function N_1 involves $\phi_2\psi_3$, thus it exists in the union of the domains of existence of ϕ_2 and ψ_3, i.e. in the union of the domains $\arg k \in [-\pi/3, \pi/3]$ and $\mathbb{C}$, which is the domain $\{k \mid \operatorname{Im}\alpha^2 k \leqslant 0\}$. The function M_2 involves $\psi_1\psi_3$, thus it exists in the union of the domains $\arg k \in [-\pi/3, 2\pi/3]$ and $\mathbb{C}$, which is the domain $\{k \mid \operatorname{Im}\alpha^2 k \leqslant 0\}$.

The proof of the analogous statements involving M_1 and N_2 is similar.

https://doi.org/10.1017/S0305004107000084 Published online by Cambridge University Press

The representation for $q(x,t)$ given by equation (2·10) involves $\phi_1\tilde{G}_1$ and $\psi_2\hat{G}_2$ evaluated at k and $\alpha^2 k$, respectively. Since $\phi_1(x,k)\tilde{G}_1(k)$ involves the function

$$\phi_1(x,k)\phi_2(0,k)\psi_3(0,k) = (\phi_1(x,k)e^{-ikx})(\phi_2(0,k)e^{-i\alpha kx})(\psi_3(0,k)e^{-i\alpha^2 kx}),$$

it follows that $\phi_1(x,k)\tilde{G}_1(k)$ is bounded and analytic in the union of the domains $D_3 \cup D_4 \cup D_5 \cup D_6$, $D_3 \cup D_4$ and $D_1 \cup D_2 \cup D_3 \cup D_4$, i.e. in the domain $D_3 \cup D_4$, which contains the ray L_1. Similarly, using

$$\psi_2(x,k)\psi_1(0,k)\psi_3(0,k) = (\psi_2(x,k)e^{-i\alpha kx})(\psi_1(0,k)e^{-ikx})(\psi_3(0,k)e^{-i\alpha^2 kx}),$$

it follows that $\psi_2(x,k)\hat{G}_2(k)$ is bounded and analytic in the domain $D_1 \cup D_2$, and thus $\psi_2(x,\alpha^2 k)\hat{G}_2(\alpha^2 k)$ is bounded and analytic in the domain $D_5 \cup D_6$, which contains the ray L_2.

A·3. *Some useful identities*

In this appendix we derive some identities satisfied by the functions $\{\psi_j\}_{j=1}^3$ and $\{\phi_j\}_{j=1}^2$ defined in equations (2·3) and (2·7):

$$W[\phi_1,\psi_2,\psi_3]W[\psi_1,\phi_2,\psi_3] - W[\phi_2,\psi_2,\psi_3]W[\psi_1,\phi_1,\psi_3] = (W[\psi_1,\psi_2,\psi_3])^2 \quad \text{(A 3)}$$

and

$$W[\psi_1,\psi_2,\psi_3]\phi_j = W[\phi_j,\psi_2,\psi_3]\psi_1 + W[\psi_1,\phi_j,\psi_3]\psi_2, \quad j=1,2, \quad \text{(A 4)}$$

where $W[a,b,c]$ is the Wronskian

$$W[a,b,c] = a(b'c'' - b''c') + b(c'a'' - c''a') + c(a'b'' - a''b').$$

The first of these expressions (A 3) follows from simple algebra, and makes use of the fact that $W[\psi_1,\psi_2,\psi_3] = W[\phi_1,\phi_2,\psi_3]$. To prove the second identity (A 4) we first derive the following identity using simple algebra

$$W[\psi_1,\psi_2,\psi_3]\phi_j = W[\phi_j,\psi_2,\psi_3]\psi_1 + W[\psi_1,\phi_j,\psi_3]\psi_2 + W[\psi_1,\psi_2,\phi_j]\psi_3, \quad j=1,2.$$

Then, by considering the functions $\{\psi_j(x,k)\}_{j=1}^3$ and $\{\phi_j(x,k)\}_{j=1}^2$ as $x \to \infty$, it follows that the last term on the right-hand side vanishes and we find equation (A 4).

A·4. *There is no jump across* Σ_2 *and* Σ_4

The function $\mu_2(x,t,k)$ is defined in (3·14) in terms of the space-dependent eigenfunctions $\{\psi_j(x,k)\}_{j=1}^3$, whereas the function $\mu_3(x,t,k)$ is defined in (3·15) in terms of the functions $\{\phi_j(x,k)\}_{j=1}^2$ and $\psi_3(x,k)$. In order to prove that there is no jump across the contour Σ_2, we will express both μ_2 and μ_3 in terms of the functions $\{\psi_j(x,k)\}_{j=1}^3$ and will prove that they are identical for $k \in \Sigma_2$. Using the second identity (A 4) we find the following expression for μ_3 for $k \in \Sigma_2$,

$$\mu_3(x,t,k) = \frac{W[\phi_1,\psi_2,\psi_3]W[\psi_1,\phi_2,\psi_3] - W[\phi_2,\psi_2,\psi_3]W[\psi_1,\phi_1,\psi_3]}{(W[\psi_1,\psi_2,\psi_3])^2}\mu_2(x,t,k).$$

Thus the first identity (A 3) implies that $\mu_3 = \mu_2$ along Σ_2, and hence the jump $k^2\mu_2 - k^2\mu_3$ vanishes.

Similar arguments demonstrate that the jump across the contour Σ_4 is also zero.

https://doi.org/10.1017/S0305004107000084 Published online by Cambridge University Press

A·5. *Solution of the global relation*

In this appendix we derive the expression for $\hat{G}_2(\alpha^2 k)$ given in equation (4·3). We solve the two global relations (2·16*a*) and (2·19*a*) for the two unknown functions $\hat{g}_1(k)$ and $\hat{g}_2(k)$, and find the following expressions:

$$\begin{aligned}\hat{g}_2(k) = {} & \frac{1}{b(k)}[\hat{q}_1(k)M_3'(0,k) - \hat{q}_3(k)M_1'(0,k)] \\ & + \frac{\hat{g}_0(k)}{b(k)}[M_1''(0,k)M_3'(0,k) - M_1'(0,k)M_3''(0,k)] \\ & - \frac{e^{-ik^3T}}{b(k)}[\hat{q}_1(T,k)M_3'(0,k) - \hat{q}_3(T,k)M_1'(0,k)],\end{aligned}$$

$$\begin{aligned}\hat{g}_1(k) = {} & \frac{1}{b(k)}[\hat{q}_1(k)M_3(0,k) - \hat{q}_3(k)M_1(0,k)] \\ & + \frac{\hat{g}_0(k)}{b(k)}[M_1''(0,k)M_3(0,k) - M_1(0,k)M_3''(0,k)] \\ & - \frac{e^{-ik^3T}}{b(k)}[\hat{q}_1(T,k)M_3(0,k) - \hat{q}_3(T,k)M_1(0,k)],\end{aligned}$$

where

$$\begin{aligned}b(k) &= M_1'(0,k)M_3(0,k) - M_1(0,k)M_3'(0,k) \\ &= \psi_2(0,k)[\psi_1(\psi_2'\psi_3'' - \psi_2''\psi_3') + \psi_2(\psi_3'\psi_1'' - \psi_3''\psi_1') + \psi_3(\psi_1'\psi_2'' - \psi_1''\psi_2')](0,k) \\ &= \psi_1(0,k)W[\psi_1,\psi_2,\psi_3].\end{aligned}$$

In the last equation above, we have used the fact that the expression in the middle equation is the Wronskian of the solutions $\{\psi_1, \psi_2, \psi_3\}$ which equals $3k^2\Delta(k)$.

Substituting these expressions into equation (2·13*b*) and using the identities

$$\begin{aligned}M_1'(0,k)M_3(0,k) - M_1(0,k)M_3'(0,k) &= \psi_2(0,k)W[\psi_1,\psi_2,\psi_3], \\ M_2'(0,k)M_1(0,k) - M_3(0,k)M_1'(0,k) &= \psi_3(0,k)W[\psi_1,\psi_2,\psi_3], \\ M_3'(0,k)M_2(0,k) - M_1(0,k)M_2'(0,k) &= \psi_1(0,k)W[\psi_1,\psi_2,\psi_3]\end{aligned}$$

and

$$\begin{aligned}&M_1''(M_2(0,k)M_3'(0,k) - M_2'(0,k)M_3(0,k)) + M_2''(M_3(0,k)M_1'(0,k) \\ &\quad - M_3'(0,k)M_1(0,k)) + M_3''(M_1(0,k)M_2'(0,k) - M_1'(0,k)M_2(0,k)) = W^2[\psi_1,\psi_2,\psi_3],\end{aligned}$$

which follow from the definitions of M_1, M_2, M_3 and $W[\psi_1, \psi_2, \psi_3]$, we find

$$\begin{aligned}\hat{G}_2(k) = {} & \frac{\psi_1(0,k)}{\psi_2(0,k)}\hat{q}_1(k) + \frac{\psi_3(0,k)}{\psi_2(0,k)}\hat{q}_3(k) + \frac{3k^2\Delta(k)}{\psi_2(0,k)}\hat{g}_0(k) \\ & - e^{-ik^3T}\left[\frac{\psi_1(0,k)}{\psi_2(0,k)}\hat{q}_1(T,k) + \frac{\psi_3(0,k)}{\psi_2(0,k)}\hat{q}_3(T,k)\right], \quad k \in D_1.\end{aligned}$$

Under the transformation $k \mapsto \alpha^2 k$ we find equation (4·3).

Acknowledgements. This work was partially supported by the Department of Applied Mathematics and Theoretical Physics at the University of Cambridge, by Trinity College, Cambridge, and by the EPSRC.

https://doi.org/10.1017/S0305004107000084 Published online by Cambridge University Press

REFERENCES

[**1**] A. S. FOKAS. A unified transform method for solving linear and certain nonlinear PDEs. *Proc. R. Soc. Lond. A* **453** (1997), 1411–1443.
[**2**] A. S. FOKAS. Two-dimensional linear PDEs in a convex polygon. *Proc. R. Soc. Lond. A*. **457** (2001), 371–393.
[**3**] A. S. FOKAS. A new transform method for evolution equations on the half-line. *IMA J. Math.* **67** (2002), 559–590.
[**4**] A. S. FOKAS and B. PELLONI. A transform method for linear evolution PDEs on a finite interval. *IMA J. Appl. Math.* **70** (2005), 564–587.
[**5**] G. DASSIOS and A. S. FOKAS. The basic elliptic equations in an equilateral triangle. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **461** (2005), 2721–2748.
[**6**] P. A. TREHARNE and A. S. FOKAS. Boundary value problems for systems of linear evolution equations. *IMA J. Appl. Math.* **69** (2004), 539–555.
[**7**] A. S. FOKAS, A. R. ITS and L-Y. SUNG. The nonlinear Schrödinger equation on the half-line. *Nonlinearity* **18** (2005), 1771–1822.
[**8**] A. S. FOKAS. Integrable nonlinear evolution equations on the half-line. *Comm. Math. Phys.* **230** (2002), 1–39.
[**9**] A. S. FOKAS. The generalized Dirichlet-to-Neumann map for certain nonlinear evolution PDEs. *Comm. Pure Appl. Math.* **58** (2005), 639–670.
[**10**] A. S. FOKAS. Boundary Value Problems for Linear PDEs with Variable Coefficients. *Proc. R. Soc. Lond. A* **460** (2004), 1131–1151.
[**11**] R. BEALS, P. DEIFT and X. ZHOU. The inverse scattering transform on the line, in *Important Developments in Soliton Theory*, A. S. Fokas, V. E. Zakharov, eds, (Springer, Berlin, 1993), 7–37.
[**12**] R. BEALS, P. DEIFT and C. TOMEI. Direct and inverse scattering on the line. *Mathematical Surveys and Monographs* **28** (American Mathematical Society, Providence, RI, 1988).
[**13**] T. AKTOSUN and M. KLAUS. Inverse theory: problem on the line, in *Scattering*, E. R. Pike, P. C. Sabatier, eds, (Academic Press, 2001), 770–785.
[**14**] A. BOUTET DE MONVEL, A. S. FOKAS and D. SHEPELSKY. The mKdV equation on the half-line. *J. Inst. Math. Jussieu* **3** (2004), 139–164.
[**15**] M. J. ABLOWITZ and A. S. FOKAS. *Complex Variables and Applications* (Cambridge University Press, 2nd ed., 2003).
[**16**] P. DEIFT and E. TRUBOWITZ. Inverse scattering on the line. *Comm. Pure Appl. Math.* **23** (1979), 121–151.
[**17**] M. J. ABLOWITZ and S. DE LILLO. Parametric forcing, bound states and solutions of a nonlinear Schrödinger type equation. *Nonlinearity* **7** (1994), 1143–1153.
[**18**] A. S. FOKAS and L–Y. SUNG. Generalised Fourier Transforms, their nonlinearisation and the imaging of the brain. *Notices of the AMS Feature Article* **52** (2005), 1176–1190.

https://doi.org/10.1017/S0305004107000084 Published online by Cambridge University Press